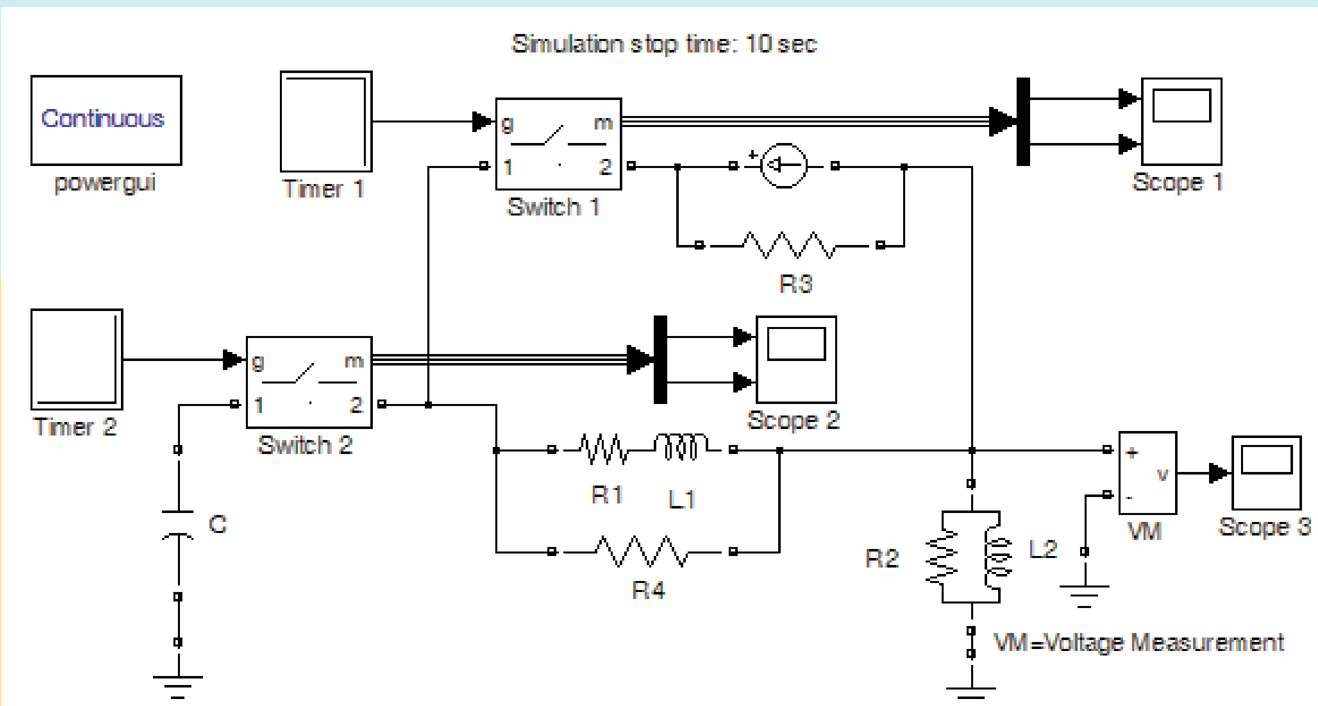
Circuit Analysis II

with MATLAB® Computing and Simulink®/SimPowerSystems® Modeling

Steven T. Karris



MATLAB® and Simulink® examples

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Preface

This text is written for use in a second course in circuit analysis. It encompasses a spectrum of subjects ranging from the most abstract to the most practical, and the material can be covered in one semester or two quarters.The reader of this book should have the traditional undergraduate knowledge of an introductory circuit analysis material such as *Circuit Analysis I with MATLAB® Computing and Simulink® / SimPowerSystems® Modeling, ISBN 978-1-934404-17-1*. Another prerequisite would be a basic knowledge of differential equations, and in most cases, engineering students at this level have taken all required mathematics courses. Appendix H serves as a review of differential equations with emphasis on engineering related topics and it is recommended for readers who may need a review of this subject.

There are several textbooks on the subject that have been used for years. The material of this book is not new, and this author claims no originality of its content. This book was written to fit the needs of the average student. Moreover, it is not restricted to computer oriented circuit analysis. While it is true that there is a great demand for electrical and computer engineers, especially in the internet field, the demand also exists for power engineers to work in electric utility companies, and facility engineers to work in the industrial areas.

Chapter 1 is an introduction to second order circuits and it is essentially a sequel to first order circuits discussed in the last chapter of *Circuit Analysis I with MATLAB® Computing and Simulink® / SimPowerSystems® Modeling, ISBN 978-1-934404-17-1*. Chapter 2 is devoted to resonance, and Chapter 3 presents practical methods of expressing signals in terms of the elementary functions, i.e., unit step, unit ramp, and unit impulse functions. Accordingly, any signal can be represented in the complex frequency domain using the Laplace transformation.

Chapters 4 and 5 are introductions to the unilateral Laplace transform and Inverse Laplace transform respectively, while Chapter 6 presents several examples of analyzing electric circuits using Laplace transformation methods. Chapter 7 is an introduction to state space and state equations. Chapter 8 begins with the frequency response concept and Bode magnitude and frequency plots. Chapter 9 is devoted to transformers with an introduction to self and mutual inductances. Chapter 10 is an introduction to one- and two-terminal devices and presents several practical examples. Chapters 11 and 12 are introductions to three-phase circuits.

It is not necessary that the reader has previous knowledge of MATLAB® . The material of this text can be learned without MATLAB. However, this author highly recommends that the reader studies this material in conjunction with the inexpensive MATLAB Student Version package that is available at most college and university bookstores. Appendix A of this text provides a practical introduction to MATLAB, Appendix B is an introduction to Simulink, and Appendix C introduces SimPowerSystems. The pages where MATLAB scripts, Simulink / SimPowerSystems models appear are indicated in the Table of Contents.

The author highly recommends that the reader studies this material in conjunction with the inexpensive Student Versions of The MathWorks™ Inc., the developers of these outstanding products, available from:

The MathWorks, Inc.
3 Apple Hill Drive
Natick, MA, 01760
Phone: 508-647-7000,
www.mathworks.com
info@mathworks.com.

Appendix D is a review of complex numbers, Appendix E is an introduction to matrices, Appendix F discusses scaling methods, Appendix G introduces the per unit system used extensively in power systems and in SimPwerSystems examples and demos. As stated above, Appendix H is a review of differential equations. Appendix I provides instructions for constructing semilog templates to be used with Bode plots.

In addition to numerous examples, this text contains several exercises at the end of each chapter. Detailed solutions of all exercises are provided at the end of each chapter. The rationale is to encourage the reader to solve all exercises and check his effort for correct solutions and appropriate steps in obtaining the correct solution. And since this text was written to serve as a self-study or supplementary textbook, it provides the reader with a resource to test his knowledge.

The author is indebted to several readers who have brought some errors to our attention. Additional feedback with other errors, advice, and comments will be most welcomed and greatly appreciated.

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Table of Contents

Chapter 1

Second Order Circuits

This chapter discusses the natural, forced and total responses in circuits that contain resistors, inductors and capacitors. These circuits are characterized by linear second–order differential equations whose solutions consist of the natural and the forced responses. We will consider both DC (constant) and AC (sinusoidal) excitations.

1.1 Response of a Second Order Circuit

A circuit that contains n energy storage devices (inductors and capacitors) is said to be an *nth–order circuit*, and the differential equation describing the circuit is an *nth–order differential equation*. For example, if a circuit contains an inductor and a capacitor, or two capacitors or two inductors, along with other devices such as resistors, it is said to be a second–order circuit and the differential equation that describes it will be a second order differential equation. It is possible, however, to describe a circuit having two energy storage devices with a set of two first–order differential equations, a circuit which has three energy storage devices with a set of three first–order differential equations and so on. These are called *state equations* and are discussed in Chapter 7.

As we know from previous studies,* the response is found from the differential equation describing the circuit, and its solution is obtained as follows:

1. We write the differential or integrodifferential (nodal or mesh) equation describing the circuit. We differentiate, if necessary, to eliminate the integral.
2. We obtain the forced (steady–state) response. Since the excitation in our work here will be either a constant (DC) or sinusoidal (AC) in nature, we expect the forced response to have the same form as the excitation. We evaluate the constants of the forced response by substitution of the assumed forced response into the differential equation and equate terms of the left side with the right side. The form of the forced response (particular solution), is described in Appendix H.
3. We obtain the general form of the natural response by setting the right side of the differential equation equal to zero, in other words, solve the homogeneous differential equation using the characteristic equation.
4. We add the forced and natural responses to form the complete response.
5. Using the initial conditions, we evaluate the constants from the complete response.

* *The natural and forced responses for first–order circuits are discussed in Circuit Analysis I with MATLAB® Computing and Simulink® / SimPowerSystems® Modeling, ISBN 978–1–934404–17–1.*

1.2 Series RLC Circuit with DC Excitation

Consider the circuit of Figure 1.1 where the initial conditions are $i_L(0) = I_0$, $v_C(0) = V_0$, and $u_0(t)$ is the unit step function.* We want to find an expression for the current $i(t)$ for $t > 0$.

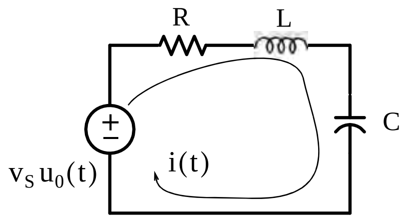


Figure 1.1. Series RLC Circuit

For this circuit

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int_0^t i\,dt + V_0 = v_S \quad t > 0 \tag{1.1}$$

and by differentiation

$$R\frac{di}{dt} + L\frac{d^2i}{dt^2} + \frac{i}{C} = \frac{dv_S}{dt}, \quad t > 0$$

To find the forced response, we must first specify the nature of the excitation v_S, that is DC or AC.

If v_S is DC ($v_S = \text{constant}$), the right side of (1.1) will be zero and thus the forced response component $i_f = 0$. If v_S is AC ($v_S = V\cos(\omega t + \theta)$, the right side of (1.1) will be another sinusoid and therefore $i_f = I\cos(\omega t + \varphi)$. Since in this section we are concerned with DC excitations, the right side will be zero and thus the total response will be just the natural response.

The natural response is found from the homogeneous equation of (1.1), that is,

$$R\frac{di}{dt} + L\frac{d^2i}{dt^2} + \frac{i}{C} = 0 \tag{1.2}$$

whose characteristic equation is

$$Ls^2 + Rs + \frac{1}{C} = 0$$

or

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

from which

* *The unit step function and other elementary functions used in science and engineering are discussed in Chapter 3.*

$$s_1, s_2 = -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} \tag{1.3}$$

We will use the following notations:

$$\underbrace{\alpha_S = \frac{R}{2L}}_{\substack{\alpha \text{ or Damping} \\ \text{Coefficient}}} \quad \underbrace{\omega_0 = \frac{1}{\sqrt{LC}}}_{\substack{\text{Resonant} \\ \text{Frequency}}} \quad \underbrace{\beta_S = \sqrt{\alpha_S{}^2 - \omega_0^2}}_{\substack{\text{Beta} \\ \text{Coefficient}}} \quad \underbrace{\omega_{nS} = \sqrt{\omega_0^2 - \alpha_S{}^2}}_{\substack{\text{Damped Natural} \\ \text{Frequency}}} \tag{1.4}$$

where the subscript s stands for series circuit. Then, we can express (1.3) as

$$s_1, s_2 = -\alpha_S \pm \sqrt{\alpha_S^2 - \omega_0^2} = -\alpha_S \pm \beta_S \quad \text{if} \quad \alpha_S^2 > \omega_0^2 \tag{1.5}$$

or

$$s_1, s_2 = -\alpha_S \pm \sqrt{\omega_0^2 - \alpha_S^2} = -\alpha_S \pm \omega_{nS} \quad \text{if} \quad \omega_0^2 > \alpha_S^2 \tag{1.6}$$

Case I: If $\alpha_S^2 > \omega_0^2$, the roots s_1 and s_2 are real, negative, and unequal. This results in the *over-damped* natural response and has the form

$$i_n(t) = k_1 e^{s_1 t} + k_2 e^{s_2 t} \tag{1.7}$$

Case II: If $\alpha_S^2 = \omega_0^2$, the roots s_1 and s_2 are real, negative, and equal. This results in the *critically damped* natural response and has the form

$$i_n(t) = Ae^{-\alpha_S t}(k_1 + k_2 t) \tag{1.8}$$

Case III: If $\omega_0^2 > \alpha_S^2$, the roots s_1 and s_2 are complex conjugates. This is known as the *under-damped or oscillatory* natural response and has the form

$$i_n(t) = e^{-\alpha_S t}(k_1 \cos\omega_{nS} t + k_2 \sin\omega_{nS} t) = k_3 e^{-\alpha_S t}(\cos\omega_{nS} t + \varphi) \tag{1.9}$$

Typical overdamped, critically damped and underdamped responses are shown in Figure 1.2, 1.3, and 1.4 respectively where it is assumed that $i_n(0) = 0$.

1.2.1 Response of Series RLC Circuits with DC Excitation

Depending on the circuit constants R, L, and C, the total response of a series RLC circuit which is excited by a DC source, may be overdamped, critically damped or underdamped. In this section we will derive the total response of series RLC circuits that are excited by DC sources.

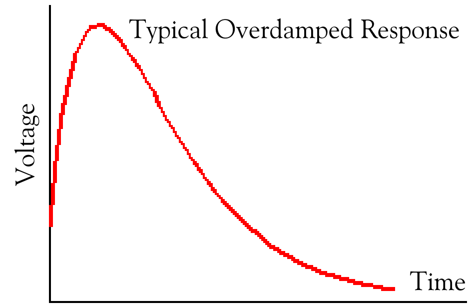


Figure 1.2. Typical overdamped response

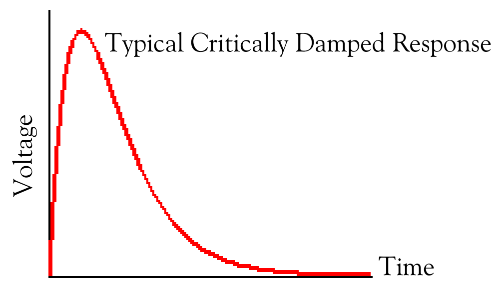


Figure 1.3. Typical critically damped response

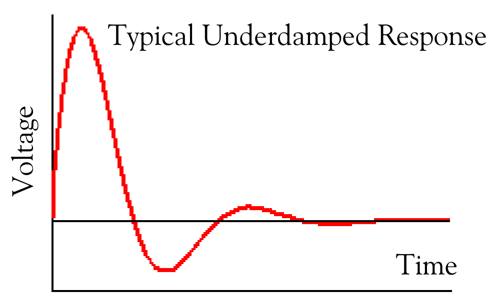


Figure 1.4. Typical underdamped (oscillatory) response

Example 1.1

For the circuit of Figure 1.5, $i_L(0) = 5\ A$, $v_C(0) = 2.5\ V$, and the $0.5\ \Omega$ resistor represents the resistance of the inductor. Compute and sketch $i(t)$ for $t > 0$.

Solution:

This circuit can be represented by the integrodifferential equation

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int_0^t i\,dt + v_C(0) = 15, \quad t > 0 \tag{1.10}$$

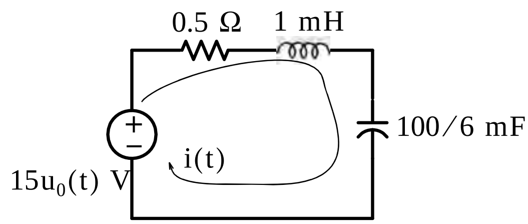


Figure 1.5. Circuit for Example 1.1

Differentiating and noting that the derivatives of the constants $v_C(0)$ and 15 are zero, we obtain the homogeneous differential equation

$$R\frac{di}{dt}+L\frac{d^2i}{dt^2}+\frac{i}{C}=0$$

or

$$\frac{d^2i}{dt^2}+\frac{R}{L}\frac{di}{dt}+\frac{i}{LC}=0$$

and by substitution of the known values R, L, and C

$$\frac{d^2i}{dt^2}+500\frac{di}{dt}+60000i=0 \tag{1.11}$$

The roots of the characteristic equation of (1.11) are $s_1=-200$ and $s_2=-300$. The total response is just the natural response and for this example it is overdamped. Therefore, from (1.7),

$$i(t)=i_n(t)=k_1e^{s_1t}+k_2e^{s_2t}=k_1e^{-200t}+k_2e^{-300t} \tag{1.12}$$

The constants k_1 and k_2 can be evaluated from the initial conditions. Thus from the first initial condition $i_L(0)=i(0)=5\text{ A}$ and (1.12) we obtain

$$i(0)=k_1e^0+k_2e^0=5$$

or

$$k_1+k_2=5 \tag{1.13}$$

We need another equation in order to compute the values of k_1 and k_2. This equation will make use of the second initial condition, that is, $v_C(0)=2.5\text{ V}$. Since $i_C(t)=i(t)=C\frac{dv_C}{dt}$, we differentiate (1.12), we evaluate it at $t=0^+$, and we equate it with this initial condition. Then,

$$\frac{di}{dt}=-200k_1e^{-200t}-300k_2e^{-300t} \quad \text{and} \quad \left.\frac{di}{dt}\right|_{t=0^+}=-200k_1-300k_2 \tag{1.14}$$

Also, at $t=0^+$,

$$Ri(0^+) + L\frac{di}{dt}\Big|_{t=0^+} + v_c(0^+) = 15$$

and solving for $\frac{di}{dt}\Big|_{t=0^+}$ we obtain

$$\frac{di}{dt}\Big|_{t=0^+} = \frac{15-0.5\times5-2.5}{10^{-3}} = 10000 \qquad (1.15)$$

Next, equating (1.14) with (1.15) we obtain:

$$-200k_1 - 300k_2 = 10000$$

$$-k_1 - 1.5k_2 = 50 \qquad (1.16)$$

Simultaneous solution of (1.13) and (1.16) yields $k_1 = 115$ and $k_2 = -110$. By substitution into (1.12) we find the total response as

$$i(t) = i_n(t) = 115e^{-200t} - 110e^{-300t} \qquad (1.17)$$

Check with MATLAB*:

```
syms t;                                    % Define symbolic variable t
                                           % Must have Symbolic Math Toolbox installed
R=0.5; L=10^(–3); C=100*10^(–3)/6;         % Circuit constants
y0=115*exp(–200*t)–110*exp(–300*t);        % Let solution i(t)=y0
y1=diff(y0);                               % Compute the first derivative of y0, i.e., di/dt
y2=diff(y0,2);                             % Compute the second derivative of y0, i.e, di2/dt2
                                           % Substitute the solution i(t), i.e., equ (1.17)
                                           % into differential equation of (1.11) to verify that
                                           % correct solution was obtained. We must also
                                           % verify that the initial conditions are satisfied.
y=y2+500*y1+60000*y0;
i0=115*exp(–200*0)–110*exp(–300*0);
vC0=–R*i0–L*(–23000*exp(–200*0)+33000*exp(–300*0))+15;
fprintf(' \n');...
disp('Solution was entered as y0 = '); disp(y0);...
disp('1st derivative of solution is y1 = '); disp(y1);...
disp('2nd derivative of solution is y2 = '); disp(y2);...
disp('Differential equation is satisfied since y = y2+y1+y0 = '); disp(y);...
disp('1st initial condition is satisfied since at t = 0, i0 = '); disp(i0);...
disp('2nd initial condition is also satisfied since vC+vL+vR=15 and vC0 = ');...
disp(vC0);...
fprintf(' \n')
```

* *An introduction to MATLAB is presented in Appendix A.*

```
Solution was entered as y0 =
115*exp(-200*t)-110*exp(-300*t)
1st derivative of solution is y1 =
-23000*exp(-200*t)+33000*exp(-300*t)
2nd derivative of solution is y2 =
4600000*exp(-200*t)-9900000*exp(-300*t)
Differential equation is satisfied since y = y2+y1+y0 = 0
1st initial condition is satisfied since at t = 0, i0 = 5
2nd initial condition is also satisfied since vC+vL+vR=15 and vC0
= 2.5000
```

We denote the first term as $i_1(t) = 115e^{-200t}$, the second term as $i_2(t) = 110e^{-300t}$, and the total current $i(t)$ as the difference of these two terms. The response is shown in Figure 1.6.

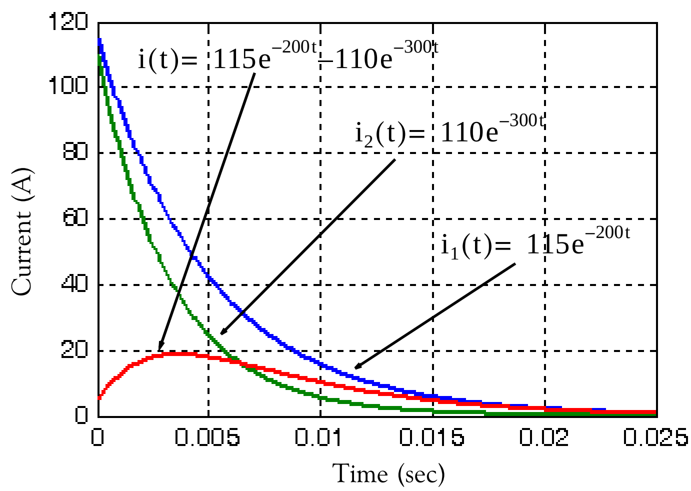


Figure 1.6. Plot for $i(t)$ *of Example 1.1*

In the above example, differentiation eliminated (set equal to zero) the right side of the differential equation and thus the total response was just the natural response. A different approach however, may not set the right side equal to zero, and therefore the total response will contain both the natural and forced components. To illustrate, we will use the following approach.

The capacitor voltage, for all time *t*, may be expressed as $v_C(t) = \frac{1}{C}\int_{-\infty}^{t} i\,dt$ and as before, the circuit can be represented by the integrodifferential equation

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{t} i\,dt = 15u_0(t) \tag{1.18}$$

and since

$$i = i_C = C\frac{dv_C}{dt}$$

we rewrite (1.18) as

$$RC\frac{dv_C}{dt} + LC\frac{dv_C^2}{dt^2} + v_C = 15u_0(t) \qquad (1.19)$$

We observe that this is a non–homogeneous differential equation whose solution will have both the natural and the forced response components. Of course, the solution of (1.19) will give us the capacitor voltage $v_C(t)$. This presents no problem since we can obtain the current by differentiation of the expression for $v_C(t)$.

Substitution of the given values into (1.19) yields

$$\frac{50}{6} \times 10^{-3}\frac{dv_C}{dt} + 1 \times 10^{-3} \times \frac{100}{6}10^{-3}\frac{dv_C^2}{dt^2} + v_C = 15u_0(t)$$

or

$$\frac{dv_C^2}{dt^2} + 500\frac{dv_C}{dt} + 60000v_C = 9 \times 10^5 u_0(t) \qquad (1.20)$$

The characteristic equation of (1.20) is the same as of that of (1.11) and thus the natural response is

$$v_{Cn}(t) = k_1e^{s_1t} + k_2e^{s_2t} = k_1e^{-200t} + k_2e^{-300t} \qquad (1.21)$$

Since the right side of (1.20) is a constant, the forced response will also be a constant and we denote it as $v_{Cf} = k_3$. By substitution into (1.20) we obtain

$$0 + 0 + 60000k_3 = 900000$$

or

$$v_{Cf} = k_3 = 15 \qquad (1.22)$$

The total solution then is the summation of (1.21) and (1.22), that is,

$$v_C(t) = v_{Cn}(t) + v_{Cf} = k_1e^{-200t} + k_2e^{-300t} + 15 \qquad (1.23)$$

As before, the constants k_1 and k_2 will be evaluated from the initial conditions. First, using $v_C(0) = 2.5\ V$ and evaluating (1.23) at $t = 0$, we obtain

$$v_C(0) = k_1e^0 + k_2e^0 + 15 = 2.5$$

or

$$k_1 + k_2 = -12.5 \qquad (1.24)$$

Also,

$$i_L = i_C = C\frac{dv_C}{dt}, \quad \frac{dv_C}{dt} = \frac{i_L}{C} \quad \text{and} \quad \left.\frac{dv_C}{dt}\right|_{t=0} = \frac{i_L(0)}{C} = \frac{5}{100/6 \times 10^{-3}} = 300 \qquad (1.25)$$

Next, we differentiate (1.23), we evaluate it at $t = 0$ and equate it with (1.25). Then,

$$\frac{dv_C}{dt} = -200k_1e^{-200t} - 300k_2e^{-300t} \quad \text{and} \quad \left.\frac{dv_C}{dt}\right|_{t=0} = -200k_1 - 300k_2 \tag{1.26}$$

Equating the right sides of (1.25) and (1.26) we obtain

$$-200k_1 - 300k_2 = 300$$

or

$$-k_1 - 1.5k_2 = 1.5 \tag{1.27}$$

From (1.24) and (1.27), we obtain $k_1 = -34.5$ and $k_2 = 22$. By substitution into (1.23), we obtain the total solution as

$$v_C(t) = (22e^{-300t} - 34.5e^{-200t} + 15)u_0(t) \tag{1.28}$$

Check with MATLAB:

```
syms t   % Define symbolic variable t. Must have Symbolic Math Toolbox installed
y0=22*exp(–300*t)–34.5*exp(–200*t)+15;    % The total solution y(t)
y1=diff(y0)                               % The first derivative of y(t)
```

```
y1 = -6600*exp(-300*t)+6900*exp(-200*t)
```

```
y2=diff(y0,2)                             % The second derivative of y(t)
```

```
y2 = 1980000*exp(-300*t)-1380000*exp(-200*t)
```

```
y=y2+500*y1+60000*y0                      % Summation of y and its derivatives
```

```
y = 900000
```

Using the expression for $v_C(t)$ we can find the current as

$$i = i_L = i_C = C\frac{dv_C}{dt} = \frac{100}{6} \times 10^{-3}(6900e^{-200t} - 6600e^{-300t}) = 115e^{-200t} - 110e^{-300t} \text{ A} \tag{1.29}$$

We observe that (1.29) is the same as (1.17). The plot for (1.28) is shown in Figure 1.7.

The same results are obtained with the Simulink/SimPowerSystems* model shown in Figure 1.8.

The waveforms for the current and the voltage across the capacitor are shown in Figure 1.9.

* *For an introduction to Simulink SimPowerSystems please refer to Appendices B and C respectively.*

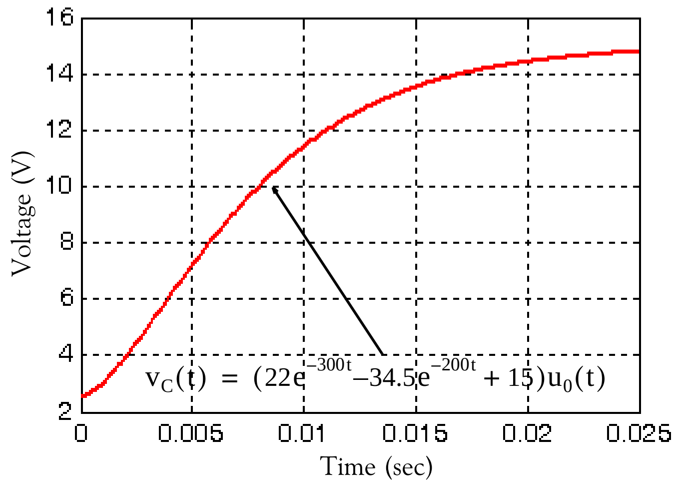


Figure 1.7. Plot for $v_C(t)$ *of Example 1.1*

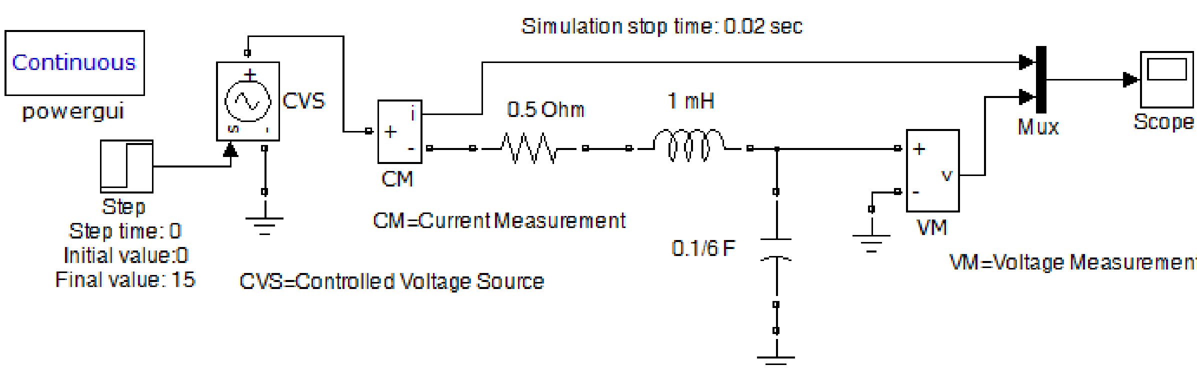


Figure 1.8. Simulink/SimPowerSystems model for the circuit in Figure 1.5

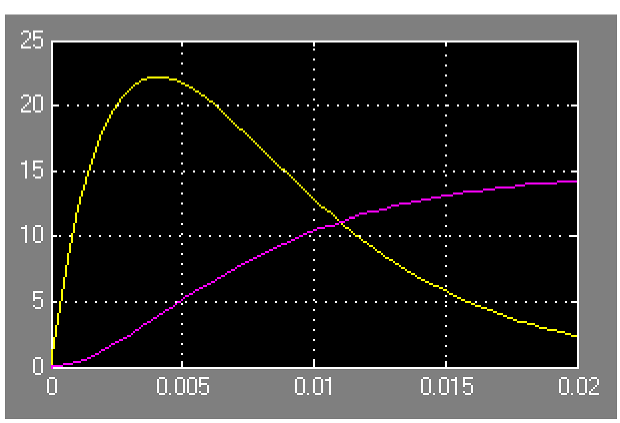


Figure 1.9. Waveforms produced by the Simulink/SimPowerSystems model in Figure 1.8

1.2.2 Response of Series RLC Circuits with AC Excitation

The total response of a series RLC circuit, which is excited by a sinusoidal source, will also consist of the natural and forced response components. As we found in the previous section, the natural response can be overdamped, or critically damped, or underdamped. The forced component will be a sinusoid of the same frequency as that of the excitation, and since it represents the AC steady–state condition, we can use phasor analysis to find it. The following example illustrates the procedure.

Example 1.2

For the circuit in Figure 1.10, $i_L(0) = 5\ A$, $v_C(0) = 2.5\ V$, and the $0.5\ \Omega$ resistor represents the resistance of the inductor. Compute and sketch $i(t)$ for $t > 0$.

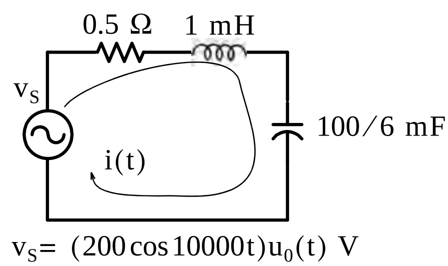


Figure 1.10. Circuit for Example 1.2

Solution:

This circuit is the same as that in Example 1.1 except that the circuit is excited by a sinusoidal source; therefore it can be represented by the integrodifferential equation

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int_0^t i\,dt + v_C(0) = 200\cos 10000t \quad t > 0 \tag{1.30}$$

whose solution consists of the summation of the natural and forced responses. We know its natural response from the previous example. We begin with

$$i(t) = i_n(t) + i_f(t) = k_1 e^{-200t} + k_2 e^{-300t} + i_f(t) \tag{1.31}$$

where the constants k_1 and k_2 will be evaluated from the initial conditions after $i_f(t)$ has been found. The steady state (or forced) response will have the form $i_f(t) = k_3\cos(10,000t + \theta)$ in the time domain $(t - \text{domain})$ and the form $k_3\angle\theta$ in the frequency domain $(j\omega - \text{domain})$.

To find $i_f(t)$ we will use the phasor analysis relation $\mathbf{I} = \mathbf{V}/Z$ where $\mathbf{I}$ is the phasor current, $\mathbf{V}$ is the phasor voltage, and Z is the impedance of the phasor circuit which, as we know, is

$$Z = R + j\left(\omega L - \frac{1}{\omega C}\right) = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}\angle \tan^{-1}\left(\omega L - \frac{1}{\omega C}\right)/R \tag{1.32}$$

The inductive and capacitive reactances are

$$X_L = \omega L = 10^4 \times 10^{-3} = 10 \ \ \Omega$$

and

$$X_C = \frac{1}{\omega C} = \frac{1}{10^4 \times (100/6)10^{-3}} = 6 \times 10^{-3} \ \ \Omega$$

Then,

$$R^2 = (0.5)^2 = 0.25 \ \ \text{and} \ \ \left(\omega L - \frac{1}{\omega C}\right)^2 = (10 - 6 \times 10^{-3})^2 = 99.88$$

Also,

$$\tan^{-1}\left(\omega L - \frac{1}{\omega C}\right)/R = \tan^{-1}\frac{(10 - 6 \times 10^{-3})}{0.5} = \tan^{-1}\left(\frac{9.994}{0.5}\right)$$

and this yields $\theta = 1.52 \text{ rads} = 87.15°$. Then, by substitution into (1.32),

$$Z = \sqrt{0.25 + 99.88}\angle\theta^{\circ} = 10\angle 87.15^{\circ}$$

and thus

$$I = \frac{V}{Z} = \frac{200\angle 0^{\circ}}{10\angle 87.15^{\circ}} = 20\angle -87.15^{\circ} \Leftrightarrow 20\cos(10000t - 87.15^{\circ}) = i_f(t)$$

The total solution is

$$i(t) = i_n(t) + i_f(t) = k_1 e^{-200t} + k_2 e^{-300t} + 20\cos(10000t - 87.15^{\circ}) \tag{1.33}$$

As before, the constants k_1 and k_2 are evaluated from the initial conditions. From (1.33) and the first initial condition $i_L(0) = 5 \text{ A}$ we obtain

$$i(0) = k_1 e^0 + k_2 e^0 + 20\cos(-87.15^{\circ}) = 5$$

or

$$i(0) = k_1 + k_2 + 20 \times 0.05 = 5$$

or

$$k_1 + k_2 = 4 \tag{1.34}$$

We need another equation in order to compute the values of k_1 and k_2. This equation will make use of the second initial condition, that is, $v_C(0) = 2.5 \text{ V}$. Since $i_C(t) = i(t) = C\frac{dv_C}{dt}$, we differentiate (1.33), we evaluate it at $t = 0$, and we equate it with this initial condition. Then,

$$\frac{di}{dt} = -200k_1 e^{-200t} - 300k_2 e^{-300t} - 2 \times 10^5 \sin(10000t - 87.15^{\circ}) \tag{1.35}$$

and at $t = 0$,

$$\left.\frac{di}{dt}\right|_{t=0} = -200k_1 - 300k_2 - 2\times10^6\sin(-87.15°) = -200k_1 - 300k_2 + 2\times10^5 \qquad (1.36)$$

Also, at $t = 0^+$

$$Ri(0^+) + L\left.\frac{di}{dt}\right|_{t=0^+} + v_c(0^+) = 200\cos(0) = 200$$

and solving for $\left.\frac{di}{dt}\right|_{t=0^+}$ we obtain

$$\left.\frac{di}{dt}\right|_{t=0^+} = \frac{200 - 0.5\times5 - 2.5}{10^{-3}} = 195000 \qquad (1.37)$$

Next, equating (1.36) with (1.37) we obtain

$$-200k_1 - 300k_2 = -5000$$

or

$$k_1 + 1.5k_2 = 25 \qquad (1.38)$$

Simultaneous solution of (1.34) and (1.38) yields $k_1 = -38$ and $k_2 = 42$. Then, by substitution into (1.31), the total response is

$$i(t) = -38e^{-200t} + 42e^{-300t} + 20\cos(10000t - 87.15°)\ A \qquad (1.39)$$

The plot is shown in Figure 1.11 and it was created with the following MATLAB script:

```
t=0:0.005:0.25; t1=–38.*exp(–200.*t); t2=42.*exp(–300.*t); t3=20.*cos(10000.*t–87.5*pi/180);
x=t1+t2+t3; plot(t,t1,t,t2,t,t3,t,x); grid
```

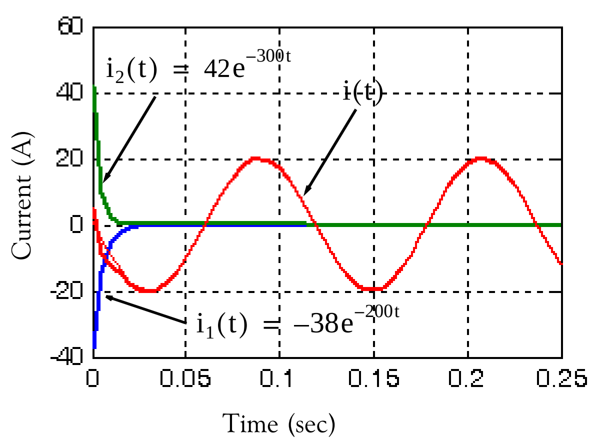


Figure 1.11. Plot for $i(t)$ of Example 1.2

The same results are obtained with the Simulink/SimPowerSystems model shown in Figure 1.12.

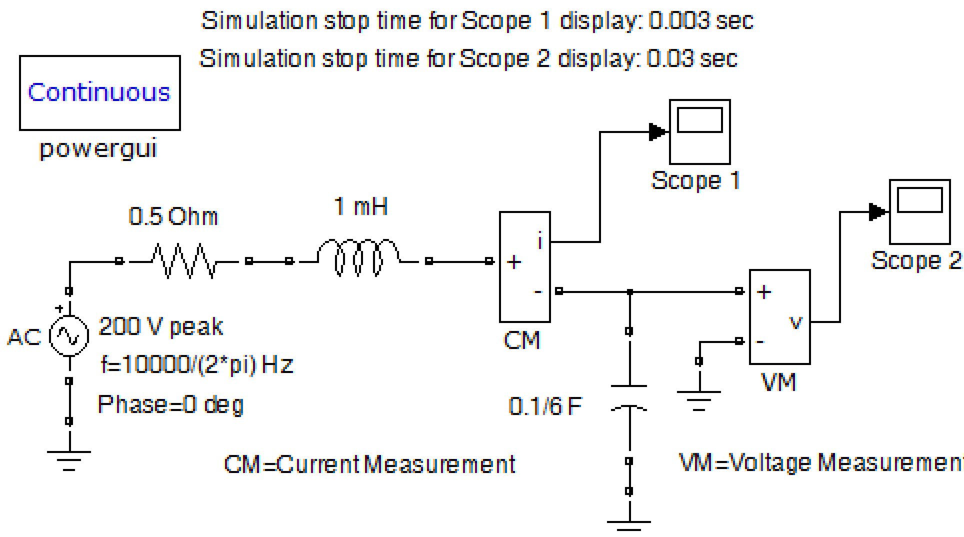


Figure 1.12. Simulink/SimPowerSystems model for the circuit in Figure 1.10

The waveforms for the current and the voltage across the capacitor are shown in Figures 1.13 and 1.14 respectively. We observe that the steady-state current is consistent with the waveform shown in Figure 1.11, and the steady state voltage across the capacitor is small since the magnitude of the capacitive reactance is $X_C = 6 \times 10^{-3}\ \Omega$.

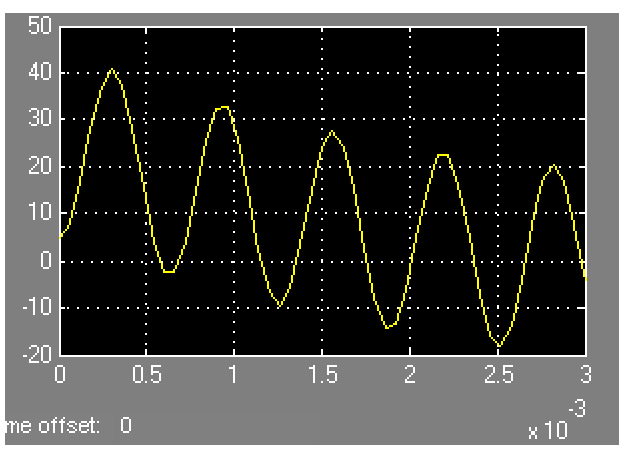


Figure 1.13. Waveform displayed in Scope 1 for the Simulink/SimPowerSystems model in Figure 1.12

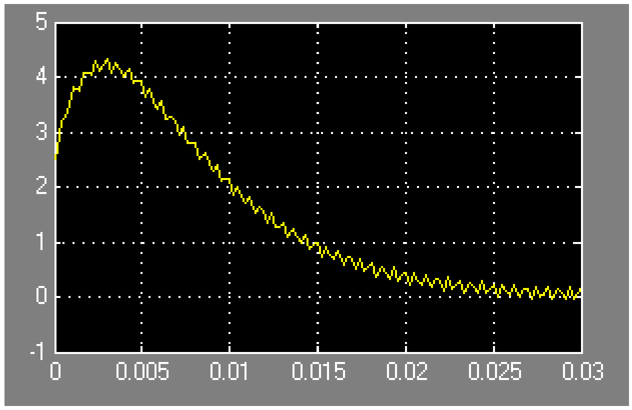


Figure 1.14. Waveform displayed in Scope 2 for the Simulink/SimPowerSystems model in Figure 1.12

1.3 Parallel RLC Circuit

Consider the circuit of Figure 1.10 where the initial conditions are $i_L(0) = I_0$, $v_C(0) = V_0$, and $u_0(t)$ is the unit step function. We want to find an expression for the voltage $v(t)$ for $t > 0$.

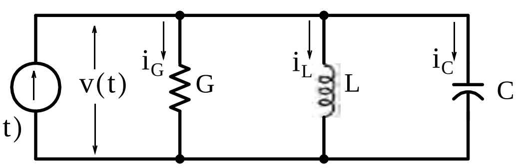


Figure 1.15. Parallel RLC circuit

For this circuit

$$i_G(t) + i_L(t) + i_C(t) = i_S(t)$$

or

$$Gv + \frac{1}{L}\int_0^t v\,dt + I_0 + C\frac{dv}{dt} = i_S \qquad t > 0$$

By differentiation,

$$C\frac{dv^2}{dt^2} + G\frac{dv}{dt} + \frac{v}{L} = \frac{di_S}{dt} \qquad t > 0 \tag{1.40}$$

To find the forced response, we must first specify the nature of the excitation i_S, that is DC or AC.

If i_S is DC ($v_S = constant$), the right side of (1.40) will be zero and thus the forced response component $v_f = 0$. If i_S is AC ($i_S = I\cos(\omega t + \theta)$, the right side of (1.40) will be another sinusoid and therefore $v_f = V\cos(\omega t + \varphi)$. Since in this section we are concerned with DC excitations, the right side will be zero and thus the total response will be just the natural response.

The natural response is found from the homogeneous equation of (1.40), that is,

$$C\frac{dv^2}{dt^2} + G\frac{dv}{dt} + \frac{v}{L} = 0 \tag{1.41}$$

whose characteristic equation is

$$Cs^2 + Gs + \frac{1}{L} = 0$$

or

$$s^2 + \frac{G}{C}s + \frac{i}{LC} = 0$$

from which

$$s_1, s_2 = -\frac{G}{2C} \pm \sqrt{\frac{G^2}{4C^2} - \frac{1}{LC}} \tag{1.42}$$

and with the following notations,

$$\underbrace{\alpha_P = \frac{G}{2C}}_{\alpha \text{ or Damping Coefficient}} \qquad \underbrace{\omega_0 = \frac{1}{\sqrt{LC}}}_{\text{Resonant Frequency}} \qquad \underbrace{\beta_P = \sqrt{\alpha_P{}^2 - \omega_0^2}}_{\text{Beta Coefficient}} \qquad \underbrace{\omega_{nP} = \sqrt{\omega_0^2 - \alpha_P{}^2}}_{\text{Damped Natural Frequency}} \tag{1.43}$$

where the subscript p stands for parallel circuit, we can express (1.42) as

$$s_1, s_2 = -\alpha_P \pm \sqrt{\alpha_P^2 - \omega_0^2} = -\alpha_P \pm \beta_P \quad \text{if} \quad \alpha_P^2 > \omega_0^2 \tag{1.44}$$

or

$$s_1, s_2 = -\alpha_P \pm \sqrt{\omega_0^2 - \alpha_P^2} = -\alpha_P \pm \omega_{nP} \quad \text{if} \quad \omega_0^2 > \alpha_P^2 \tag{1.45}$$

Note: From (1.4), Page 1−3, and (1.43), Page 1−14, we observe that $\alpha_S \neq \alpha_P$

As in the series circuit, the natural response $v_n(t)$ can be overdamped, critically damped, or underdamped.

Case I: If $\alpha_P^2 > \omega_0^2$, the roots s_1 and s_2 are real, negative, and unequal. This results in the overdamped natural response and has the form

$$v_n(t) = k_1 e^{s_1 t} + k_2 e^{s_2 t} \tag{1.46}$$

Case II: If $\alpha_P^2 = \omega_0^2$, the roots s_1 and s_2 are real, negative, and equal. This results in the critically damped natural response and has the form

$$v_n(t) = e^{-\alpha_P t}(k_1 + k_2 t) \tag{1.47}$$

Case III: If $\omega_0^2 > \alpha_P^2$, the roots s_1 and s_2 are complex conjugates. This results in the underdamped or oscillatory natural response and has the form

$$v_n(t) = e^{-\alpha_P t}(k_1 \cos\omega_{nP} t + k_2 \sin\omega_{nP} t) = k_3 e^{-\alpha_P t}(\cos\omega_{nP} t + \varphi) \qquad (1.48)$$

1.3.1 Response of Parallel RLC Circuits with DC Excitation

Depending on the circuit constants G (or R), L, and C, the natural response of a parallel RLC circuit may be overdamped, critically damped or underdamped. In this section we will derive the total response of a parallel RLC circuit which is excited by a DC source for the example which follows.

Example 1.3

For the circuit of Figure 1.16, $i_L(0) = 2\ A$ and $v_C(0) = 5\ V$. Compute and sketch $v(t)$ for $t > 0$.

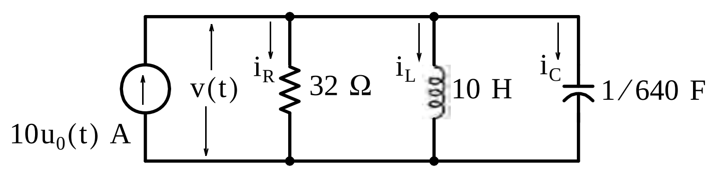


Figure 1.16. Circuit for Example 1.3

Solution:

We could write the integrodifferential equation that describes the given circuit, differentiate, and find the roots of the characteristic equation from the homogeneous differential equation as we did in the previous section. However, we will skip these steps and begin with

$$v(t) = v_f(t) + v_n(t) \qquad (1.49)$$

and when steady–state conditions have been reached, we will have $v = v_L = L\frac{di}{dt} = 0$, $v_f = 0$ and $v(t) = v_n(t)$.

To find out whether the natural response is overdamped, critically damped, or oscillatory, we need to compute the values of α_P and ω_0 using (1.43) and the values of s_1 and s_2 using (1.44) or (1.45). Then we will use (1.46), or (1.47), or (1.48) as appropriate. For this example,

$$\alpha_P = \frac{G}{2C} = \frac{1}{2RC} = \frac{1}{2 \times 32 \times 1/640} = 10$$

or

$$\alpha_P^2 = 100$$

and

$$\omega_0^2 = \frac{1}{LC} = \frac{1}{10 \times 1/640} = 64$$

Then

$$s_1, s_2 = -\alpha_P \pm \sqrt{\alpha_P^2 - \omega_0^2} = -10 \pm 6$$

or $s_1 = -4$ and $s_2 = -16$. Therefore, the natural response is overdamped and from (1.46) we obtain

$$v(t)= v_n(t) = k_1 e^{s_1 t} + k_2 e^{s_2 t} = k_1 e^{-4t} + k_2 e^{-16t} \tag{1.50}$$

and the constants k_1 and k_2 will be evaluated from the initial conditions.

With the initial condition $v_C(0) = v(0) = 5\ V$ and (1.50) we obtain

$$v(0) = k_1 e^0 + k_2 e^0 = 5$$

or

$$k_1 + k_2 = 5 \tag{1.51}$$

The second equation that is needed for the computation of the values of k_1 and k_2 is found from the other initial condition, that is, $i_L(0) = 2\ A$. Since $i_C(t) = C\dfrac{dv_C}{dt} = C\dfrac{dv}{dt}$, we differentiate (1.50), we evaluate it at $t = 0^+$, and we equate it with this initial condition.Then,

$$\frac{dv}{dt} = -4k_1 e^{-4t} - 16k_2 e^{-16t} \quad \text{and} \quad \left.\frac{dv}{dt}\right|_{t=0^+} = -4k_1 - 16k_2 \tag{1.52}$$

Also, at $t = 0^+$

$$\frac{1}{R}v(0^+) + i_L(0^+) + C\left.\frac{dv}{dt}\right|_{t=0^+} = 10$$

and solving for $\left.\dfrac{dv}{dt}\right|_{t=0^+}$ we obtain

$$\left.\frac{dv}{dt}\right|_{t=0^+} = \frac{10 - 5/32 - 2}{1/640} = 502 \tag{1.53}$$

Next, equating (1.52) with (1.53) we obtain

$$-4k_1 - 16k_2 = 502$$

or

$$-2k_1 - 8k_2 = 251 \tag{1.54}$$

Simultaneous solution of (1.51) and (1.54) yields $k_1 = 291/6$, $k_2 = -261/6$, and by substitution into (1.50) we obtain the total response as

$$v(t)= v_n(t) = \frac{291}{6}e^{-4t} - \frac{261}{6}e^{-16t} = 48.5e^{-4t} - 43.5e^{-16t}\ V \tag{1.55}$$

Check with MATLAB:

```
syms t    %  Define symbolic variable t. Must have Symbolic Math Toolbox installed
y0=291*exp(–4*t)/6–261*exp(–16*t)/6;          %  Let solution v(t) = y0
```

```
y1=diff(y0)                         % Compute and display first derivative
y1 = -194*exp(-4*t)+696*exp(-16*t)
y2=diff(y0,2)                       % Compute and display second derivative
y2 = 776*exp(-4*t)-11136*exp(-16*t)
y=y2/640+y1/32+y0/10                % Verify that (1.40) is satisfied
y = 0
```

The plot is shown in Figure 1.17.

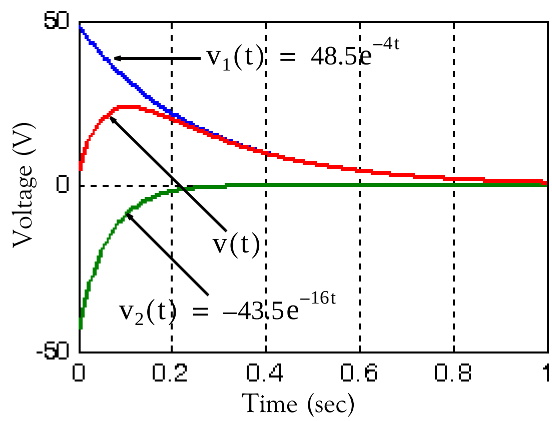


Figure 1.17. Plot for $v(t)$ *of Example 1.3*

From the plot of Figure 1.17, we observe that $v(t)$ attains its maximum value somewhere in the interval 0.10 and 0.12 sec., and the maximum voltage is approximately 24 V. If we desire to compute precisely the maximum voltage and the exact time it occurs, we can compute the derivative of (1.55), set it equal to zero, and solve for t. Thus,

$$\left.\frac{dv}{dt}\right|_{t=0} = -1164e^{-4t} + 4176e^{-16t} = 0 \tag{1.56}$$

Division of (1.56) by e^{-16t} yields

$$-1164e^{12t} + 4176 = 0$$

or

$$e^{12t} = \frac{348}{97}$$

or

$$12t = \ln\left(\frac{348}{97}\right) = 1.2775$$

and

$$t = t_{max} = \frac{1.2775}{12} = 0.106\ s$$

By substitution into (1.55)

$$v_{max} = 48.5e^{-4x0.106} - 43.5e^{-16x0.106} = 23.76\ V \tag{1.57}$$

A useful quantity, especially in electronic circuit analysis, is the *settling time*, denoted as t_S, and it is defined as the time required for the voltage to drop to 1% of its maximum value. Therefore, t_S is an indication of the time it takes for $v(t)$ to damp–out, meaning to decrease the amplitude of $v(t)$ to approximately zero. For this example, $0.01 \times 23.76 = 0.2376\ V$, and we can find t_S by substitution into (1.55). Then,

$$0.01v_{max} = 0.2376 = 48.5e^{-4t} - 43.5e^{-16t} \tag{1.58}$$

and we need to solve for the time t. To simplify the computation, we neglect the second term on the right side of (1.58) since this component of the voltage damps out much faster than the other component. This expression then simplifies to

$$0.2376 = 48.5e^{-4t_s}$$

or

$$-4t_S = \ln(0.005) = (-5.32)$$

or

$$t_S = 1.33\ s \tag{1.59}$$

Example 1.4

For the circuit of Figure 1.18, $i_L(0) = 2\ A$ and $v_C(0) = 5\ V$, and the resistor is to be adjusted so that the natural response will be critically damped.Compute and sketch $v(t)$ for $t > 0$.

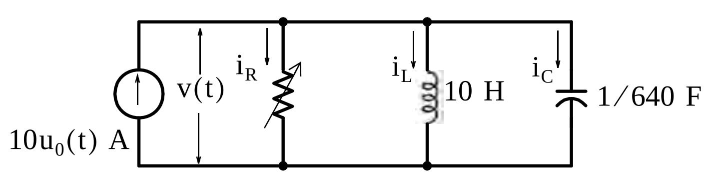


Figure 1.18. Circuit for Example 1.4

Solution:

Since the natural response is to be critically damped, we must have $\omega_0^2 = 64$ because the L and C values are the same as in the previous example. Please refer to (1.43), Page 1–16. We must also have

$$\alpha_P = \frac{G}{2C} = \frac{1}{2RC} = \omega_0 = \sqrt{\frac{1}{LC}} = 8$$

or

$$\frac{1}{R} = 8 \times \frac{2}{640} = \frac{1}{40}$$

or $R = 40\ \Omega$ and thus $s_1 = s_2 = -\alpha_P = -8$. The natural response will have the form

$$v(t) = v_n(t) = e^{-\alpha_P t}(k_1 + k_2 t) \text{ or } v(t) = v_n(t) = e^{-8t}(k_1 + k_2 t) \qquad (1.60)$$

Using the initial condition $v_C(0) = 5\ V$, and evaluating (1.60) at $t = 0$, we obtain

$$v(0) = e^0(k_1 + k_2 0) = 5$$

or

$$k_1 = 5 \qquad (1.61)$$

and (1.60) simplifies to

$$v(t) = e^{-8t}(5 + k_2 t) \qquad (1.62)$$

As before, we need to compute the derivative dv/dt in order to apply the second initial condition and find the value of the constant k_2.

We obtain the derivative using MATLAB as follows:

```
syms t k2; v0=exp(–8*t)*(5+k2*t); v1=diff(v0);      % v1 is 1st derivative of v0
                                                     % Must have Symbolic Math Toolbox installed

v1 = -8*exp(-8*t)*(5+k2*t)+exp(-8*t)*k2
```

Thus,

$$\frac{dv}{dt} = -8e^{-8t}(5 + k_2 t) + k_2 e^{-8t}$$

and

$$\left.\frac{dv}{dt}\right|_{t=0} = -40 + k_2 \qquad (1.63)$$

Also, $i_C = C\frac{dv}{dt}$ or $\frac{dv}{dt} = \frac{i_C}{C}$ and

$$\left.\frac{dv}{dt}\right|_{t=0^+} = \frac{i_C(0^+)}{C} = \frac{I_S - i_R(0^+) - i_L(0^+)}{C} \qquad (1.64)$$

or

$$\left.\frac{dv}{dt}\right|_{t=0} = \frac{I_S - v_C(0)/R - i_L(0)}{C} = \frac{10 - 5/40 - 2}{1/640} = \frac{7.875}{1/640} = 5040 \qquad (1.65)$$

Equating (1.63) with (1.65) and solving for k_2 we obtain

$$-40 + k_2 = 5040$$

or

$$k_2 = 5080 \qquad (1.66)$$

and by substitution into (1.62), we obtain the total solution as

$$v(t) = e^{-8t}(5 + 5080t)\ V \tag{1.67}$$

Check with MATLAB:

```
syms t; y0=exp(–8*t)*(5+5080*t); y1=diff(y0)  % Compute 1st derivative
                                              % Must have Symbolic Math Toolbox installed
y1 = -8*exp(-8*t)*(5+5080*t)+5080*exp(-8*t)
y2=diff(y0,2)                                 % Compute 2nd derivative
y2 = 64*exp(-8*t)*(5+5080*t)-81280*exp(-8*t)
y=y2/640+y1/40+y0/10                          % Verify differential equation, see (1.40), Pg 1-15
y = 0
```

The plot is shown in Figure 1.19.

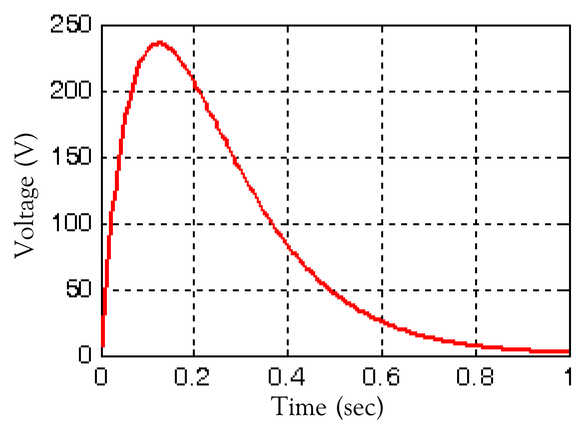


Figure 1.19. Plot for $v(t)$ *of Example 1.4*

By inspection of (1.67), we see that at $t = 0$, $v(t) = 5\ V$ and thus the second initial condition is satisfied. We can verify that the first initial condition is also satisfied by differentiation of (1.67). We can also show that $v(t)$ approaches zero as t approaches infinity with L'Hôpital's rule, i.e.,

$$\lim_{t \to \infty} v(t) = \lim_{t \to \infty} e^{-8t}(5 + 5080t) = \lim_{t \to \infty} \frac{(5 + 5080t)}{e^{8t}} = \lim_{t \to \infty} \frac{\frac{d}{dt}(5 + 5080t)}{\frac{d}{dt}e^{8t}} = \lim_{t \to \infty} \frac{5080}{8e^{8t}} = 0 \tag{1.68}$$

Example 1.5

For the circuit of Figure 1.20, $i_L(0) = 2\ A$ and $v_C(0) = 5\ V$. Compute and sketch $v(t)$ for $t > 0$.

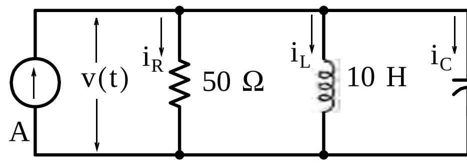


Figure 1.20. Circuit for Example 1.5

Solution:

This is the same circuit as the that of the two previous examples except that the resistance has been increased to $50\ \Omega$. For this example,

$$\alpha_P = \frac{G}{2C} = \frac{1}{2RC} = \frac{1}{2\times 50\times 1/640} = 6.4$$

or

$$\alpha_P^2 = 40.96$$

and as before,

$$\omega_0^2 = \frac{1}{LC} = \frac{1}{10\times 1/640} = 64$$

Also, $\omega_0^2 > \alpha_P^2$. Therefore, the natural response is underdamped with natural frequency

$$\omega_{nP} = \sqrt{\omega_0^2 - \alpha_P^2} = \sqrt{64 - 40.96} = \sqrt{23.04} = 4.8$$

Since $v_f = 0$, the total response is just the natural response. Then, from (1.48),

$$v(t) = v_n(t) = ke^{-\alpha_P t}\cos(\omega_{nP}t + \varphi) = ke^{-6.4t}\cos(4.8t + \varphi) \tag{1.69}$$

and the constants k and φ will be evaluated from the initial conditions.

From the initial condition $v_C(0) = v(0) = 5\text{ V}$ and (1.69) we obtain

$$v(0) = ke^0\cos(0 + \varphi) = 5$$

or

$$k\cos\varphi = 5 \tag{1.70}$$

To evaluate the constants k and φ we differentiate (1.69), we evaluate it at $t = 0$, we write the equation which describes the circuit at $t = 0^+$, and we equate these two expressions. Using MATLAB we obtain:

```
syms t k phi; y0=k*exp(–6.4*t)*cos(4.8*t+phi); y1=diff(y0)
% Must have Symbolic Math Toolbox installed
```

```
y1 = -32/5*k*exp(-32/5*t)*cos(24/5*t+phi)
-24/5*k*exp(-32/5*t)*sin(24/5*t+phi)
```

```
pretty(y1)
```

```
- 32/5 k exp(- 32/5 t) cos(24/5 t + phi)
- 24/5 k exp(- 32/5 t) sin(24/5 t + phi)
```

Thus,

$$\frac{dv}{dt} = -6.4ke^{-6.4t}\cos(4.8t + \varphi) - 4.8ke^{-6.4t}\sin(4.8t + \varphi) \qquad (1.71)$$

and

$$\left.\frac{dv}{dt}\right|_{t=0} = -6.4k\cos\varphi - 4.8k\sin\varphi$$

By substitution of (1.70), the above expression simplifies to

$$\left.\frac{dv}{dt}\right|_{t=0} = -32 - 4.8k\sin\varphi \qquad (1.72)$$

Also, $i_C = C\frac{dv}{dt}$ or $\frac{dv}{dt} = \frac{i_C}{C}$ and

$$\left.\frac{dv}{dt}\right|_{t=0^+} = \frac{i_C(0^+)}{C} = \frac{I_S - i_R(0^+) - i_L(0^+)}{C}$$

or

$$\left.\frac{dv}{dt}\right|_{t=0} = \frac{I_S - v_C(0)/R - i_L(0)}{C} = \frac{10 - 5/50 - 2}{1/640} = 7.9 \times 640 = 5056 \qquad (1.73)$$

Equating (1.72) with (1.73) we obtain

$$-32 - 4.8k\sin\varphi = 5056$$

or

$$k\sin\varphi = -1060 \qquad (1.74)$$

The phase angle φ can be found by dividing (1.74) by (1.70). Then,

$$\frac{k\sin\varphi}{k\cos\varphi} = \tan\varphi = \frac{-1060}{5} = -212$$

or

$$\varphi = \tan^{-1}(-212) = -1.566 \text{ rads} = -89.73 \text{ deg}$$

The value of the constant k is found from (1.70) as

$$k\cos(-1.566) = 5$$

or

$$k = \frac{5}{\cos(-1.566)} = 1042$$

and by substitution into (1.69), the total solution is

$$v(t) = 1042e^{-6.4t}\cos(4.8t - 89.73°) \qquad (1.75)$$

The plot is shown in Figure 1.21.

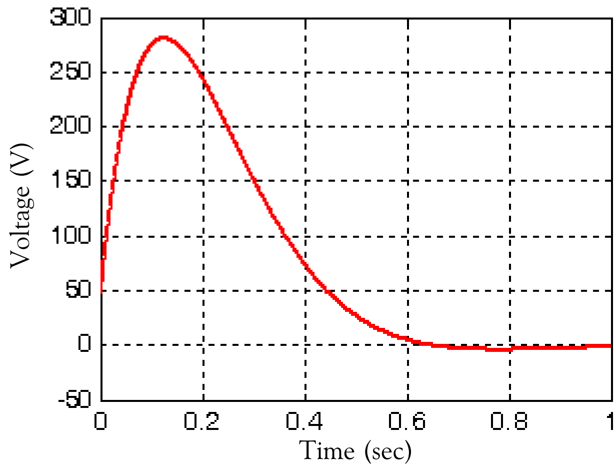


Figure 1.21. Plot for $v(t)$ *of Example 1.5*

From the plot of Figure 1.21 we observe that the maximum value occurs somewhere between $t = 0.10\ sec$ and $t = 0.20\ sec$, while the minimum value occurs somewhere between $t = 0.73\ sec$ and $t = 0.83\ sec$. Values for the maximum and minimum accurate to 3 decimal places are determined with the MATLAB script below.

```
fprintf(' \n');
disp('  t        Vc');
disp('-----------------');
t=0.10:0.01:0.20; Vc=zeros(11,2); Vc(:,1)=t';
Vc(:,2)=1042.*exp(–6.4.*t).*cos(4.8.*t–87.5*pi./180);
fprintf('%0.2f\t %8.3f\n',Vc')
```

```
  t       Vc
-----------------
0.10   274.736
0.11   278.822
0.12   280.743
0.13   280.748
0.14   279.066
0.15   275.911
0.16   271.478
0.17   265.948
0.18   259.486
0.19   252.242
0.20   244.354
```

```
fprintf(' \n');
disp('  t        Vc');
disp('-----------------');
```

```
t=0.73:0.01:0.83; Vc=zeros(11,2); Vc(:,1)=t';
Vc(:,2)=1042.*exp(–6.4.*t).*cos(4.8.*t–87.5*pi./180);
fprintf('%0.2f\t %8.3f\n',Vc')
```

```
  t        Vc
-----------------
0.73   -3.850
0.74   -4.010
0.75   -4.127
0.76   -4.205
0.77   -4.248
0.78   -4.261
0.79   -4.246
0.80   -4.208
0.81   -4.149
0.82   -4.073
0.83   -3.981
```

The maximum and minimum values and the times at which they occur are listed in the table below.

	t (sec)	v (V)
Maximum	0.13	280.748
Minimum	0.78	–4.261

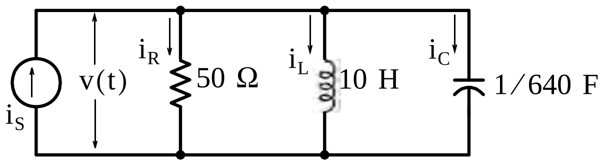
Alternately, we can find the maxima and minima by differentiating the response of (1.75) and setting it equal to zero.

1.3.2 Response of Parallel RLC Circuits with AC Excitation

The total response of a parallel RLC circuit that is excited by a sinusoidal source also consists of the natural and forced response components. The natural response will be overdamped, critically damped or underdamped. The forced component will be a sinusoid of the same frequency as that of the excitation, and since it represents the AC steady–state condition, we can use phasor analysis to find the forced response. We will derive the total response of a parallel RLC circuit which is excited by an AC source with the following example.

Example 1.6

For the circuit of Figure 1.22, $i_L(0) = 2\ A$ and $v_C(0) = 5\ V$. Compute and sketch $v(t)$ for $t > 0$.



$i_S = 20\sin(6400t + 90°)u_0(t)$ A

Figure 1.22. Circuit for Example 1.6

Solution:

This is the same circuit as the previous example where the DC source has been replaced by an AC source. The total response will consist of the natural response $v_n(t)$ which we already know from the previous example, and the forced response $v_f(t)$ which is the AC steady–state response, will be found by phasor analysis.

The $t-\text{domain}$ to $j\omega-\text{domain}$j transformation yields

$$i_s(t) = 20\sin(6400t + 90°) = 20\cos 6400t \Leftrightarrow I = 20\angle 0°$$

The admittance Y is

$$Y = G + j\left(\omega C - \frac{1}{\omega L}\right) = \sqrt{G^2 + \left(\omega C - \frac{1}{\omega L}\right)^2}\angle\tan^{-1}\left(\omega C - \frac{1}{\omega L}\right)/G$$

where

$$G = \frac{1}{R} = \frac{1}{50},\quad \omega C = 6400 \times \frac{1}{640} = 10 \text{ and } \frac{1}{\omega L} = \frac{1}{6400 \times 10} = \frac{1}{64000}$$

and thus

$$Y = \sqrt{\left(\frac{1}{50}\right)^2 + \left(10 - \frac{1}{64000}\right)^2}\angle\tan^{-1}\left(\left(10 - \frac{1}{64000}\right)/\frac{1}{50}\right) = 10\angle 89.72°$$

Now, we find the phasor voltage $\mathbf{V}$ as

$$\mathbf{V} = \frac{\mathbf{I}}{Y} = \frac{20\angle 0°}{10\angle 89.72°} = 2\angle -89.72°$$

and $j\omega-\text{domain}$ to $t-\text{domain}$ transformation yields

$$V = 2\angle -89.72° \Leftrightarrow v_f(t) = 2\cos(6400t - 89.72°)$$

The total solution is

$$v(t) = v_n(t) + v_f(t) = ke^{-6.4t}\cos(4.8t + \varphi) + 2\cos(6400t - 89.72°) \tag{1.76}$$

Now, we need to evaluate the constants k and φ.

With the initial condition $v_C(0) = 5$ V (1.76) becomes

$$v(0) = v_C(0) = ke^0 \cos\varphi + 2\cos(-89.72°) = 5$$

or

$$k\cos\varphi \approx 5 \tag{1.77}$$

To make use of the second initial condition, we differentiate (1.76) using MATLAB as follows, and then we evaluate it at $t = 0$.

```
syms t k phi; y0=k*exp(-6.4*t)*cos(4.8*t+phi)+2*cos(6400*t-1.5688);  % Must have Sym Math
y1=diff(y0);                          % Differentiate v(t) of (1.76)
```

```
y1 = -32/5*k*exp(-32/5*t)*cos(24/5*t+phi)-24/5*k*exp(-32/
5*t)*sin(24/5*t+phi)-12800*sin(6400*t-1961/1250)
```

or

$$\frac{dv}{dt} = -6.4ke^{-6.4t}\cos(4.8t+\varphi) - 4.8ke^{-6.4t}\sin(4.8t+\varphi) - 12800\sin(6400t - 1.5688)$$

and

$$\left.\frac{dv}{dt}\right|_{t=0} = -6.4k\cos\varphi - 4.8k\sin\varphi - 12800\sin(-1.5688) = -6.4k\cos\varphi - 4.8k\sin\varphi + 12800 \tag{1.78}$$

With (1.77) we obtain

$$\left.\frac{dv}{dt}\right|_{t=0} = -32 - 4.8k\sin\varphi + 12800 \approx -4.8k\sin\varphi + 12832 \tag{1.79}$$

Also, $i_C = C\frac{dv}{dt}$ or $\frac{dv}{dt} = \frac{i_C}{C}$ and

$$\left.\frac{dv}{dt}\right|_{t=0^+} = \frac{i_C(0^+)}{C} = \frac{i_S(0^+) - i_R(0^+) - i_L(0^+)}{C}$$

or

$$\left.\frac{dv}{dt}\right|_{t=0} = \frac{i_S(0^+) - v_C(0)/R - i_L(0)}{C} = \frac{20 - 5/50 - 2}{1/640} = 11456 \tag{1.80}$$

Equating (1.79) with (1.80) and solving for k we obtain

$$-4.8k\sin\varphi + 12832 = 11456$$

or

$$k\sin\varphi = 287$$

Then with (1.77) and (1.81),

$$\frac{k\sin\varphi}{k\cos\varphi} = \tan\varphi = \frac{287}{5} = 57.4$$

or

$$\varphi = 1.53 \text{ rad} = 89°$$

The value of the constant k is found from (1.77), that is,

$$k = 5/(\cos 89°) = 279.4$$

By substitution into (1.76), we obtain the total solution as

$$v(t) = 279.4e^{-6.4t}\cos(4.8t + 89°) + 2\cos(6400t - 89.72°) \tag{1.81}$$

With MATLAB we obtain the plot shown in Figure 1.23. The plot was created with the MATLAB script below.

```
t=0: 0.01: 1; vt=279.4.*exp(-6.4.*t).*cos(4.8.*t+89*pi./180)+2.*cos(6400.*t-89.72.*pi./180);
plot(t,vt); grid
```

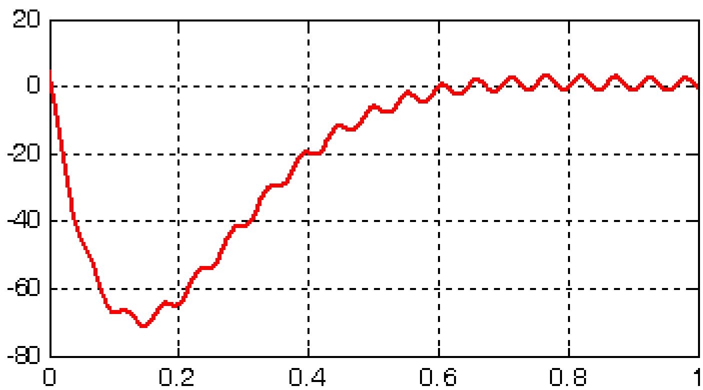


Figure 1.23. Plot for $v(t)$ *of Example 1.6*

The same results are obtained with the Simulink/SimPowerSystems model shown in Figure 1.24.

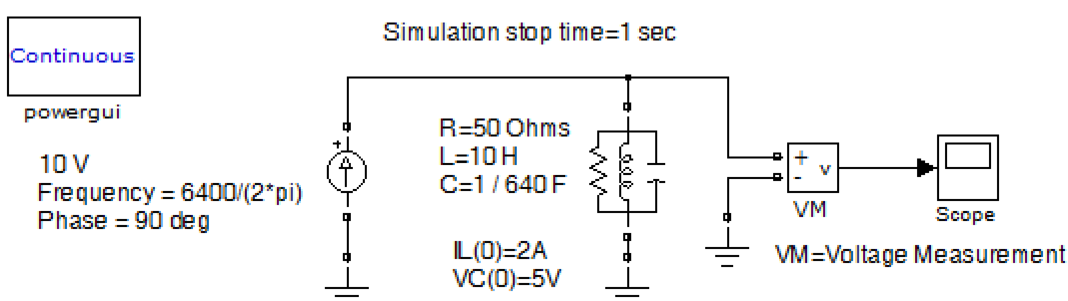


Figure 1.24. Simulink/SimPowerSystems model for the circuit in Figure 1.23

The waveform displayed by the Scope block is shown in Figures 1.25, and we observe that it is consistent with the waveform shown in Figure 1.23.

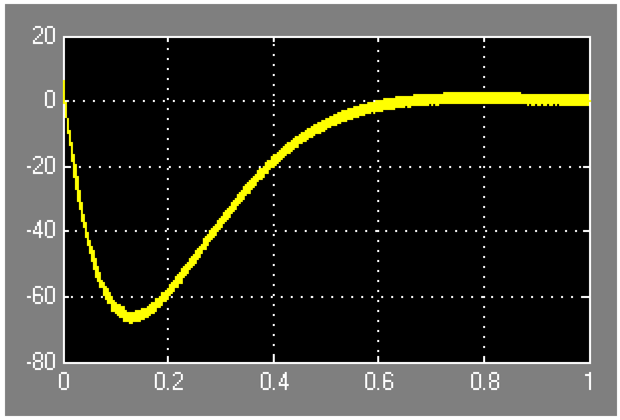


Figure 1.25. Waveform displayed by the Scope block in Figure 1.24

1.4 Other Second Order Circuits

Second order circuits are not restricted to RLC circuits. They include amplifiers and filter among others, and since it is beyond the scope of this text to analyze such circuits in detail, we will illustrate the transient analysis of a second order active low–pass filter.

Example 1.7

The circuit of Figure 1.26 a known as a Multiple Feed Back (MFB) active low–pass filter. For this circuit, the initial conditions are $v_{C1} = v_{C2} = 0$. Compute and sketch $v_{out}(t)$ for $t > 0$.

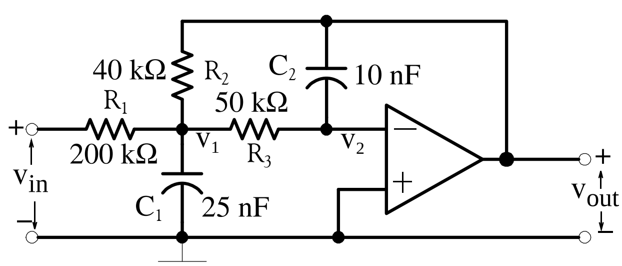


Figure 1.26. Circuit for Example 1.7

Solution:

At node V_1:

$$\frac{v_1 - v_{in}}{R_1} + C_1\frac{dv_1}{dt} + \frac{v_1 - v_{out}}{R_2} + \frac{v_1 - v_2}{R_3} = 0 \quad t > 0 \tag{1.82}$$

At node V_2:

$$\frac{v_2 - v_1}{R_3} = C_2\frac{dv_{out}}{dt} \tag{1.83}$$

We observe that $v_2 = 0$ (virtual ground).

Collecting like terms and rearranging (1.83) and (1.84) we obtain

$$\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\right)v_1 + C_1\frac{dv_1}{dt} - \frac{1}{R_2}v_{out} = \frac{1}{R_1}v_{in} \tag{1.84}$$

and

$$v_1 = -R_3C_2\frac{dv_{out}}{dt} \tag{1.85}$$

Differentiation of (1.86) yields

$$\frac{dv_1}{dt} = -R_3C_2\frac{dv^2_{out}}{dt^2} \tag{1.86}$$

and by substitution of given numerical values into (1.85) through (1.87), we obtain

$$\left(\frac{1}{2\times 10^5}+\frac{1}{4\times 10^4}+\frac{1}{5\times 10^4}\right)v_1 + 25\times 10^{-9}\frac{dv_1}{dt} - \frac{1}{4\times 10^4}v_{out} = \frac{1}{2\times 10^5}v_{in}$$

or

$$(0.05\times 10^{-3})v_1 + 25\times 10^{-9}\frac{dv_1}{dt} - (0.25\times 10^{-4})v_{out} = (0.5\times 10^{-5})v_{in} \tag{1.87}$$

$$v_1 = -5\times 10^{-4}\frac{dv_{out}}{dt} \tag{1.88}$$

$$\frac{dv_1}{dt} = -5\times 10^{-4}\frac{d^2v_{out}}{dt^2} \tag{1.89}$$

Next, substitution of (1.89) and (1.90) into (1.88) yields

$$0.05\times 10^{-3}\left(-5\times 10^{-4}\frac{dv_{out}}{dt}\right) + 25\times 10^{-9}(-5\times 10^{-4})\frac{d^2v_{out}}{dt^2} \tag{1.90}$$
$$-(0.25\times 10^{-4})v_{out} = (0.5\times 10^{-5})v_{in}$$

or

$$-125\times 10^{-13}\frac{d^2v_{out}}{dt^2} - 0.25\times 10^{-7}\frac{dv_{out}}{dt} - (0.25\times 10^{-4})v_{out} = 10^{-4}v_{in}$$

and division by -125×10^{-13} yields

$$\frac{d^2v_{out}}{dt^2} + 2\times 10^3\frac{dv_{out}}{dt} + 2\times 10^6 v_{out} = (-1.6\times 10^5)v_{in}$$

or

$$\frac{d^2v_{out}}{dt^2} + 2\times 10^3\frac{dv_{out}}{dt} + 2\times 10^6 v_{out} = -10^6\cos 6280t \tag{1.91}$$

We use MATLAB to find the roots of the characteristic equation of (1.92).

```
syms s; y0=solve('s^2+2*10^3*s+2*10^6')  % Must have Symbolic Math Toolbox installed
```

```
y0 =
[-1000+1000*i]
[-1000-1000*i]
```

that is,

$$s_1, s_2 = -\alpha \pm j\beta = -1000 \pm j1000 = 1000(-1 \pm j1)$$

We cannot classify the given circuit as series or parallel and therefore, we should not use the damping ratio α_S or α_P. Instead, for the natural response $v_n(t)$ we will use the general expression

$$v_n(t) = Ae^{s_1 t} + Be^{s_2 t} = e^{-\alpha t}(k_1 \cos\beta t + k_2 \sin\beta t) \tag{1.92}$$

where

$$s_1, s_2 = -\alpha \pm j\beta = -1000 \pm j1000$$

Therefore, the natural response is oscillatory and has the form

$$v_n(t) = e^{-1000t}(k_1 \cos 1000t + k_2 \sin 1000t) \tag{1.93}$$

Since the right side of (1.92) is a sinusoid, the forced response has the form

$$v_f(t) = k_3 \cos 6280t + k_4 \sin 6280t \tag{1.94}$$

Of course, for the derivation of the forced response we could use phasor analysis but we must first derive an expression for the impedance or admittance, since the expressions we used earlier were for series and parallel circuits only.

The coefficients k_3 and k_4 will be found by substitution of (1.95) into (1.92) and then by equating like terms. Using MATLAB we obtain:

```
syms t k3 k4; y0=k3*cos(6280*t)+k4*sin(6280*t); y1=diff(y0)
```

```
y1 =
-6280*k3*sin(6280*t)+6280*k4*cos(6280*t)
```

```
y2=diff(y0,2)
```

```
y2 =
-39438400*k3*cos(6280*t)-39438400*k4*sin(6280*t)
```

```
y=y2+2*10^3*y1+2*10^6*y0
```

```
y =
-37438400*k3*cos(6280*t)-37438400*k4*sin(6280*t)-
12560000*k3*sin(6280*t)+12560000*k4*cos(6280*t)
```

Equating like terms with (1.92) we obtain

$$(-37438400 \cdot k_3 + 12560000 \cdot k_4)\cos 6280t = -10^6 \cos 6280t$$
$$(-12560000 \cdot k_3 - 37438400 \cdot k_4)\sin 6280t = 0 \qquad (1.95)$$

Simultaneous solution of the equations of (1.96) is done with MATLAB.

```
syms k3 k4
eq1=–37438400*k3+12560000*k4+10^6;
eq2=–12560000*k3–37438400*k4+0;
y=solve(eq1,eq2)
```

```
y =
    k3: [1x1 sym]
    k4: [1x1 sym]
```

```
y.k3
```

```
ans =
    0.0240
```

```
y.k4
```

```
ans =
   -0.0081
```

that is, $k_3 = 0.024$ and $k_4 = -0.008$. Then, by substitution into (1.95)

$$v_f(t) = 0.024\cos 6280t - 0.008\sin 6280t \qquad (1.96)$$

The total response is

$$v_{out}(t) = v_n(t) + v_f(t) = e^{-1000t}(k_1\cos 1000t + k_2\sin 1000t) + 0.024\cos 6280t - 0.008\sin 6280t \qquad (1.97)$$

We will use the initial conditions $v_{C1} = v_{C2} = 0$ to evaluate k_1 and k_2. We observe that $v_{C2} = v_{out}$ and at $t = 0$ relation (1.98) becomes

$$v_{out}(0) = e^0(k_1\cos 0 + 0) + 0.024\cos 0 - 0 = 0$$

or $k_1 = -0.024$ and thus (1.98) simplifies to

$$v_{out}(t) = e^{-1000t}(-0.024\cos 1000t + k_2\sin 1000t) + 0.024\cos 6280t - 0.008\sin 6280t \qquad (1.98)$$

To evaluate the constant k_2, we make use of the initial condition $v_{C1}(0) = 0$. We observe that $v_{C1} = v_1$ and by KCL at node v_1 we have:

$$\frac{v_1 - v_2}{R_3} + C_2\frac{dv_{out}}{dt} = 0$$

or

$$\frac{v_1 - 0}{5 \times 10^4} = -10^{-8} \frac{dv_{out}}{dt}$$

or

$$v_1 = -5 \times 10^{-4} \frac{dv_{out}}{dt}$$

and since $v_{C1}(0) = v_1(0) = 0$, it follows that

$$\left.\frac{dv_{out}}{dt}\right|_{t=0} = 0 \tag{1.99}$$

The last step in finding the constant k_2 is to differentiate (1.99), evaluate it at $t = 0$, and equate it with (1.100). This is done with MATLAB as follows:

```
y0=exp(–1000*t)*(–0.024*cos(1000*t)+k2*sin(1000*t))...
+0.024*cos(6280*t)–0.008*sin(6280*t);
```

```
y1=diff(y0)
```

```
y1 =
-1000*exp(-1000*t)*(-3/125*cos(1000*t)+k2*sin(1000*t))+exp(-
1000*t)*(24*sin(1000*t)+1000*k2*cos(1000*t))-3768/
25*sin(6280*t)-1256/25*cos(6280*t)
```

or

$$\frac{dv_{out}}{dt} = -1000e^{-1000t}\left(\frac{-3}{125}\cos 1000t + k_2 \sin 1000t\right) + e^{-1000t}(24\sin 1000t + 1000k_2\cos 1000t)$$
$$-\frac{3768}{25}\sin(6280t) - \frac{1256}{25}\cos 6280t$$

and

$$\left.\frac{dv_{out}}{dt}\right|_{t=0} = -1000\left(\frac{-3}{125}\right) + 1000k_2 - \frac{1256}{25} \tag{1.100}$$

Simplifying and equating (1.100) with (1.101) we obtain

$$1000k_2 - 26.24 = 0$$

or

$$k_2 = 0.026$$

and by substitution into (1.99),

$$v_{out}(t) = e^{-1000t}(-0.024\cos 1000t + 0.026\sin 1000t) + 0.024\cos 6280t - 0.008\sin 6280t \tag{1.101}$$

The plot is shown in Figure 1.27.

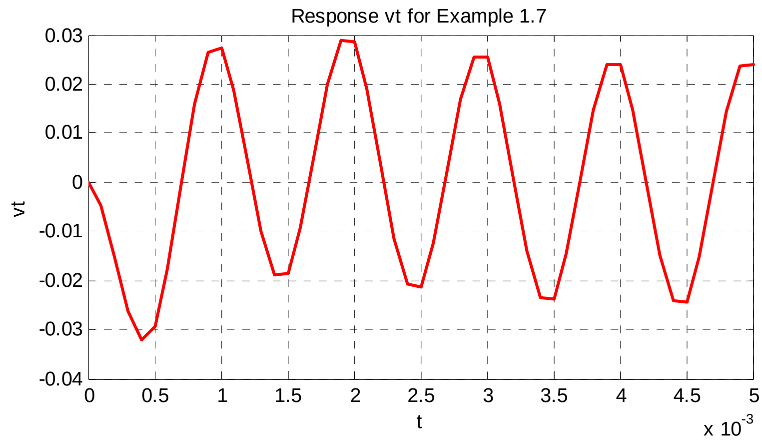


Figure 1.27. Plot of $v_{out}(t)$ *for Example 1.7*

1.5 Summary

- Circuits that contain energy storing devices can be described by integrodifferential equations and upon differentiation can be simplified to differential equations with constant coefficients.
- A second order circuit contains two energy storing devices. Thus, an RLC circuit is a second order circuit.
- The total response is the summation of the natural and forced responses.
- If the differential equation describing a series RLC circuit that is excited by a constant (DC) voltage source is written in terms of the current, the forced response is zero and thus the total response is just the natural response.
- If the differential equation describing a parallel RLC circuit that is excited by a constant (DC) current source is written in terms of the voltage, the forced response is zero and thus the total response is just the natural response.
- If a circuit is excited by a sinusoidal (AC) source, the forced response is never zero.
- The natural response of a second order circuit may be overdamped, critically damped, or underdamped depending on the values of the circuit constants.
- For a series RLC circuit, the roots s_1 and s_2 are found from

$$s_1, s_2 = -\alpha_S \pm \sqrt{\alpha_S^2 - \omega_0^2} = -\alpha_S \pm \beta_S \quad \text{if} \quad \alpha_S^2 > \omega_0^2$$

or

$$s_1, s_2 = -\alpha_S \pm \sqrt{\omega_0^2 - \alpha_S^2} = -\alpha_S \pm \omega_{nS} \quad \text{if} \quad \omega_0^2 > \alpha_S^2$$

where

$$\alpha_S = \frac{R}{2L} \qquad \omega_0 = \frac{1}{\sqrt{LC}} \qquad \beta_S = \sqrt{\alpha_S{}^2 - \omega_0^2} \qquad \omega_{nS} = \sqrt{\omega_0^2 - \alpha_S{}^2}$$

If $\alpha_S^2 > \omega_0^2$, the roots s_1 and s_2 are real, negative, and unequal. This results in the *overdamped* natural response and has the form

$$i_n(t) = k_1 e^{s_1 t} + k_2 e^{s_2 t}$$

If $\alpha_S^2 = \omega_0^2$, the roots s_1 and s_2 are real, negative, and equal. This results in the *critically damped* natural response and has the form

$$i_n(t) = e^{-\alpha_S t}(k_1 + k_2 t)$$

If $\omega_0^2 > \alpha_S^2$, the roots s_1 and s_2 are complex conjugates. This is known as the *underdamped or oscillatory* natural response and has the form

$$i_n(t) = e^{-\alpha_S t}(k_1 \cos\omega_{nS} t + k_2 \sin\omega_{nS} t) = k_3 e^{-\alpha_S t}(\cos\omega_{nS} t + \varphi)$$

- For a parallel RLC circuit, the roots s_1 and s_2 are found from

$$s_1, s_2 = -\alpha_P \pm \sqrt{\alpha_P^2 - \omega_0^2} = -\alpha_P \pm \beta_P \quad \text{if} \quad \alpha_P^2 > \omega_0^2$$

or

$$s_1, s_2 = -\alpha_P \pm \sqrt{\omega_0^2 - \alpha_P^2} = -\alpha_P \pm \omega_{nP} \quad \text{if} \quad \omega_0^2 > \alpha_P^2$$

where

$$\alpha_P = \frac{G}{2C} \qquad \omega_0 = \frac{1}{\sqrt{LC}} \qquad \beta_P = \sqrt{\alpha_P{}^2 - \omega_0^2} \qquad \omega_{nP} = \sqrt{\omega_0^2 - \alpha_P{}^2}$$

If $\alpha_P^2 > \omega_0^2$, the roots s_1 and s_2 are real, negative, and unequal. This results in the overdamped natural response and has the form

$$v_n(t) = k_1 e^{s_1 t} + k_2 e^{s_2 t}$$

If $\alpha_P^2 = \omega_0^2$, the roots s_1 and s_2 are real, negative, and equal. This results in the critically damped natural response and has the form

$$v_n(t) = e^{-\alpha_P t}(k_1 + k_2 t)$$

If $\omega_0^2 > \alpha_P^2$, the roots s_1 and s_2 are complex conjugates. This results in the underdamped or oscillatory natural response and has the form

$$v_n(t) = e^{-\alpha_P t}(k_1 \cos\omega_{nP} t + k_2 \sin\omega_{nP} t) = k_3 e^{-\alpha_P t}(\cos\omega_{nP} t + \varphi)$$

- If a second order circuit is neither series nor parallel, the natural response if found from

$$y_n = k_1 e^{s_1 t} + k_2 e^{s_2 t}$$

or

$$y_N = (k_1 + k_2 t) e^{s_1 t}$$

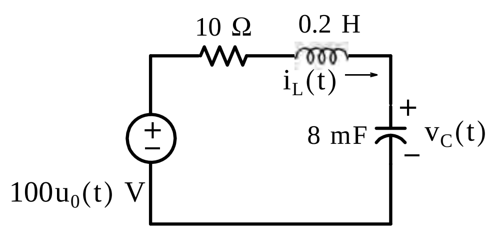
or

$$y_n = e^{-\alpha t}(k_3 \cos\beta t + k_4 \sin\beta t) = e^{-\alpha t} k_5 \cos(\beta t + \varphi)$$

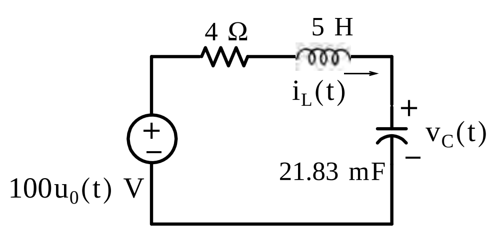
depending on the roots of the characteristic equation being real and unequal, real and equal, or complex conjugates respectively.

1.6 Exercises

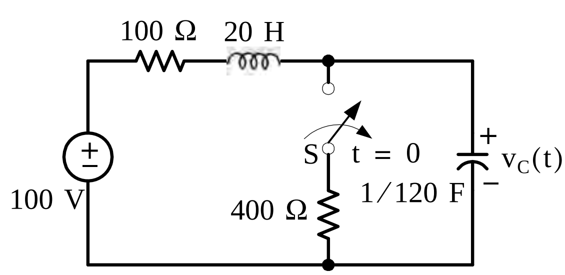
1. For the circuit below it is known that $v_C(0^-) = 0$ and $i_L(0^-) = 0$. Compute and sketch $v_C(t)$ and $i_L(t)$ for $t > 0$.



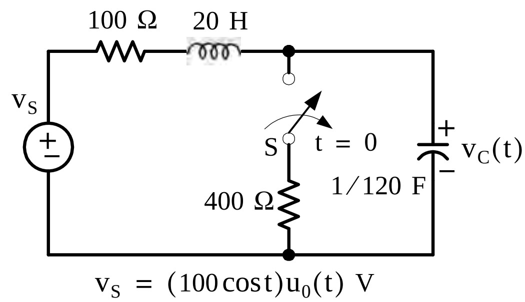
2. For the circuit below it is known that $v_C(0^-) = 0$ and $i_L(0^-) = 0$. Compute and sketch $v_C(t)$ and $i_L(t)$ for $t > 0$.



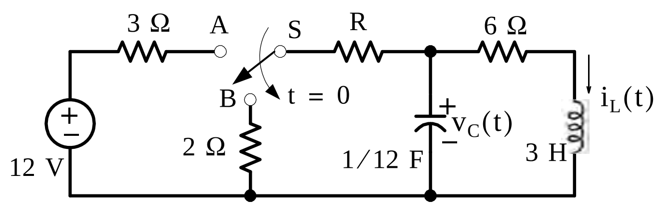
3. In the circuit below the switch S has been closed for a very long time and opens at $t = 0$. Compute $v_C(t)$ for $t > 0$.



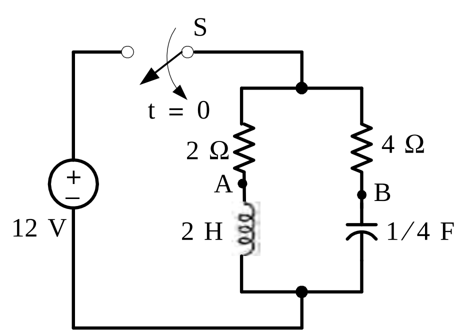
4. In the circuit below, the switch S has been closed for a very long time and opens at $t = 0$. Compute $v_C(t)$ for $t > 0$.



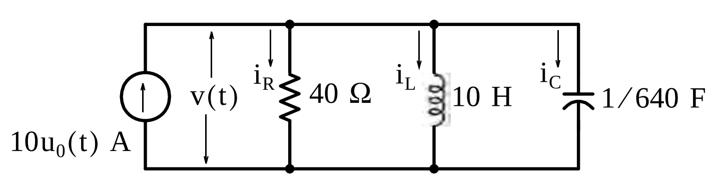
5. In the circuit below the switch S has been in position A for closed for a very long time and it is placed in position B at $t = 0$. Find the value of R that will cause the circuit to become critically damped and then compute $v_C(t)$ and $i_L(t)$ for $t > 0$



6. In the circuit below the switch S has been closed for a very long time and opens at $t = 0$. Compute $v_{AB}(t)$ for $t > 0$.



7. Create a Simulink/SimPowerSystems model for the circuit below.



This is the same circuit as in Example 1.4, Page 1−21 where we found that $R = 40\ \Omega$. The initial conditions are the same as in Example 1.4, that is, $i_L(0) = 2\ A$ and $v_C(0) = 5\ V$,

1.7 Solutions to End–of–Chapter Exercises

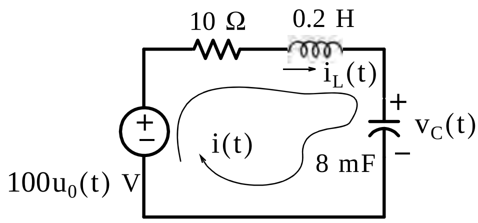
Dear Reader:

The remaining pages on this chapter contain solutions to the End–of–Chapter exercises.

You must, for your benefit, make an honest effort to solve the problems without first looking at the solutions that follow. It is recommended that first you go through and answer those you feel that you know. For the exercises that you are uncertain, review the pertinent section(s) in this chapter and try again. If your answers to the exercises do not agree with those provided, look over your procedures for inconsistencies and computational errors. Refer to the solutions as a last resort and rework those problems at a later date.

You should follow this practice with the problems in all chapters of this book.

1.



$$Ri + L\frac{di}{dt} + v_C = 100 \qquad t > 0$$

and since $i = i_C = C\frac{dv_C}{dt}$, the above becomes

$$RC\frac{dv_C}{dt} + LC\frac{d^2v_C}{dt^2} + v_C = 100$$

$$\frac{d^2v_C}{dt^2} + \frac{R}{L}\frac{dv_C}{dt} + \frac{1}{LC}v_C = \frac{100}{LC}$$

$$\frac{d^2v_C}{dt^2} + \frac{10}{0.2}\frac{dv_C}{dt} + \frac{1}{0.2 \times 8 \times 10^{-3}}v_C = \frac{100}{0.2 \times 8 \times 10^{-3}}$$

$$\frac{d^2v_C}{dt^2} + 50\frac{dv_C}{dt} + 625\,v_C = 62500$$

From the characteristic equation

$$s^2 + 50s + 625 = 0$$

we obtain $s_1 = s_2 = -25$ (critical damping) and $\alpha_S = R/2L = 25$

The total solution is

$$v_C(t) = v_{Cf} + v_{Cn} = 100 + e^{-\alpha_S t}(k_1 + k_2 t) = 100 + e^{-25t}(k_1 + k_2 t) \quad (1)$$

With the first initial condition $v_C(0^-) = 0$ the above expression becomes

$$0 = 100 + e^0(k_1 + 0)$$

$$k_1 = -100$$

and by substitution into (1) we obtain

$$v_C(t) = 100 + e^{-25t}(k_2 t - 100) \quad (2)$$

To evaluate k_2 we make use of the second initial condition $i_L(0^-) = 0$ and since $i_L = i_C$, and $i = i_C = C\frac{dv_C}{dt}$, we differentiate (2) using the following MATLAB script:

```
syms t k2  % Must have Symbolic Math Toolbox installed
v0=100+exp(–25*t)*(k2*t–100); v1=diff(v0)
```

```
v1 =
-25*exp(-25*t)*(k2*t-100)+exp(-25*t)*k2
```

Thus,

$$\frac{dv_C}{dt} = k_2 e^{-25t} - 25e^{-25t}(k_2 t - 100)$$

and

$$\left.\frac{dv_C}{dt}\right|_{t=0} = k_2 + 2500 \quad (3)$$

Also, $\frac{dv_C}{dt} = \frac{i_C}{C} = \frac{i_L}{C}$ and at $t = 0$

$$\left.\frac{dv_C}{dt}\right|_{t=0} = \frac{i_L(0^-)}{C} = 0 \quad (4)$$

From (3) and (4) $k_2 + 2500 = 0$ or $k_2 = -2500$ and by substitution into (2)

$$v_C(t) = 100 - e^{-25t}(2500t + 100) \quad (5)$$

We find $i_L(t) = i_C(t)$ by differentiating (5) and multiplication by C.

Using MATLAB we obtain:

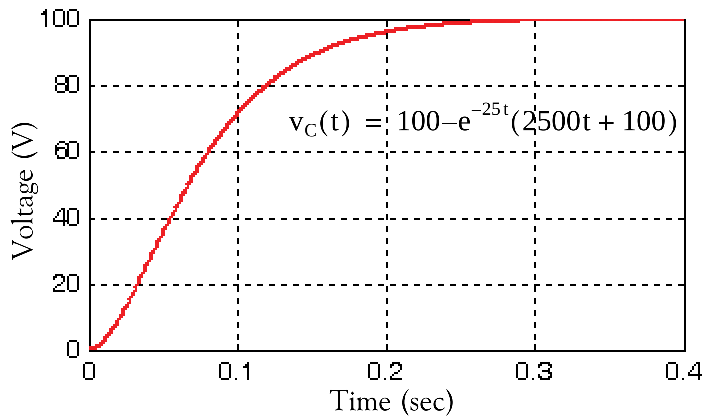
```
syms t   % Must have Symbolic Math Toolbox installed
C=8*10^(–3); i0=C*(100–exp(–25*t)*(100+2500*t)); iL=diff(i0)
```

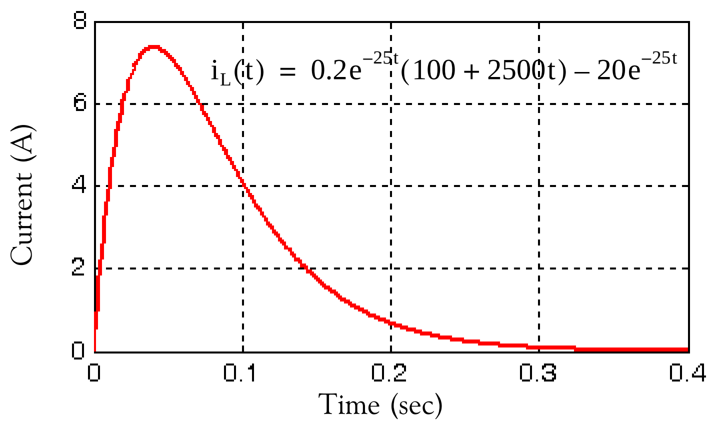
```
iL =
1/5*exp(-25*t)*(100+2500*t)-20*exp(-25*t)
```

Thus,

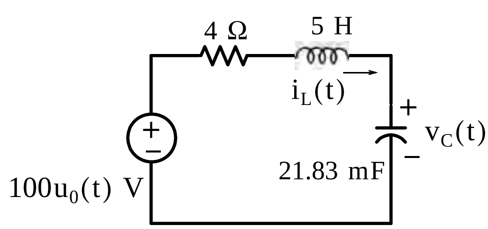
$$i_L(t) = i_C(t) = 0.2e^{-25t}(100 + 2500t) - 20e^{-25t}$$

The plots for $v_C(t)$ and $i_L(t)$ are shown below.





2.



The general form of the differential equation that describes this circuit is same as in Exercise 1, that is,

$$\frac{d^2v_C}{dt^2} + \frac{R}{L}\frac{dv_C}{dt} + \frac{1}{LC}v_C = \frac{100}{LC} \qquad t > 0$$

$$\frac{d^2v_C}{dt^2} + 0.8\frac{dv_C}{dt} + 9.16v_C = 916$$

From the characteristic equation $s^2 + 0.8s + 9.16 = 0$ and the MATLAB script below

```
s=[1  0.8  9.16]; roots(s)
```

we obtain

```
ans =
  -0.4000 + 3.0000i
  -0.4000 - 3.0000i
```

that is, $s_1 = -0.4 + j3$ and $s_2 = -0.4 - j3$. Therefore, the total solution is

$$v_C(t) = v_{Cf} + v_{Cn} = 100 + ke^{-\alpha_S t}\cos(\omega_{nS} t + \varphi)$$

where

$$\alpha_S = R/2L = 0.4$$

and

$$\omega_{nS} = \sqrt{\omega_0^2 - \alpha_S^2} = \sqrt{1/LC - R^2/4L^2} = \sqrt{9.16 - 0.16} = 3$$

Thus,

$$v_C(t) = 100 + ke^{-0.4t}\cos(3t + \varphi) \quad (1)$$

and with the initial condition $v_C(0^-) = 0$ we obtain

$$0 = 100 + k\cos(0 + \varphi)$$

or

$$k\cos\varphi = -100 \quad (2)$$

To evaluate k and φ we differentiate (1) using MATLAB and evaluate it at $t = 0$.

```
syms t  k  phi; v0=100+k*exp(–0.4*t)*cos(3*t+phi); v1=diff(v0)
% Must have Symbolic Math Toolbox installed
```

```
v1 =
-2/5*k*exp(-2/5*t)*cos(3*t+phi)-3*k*exp(-2/5*t)*sin(3*t+phi)
```

or

$$\frac{dv_C}{dt} = -0.4ke^{-0.4t}\cos(3t + \varphi) - 3ke^{-0.4t}\sin(3t + \varphi)$$

$$\left.\frac{dv_C}{dt}\right|_{t=0} = -0.4k\cos\varphi - 3k\sin\varphi$$

and with (2)

$$\left.\frac{dv_C}{dt}\right|_{t=0} = 40 - 3k\sin\varphi \quad (3)$$

Also, $\dfrac{dv_C}{dt} = \dfrac{i_C}{C} = \dfrac{i_L}{C}$ and at $t = 0$

$$\left.\frac{dv_C}{dt}\right|_{t=0} = \frac{i_L(0^-)}{C} = 0 \quad (4)$$

From (3) and (4)

$$3k\sin\varphi = 40 \quad (5)$$

and from (2) and (5)

$$\frac{3k\sin\varphi}{k\cos\varphi} = \frac{40}{-100}$$

$$3\tan\varphi = -0.4$$

$$\varphi = \tan^{-1}(-0.4/3) = -0.1326\ \text{rad} = -7.6°$$

The value of k can be found from either (2) or (5). From (2)

$$k\cos(-0.1236) = -100$$

$$k = \frac{-100}{\cos(-0.1236)} = -100.8$$

and by substitution into (1)

$$v_C(t) = 100 - 100.8e^{-0.4t}\cos(3t - 7.6°) \quad (6)$$

Since $i_L(t) = i_C(t) = C(dv_C/dt)$, we use MATLAB to differentiate (6).

```
syms t; vC=100–100.8*exp(–0.4*t)*cos(3*t-0.1326); C=0.02183; iL=C*diff(vC)
 % Must have Symbolic Math Toolbox installed
```

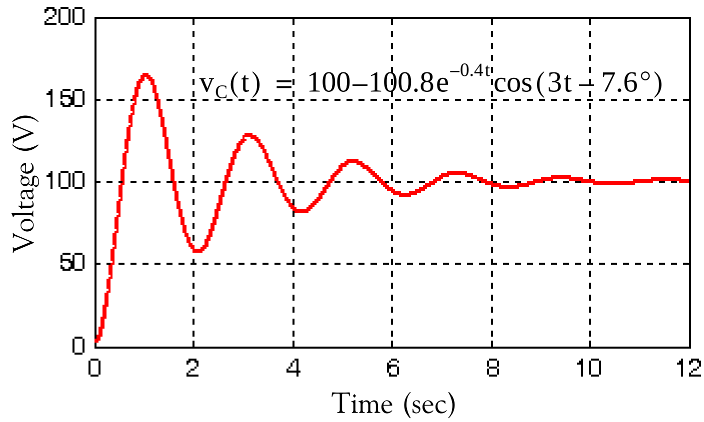
```
iL =
137529/156250*exp(-2/5*t)*cos(3*t-663/5000)+412587/62500*exp(-
2/5*t)*sin(3*t-663/5000)
```

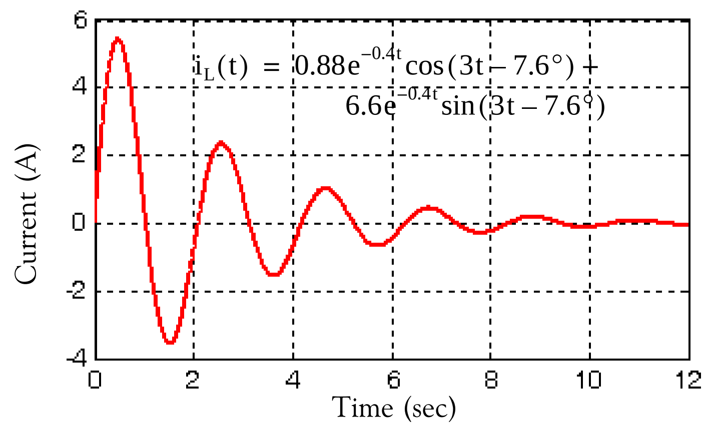
```
137529/156250, 412587/62500
```

```
ans =
   0.8802
ans =
   6.6014
```

$$i_L(t) = 0.88e^{-0.4t}\cos(3t - 7.6°) + 6.6e^{-0.4t}\sin(3t - 7.6°)$$

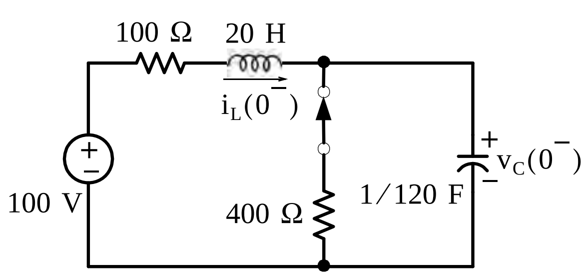
The plots for $v_C(t)$ and $i_L(t)$ are shown below.





3.

At $t = 0^-$ the circuit is as shown below.



At this time the inductor behaves as a short and the capacitor as an open. Then,

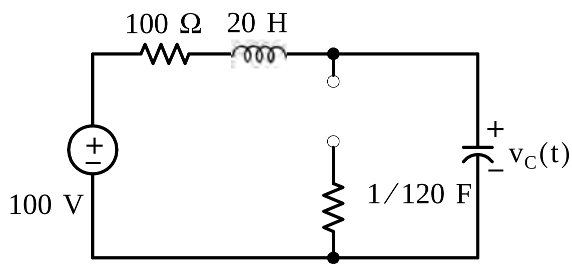
$$i_L(0^-) = 100/(100+400) = I_0 = 0.2 \text{ A}$$

and this establishes the first initial condition as $I_0 = 0.2$ A. Also,

$$v_C(0^-) = v_{400\ \Omega} = 400 \times i_L(0^-) = 400 \times 0.2 = V_0 = 80 \text{ V}$$

and this establishes the second initial condition as $V_0 = 80\ V$.

For $t > 0$ the circuit is as shown below.



The general form of the differential equation that describes this circuit is same as in Exercise 1, that is,

$$\frac{d^2v_C}{dt^2} + \frac{R}{L}\frac{dv_C}{dt} + \frac{1}{LC}v_C = \frac{100}{LC} \qquad t > 0$$

$$\frac{d^2v_C}{dt^2} + 5\frac{dv_C}{dt} + 6v_C = 600$$

From the characteristic equation $s^2 + 5s + 6 = 0$ we find that $s_1 = -2$ and $s_2 = -3$ and the total response for the capacitor voltage is

$$v_C(t) = v_{Cf} + v_{Cn} = 100 + k_1e^{s_1t} + k_2e^{s_2t} = 100 + k_1e^{-2t} + k_2e^{-3t} \quad (1)$$

Using the initial condition $V_0 = 80\ V$ we obtain

$$v_C(0^-) = V_0 = 80\ V = 100 + k_1e^0 + k_2e^0$$

or

$$k_1 + k_2 = -20 \quad (2)$$

Differentiation of (1) and evaluation at $t = 0$ yields

$$\left.\frac{dv_C}{dt}\right|_{t=0} = -2k_1 - 3k_2 \quad (3)$$

Also, $\dfrac{dv_C}{dt} = \dfrac{i_C}{C} = \dfrac{i_L}{C}$ and at $t = 0$

$$\left.\frac{dv_C}{dt}\right|_{t=0} = \frac{i_L(0^-)}{C} = \frac{0.2}{1/120} = 24 \quad (4)$$

Equating (3) and (4) we obtain

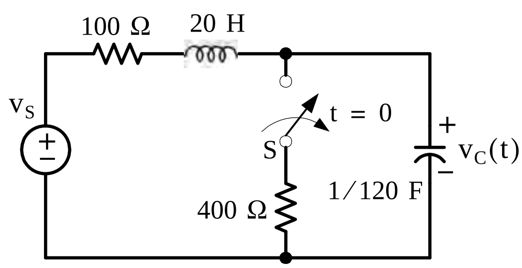
$$-2k_1 - 3k_2 = 24 \quad (5)$$

and simultaneous solution of (2) and (5) yields $k_1 = -36$ and $k_2 = 16$

By substitution into (1) we find the total solution

$$v_C(t) = v_{Cf} + v_{Cn} = 100 - 36e^{-2t} + 16e^{-3t}$$

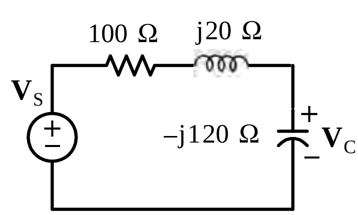
4.



$v_S = (100\cos t)u_0(t)$ V

This is the same circuit as in Exercise 3 where the DC voltage source has been replaced by an AC source that is being applied at $t = 0^+$. No initial conditions were given so we will assume that $i_L(0^-) = 0$ and $v_C(0^-) = 0$. Also, the circuit constants are the same and thus the natural response has the form $v_{Cn} = k_1e^{-2t} + k_2e^{-3t}$.

We will find the forced (steady-state) response using phasor circuit analysis where $\omega = 1$, $j\omega L = j20$, $-j/\omega C = -j120$, and $100\cos t \Leftrightarrow 100\angle 0°$. The phasor circuit is shown below.



$\mathbf{V}_S = 100\angle 0°$ V

Using the voltage division expression we obtain

$$\mathbf{V}_C = \frac{-j120}{100 + j20 - j120}100\angle 0° = \frac{-j120}{100 + j100}100\angle 0° = \frac{120\angle -90° \times 100\angle 0°}{100\sqrt{2}\angle 45°} = 60\sqrt{2}\angle -135°$$

and in the t-domain $v_{Cf} = 60\sqrt{2}\cos(t - 135°)$. Therefore, the total response is

$$v_C(t) = 60\sqrt{2}\cos(t - 135°) + k_1e^{-2t} + k_2e^{-3t} \quad (1)$$

Using the initial condition $v_C(0^-) = 0$ and (1) we obtain

$$v_C(0^-) = 0 = 60\sqrt{2}\cos(-135°) + k_1 + k_2$$

and since $\cos(-135°) = -\sqrt{2}/2$, the above expression reduces to

$$k_1 + k_2 = 60 \quad (2)$$

Differentiating (1) we obtain

$$\frac{dv_C}{dt} = 60\sqrt{2}\sin(t + 45°) + -2k_1e^{-2t} - 3k_2e^{-3t}$$

and

$$\left.\frac{dv_C}{dt}\right|_{t=0} = 60\sqrt{2}\sin(45°) - 2k_1 - 3k_2$$

or

$$\left.\frac{dv_C}{dt}\right|_{t=0} = 60 - 2k_1 - 3k_2 \quad (3)$$

Also, $\frac{dv_C}{dt} = \frac{i_C}{C} = \frac{i_L}{C}$ and at $t = 0$

$$\left.\frac{dv_C}{dt}\right|_{t=0} = \frac{i_L(0^-)}{C} = 0 \quad (4)$$

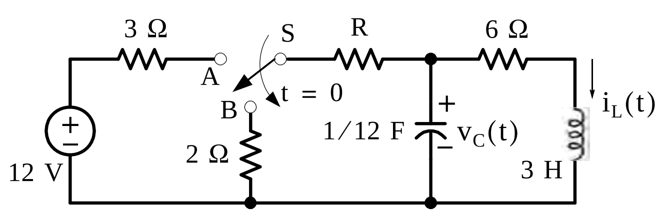
Equating (3) and (4) we obtain

$$2k_1 + 3k_2 = 60 \quad (5)$$

Simultaneous solution of (2) and (5) yields $k_1 = 120$ and $k_2 = -60$. Then, by substitution into (1) we obtain

$$v_C(t) = 60\sqrt{2}\cos(t - 135°) + 120e^{-2t} - 60e^{-3t}$$

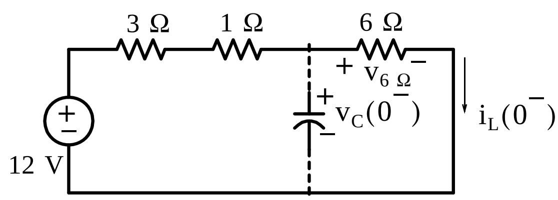
5.



We must first find the value of R before we can establish initial conditions for $i_L(0^-) = 0$ and $v_C(0^-) = 0$.

The condition for critical damping is $\sqrt{\alpha_P^2 - \omega_0^2} = 0$ where $\alpha_P = G/2C = 1/2R'C$ and $\omega_0^2 = 1/LC$. Then, $\alpha_P^2 = \left(\frac{1}{2R' \times 1/12}\right)^2 = \omega_0^2 = \frac{1}{3 \times 1/12}$ where $R' = R + 2\ \Omega$. Therefore, $\left(\frac{12}{2(R+2)}\right)^2 = 4$, or $\left(\frac{6}{R+2}\right)^2 = 4$, or $(R+2)^2 = 36/4 = 9$, or $R + 2 = 3$ and thus $R = 1$.

At $t = 0^-$ the circuit is as shown below.



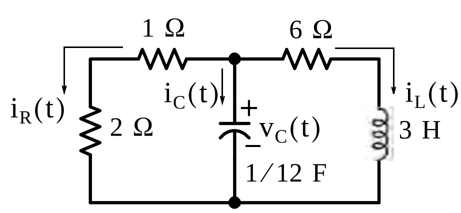
From the circuit above

$$v_C(0^-) = v_{6\,\Omega} = \frac{6}{3+1+6} \times 12 = 7.2\ \text{V}$$

and

$$i_L(0^-) = \frac{v_{6\,\Omega}}{6} = \frac{7.2}{6} = 1.2\ \text{A}$$

At $t = 0^+$ the circuit is as shown below.



Since the circuit is critically damped, the solution has the form

$$v_C(t) = e^{-\alpha_P t}(k_1 + k_2 t)$$

where $\alpha_P = \left(\frac{1}{2(1+2) \times 1/12}\right) = 2$ and thus

$$v_C(t) = e^{-2t}(k_1 + k_2 t) \quad (1)$$

With the initial condition $v_C(0^-) = 7.2\ \text{V}$ relation (1) becomes $7.2 = e^0(k_1 + 0)$ or $k_1 = 7.2\ \text{V}$ and (1) simplifies to

$$v_C(t) = e^{-2t}(7.2 + k_2 t) \quad (2)$$

Differentiating (2) we obtain

$$\frac{dv_C}{dt} = k_2 e^{-2t} - 2e^{-2t}(7.2 + k_2 t)$$

and

$$\left.\frac{dv_C}{dt}\right|_{t=0} = k_2 - 2(7.2 + 0) = k_2 - 14.4 \quad (3)$$

Also, $\frac{dv_C}{dt} = \frac{i_C}{C}$ and at $t = 0$

$$\left.\frac{dv_C}{dt}\right|_{t=0} = \frac{i_C(0)}{C} = \frac{0}{C} = 0 \quad (4)$$

because at $t = 0$ the capacitor is an open circuit.

Equating (3) and (4) we obtain $k_2 - 14.4 = 0$ or $k_2 = 14.4$ and by substitution into (2)

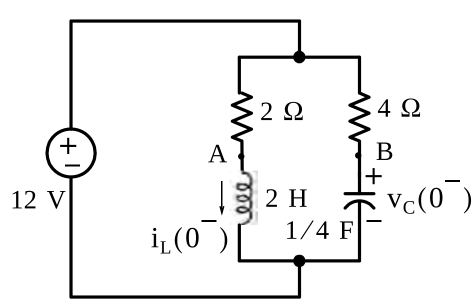
$$v_C(t) = e^{-2t}(7.2 + 14.4t) = 7.2e^{-2t}(2t + 1)$$

We find $i_L(t)$ from $i_R(t) + i_C(t) + i_L(t) = 0$ or $i_L(t) = -i_C(t) - i_R(t)$ where $i_C(t) = C(dv_C/dt)$ and $i_R(t) = v_R(t)/(1 + 2) = v_C(t)/3$. Then,

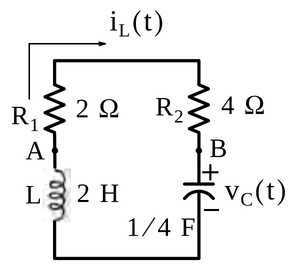
$$i_L(t) = -\frac{1}{12}(-14.4e^{-2t}(2t + 1) + 14.4e^{-2t}) - \frac{7.2}{3}e^{-2t}(2t + 1) = -2.4e^{-2t}(t + 1)$$

6.

At $t = 0^-$ the circuit is as shown below where $i_L(0^-) = 12/2 = 6\ A$, $v_C(0^-) = 12\ V$, and thus the initial conditions have been established.



For $t > 0$ the circuit is as shown below.



For this circuit

$$(R_1 + R_2) i_L + v_C + L\frac{di_L}{dt} = 0$$

and with $i_L = i_C = C(dv_C/dt)$ the above relation can be written as

$$(R_1 + R_2) C\frac{dv_C}{dt} + LC\frac{d^2 v_C}{dt^2} + v_C = 0$$

$$\frac{d^2 v_C}{dt^2} + \frac{(R_1 + R_2)}{L}\frac{dv_C}{dt} + \frac{1}{LC} v_C = 0$$

$$\frac{d^2 v_C}{dt^2} + 3\frac{dv_C}{dt} + 2v_C = 0$$

The characteristic equation of the last expression above yields $s_1 = -1$ and $s_2 = -2$ and thus

$$v_C(t) = k_1 e^{-t} + k_2 e^{-2t} \quad (1)$$

With the initial condition $v_C(0^-) = 12\ V$ and (1) we obtain

$$k_1 + k_2 = 12 \quad (2)$$

Differentiating (1) we obtain

$$\frac{dv_C}{dt} = -k_1 e^{-t} - 2k_2 e^{-2t}$$

and

$$\left.\frac{dv_C}{dt}\right|_{t=0} = -k_1 - 2k_2 \quad (3)$$

Also, $\frac{dv_C}{dt} = \frac{i_C}{C} = \frac{i_L}{C}$ and at $t = 0$

$$\left.\frac{dv_C}{dt}\right|_{t=0} = \frac{i_L(0)}{C} = \frac{6}{1/4} = 24 \quad (4)$$

From (3) and (4)

$$-k_1 - 2k_2 = 24 \quad (5)$$

and from (2) and (5) $k_1 = 48$ and $k_2 = -36$. By substitution into (1) we obtain

$$v_C(t) = 48e^{-t} - 36e^{-2t}$$

Thus,

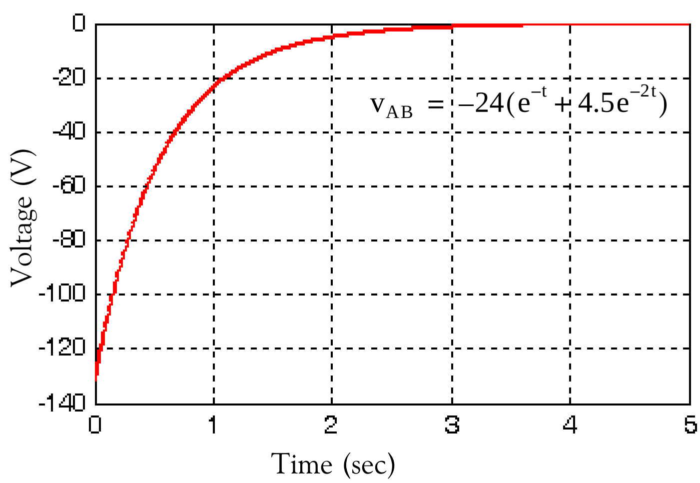
$$v_{AB} = v_L(t) - v_C(t) = L\frac{di_L}{dt} - v_C(t) = LC\frac{d^2 i_C}{dt^2} - v_C(t)$$

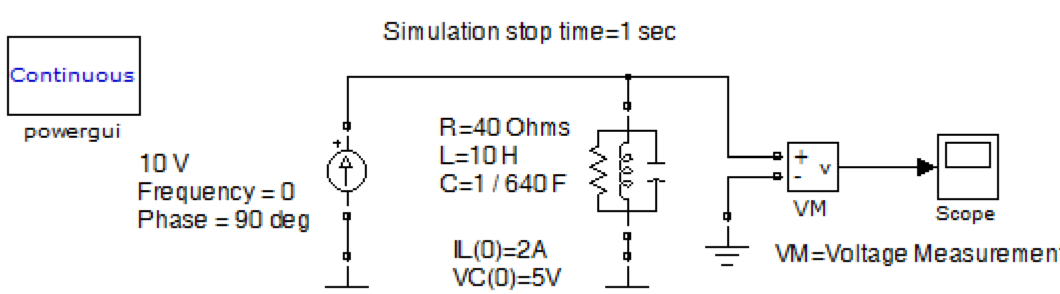
$$= 0.5\left(\frac{d^2}{dt^2}(48e^{-t} - 36e^{-2t})\right) - 48e^{-t} - 36e^{-2t}$$

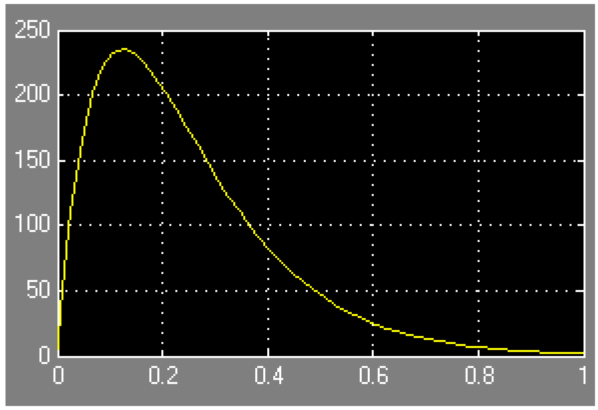
$$= 0.5(48e^{-t} - 144e^{-2t}) - 48e^{-t} - 36e^{-2t}$$

$$= -24e^{-t} - 108e^{-2t} = -24(e^{-t} + 4.5e^{-2t})$$

The plot for v_{AB} is shown below.






250
200
150
100
50
0
0
0.2
0.4
0.6
0.8
1

Chapter 2

Resonance

This chapter defines series and parallel resonance. The quality factor Q is then defined in terms of the series and parallel resonant frequencies. The half–power frequencies and bandwidth are also defined in terms of the resonant frequency.

2.1 Series Resonance

Consider phasor series RLC circuit of Figure 2.1.

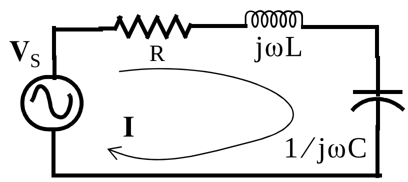


Figure 2.1. Series RLC phasor circuit

The impedance Z is

$$\text{Impedance} = Z = \frac{\text{Phasor Voltage}}{\text{Phasor Current}} = \frac{\mathbf{V_S}}{\mathbf{I}} = R + j\omega L + \frac{1}{j\omega C} = R + j\left(\omega L - \frac{1}{\omega C}\right) \tag{2.1}$$

or

$$Z = \sqrt{R^2 + (\omega L - 1/\omega C)^2}\angle\tan^{-1}(\omega L - 1/\omega C)/R \tag{2.2}$$

Therefore, the magnitude and phase angle of the impedance are:

$$|Z| = \sqrt{R^2 + (\omega L - 1/\omega C)^2} \tag{2.3}$$

and

$$\theta_Z = \tan^{-1}(\omega L - 1/\omega C)/R \tag{2.4}$$

The components of $|Z|$ are shown on the plot in Figure 2.2.

The frequency at which the capacitive reactance $X_C = 1/\omega C$ and the inductive reactance $X_L = \omega L$ are equal is called the *resonant frequency*. The resonant frequency is denoted as ω_0 or f_0 and these can be expressed in terms of the inductance L and capacitance C by equating the reactances, that is,

$$\omega_0 L = \frac{1}{\omega_0 C}$$

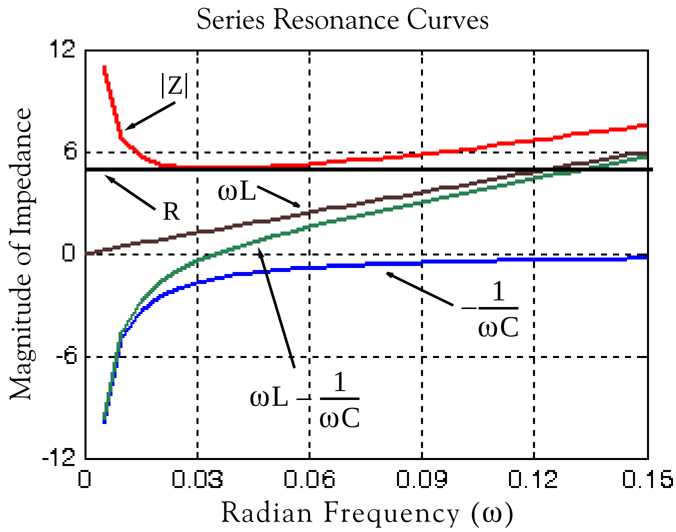


Figure 2.2. The components of $|Z|$ *in a series RLC circuit*

or

$$\omega_0^2 = \frac{1}{LC}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} \tag{2.5}$$

and

$$f_0 = \frac{1}{2\pi\sqrt{LC}} \tag{2.6}$$

We observe that at resonance $Z_0 = R$ where Z_0 denotes the impedance value at resonance, and $\theta_Z = 0$. In our subsequent discussion the subscript zero will be used to indicate that the circuit variables are at resonance.

Example 2.1

For the circuit shown in Figure 2.3, compute $\mathbf{I}_0$, ω_0, C, $\mathbf{V}_{R0}$, $|\mathbf{V}_{L0}|$, and $|\mathbf{V}_{C0}|$. Then, draw a phasor diagram showing $\mathbf{V}_{R0}$, $|\mathbf{V}_{L0}|$, and $|\mathbf{V}_{C0}|$.

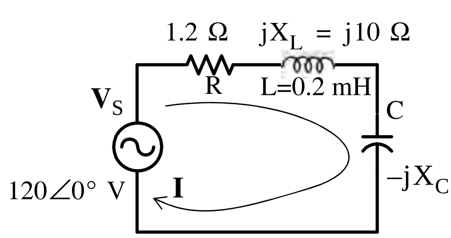


Figure 2.3. Circuit for Example 2.1

Solution:

At resonance,

$$jX_L = -jX_C$$

and thus

$$Z_0 = R = 1.2\ \Omega$$

Then,

$$\mathbf{I_0} = \frac{120\ V}{1.2\ \Omega} = 100\ A$$

Since

$$X_{L0} = \omega_0 L = 10\ \Omega$$

it follows that

$$\omega_\mathbf{0} = \frac{10}{L} = \frac{10}{0.2 \times 10^{-3}} = 50000\ rad/s$$

Therefore,

$$X_{C0} = X_{L0} = 10 = \frac{1}{\omega_0 C}$$

or

$$C = \frac{1}{10 \times 50000} = 2\ \mu F$$

Now,

$$\mathbf{V_{R0}} = RI_0 = 1.2 \times 100 = 120\ V$$

$$|\mathbf{V_{L0}}| = \omega_0 L I_0 = 50000 \times 0.2 \times 10^{-3} \times 100 = 1000$$

and

$$|\mathbf{V_{C0}}| = \frac{1}{\omega_0 C} I_0 = \frac{1}{50000 \times 2 \times 10^{-6}} \times 100 = 1000\ V$$

The phasor diagram showing $\mathbf{V}_{R0}$, $|\mathbf{V}_{L0}|$, and $|\mathbf{V}_{C0}|$ is shown in Figure 2.4.

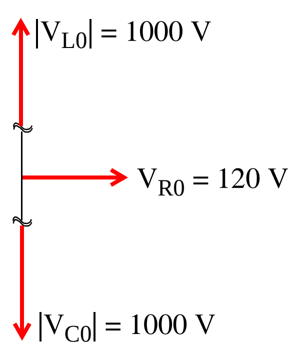


Figure 2.4. Phasor diagram for Example 2.1

Figure 2.4 reveals that $|\mathbf{V}_{L0}| = |\mathbf{V}_{C0}| = 1000\ V$ and these voltages are much higher than the applied voltage of $120\ V$. This illustrates the useful property of resonant circuits to develop high voltages across capacitors and inductors.

2.2 Quality Factor Q_{0S} in Series Resonance

The *quality factor* * is an important parameter in resonant circuits. Its definition is derived from the following relations:

At resonance,

$$\omega_0 L = \frac{1}{\omega_0 C}$$

and

$$\mathbf{I}_0 = \frac{|\mathbf{V}_S|}{R}$$

Then

$$|\mathbf{V}_{L0}| = \omega_0 L \mathbf{I}_0 = \omega_0 L \frac{|\mathbf{V}_S|}{R} = \frac{\omega_0 L}{R}|\mathbf{V}_S| \tag{2.7}$$

and

$$|\mathbf{V}_{C0}| = \frac{1}{\omega_0 C}\mathbf{I}_0 = \frac{1}{\omega_0 C}\frac{|\mathbf{V}_S|}{R} = \frac{1}{\omega_0 RC}|\mathbf{V}_S| \tag{2.8}$$

At series resonance the left sides of (2.7) and (2.8) are equal and therefore,

$$\frac{\omega_0 L}{R} = \frac{1}{\omega_0 RC}$$

Then, by definition

$$Q_{0S} = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 RC} \tag{2.9}$$

Quality Factor at Series Resonance

In a practical circuit, the resistance R in the definition of Q_{0S} above, represents the resistance of the inductor and thus the quality factor Q_{0S} *is a measure of the energy storage property of the inductance* L *in relation to the energy dissipation property of the resistance* R *of that inductance.*

In terms of Q_{0S}, the magnitude of the voltages across the inductor and capacitor are

$$|\mathbf{V}_{L0}| = |\mathbf{V}_{C0}| = Q_{0S}|\mathbf{V}_S| \tag{2.10}$$

and therefore, we say that there is a "resonant" rise in the voltage across the reactive devices and it is equal to the Q_{0S} times the applied voltage. Thus in Example 2.1,

$$Q_{0S} = \frac{|\mathbf{V}_{L0}|}{|\mathbf{V}_S|} = \frac{|\mathbf{V}_{C0}|}{|\mathbf{V}_S|} = \frac{1000}{120} = \frac{25}{3}$$

* *We denote the quality factor for series resonant circuits as* Q_{0S}, *and the quality factor for parallel resonant circuits as* Q_{0P}.

The quality factor Q is also a measure of *frequency selectivity*. Thus, we say that a circuit with a high Q has a high selectivity, whereas a low Q circuit has low selectivity. The high frequency selectivity is more desirable in parallel circuits as we will see in the next section.

We will see later that

$$Q = \frac{\omega_0}{\omega_2 - \omega_1} = \frac{\text{Resonant Frequency}}{\text{Bandwidth}} \tag{2.11}$$

Figure 2.5 shows the relative response versus ω for $Q = 25, 50$, and 100 where we observe that highest Q provides the best frequency selectivity, i.e., higher rejection of signal components outside the bandwidth $BW = \omega_2 - \omega_1$ which is the difference in the $3\ dB$ frequencies. The curves were created with the MATLAB script below.

```
w=450:1:550; x1=1./(1+25.^2*(w./500–500./w).^2); plot(w,x1);...
x2=1./(1+50.^2*(w./500–500./w).^2); plot(w,x2);...
x3=1./(1+100.^2*(w./500–500./w).^2); plot(w,x3);...
plot(w,x1,w,x2,w,x3); grid
```

We also observe from (2.9) that selectivity depends on R and this dependence is shown on the plot of Figure 2.6.

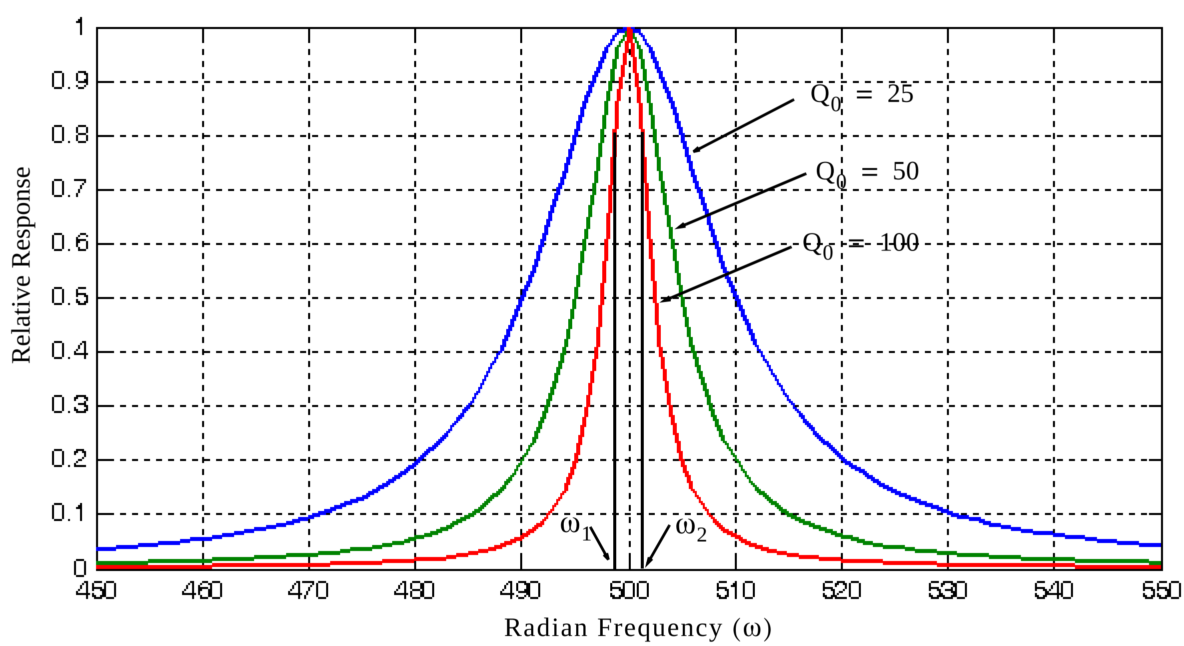


Figure 2.5. Selectivity curves with $Q = 25, 50$, and 100

The curves in Figure 2.6 were created with the MATLAB script below.

```
w=0:10:6000; R1=0.5; R2=1; L=10^(–3); C=10^(–4); Y1=1./sqrt(R1.^2+(w.*L–1./(w.*C)).^2);...
Y2=1./sqrt(R2.^2+(w.*L–1./(w.*C)).^2); plot(w,Y1,w,Y2)
```

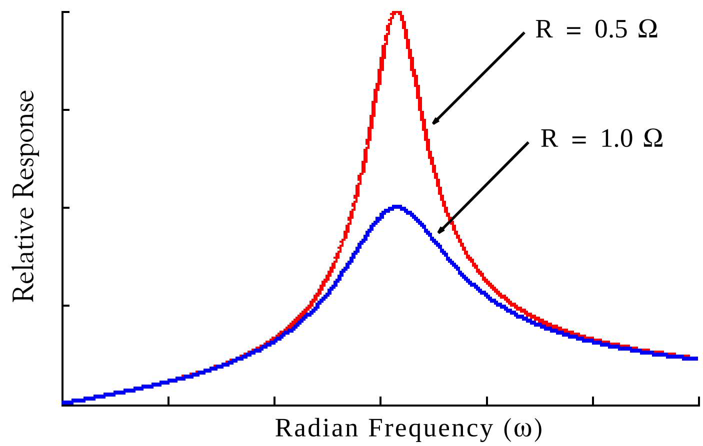


Figure 2.6. Selectivity curves with different values of R

If we keep one reactive device, say L, constant while varying C, the relative response "shifts" as shown in Figure 2.7, but the general shape does not change.

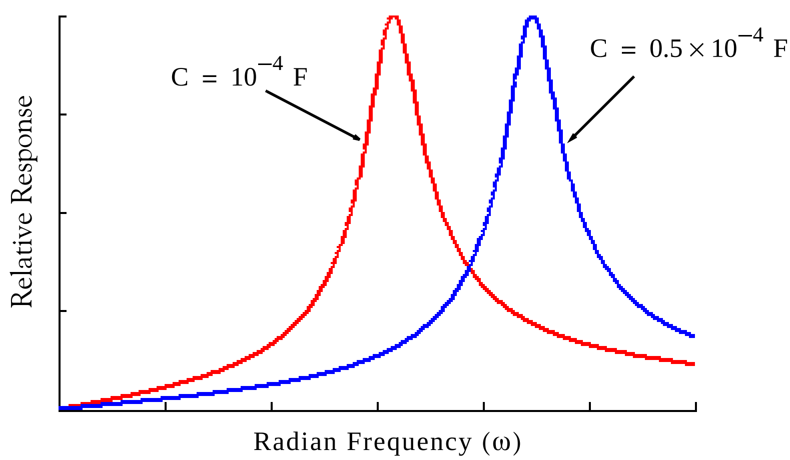


Figure 2.7. Relative response with constant L and variable C

The curves in Figure 2.7 were created with the MATLAB script below.

```
w=0:10:6000; R=0.5; L=10^(−3); C1=10^(−4); C2=0.5*10^(−4);...
Y1=1./sqrt(R.^2+(w.*L−1./(w.*C1)).^2);...
Y2=1./sqrt(R.^2+(w.*L−1./(w.*C2)).^2); plot(w,Y1,w,Y2)
```

2.3 Parallel Resonance

Parallel resonance (*antiresonance*) applies to parallel circuits such as that shown in Figure 2.8. The admittance Y for this circuit is given by

$$\text{Admittance} = Y = \frac{\text{Phasor Current}}{\text{Phasor Voltage}} = \frac{\mathbf{I}_S}{\mathbf{V}} = G + j\omega C + \frac{1}{j\omega L} = G + j\left(\omega C - \frac{1}{\omega L}\right)$$

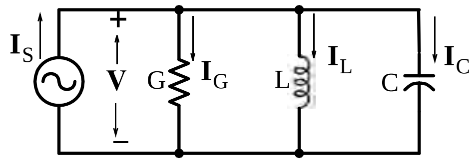


Figure 2.8. Parallel GLC circuit for defining parallel resonance

or

$$Y = \sqrt{G^2 + (\omega C - 1/\omega L)^2} \angle \tan^{-1}(\omega C - 1/\omega L)/G \qquad (2.12)$$

Therefore, the magnitude and phase angle of the admittance Y are:

$$|Y| = \sqrt{G^2 + (\omega C - 1/(\omega L))^2} \qquad (2.13)$$

and

$$\theta_Y = \tan^{-1}\frac{(\omega C - 1/\omega L)}{G} \qquad (2.14)$$

The frequency at which the inductive susceptance $B_L = 1/\omega L$ and the capacitive susceptance $B_C = \omega C$ are equal is, again, called the *resonant frequency* and it is also denoted as ω_0. We can find ω_0 in terms of L and C as before.

Since

$$\omega_0 C - \frac{1}{\omega_0 L}$$

then,

$$\boxed{\omega_0 = \frac{1}{\sqrt{LC}}} \qquad (2.15)$$

as before. The components of $|Y|$ are shown on the plot of Figure 2.9.

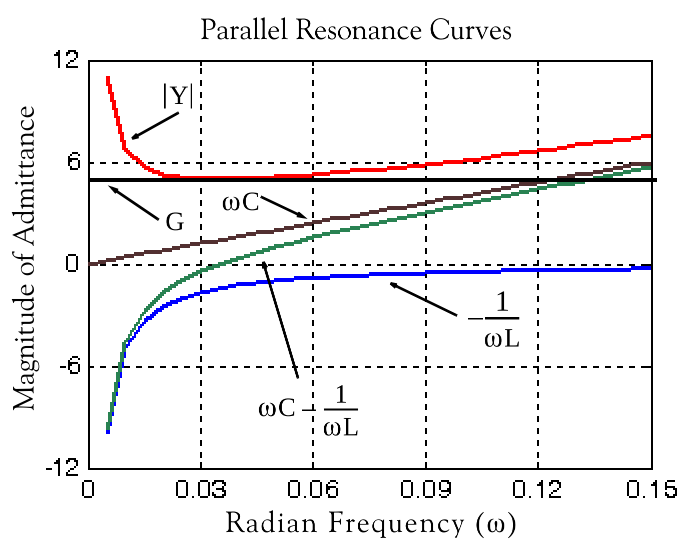


Figure 2.9. The components of $|Y|$ in a parallel RLC circuit

We observe that at this parallel resonant frequency,

$$Y_0 = G \tag{2.16}$$

and

$$\theta_Y = 0 \tag{2.17}$$

Example 2.2

For the circuit of Figure 2.10, $i_S(t) = 10\cos 5000t\ mA$. Compute $i_G(t)$, $i_L(t)$, and $i_C(t)$.

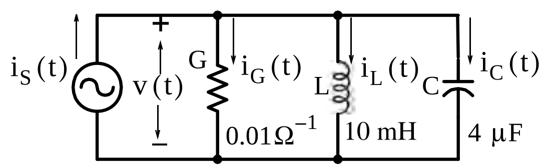


Figure 2.10. Circuit for Example 2.2

Solution:

The capacitive and inductive susceptances are

$$B_C = \omega C = 5000 \times 4 \times 10^{-6} = 0.02\ \Omega^{-1}$$

and

$$B_L = \frac{1}{\omega L} = \frac{1}{5000 \times 10 \times 10^{-3}} = 0.02\ \Omega^{-1}$$

and since $B_L = B_C$, the given circuit operates at parallel resonance with $\omega_0 = 5000\ rad/s$. Then,

$$Y_0 = G = 0.01\ \Omega^{-1}$$

and

$$i_G(t) = i_S(t) = 10\cos 5000t\ mA$$

Next, to compute $i_L(t)$ and $i_C(t)$, we must first find $v_0(t)$. For this example,

$$v_0(t) = \frac{i_G(t)}{G} = \frac{10\cos 5000t\ mA}{0.01\ \Omega^{-1}} = 1000\cos 5000t\ mV = \cos 5000t\ V$$

In phasor form,

$$v_0(t) = \cos 5000t\ V \Leftrightarrow \mathbf{V}_0 = 1\angle 0°$$

Now,

$$\mathbf{I}_{L0} = (-jB_L)\mathbf{V}_0 = (1\angle -90°)(0.02)(1\angle 0°) = 0.02\angle -90°\ A$$

and in the t-domain,

$$\mathbf{I}_{L0} = 0.02\angle -90°\ A \Leftrightarrow i_{L0}(t) = 0.02\cos(5000t - 90°)\ A$$

or

$$i_{L0}(t) = 20\sin 5000t \text{ mA}$$

Similarly,

$$\mathbf{I}_{C0} = jB_C\mathbf{V}_0 = (1\angle 90°)(0.02)(1\angle 0°) = 0.02\angle 90° \text{ A}$$

and in the t –domain,

$$\mathbf{I}_{C0} = 0.02\angle 90° \text{ A} \Leftrightarrow i_{C0}(t) = 0.02\cos(5000t + 90°) \text{ A}$$

or

$$i_{C0}(t) = -20\sin 5000t \text{ mA}$$

We observe that $\mathbf{i_{L0}(t) + i_{C0}(t)} = 0$ as expected.

2.4 Quality Factor Q_{0P} in Parallel Resonance

At parallel resonance,

$$\omega_0 C = \frac{1}{\omega_0 L}$$

and

$$\mathbf{V}_0 = \frac{|\mathbf{I}_S|}{G}$$

Then,

$$|\mathbf{I}_{C0}| = \omega_0 C\mathbf{V}_0 = \omega_0 C\frac{|\mathbf{I}_S|}{G} = \frac{\omega_0 C}{G}|\mathbf{I}_S| \tag{2.18}$$

Also,

$$|\mathbf{I}_{L0}| = \frac{1}{\omega_0 L}\mathbf{V}_0 = \frac{1}{\omega_0 L}\frac{|\mathbf{V}_S|}{G} = \frac{1}{\omega_0 GL}|\mathbf{I}_S| \tag{2.19}$$

At parallel resonance the left sides of (2.18) and (2.19) are equal and therefore,

$$\frac{\omega_0 C}{G} = \frac{1}{\omega_0 GL}$$

Now, by definition

$$Q_{0P} = \frac{\omega_0 C}{G} = \frac{1}{\omega_0 GL} \tag{2.20}$$

Quality Factor at Parallel Resonance

The above expressions indicate that at parallel resonance, it is possible to develop high currents through the capacitors and inductors. This was found to be true in Example 2.2.

2.5 General Definition of Q

The general (and best) definition of Q is

$$Q = 2\pi\frac{\text{Maximum Energy Stored}}{\text{Energy Dissipated per Cycle}} \tag{2.21}$$

Essentially, the resonant frequency is the frequency at which the inductor gives up energy just as fast as the capacitor requires it during one quarter cycle, and absorbs energy just as fast as it is released by the capacitor during the next quarter cycle. This can be seen from Figure 2.11 where at the instant of maximum current the energy is all stored in the inductance, and at the instant of zero current all the energy is stored in the capacitor.

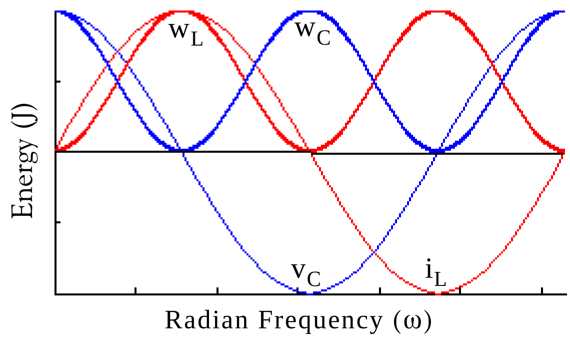


Figure 2.11. Waveforms for W_L *and* W_C *at resonance*

2.6 Energy in L and C at Resonance

For a series RLC circuit we let

$$i = I_p \cos\omega t = C\frac{dv_C}{dt}$$

Then,

$$v_C = \frac{I_p}{\omega C}\sin\omega t$$

Also,

$$W_L = \frac{1}{2}Li^2 = \frac{1}{2}LI_p^2\cos^2\omega t \tag{2.22}$$

and

$$W_C = \frac{1}{2}Cv^2 = \frac{1}{2}\frac{I_p^2}{\omega^2 C}\sin^2\omega t \tag{2.23}$$

Therefore, by (2.22) and (2.23), the total energy W_T at any instant is

$$\boxed{W_T = W_L + W_C = \frac{1}{2}I_p^2\left[L\cos^2\omega t + \frac{1}{\omega^2 C}\sin^2\omega t\right]} \tag{2.24}$$

and this expression is true for any series circuit, that is, the circuit need not be at resonance. However, at resonance,

$$\omega_0 L = \frac{1}{\omega_0 C}$$

or

$$L = \frac{1}{\omega_0^2 C}$$

By substitution into (2.24),

$$W_T = \frac{1}{2}I_p^2[L\cos^2\omega_0 t + L\sin^2\omega_0 t] = \frac{1}{2}I_p^2 L = \frac{1}{2}I_p^2\frac{1}{\omega_0^2 C} \qquad (2.25)$$

and (2.25) shows that the total energy W_T is dependent only on the circuit constants L, C and resonant frequency, but it is independent of time.

Next, using the general definition of Q we obtain:

$$Q_{0S} = 2\pi\frac{\text{Maximum Energy Stored}}{\text{Energy Dissipated per Cycle}} = 2\pi\frac{(1/2)I_p^2 L}{(1/2)I_p^2 R/f_0} = 2\pi\frac{f_0 L}{R}$$

or

$$Q_{0S} = \frac{\omega_0 L}{R} \qquad (2.26)$$

and we observe that (2.26) is the same as (2.9). Similarly,

$$Q_{0S} = 2\pi\frac{\text{Maximum Energy Stored}}{\text{Energy Dissipated per Cycle}} = 2\pi\frac{(1/2)I_p^2(1/\omega_0^2 C)}{(1/2)I_p^2 R/f_0} = 2\pi\frac{f_0}{\omega_0^2 RC}$$

or

$$Q_{0S} = \frac{\omega_0}{\omega_0{}^2 RC} = \frac{1}{\omega_0 RC} \qquad (2.27)$$

and this is also the same as (2.9).

Following the same procedure for a simple GLC (or RLC) parallel circuit we can show that:

$$Q_{0P} = \frac{\omega_0 C}{G} = \frac{1}{\omega_0 LG} \qquad (2.28)$$

and this is the same as (2.20).

2.7 Half−Power Frequencies − Bandwidth

Parallel resonance is by far more important and practical than series resonance and therefore, the remaining discussion will be on parallel GLC (or RLC) circuits. The plot in Figure 2.12 shows the magnitude of the voltage response versus radian frequency for a typical parallel RLC circuit.

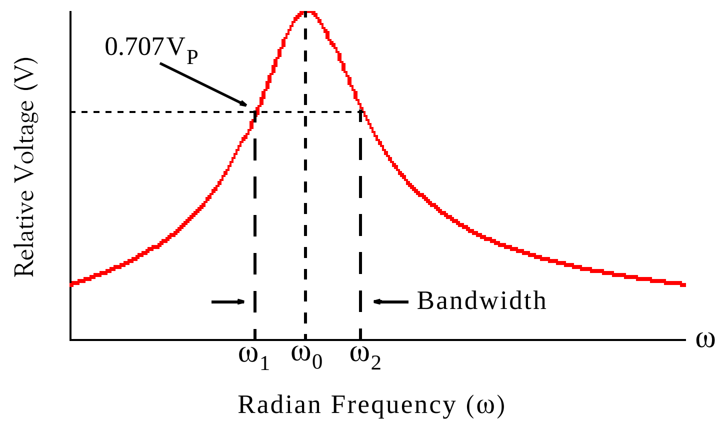


Figure 2.12. Relative voltage vs.radian frequency in a parallel RLC *circuit*

By definition, the *half–power frequencies* ω_1 and ω_2 in Figure 2.12 are the frequencies at which the magnitude of the input admittance of a parallel resonant circuit, is *greater* than the magnitude at resonance by a factor of $\sqrt{2}$, or equivalently, the frequencies at which the magnitude of the input impedance of a parallel resonant circuit, is *less* than the magnitude at resonance by a factor of $\sqrt{2}$ as shown above. We observe also, that ω_1 and ω_2 are not exactly equidistant from ω_0. However, it is convenient to assume that they are equidistant, and unless otherwise stated, this assumption will be followed in the subsequent discussion.

We call ω_1 the *lower half–power point*, and ω_2 the *upper half–power point*. The difference $\omega_2 - \omega_1$ is the *half–power bandwidth* BW, that is,

$$\boxed{\text{Bandwidth} = BW = \omega_2 - \omega_1} \tag{2.29}$$

The names half–power frequencies and half–power bandwidth arise from the fact that the power at these frequencies drop to 0.5 since $(\sqrt{2}/2)^2 = 0.5$.

The bandwidth BW can also be expressed in terms of the quality factor Q as follows:

Consider the admittance

$$Y = G + j\left(\omega C - \frac{1}{\omega L}\right)$$

Multiplying the j term by $G\left(\dfrac{\omega_0}{\omega_0 G}\right)$, we obtain

$$Y = G + jG\left(\frac{\omega\omega_0 C}{\omega_0 G} - \frac{\omega_0}{\omega\omega_0 LG}\right)$$

Recalling that for parallel resonance

$$Q_{0P} = \frac{\omega_0 C}{G} = \frac{1}{\omega_0 LG}$$

by substitution we obtain

$$Y = G\left[1 + jQ_{0P}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)\right] \tag{2.30}$$

and if $\omega = \omega_0$, then

$$Y = G$$

Next, we want to find the bandwidth $\omega_2 - \omega_1$ in terms of the quality factor Q_{0P}. At the half–power points, the magnitude of the admittance is $(\sqrt{2}/2)|Y_p|$ and, if we use the half–power points as reference, then to obtain the admittance value of

$$||Y_{max}| = \sqrt{2}G$$

we must set

$$Q_{0P}\left(\frac{\omega_2}{\omega_0} - \frac{\omega_0}{\omega_2}\right) = 1$$

for $\omega = \omega_2$.

We must also set

$$Q_{0P}\left(\frac{\omega_1}{\omega_0} - \frac{\omega_0}{\omega_1}\right) = -1$$

for $\omega = \omega_1$.

Recalling that $|(1 \pm j1)| = \sqrt{2}$ and solving the above expressions for ω_1 and ω_2, we obtain

$$\omega_2 = \left[\sqrt{1 + \left(\frac{1}{2Q_{0P}}\right)^2} + \frac{1}{2Q_{0P}}\right] \tag{2.31}$$

and

$$\omega_1 = \left[\sqrt{1 + \left(\frac{1}{2Q_{0P}}\right)^2} - \frac{1}{2Q_{0P}}\right] \tag{2.32}$$

Subtraction of (2.32) from (2.31) yields

$$\boxed{BW = \omega_2 - \omega_1 = \frac{\omega_0}{Q_{0P}}} \tag{2.33}$$

or

$$\boxed{BW = f_2 - f_1 = \frac{f_0}{Q_{0P}}} \tag{2.34}$$

As mentioned earlier, ω_1 and ω_2 are not equidistant from ω_0. In fact, the resonant frequency

ω_0 is the *geometric mean** of ω_1 and ω_2, that is,

$$\omega_0 = \sqrt{\omega_1 \omega_2} \tag{2.35}$$

This can be shown by multiplication of the two expressions in (2.31) and (2.32) and substitution into (2.33).

Example 2.3

For the network of Figure 2.13, find:

a. ω_0

b. Q_{0P}

c. BW

d. ω_1

e. ω_2

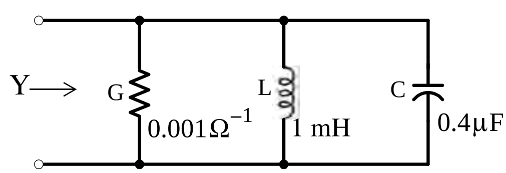


Figure 2.13. Network for Example 2.3

Solution:

a.

$$\omega_0^2 = \frac{1}{LC} = \frac{1}{1 \times 10^{-3} \times 0.4 \times 10^{-6}} = 25 \times 10^8$$

or

$$\omega_0 = 50000 \text{ r/s} \qquad f_0 \approx 8000 \text{ Hz}$$

b.

$$Q_{0P} = \frac{\omega_0 C}{G} = \frac{5 \times 10^4 \times 0.4 \times 10^{-6}}{10^{-3}} = 20$$

c.

$$BW = \frac{\omega_0}{Q_{0P}} = \frac{50000}{20} = 2500 = \text{ rad/s}$$

d.

$$\omega_1 = \omega_0 - \frac{BW}{2} = 50000 - 1250 = 48750 \text{ rad/s}$$

* *The geometric mean of n positive numbers* $a_1, a_2, \ldots, a_n$ *is the nth root of the product.* $a_1 \cdot a_2 \cdot \ldots \cdot a_n$

e.

$$\omega_2 = \omega_0 + \frac{BW}{2} = 50000 + 1250 = 51250 \text{ rad/s}$$

The SimPowerSystems model for the circuit in Figure 2.13 is shown in Figure 2.14.

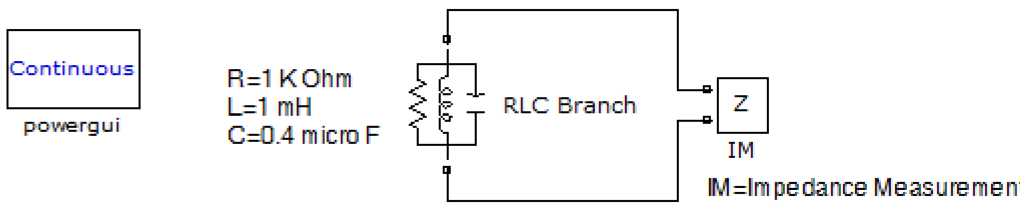


Figure 2.14. SimPowerSystems model for the circuit in Figure 2.13

To observe the impedance of the parallel RLC circuit in Figure 2.14 we double-click the powergui block to open the Simulation and configuration options window shown in Figure 2.15, we click the Impedance vs Frequency option, and the magnitude an phase of the impedance as a function of frequency are shown in Figure 2.16.

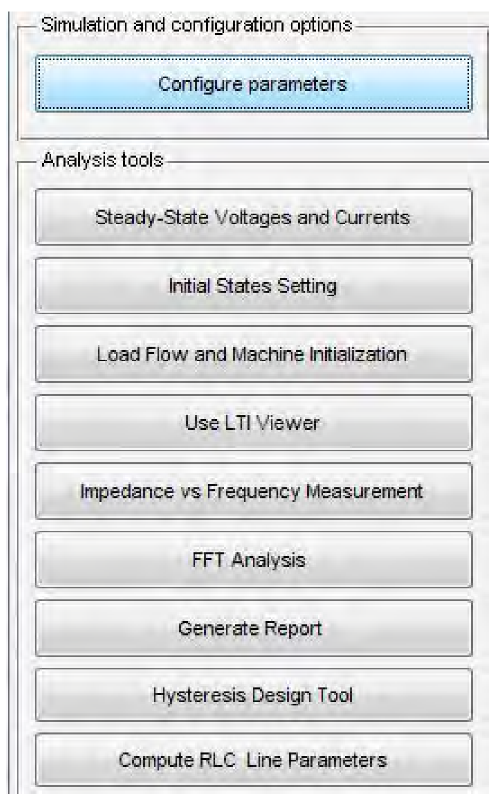


Figure 2.15. Simulation and configuration options in the powergui

In Figure 2.16, the frequency is in logarithmic scale for the frequency range 10^3 Hz to 10^5 Hz as shown on the right pane. The resonant frequency is about 8 KHz and at that frequency the magnitude of the impedance is 1 KΩ (purely resistive) and the phase is 0 degrees.

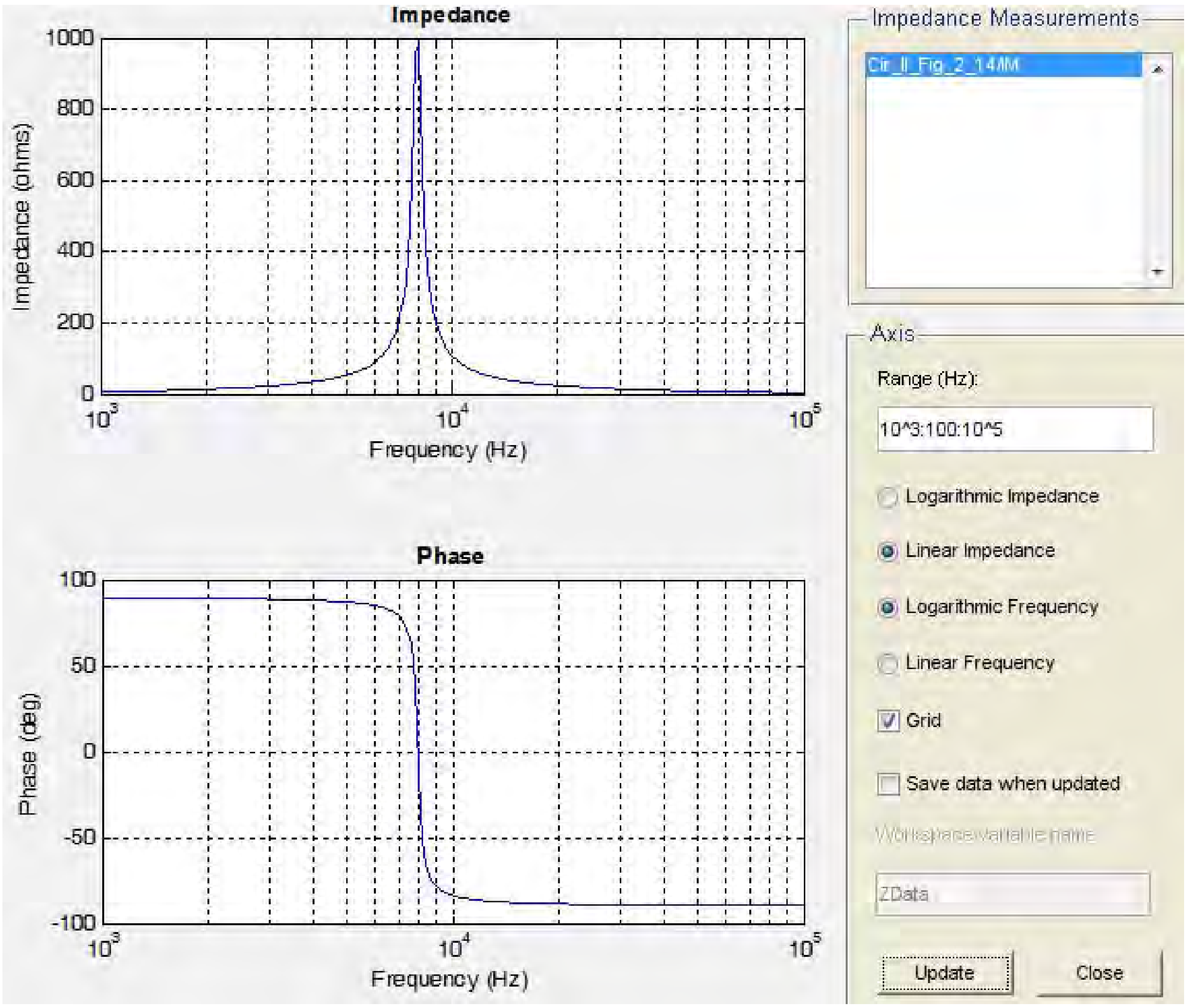


Figure 2.16. Plots for the magnitude and phase for the model in Figure 2.14

2.8 A Practical Parallel Resonant Circuit

In our previous discussion, we assumed that the inductors are ideal, but a *real* inductor has some resistance. The circuit shown in Figure 2.17 is a practical parallel resonant circuit. To derive an expression for its resonant frequency, we make use of the fact that the resonant frequency is independent of the conductance G and, for simplicity, it is omitted from the network of Figure 2.17. We will therefore, find an expression for the network of Figure 2.18.

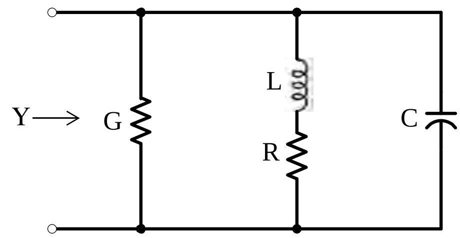


Figure 2.17. A practical parallel resonant circuit

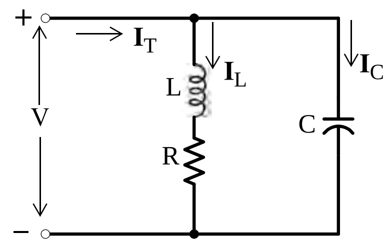


Figure 2.18. Simplified network for derivation of the resonant frequency

For the network of Figure 2.18,

$$I_L = \frac{V}{R + j\omega L} = \frac{(R - j\omega L)}{R^2 + (\omega L)^2}V$$

and

$$I_C = \frac{V}{1/(j\omega C)} = (j\omega C)V$$

where

$$Re\{I_L\} = \frac{R}{R^2 + (\omega L)^2}V$$

and

$$Im\{I_L\} = \frac{-\omega L}{R^2 + (\omega L)^2}V$$

Also,

$$Re\{I_C\} = 0$$

and

$$Im\{I_C\} = (\omega C)V$$

Then,

$$\begin{aligned} I_T &= I_L + I_C = [Re\{I_L\} + Im\{I_L\}]V + [Re\{I_C\} + Im\{I_C\}]V \\ &= [Re\{I_L\} + Re\{I_C\} + Im\{I_L\} + Im\{I_C\}]V \\ &= [Re\{I_T\} + Im\{I_T\}]V \end{aligned} \quad (2.36)$$

Now, at resonance, the imaginary component of I_T must be zero, that is,

$$Im\{I_T\} = Im\{I_L\} + Im\{I_C\} = \left(\omega_0 C - \frac{\omega_0 L}{R^2 + (\omega_0 L)^2}\right)V = 0$$

and solving for ω_0 we obtain

$$\boxed{\omega_0 = \sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}} \qquad (2.37)$$

or

$$\boxed{f_0 = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}} \qquad (2.38)$$

We observe that for $R = 0$, (2.37) reduces to $\omega_0 = \frac{1}{\sqrt{LC}}$ as before.

2.9 Radio and Television Receivers

When a radio or TV receiver is tuned to a particular station or channel, it is set to operate at the resonant frequency of that station or channel. As we have seen, a parallel circuit has high impedance (low admittance) at its resonant frequency. Therefore, it attenuates signals at all frequencies except the resonant frequency.

We have also seen that one particular inductor and one particular capacitor will resonate to one frequency only. Varying either the inductance or the capacitance of the tuned circuit, will change the resonant frequency. Generally, the inductance is kept constant and the capacitor value is changed as we select different stations or channels.

The block diagram of Figure 2.19 is a typical AM *(Amplitude Modulation)* radio receiver.

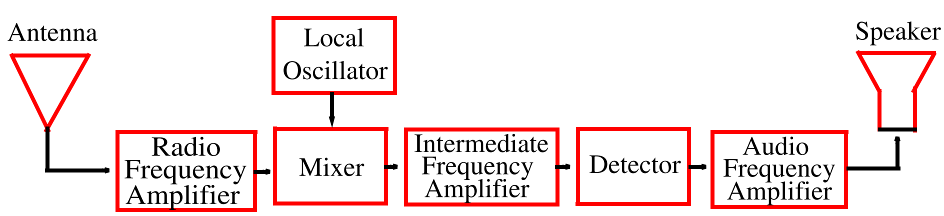


Figure 2.19. Block diagram of a typical AM radio receiver

The antenna picks up signals from several stations and these are fed into the Radio Frequency (RF) Amplifier which improves the *Signal–to–Noise* (S/N) ratio. The RF amplifier also serves as a preselector. This preselection suppresses the *image–frequency interference* as explained below.

When we tune to a station of, say $740\ \mathrm{KHz}$, we are setting the RF circuit to $740\ \mathrm{KHz}$ and at the same time the local oscillator is set at $740\ \mathrm{KHz} + 456\ \mathrm{KHz} = 1196\ \mathrm{KHz}$. This is accomplished by the capacitor in the RF amplifier which is also ganged to the local oscillator. These two signals, one of $740\ \mathrm{KHz}$ and the other of $1196\ \mathrm{KHz}$, are fed into the mixer whose output into the Intermediate Frequency (IF) amplifier is $456\ \mathrm{KHz}$; this is the difference between these two frequencies $(1196\ \mathrm{KHz} - 740\ \mathrm{KHz} = 456\ \mathrm{KHz})$.

The IF amplifier is always set at $456\ \mathrm{KHz}$ and therefore if the antenna picks another signal from another station, say $850\ \mathrm{KHz}$, it would be mixed with the local oscillator to produce a frequency of $1196\ \mathrm{KHz} - 850\ \mathrm{KHz} = 346\ \mathrm{KHz}$ but since the IF amplifier is set at $456\ \mathrm{KHz}$, the unwanted $850\ \mathrm{KHz}$ signal will not be amplified. Of course, in order to hear the signal at $850\ \mathrm{KHz}$ the radio receiver must be retuned to that frequency and the local oscillator frequency will be changed to $850\ \mathrm{KHz} + 456\ \mathrm{KHz} = 1306\ \mathrm{KHz}$ so that the difference of these frequencies will be again $456\ \mathrm{KHz}$.

Now let us assume that we select a station at $600\ \mathrm{KHz}$. Then, the local oscillator will be set to $600\ \mathrm{KHz} + 456\ \mathrm{KHz} = 1056\ \mathrm{KHz}$ so that the IF signal will again be $456\ \mathrm{KHz}$. Now, let us suppose that a powerful nearby station broadcasts at $1512\ \mathrm{KHz}$ and this signal is picked up by the mixer circuit. The difference between this signal and the local oscillator will also be $456\ \mathrm{KHz}$ $1512\ \mathrm{KHz} - 1056\ \mathrm{KHz} = 456\ \mathrm{KHz}$. The IF amplifier will then amplify both signals and the result will be a strong interference so that the radio speaker will produce unintelligent sounds. This interference is called *image–frequency interference* and it is reduced by the RF amplifier before entering the mixer circuit and for this reason the RF amplifier is said to act as a *preselector*.

The function of the detector circuit is to convert the IF signal which contains both the carrier and the desired signal to an audio signal and this signal is amplified by the Audio Frequency (AF) Amplifier whose output appears at the radio speaker.

Example 2.4

A radio receiver with a parallel GLC circuit whose inductance is $L = 0.5\ \mathrm{mH}$ is tuned to a radio station transmitting at $810\ \mathrm{KHz}$ frequency.

a. What is the value of the capacitor of this circuit at this resonant frequency?

b. What is the value of conductance G if $Q_{0P} = 75$?

c. If a nearby radio station transmits at $740\ \mathrm{KHz}$ and both signals picked up by the antenna have the same current amplitude $I\ (\mu A)$, what is the ratio of the voltage at $810\ \mathrm{KHz}$ to the voltage at $740\ \mathrm{KHz}$?

Solution:

a.

$$\omega_0^2 = \frac{1}{LC}$$

or

$$f_0^2 = \frac{1}{4\pi^2 LC}$$

Then,

$$C = \frac{1}{4\pi^2 0.5\times 10^{-3}\times(810\times 10^3)^2} = 77.2\ \text{pF}$$

b.

$$Q_{0P} = \frac{\omega_0 C}{G}$$

or

$$G = \frac{2\pi f_0 C}{Q_{0P}} = \frac{2\pi\times 8.1\times 10^5\times 77.2\times 10^{-12}}{75} = 5.4\ \mu\Omega^{-1}$$

c.

$$|V_{810\ KHz}| = \frac{I}{|Y_{810\ KHz}|} = \frac{I}{Y_0} = \frac{I}{G} = \frac{I}{5.24\times 10^{-6}} \tag{2.39}$$

Also,

$$|V_{740\ KHz}| = \frac{I}{|Y_{740\ KHz}|}$$

where

$$|Y_{740\ KHz}| = \sqrt{G^2 + \left(\omega C - \frac{1}{\omega L}\right)^2}$$

or

$$|Y_{740\ KHz}| = \sqrt{(5.24\times 10^{-6})^2 + \left(2\pi\times 740\times 10^3\times 77.2\times 10^{-12} - \frac{1}{2\pi\times 740\times 10^3\times 0.5\times 10^{-3}}\right)^2}$$

or

$$|Y_{740\ KHz}| = 71.2\ \mu\Omega^{-1}$$

and

$$|V_{740\ KHz}| = \frac{I}{71.2\times 10^{-6}} \tag{2.40}$$

Then from (2.39) and (2.40),

$$\frac{|\mathbf{V}_{810\ KHz}|}{|\mathbf{V}_{740\ KHz}|} = \frac{I/5.24\times 10^{-6}}{I/71.2\times 10^{-6}} = \frac{71.2\times 10^{-6}}{5.24\times 10^{-6}} = 13.6 \tag{2.41}$$

that is, the voltage developed across the parallel circuit when it is tuned at $f = 810\ KHz$ is 13.6 times larger than the voltage developed at $f = 740\ KHz$.

2.10 Summary

- In a series RLC circuit, the frequency at which the capacitive reactance $X_C = 1/\omega C$ and the inductive reactance $X_L = \omega L$ are equal, is called the resonant frequency.
- The resonant frequency is denoted as ω_0 or f_0 where

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

and

$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$

- The quality factor Q_{0S} at series resonance is defined as

$$Q_{0S} = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 RC}$$

- In a parallel GLC circuit, the frequency at which the inductive susceptance $B_L = 1/\omega L$ and the capacitive susceptance $B_C = \omega C$ are equal is, again, called the resonant frequency and it is also denoted as ω_0. As in a series RLC circuit, the resonant frequency is

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

- The quality factor Q_{0P} at parallel resonance is defined as

$$Q_{0P} = \frac{\omega_0 C}{G} = \frac{1}{\omega_0 GL}$$

- The general definition of Q is

$$Q = 2\pi\frac{\text{Maximum Energy Stored}}{\text{Energy Dissipated per Cycle}}$$

- In a parallel RLC circuit, the half–power frequencies ω_1 and ω_2 are the frequencies at which the magnitude of the input admittance of a parallel resonant circuit, is greater than the magnitude at resonance by a factor of $\sqrt{2}$, or equivalently, the frequencies at which the magnitude of the input impedance of a parallel resonant circuit, is less than the magnitude at resonance by a factor of $\sqrt{2}$.

- We call ω_1 the lower half–power point, and ω_2 the upper half–power point. The difference $\omega_2 - \omega_1$ is the half–power bandwidth BW, that is,

$$\text{Bandwidth} = BW = \omega_2 - \omega_1$$

- The bandwidth BW can also be expressed in terms of the quality factor Q as

$$BW = \omega_2 - \omega_1 = \frac{\omega_0}{Q_{0P}}$$

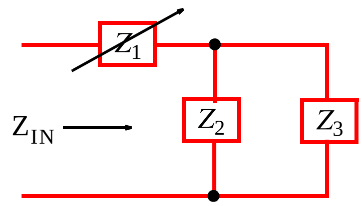
or

$$BW = f_2 - f_1 = \frac{f_0}{Q_{0P}}$$

2.11 Exercises

1. A series RLC circuit is resonant at $f_0 = 1 \text{ MHz}$ with $Z_0 = 100\ \Omega$ and its half–power bandwidth is $BW = 20 \text{ KHz}$. Find R, L, and C for this circuit.

2. For the network below the impedance Z_1 is variable, $Z_2 = 3 + j4$ and $Z_3 = 4 - j3$. To what value should Z_1 be adjusted so that the network will operate at resonant frequency?

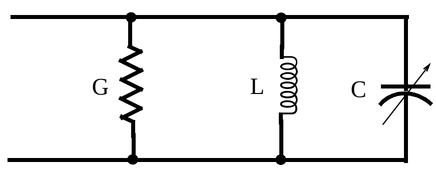


3. For the circuit below with the capacitance C adjusted to $1\ \mu F$, the half–power frequencies are $f_1 = 925 \text{ KHz}$ and $f_2 = 1075 \text{ KHz}$.

   a. Compute the *approximate* resonant frequency.

   b. Compute the *exact* resonant frequency.

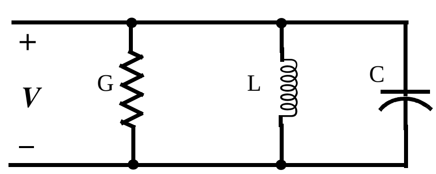
   c. Using the approximate value of the resonant frequency, compute the values of Q_{op}, G, and L.



4. The GLC circuit below is resonant at $f_0 = 500 \text{ KHz}$ with $\mathbf{V}_0 = 20 \text{ V}$ and its half–power bandwidth is $BW = 20 \text{ KHz}$.

   a. Compute L, C, and I_0 for this circuit.

   b. Compute the magnitude of the admittances $|Y_1|$ and $|Y_2|$ corresponding to the half–power frequencies f_1 and f_2. Use MATLAB to plot $|Y|$ in the $100 \text{ KHz} \le f \le 1000 \text{ KHz}$ range.



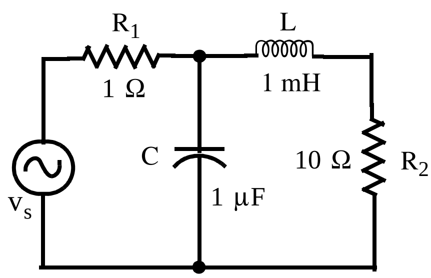
5. For the circuit below $v_s = 170\cos\omega t$ and $Q_0 = 50$. Find:

a. ω_0

b. BW

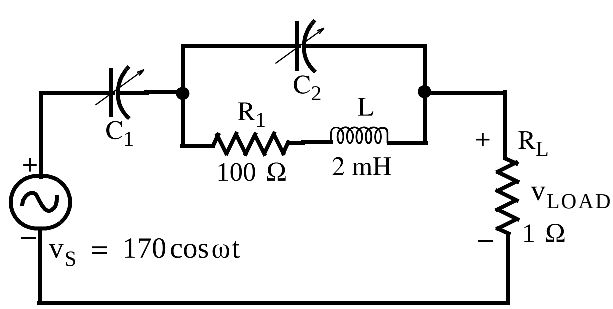
c. ω_1 and ω_2

d. $|\mathbf{V}_{C0}|$



6. The series–parallel circuit below will behave as a filter if the parallel part is made resonant to the frequency we want to suppress, and the series part is made resonant to the frequency we wish to pass. Accordingly, we can adjust capacitor C_2 to achieve parallel resonance which will reject the unwanted frequency by limiting the current through the resistive load to its minimum value. Afterwards, we can adjust C_1 to make the entire circuit series resonant at the desired frequency thus making the total impedance minimum so that maximum current will flow into the load.

For this circuit, we want to set the values of capacitors so that v_{LOAD} will be maximum at $f_1 = 10\ \text{KHz}$ and minimum at $f_2 = 43\ \text{KHz}$. Compute the values of C_1 and C_2 that will achieve these values. It is suggested that you use MATLAB to plot $|v_{LOAD}|$ versus frequency f in the interval $1\ \text{KHz} \le f \le 100\ \text{KH}$ to verify your answers.



2.12 Solutions to End–of–Chapter Exercises

1. At series resonance $Z_0 = R = 100$ and thus $R = 100\ \Omega$. We find L from $Q_{0S} = \omega_0 L/R$ where $\omega_0 = 2\pi f_0$. Also,

$$Q_{0S} = \frac{\omega_0}{\omega_2 - \omega_1} = \frac{\omega_0}{BW} = \frac{2\pi \times 10^6}{2\pi \times 20 \times 10^3} = 50$$

Then,

$$L = \frac{R \cdot Q_{0S}}{\omega_0} = \frac{100 \times 50}{2\pi \times 10^6} = 0.796\ mH$$

and from $\omega_0^2 = 1/LC$

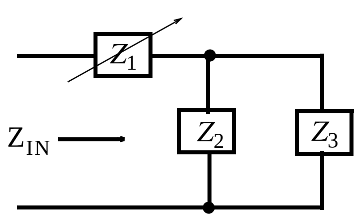
$$C = \frac{1}{\omega_0^2 L} = \frac{1}{(2\pi \times 10^6)^2 \times 7.96 \times 10^{-4}} = 31.8\ pF$$

Check with MATLAB:

```
f0=10^6; w0=2*pi*f0; Z0=100; BW=2*pi*20000; w1=w0–BW/2; w2=w0+BW/2;...
R=Z0; Qos=w0/BW; L=R*Qos/w0; C=1/(w0^2*L); fprintf(' \n');...
fprintf('R = %5.2f Ohms \t', R); fprintf('L = %5.2e H \t', L);...
fprintf('C = %5.2e F \t', C); fprintf(' \n'); fprintf(' \n');
```

```
R = 100.00 Ohms   L = 7.96e-004 H    C = 3.18e-011 F
```

2.



$$Z_{IN} = Z_1 + Z_2 \| Z_3$$

where

$$Z_2 \| Z_3 = \frac{(3 + j4) \cdot (4 - j3)}{3 + j4 + 4 - j3} = \frac{12 - j9 + j16 + 12}{7 + j} \cdot \frac{7 - j}{7 - j}$$

$$= \frac{168 + j49 - j24 + 7}{7^2 + 1^2} = \frac{175 + j25}{50} = 3.5 + j0.5$$

We let $Z_{IN} = R_{IN} + jX_{IN}$ and $Z_1 = R_1 + jX_1$. For resonance we must have

$$Z_{IN} = R_{IN} + jX_{IN} = R_1 + jX_1 + 3.5 + j0.5 = R_{IN} + 0 = R_1 + jX_1 + 3.5 + j0.5$$

Equating real and imaginary parts we obtain

$$R_{IN} = R_1 + 3.5$$
$$0 = jX_1 + j0.5$$

and while R_1 can be any real number, we must have $jX_1 = -j0.5$ and thus

$$Z_1 = R_1 - j0.5\ \Omega$$

3.

a.

$$BW = f_2 - f_1 = 1075 - 925 = 150\ KHz$$

Then,

$$f_0 = f_1 + BW/2 = 925 + 150/2 = 1000\ KHz$$

b. The exact value of f_0 is the geometric mean of f_1 and f_2 and thus

$$f_0 = \sqrt{f_1 \cdot f_2} = \sqrt{(925 + 1075)10^3} = 997.18\ KHz$$

c.

$$Q_{0P} = \frac{f_0}{f_2 - f_1} = \frac{1000}{150} = 20/3. \text{ Also, } Q_{0P} = \frac{\omega_0 C}{G}$$

Then

$$G = \frac{\omega_0 C}{Q_{0P}} = \frac{2\pi f_0 C}{Q_{0P}} = \frac{2\pi \times 10^6 \times 10^{-6}}{20/3} = \frac{3\pi}{10} = 0.94\ \Omega^{-1}$$

and

$$L = \frac{1}{\omega_0 C} = \frac{1}{4\pi^2 f_0^2 C} = \frac{1}{4\pi^2 \times 10^{12} \times 10^{-6}} = 0.025\ \mu H$$

4.

a.

$$Q_{0P} = \frac{f_0}{BW} = \frac{500}{20} = 25$$

Also, $Q_{0P} = \frac{\omega_0 C}{G}$ or

$$C = \frac{Q_{0P} \cdot G}{\omega_0} = \frac{25 \times 10^{-3}}{2\pi \times 5 \times 10^5} = 7.96 \times 10^{-9}\ F = 7.96\ nF$$

$$L = \frac{1}{\omega_0 C} = \frac{1}{4\pi^2 f_0^2 C} = \frac{1}{4\pi^2 \times 25 \times 10^{10} \times 7.96 \times 10^{-9}} = 12.73 \times 10^{-6}\ H = 12.73\ \mu H$$

$$I_0 = V_0 Y_0 = V_0 G = 20 \times 10^{-3}\ A = 20\ mA$$

b. $f_1 = f_0 - BW/2 = 500 - 10 = 490\ KHz$ and $f_2 = f_0 + BW/2 = 500 + 10 = 510\ KHz$

$$Y|_{f = f_1} = G + j\left(\omega_1 C - \frac{1}{\omega_1 L}\right)$$

$$= 10^{-3} + j\left(2\pi \times 490 \times 10^{3} \times 7.96 \times 10^{-9} - \frac{1}{2\pi \times 490 \times 10^{3} \times 12.73 \times 10^{-6}}\right)$$

Likewise,

$$Y|_{f = f_2} = G + j\left(\omega_1 C - \frac{1}{\omega_1 L}\right)$$

$$= 10^{-3} + j\left(2\pi \times 510 \times 10^{3} \times 7.96 \times 10^{-9} - \frac{1}{2\pi \times 510 \times 10^{3} \times 12.73 \times 10^{-6}}\right)$$

We will use MATLAB to do the computations.

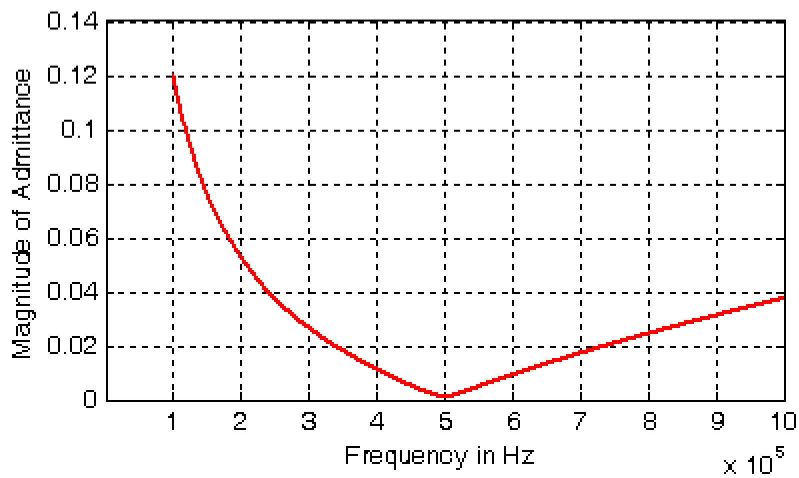
```
G=10^(–3); BC1=2*pi*490*10^3*7.96*10^(–9);...
BL1=1/(2*pi*490*10^3*12.73*10^(–6)); Y1=G+j*(BC1–BL1);...
BC2=2*pi*510*10^3*7.96*10^(–9); BL2=1/(2*pi*510*10^3*12.73*10^(–6));...
Y2=G+j*(BC2–BL2); fprintf(' \n'); fprintf('magY1 = %5.2e mho \t', abs(Y1));...
fprintf('magY2 = %5.2e mho \t', abs(Y2)); fprintf(' \n'); fprintf(' \n')
```

```
magY1 = 1.42e-003 mho magY2 = 1.41e-003 mho
```

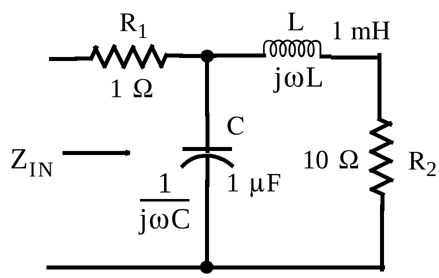
We will use the following MATLAB script for the plot

```
f=100*10^3: 10^3: 1000*10^3; w=2*pi*f;...
G=10^(–3); C=7.96*10^(–9); L=12.73*10^(–6);...
BC=w.*C; BL=1./(w.*L); Y=G+j*(BC–BL); plot(f,abs(Y));...
xlabel('Frequency in Hz'); ylabel('Magnitude of Admittance');grid
```

The plot is shown below.



5.



a. It is important to remember that the relation $\omega_0 = 1/\sqrt{LC}$ applies only to series RLC and parallel GLC circuits. For any other circuit we must find the input impedance Z_{IN}, set the imaginary part of Z_{IN} equal to zero, and solve for ω_0. Thus, for the given circuit

$$Z_{IN} = R_1 + \frac{1}{j\omega C} \| (R_2 + j\omega L) = 1 + \frac{1/j\omega C \cdot (10 + j\omega L)}{10 + j(\omega L - 1/\omega C)}$$

$$= \frac{10 + j(\omega L - 1/\omega C) + 10/j\omega C + L/C}{10 + j(\omega L - 1/\omega C)} \cdot \frac{10 - j(\omega L - 1/\omega C)}{10 - j(\omega L - 1/\omega C)}$$

$$= \frac{100 + j10(\omega L - 1/\omega C) + 100/(j\omega C) + 10L/C - j10(\omega L - 1/\omega C)}{100 + (\omega L - 1/\omega C)^2}$$

$$+ \frac{(\omega L - 1/\omega C)^2 - (10/\omega C)(\omega L - 1/\omega C) - jL/C(\omega L - 1/\omega C)}{100 + (\omega L - 1/\omega C)^2}$$

$$= \frac{100 + 10L/C + (\omega L - 1/\omega C)^2 - (10/\omega C)(\omega L - 1/\omega C)}{100 + (\omega L - 1/\omega C)^2}$$

$$+ \frac{100/(j\omega C) - jL/C(\omega L - 1/\omega C)}{100 + (\omega L - 1/\omega C)^2}$$

For resonance, the imaginary part of Z_{IN} must be zero, that is,

$$\frac{100}{j\omega_0 C} - \frac{jL}{C}\left(\omega_0 L - \frac{1}{\omega_0 C}\right) = 0$$

$$-\frac{j}{C}\left[\frac{100}{\omega_0} + L\left(\omega_0 L - \frac{1}{\omega_0 C}\right)\right] = 0$$

$$\frac{100}{\omega_0} + \omega_0 L^2 - \frac{L}{\omega_0 C} = 0$$

$$L^2 C\omega_0^2 + 100C - L = 0$$

$$\omega_0^2 = \frac{1}{LC} - \frac{100}{L^2} = \frac{1}{10^{-3} \times 10^{-6}} - \frac{100}{10^{-6}} = 10^9 - 10^8 = 9 \times 10^8$$

and thus

$$\omega_0 = \sqrt{9 \times 10^8} = 30,000 \text{ r/s}$$

b.

$$BW = \omega_0 / Q = 30,000/50 = 600 \text{ r/s}$$

c.

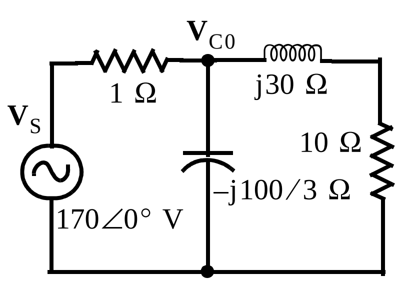
$$\omega_1 = \omega_0 - BW/2 = 30,000 - 300 = 29,700 \text{ r/s}$$

$$\omega_2 = \omega_0 + BW/2 = 30,000 + 300 = 30,300 \text{ r/s}$$

d. At resonance

$$j\omega_0 L = j3 \times 10^4 \times 10^{-3} = j30\ \Omega \text{ and } 1/j\omega_0 C = -j10^{-4} \times 10^6/3 = -j100/3$$

The phasor equivalent circuit is shown below.



We let $z_1 = 1\ \Omega$, $z_2 = -j100/3\ \Omega$, and $z_3 = 10 + j30\ \Omega$. Using nodal analysis we obtain:

$$\frac{V_{C0} - V_S}{z_1} + \frac{V_{C0}}{z_2} + \frac{V_{C0}}{z_3} = 0$$

$$\left(\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right)\mathbf{V}_{C0} = \frac{\mathbf{V}_S}{z_1}$$

We will use MATLAB to obtain the value of $\mathbf{V}_{C0}$.

```
Vs=170; z1=1; z2=–j*100/3; z3=10+j*30; Z=1/z1+1/z2+1/z3; Vc0=Vs/Z;...
fprintf(' \n'); fprintf('Vc0 = %6.2f', abs(Vc0)); fprintf(' \n'); fprintf(' \n')
```

```
Vc0 = 168.32
```

6.

First, we will find the appropriate value of C_2. We recall that at parallel resonance the voltage is maximum and the current is minimum. For this circuit the parallel resonance was found as in (2.37), that is,

$$\omega_0 = \sqrt{\frac{1}{LC}-\frac{R^2}{L^2}}$$

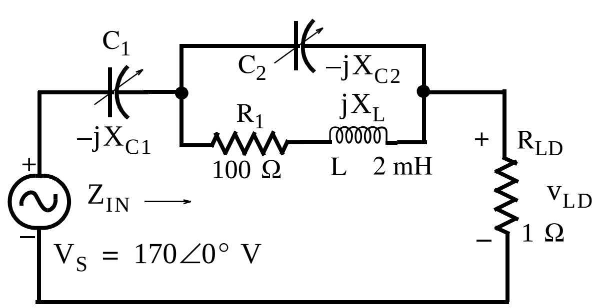
or

$$2\pi\times 43,000 = \sqrt{\frac{1}{2\times 10^{-3}C_2}-\frac{10^4}{4\times 10^{-6}}}$$

$$\frac{10^3}{2C_2} = \frac{10^4}{4\times 10^{-6}}+(2\pi\times 4.3\times 10^4)^2 = \frac{10^4+(2\pi\times 4.3\times 10^4)^2\times 4\times 10^{-6}}{4\times 10^{-6}}$$

$$C_2 = 500\left[\frac{4\times 10^{-6}}{10^4+(2\pi\times 4.3\times 10^4)^2\times 4\times 10^{-6}}\right] = 6.62\times 10^{-9}\ \text{F} = 6.62\ \text{nF}$$

Next, we must find the value of C_1 that will make the entire circuit series resonant (minimum impedance, maximum current) at $f = 10\ \text{KHz}$. In the circuit below we let $z_1 = -jX_{C1}$, $z_2 = -jX_{C2}$, $z_3 = R_1+jX_L$, and $z_{LD} = 1$.



Then,

$$Z_{IN} = z_1 + z_2 \| z_3 + z_{LOAD}$$

and

$$_{IN}(f = 10\ KHz) = z_1 + z_2 \| z_3|_{f = 10\ KHz} + z_{LOAD} = z_1 + z_2 \| z_3|_{f = 10\ KHz} + \quad (1)$$

where $z_2 \| z_3|_{f = 10\ KHz}$ is found with the MATLAB script below.

```
format short g; f=10000; w=2*pi*f; C2=6.62*10^(–9); XC2=1/(w*C2); L=2*10^(–3);...
XL=w*L; R1=100; z2=–j*XC2; z3=R1+j*XL; Zp=z2*z3/(z2+z3)
```

```
Zp =
     111.12 + 127.72i
```

and by substitution into (1)

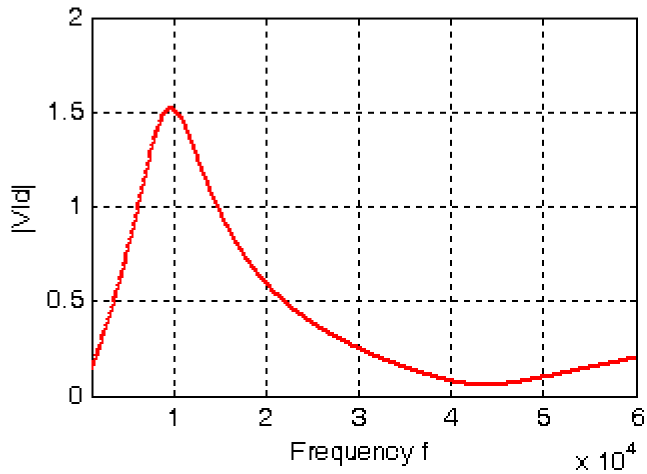
$$Z_{IN}(f = 10\ KHz) = z_1 + 111.12 + j127.72 + 1 = z_1 + 113.12 + j127.72\ \Omega \quad (2)$$

The expression of (2) will be minimum if we let $z_1 = -j127.72\ \Omega$ at $f = 10\ KHz$. Then, the capacitor C_1 value must be such that $1/\omega C = 127.72$ or

$$C_1 = \frac{1}{2\pi \times 10^4 \times 127.72} = 1.25 \times 10^{-7}\ F = 0.125\ \mu F$$

Shown below is the plot of $|V_{LD}|$ versus frequency and the MATLAB script that produces this plot.

```
f=1000: 100: 60000; w=2*pi*f; Vs=170; C1=1.25*10^(–7); C2=6.62*10^(–9); L=2.*10.^(–3);...
R1=100; Rld=1; z1=–j./(w.*C1); z2=–j./(w.*C2); z3=R1+j.*w.*L; Zld=Rld;...
Zin=z1+z2.*z3./(z2+z3); Vld=Zld.*Vs./(Zin+Zld); magVld=abs(Vld);...
plot(f,magVld); axis([1000 60000 0 2]);...
xlabel('Frequency f'); ylabel('|Vld|'); grid
```



This circuit is considered to be a special type of filter that allows a specific frequency (not a band of frequencies) to pass, and attenuates another specific frequency.

Chapter 3

Elementary Signals

This chapter begins with a discussion of elementary signals that may be applied to electric networks. The unit step, unit ramp, and delta functions are then introduced. The sampling and sifting properties of the delta function are defined and derived. Several examples for expressing a variety of waveforms in terms of these elementary signals are provided.

3.1 Signals Described in Math Form

Consider the network of Figure 3.1 where the switch is closed at time $t = 0$.

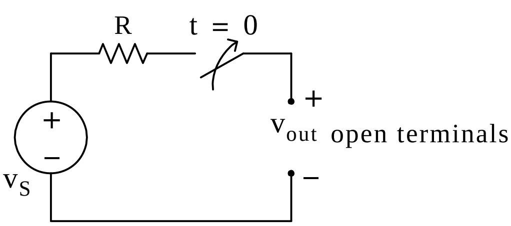


Figure 3.1. A switched network with open terminals

We wish to describe v_{out} in a math form for the time interval $-\infty < t < +\infty$. To do this, it is convenient to divide the time interval into two parts, $-\infty < t < 0$, and $0 < t < \infty$.

For the time interval $-\infty < t < 0$, the switch is open and therefore, the output voltage v_{out} is zero. In other words,

$$v_{out} = 0 \quad \text{for} \quad -\infty < t < 0 \tag{3.1}$$

For the time interval $0 < t < \infty$, the switch is closed. Then, the input voltage v_S appears at the output, i.e.,

$$v_{out} = v_S \quad \text{for} \quad 0 < t < \infty \tag{3.2}$$

Combining (3.1) and (3.2) into a single relationship, we obtain

$$\boxed{v_{out} = \begin{cases} 0 & -\infty < t < 0 \\ v_S & 0 < t < \infty \end{cases}} \tag{3.3}$$

We can express (3.3) by the waveform shown in Figure 3.2.

The waveform of Figure 3.2 is an example of a discontinuous function. A function is said to be *discontinuous* if it exhibits points of discontinuity, that is, the function jumps from one value to another without taking on any intermediate values.

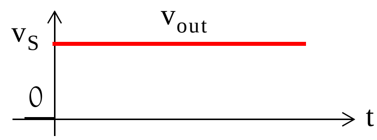


Figure 3.2. Waveform for v_{out} *as defined in relation (3.3)*

3.2 The Unit Step Function $u_0(t)$

A well known discontinuous function is the *unit step function* $u_0(t)$ * which is defined as

$$u_0(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases} \tag{3.4}$$

It is also represented by the waveform of Figure 3.3.

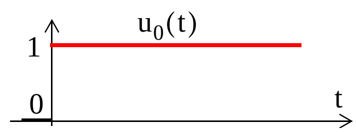


Figure 3.3. Waveform for $u_0(t)$

In the waveform in Figure 3.3, the unit step function $u_0(t)$ changes abruptly from 0 to 1 at $t = 0$. But if it changes at $t = t_0$ instead, it is denoted as $u_0(t - t_0)$. In this case, its waveform and definition are as shown in Figure 3.4 and relation (3.5) respectively.

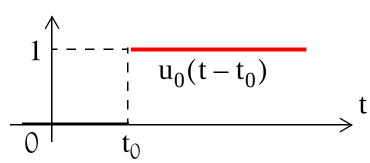


Figure 3.4. Waveform for $u_0(t - t_0)$

$$u_0(t - t_0) = \begin{cases} 0 & t < t_0 \\ 1 & t > t_0 \end{cases} \tag{3.5}$$

If the unit step function changes abruptly from 0 to 1 at $t = -t_0$, it is denoted as $u_0(t + t_0)$. In this case, its waveform and definition are as shown in Figure 3.5 and relation (3.6) respectively.

* *In some books, the unit step function is denoted as* $u(t)$, *that is, without the subscript 0. In this text, however, we will reserve the* $u(t)$ *designation for any input when we will discuss state variables in Chapter 7.*

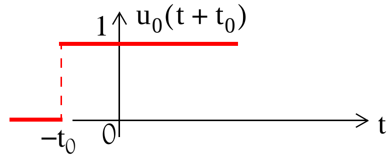


Figure 3.5. Waveform for $u_0(t+t_0)$

$$u_0(t+t_0) = \begin{cases} 0 & t < -t_0 \\ 1 & t > -t_0 \end{cases} \tag{3.6}$$

Example 3.1

Consider the network of Figure 3.6, where the switch is closed at time $t = T$.

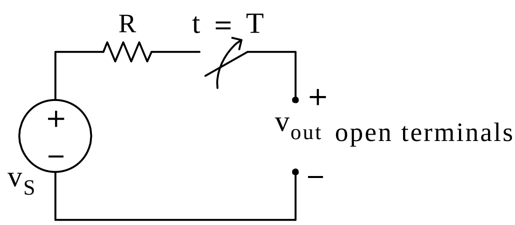


Figure 3.6. Network for Example 3.1

Express the output voltage v_{out} as a function of the unit step function, and sketch the appropriate waveform.

Solution:

For this example, the output voltage $v_{out} = 0$ for $t < T$, and $v_{out} = v_S$ for $t > T$. Therefore,

$$v_{out} = v_S u_0(t-T) \tag{3.7}$$

and the waveform is shown in Figure 3.7.

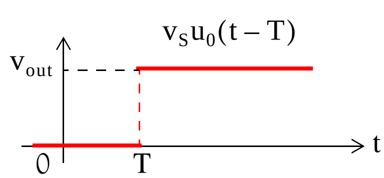


Figure 3.7. Waveform for Example 3.1

Other forms of the unit step function are shown in Figure 3.8.

Unit step functions can be used to represent other time–varying functions such as the rectangular pulse shown in Figure 3.9.

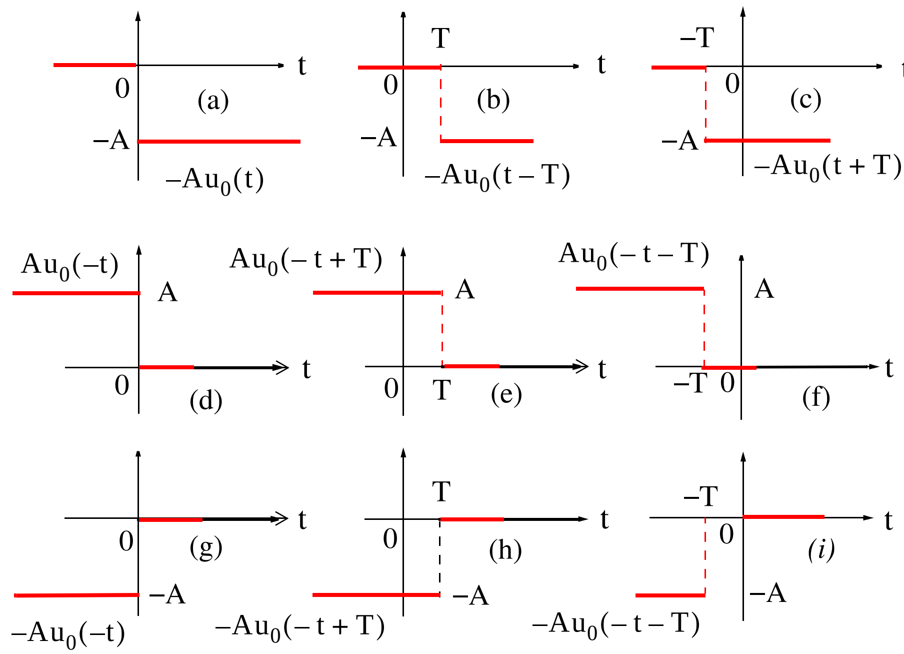


Figure 3.8. Other forms of the unit step function

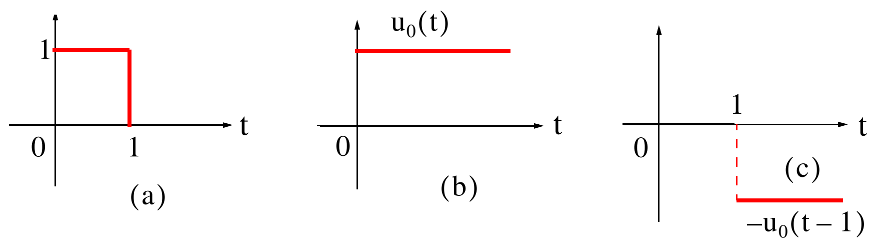


Figure 3.9. A rectangular pulse expressed as the sum of two unit step functions

Thus, the pulse of Figure 3.9(a) is the sum of the unit step functions of Figures 3.9(b) and 3.9(c) and it is represented as $u_0(t) - u_0(t-1)$.

The unit step function offers a convenient method of describing the sudden application of a voltage or current source. For example, a constant voltage source of $24\ V$ applied at $t = 0$, can be denoted as $24u_0(t)\ V$. Likewise, a sinusoidal voltage source $v(t) = V_m \cos\omega t\ V$ that is applied to a circuit at $t = t_0$, can be described as $v(t) = (V_m \cos\omega t)u_0(t - t_0)\ V$. Also, if the excitation in a circuit is a rectangular, or triangular, or sawtooth, or any other recurring pulse, it can be represented as a sum (difference) of unit step functions.

Example 3.2

Express the square waveform of Figure 3.10 as a sum of unit step functions. The vertical dotted lines indicate the discontinuities at $T, 2T, 3T$, and so on.

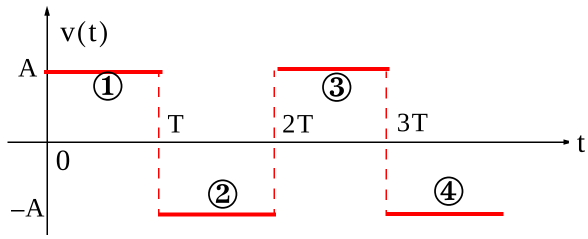


Figure 3.10. Square waveform for Example 3.2

Solution:

Line segment ① has height A, starts at $t = 0$, and terminates at $t = T$. Then, as in Example 3.1, this segment is expressed as

$$v_1(t) = A[u_0(t) - u_0(t-T)] \tag{3.8}$$

Line segment ② has height $-A$, starts at $t = T$ and terminates at $t = 2T$. This segment is expressed as

$$v_2(t) = -A[u_0(t-T) - u_0(t-2T)] \tag{3.9}$$

Line segment ③ has height A, starts at $t = 2T$ and terminates at $t = 3T$. This segment is expressed as

$$v_3(t) = A[u_0(t-2T) - u_0(t-3T)] \tag{3.10}$$

Line segment④ has height $-A$, starts at $t = 3T$, and terminates at $t = 4T$. It is expressed as

$$v_4(t) = -A[u_0(t-3T) - u_0(t-4T)] \tag{3.11}$$

Thus, the square waveform of Figure 3.10 can be expressed as the summation of (3.8) through (3.11), that is,

$$\begin{aligned} v(t) &= v_1(t) + v_2(t) + v_3(t) + v_4(t) \\ &= A[u_0(t) - u_0(t-T)] - A[u_0(t-T) - u_0(t-2T)] \\ &\quad + A[u_0(t-2T) - u_0(t-3T)] - A[u_0(t-3T) - u_0(t-4T)] \end{aligned} \tag{3.12}$$

Combining like terms, we obtain

$$v(t) = A[u_0(t) - 2u_0(t-T) + 2u_0(t-2T) - 2u_0(t-3T) + \ldots] \tag{3.13}$$

Example 3.3

Express the symmetric rectangular pulse of Figure 3.11 as a sum of unit step functions.

Solution:

This pulse has height A, starts at $t = -T/2$, and terminates at $t = T/2$. Therefore, with reference to Figures 3.5 and 3.8 (b), we obtain

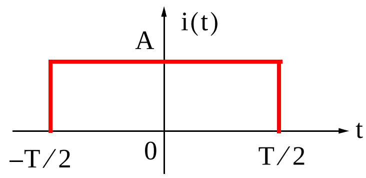


Figure 3.11. Symmetric rectangular pulse for Example 3.3

$$i(t) = Au_0\left(t+\frac{T}{2}\right) - Au_0\left(t-\frac{T}{2}\right) = A\left[u_0\left(t+\frac{T}{2}\right) - u_0\left(t-\frac{T}{2}\right)\right] \tag{3.14}$$

Example 3.4

Express the symmetric triangular waveform of Figure 3.12 as a sum of unit step functions.

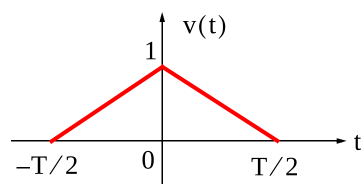


Figure 3.12. Symmetric triangular waveform for Example 3.4

Solution:

We first derive the equations for the linear segments ① and ② shown in Figure 3.13.

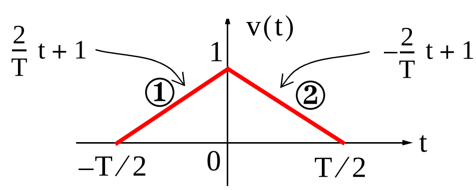


Figure 3.13. Equations for the linear segments in Figure 3.12

For line segment ①,

$$v_1(t) = \left(\frac{2}{T}t+1\right)\left[u_0\left(t+\frac{T}{2}\right) - u_0(t)\right] \tag{3.15}$$

and for line segment ②,

$$v_2(t) = \left(-\frac{2}{T}t+1\right)\left[u_0(t) - u_0\left(t-\frac{T}{2}\right)\right] \tag{3.16}$$

Combining (3.15) and (3.16), we obtain

$$v(t) = v_1(t) + v_2(t) = \left(\frac{2}{T}t+1\right)\left[u_0\left(t+\frac{T}{2}\right) - u_0(t)\right] + \left(-\frac{2}{T}t+1\right)\left[u_0(t) - u_0\left(t-\frac{T}{2}\right)\right] \tag{3.17}$$

Example 3.5

Express the waveform of Figure 3.14 as a sum of unit step functions.

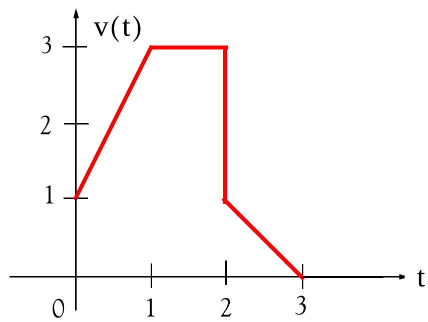


Figure 3.14. Waveform for Example 3.5

Solution:

As in the previous example, we first find the equations of the linear segments linear segments ① and ② shown in Figure 3.15.

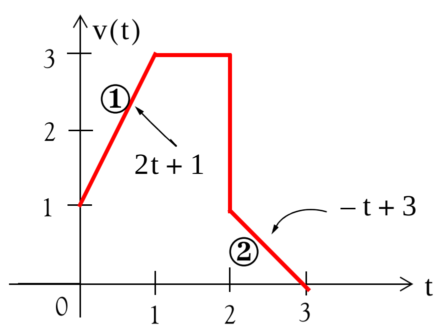


Figure 3.15. Equations for the linear segments of Figure 3.14

Following the same procedure as in the previous examples, we obtain

$$\begin{aligned} v(t) &= (2t+1)[u_0(t) - u_0(t-1)] + 3[u_0(t-1) - u_0(t-2)] \\ &\quad + (-t+3)[u_0(t-2) - u_0(t-3)] \end{aligned}$$

Multiplying the values in parentheses by the values in the brackets, we obtain

$$\begin{aligned} v(t) &= (2t+1)u_0(t) - (2t+1)u_0(t-1) + 3u_0(t-1) \\ &\quad - 3u_0(t-2) + (-t+3)u_0(t-2) - (-t+3)u_0(t-3) \end{aligned}$$

$$\begin{aligned} v(t) &= (2t+1)u_0(t) + [-(2t+1)+3]u_0(t-1) \\ &\quad + [-3+(-t+3)]u_0(t-2) - (-t+3)u_0(t-3) \end{aligned}$$

and combining terms inside the brackets, we obtain

$$v(t) = (2t+1)u_0(t) - 2(t-1)u_0(t-1) - tu_0(t-2) + (t-3)u_0(t-3) \tag{3.18}$$

Two other functions of interest are the *unit ramp function*, and the *unit impulse* or *delta function*. We will introduce them with the examples that follow.

Example 3.6

In the network of Figure 3.16 i_S is a constant current source and the switch is closed at time $t = 0$. Express the capacitor voltage $v_C(t)$ as a function of the unit step.

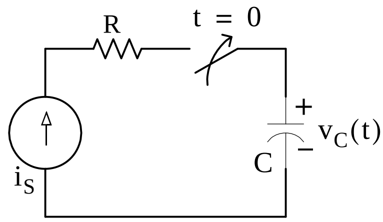


Figure 3.16. Network for Example 3.6

Solution:

The current through the capacitor is $i_C(t) = i_S = \text{constant}$, and the capacitor voltage $v_C(t)$ is

$$v_C(t) = \frac{1}{C}\int_{-\infty}^{t} i_C(\tau)d\tau^{*} \tag{3.19}$$

where τ is a dummy variable.

Since the switch closes at $t = 0$, we can express the current $i_C(t)$ as

$$i_C(t) = i_S\, u_0(t) \tag{3.20}$$

and assuming that $v_C(t) = 0$ for $t < 0$, we can write (3.19) as

$$v_C(t) = \frac{1}{C}\int_{-\infty}^{t} i_S\, u_0(\tau)d\tau = \underbrace{\frac{i_S}{C}\int_{-\infty}^{0} u_0(\tau)\,d\tau}_{0} + \frac{i_S}{C}\int_{0}^{t} u_0(\tau)d\tau \tag{3.21}$$

or

$$\boxed{v_C(t) = \frac{i_S}{C}\, tu_0(t)} \tag{3.22}$$

Therefore, we see that when a capacitor is charged with a constant current, the voltage across it is a linear function and forms a *ramp* with slope i_S/C as shown in Figure 3.17.

* *Since the initial condition for the capacitor voltage was not specified, we express this integral with* $-\infty$ *at the lower limit of integration so that any non-zero value prior to* $t < 0$ *would be included in the integration.*

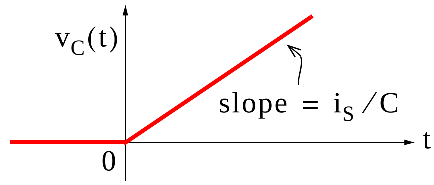


Figure 3.17. Voltage across a capacitor when charged with a constant current source

3.3 The Unit Ramp Function $u_1(t)$

The *unit ramp function*, denoted as $u_1(t)$, is defined as

$$u_1(t) = \int_{-\infty}^{t} u_0(\tau)d\tau \tag{3.23}$$

where τ is a dummy variable.

We can evaluate the integral of (3.23) by considering the area under the unit step function $u_0(t)$ from $-\infty$ to t as shown in Figure 3.18.

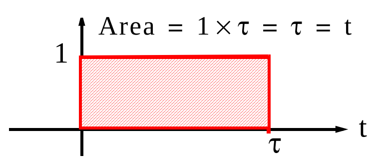


Figure 3.18. Area under the unit step function from $-\infty$ to t

Therefore, we define $u_1(t)$ as

$$u_1(t) = \begin{cases} 0 & t < 0 \\ t & t \geq 0 \end{cases} \tag{3.24}$$

Since $u_1(t)$ is the integral of $u_0(t)$, then $u_0(t)$ must be the derivative of $u_1(t)$, i.e.,

$$\frac{d}{dt}u_1(t) = u_0(t) \tag{3.25}$$

Higher order functions of t can be generated by repeated integration of the unit step function. For example, integrating $u_0(t)$ twice and multiplying by 2, we define $u_2(t)$ as

$$u_2(t) = \begin{cases} 0 & t < 0 \\ t^2 & t \geq 0 \end{cases} \quad \text{or} \quad u_2(t) = 2\int_{-\infty}^{t} u_1(\tau)d\tau \tag{3.26}$$

Similarly,

$$u_3(t) = \begin{cases} 0 & t < 0 \\ t^3 & t \ge 0 \end{cases} \quad \text{or} \quad u_3(t) = 3\int_{-\infty}^{t} u_2(\tau)d\tau \tag{3.27}$$

and in general,

$$u_n(t) = \begin{cases} 0 & t < 0 \\ t^n & t \ge 0 \end{cases} \quad \text{or} \quad u_n(t) = n\int_{-\infty}^{t} u_{n-1}(\tau)d\tau \tag{3.28}$$

Also,

$$u_{n-1}(t) = \frac{1}{n}\frac{d}{dt}u_n(t) \tag{3.29}$$

Example 3.7

In the network of Figure 3.19, the switch is closed at time $t = 0$ and $i_L(t) = 0$ for $t < 0$. Express the inductor voltage $v_L(t)$ in terms of the unit step function.

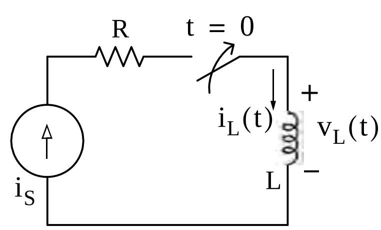


Figure 3.19. Network for Example 3.7

Solution:

The voltage across the inductor is

$$v_L(t) = L\frac{di_L}{dt} \tag{3.30}$$

and since the switch closes at $t = 0$,

$$i_L(t) = i_S\, u_0(t) \tag{3.31}$$

Therefore, we can write (3.30) as

$$\boxed{v_L(t) = Li_S \frac{d}{dt}u_0(t)} \tag{3.32}$$

But, as we know, $u_0(t)$ is constant (0 or 1) for all time except at $t = 0$ where it is discontinuous. Since the derivative of any constant is zero, the derivative of the unit step $u_0(t)$ has a non–zero value only at $t = 0$. The derivative of the unit step function is defined in the next section.

3.4 The Delta Function $\delta(t)$

The *delta function* or *unit impulse*, denoted as $\delta(t)$, is the derivative of the unit step $u_0(t)$. It is also defined as

$$\int_{-\infty}^{t} \delta(\tau)d\tau = u_0(t) \tag{3.33}$$

and

$$\delta(t) = 0 \text{ for all } t \neq 0 \tag{3.34}$$

To better understand the delta function $\delta(t)$, let us represent the unit step $u_0(t)$ as shown in Figure 3.20 (a).

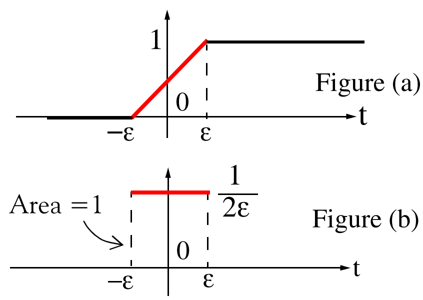


Figure 3.20. Representation of the unit step as a limit

The function of Figure 3.20 (a) becomes the unit step as $\varepsilon \to 0$. Figure 3.20 (b) is the derivative of Figure 3.20 (a), where we see that as $\varepsilon \to 0$, $1/2\varepsilon$ becomes unbounded, but the area of the rectangle remains 1. Therefore, in the limit, we can think of $\delta(t)$ as approaching a very large spike or impulse at the origin, with unbounded amplitude, zero width, and area equal to 1.

Two useful properties of the delta function are the sampling property and the sifting property.

3.4.1 The Sampling Property of the Delta Function $\delta(t)$

The *sampling property* of the delta function states that

$$f(t)\delta(t-a) = f(a)\delta(t) \tag{3.35}$$

or, when $a = 0$,

$$f(t)\delta(t) = f(0)\delta(t) \tag{3.36}$$

that is, multiplication of any function $f(t)$ by the delta function $\delta(t)$ results in sampling the function at the time instants where the delta function is not zero. The study of discrete−time systems is based on this property.

Proof:

Since $\delta(t) = 0$ for $t < 0$ and $t > 0$ then,

$$f(t)\delta(t) = 0 \quad \text{for} \quad t<0 \quad \text{and} \quad t>0 \tag{3.37}$$

We rewrite $f(t)$ as

$$f(t) = f(0) + [f(t) - f(0)] \tag{3.38}$$

Integrating (3.37) over the interval $-\infty$ to t and using (3.38), we obtain

$$\int_{-\infty}^{t} f(\tau)\delta(\tau)d\tau = \int_{-\infty}^{t} f(0)\delta(\tau)d\tau + \int_{-\infty}^{t} [f(\tau) - f(0)]\delta(\tau)d\tau \tag{3.39}$$

The first integral on the right side of (3.39) contains the constant term $f(0)$; this can be written outside the integral, that is,

$$\int_{-\infty}^{t} f(0)\delta(\tau)d\tau = f(0)\int_{-\infty}^{t} \delta(\tau)d\tau \tag{3.40}$$

The second integral of the right side of (3.39) is always zero because

$$\delta(t) = 0 \quad \text{for} \quad t<0 \quad \text{and} \quad t>0$$

and

$$[f(\tau) - f(0)]\big|_{\tau = 0} = f(0) - f(0) = 0$$

Therefore, (3.39) reduces to

$$\int_{-\infty}^{t} f(\tau)\delta(\tau)d\tau = f(0)\int_{-\infty}^{t} \delta(\tau)d\tau \tag{3.41}$$

Differentiating both sides of (3.41), and replacing τ with t, we obtain

$$\boxed{\begin{array}{c} f(t)\delta(t) = f(0)\delta(t) \\ \text{Sampling Property of } \delta(t) \end{array}} \tag{3.42}$$

3.4.2 The Sifting Property of the Delta Function $\delta(t)$

The *sifting property* of the delta function states that

$$\boxed{\int_{-\infty}^{\infty} f(t)\delta(t-\alpha)dt = f(\alpha)} \tag{3.43}$$

that is, if we multiply any function $f(t)$ by $\delta(t-\alpha)$, and integrate from $-\infty$ to $+\infty$, we will obtain the value of $f(t)$ evaluated at $t = \alpha$.

Proof:

Let us consider the integral

$$\int_{a}^{b} f(t)\delta(t-\alpha)dt \quad \text{where} \quad a<\alpha<b \tag{3.44}$$

We will use integration by parts to evaluate this integral. We recall from the derivative of products that

$$d(xy) = xdy + ydx \quad \text{or} \quad xdy = d(xy) - ydx \tag{3.45}$$

and integrating both sides we obtain

$$\int xdy = xy - \int ydx \tag{3.46}$$

Now, we let $x = f(t)$; then, $dx = f'(t)$. We also let $dy = \delta(t-\alpha)$; then, $y = u_0(t-\alpha)$. By substitution into (3.44), we obtain

$$\int_a^b f(t)\delta(t-\alpha)dt = f(t)u_0(t-\alpha)\Big|_a^b - \int_a^b u_0(t-\alpha)f'(t)dt \tag{3.47}$$

We have assumed that $a < \alpha < b$; therefore, $u_0(t-\alpha) = 0$ for $\alpha < a$, and thus the first term of the right side of (3.47) reduces to $f(b)$. Also, the integral on the right side is zero for $\alpha < a$, and therefore, we can replace the lower limit of integration a by α. We can now rewrite (3.47) as

$$\int_a^b f(t)\delta(t-\alpha)dt = f(b) - \int_\alpha^b f'(t)dt = f(b) - f(b) + f(\alpha)$$

and letting $a \to -\infty$ and $b \to \infty$ for any $|\alpha| < \infty$, we obtain

$$\int_{-\infty}^{\infty} f(t)\delta(t-\alpha)dt = f(\alpha) \tag{3.48}$$

Sifting Property of $\delta(t)$

3.5 Higher Order Delta Functions

An *nth-order delta function* is defined as the nth derivative of $u_0(t)$, that is,

$$\delta^n(t) = \frac{\delta^n}{dt}[u_0(t)] \tag{3.49}$$

The function $\delta'(t)$ is called *doublet*, $\delta''(t)$ is called *triplet*, and so on. By a procedure similar to the derivation of the sampling property of the delta function, we can show that

$$f(t)\delta'(t-a) = f(a)\delta'(t-a) - f'(a)\delta(t-a) \tag{3.50}$$

Also, the derivation of the sifting property of the delta function can be extended to show that

$$\int_{-\infty}^{\infty} f(t)\delta^n(t-\alpha)dt = (-1)^n \frac{d^n}{dt^n}[f(t)]\Big|_{t=\alpha} \tag{3.51}$$

Example 3.8

Evaluate the following expressions:

a. $3t^4\delta(t-1)$ b. $\int_{-\infty}^{\infty} t\delta(t-2)dt$ c. $t^2\delta'(t-3)$

Solution:

a. The sampling property states that $f(t)\delta(t-a) = f(a)\delta(t)$ For this example, $f(t) = 3t^4$ and $a = 1$. Then,

$$3t^4\delta(t-1) = \{3t^4|_{t=1}\}\delta(t-1) = 3\delta(t)$$

b. The sifting property states that $\int_{-\infty}^{\infty} f(t)\delta(t-\alpha)dt = f(\alpha)$. For this example, $f(t) = t$ and $\alpha = 2$. Then,

$$\int_{-\infty}^{\infty} t\delta(t-2)dt = f(2) = t|_{t=2} = 2$$

c. The given expression contains the doublet; therefore, we use the relation

$$f(t)\delta'(t-a) = f(a)\delta'(t-a) - f'(a)\delta(t-a)$$

Then, for this example,

$$t^2\delta'(t-3) = t^2|_{t=3}\delta'(t-3) - \frac{d}{dt}t^2|_{t=3}\delta(t-3) = 9\delta'(t-3) - 6\delta(t-3)$$

Example 3.9

a. Express the voltage waveform $v(t)$ shown in Figure 3.21 as a sum of unit step functions for the time interval $-1 < t < 7$ s.

b. Using the result of part (a), compute the derivative of $v(t)$ and sketch its waveform.

Solution:

a. We begin with the derivation of the equations for the linear segments of the given waveform as shown in Figure 3.22.

Next, we express $v(t)$ in terms of the unit step function $u_0(t)$, and we obtain

$$\begin{aligned} v(t) = {} & 2t[u_0(t+1) - u_0(t-1)] + 2[u_0(t-1) - u_0(t-2)] \\ & + (-t+5)[u_0(t-2) - u_0(t-4)] + [u_0(t-4) - u_0(t-5)] \\ & + (-t+6)[u_0(t-5) - u_0(t-7)] \end{aligned} \tag{3.52}$$

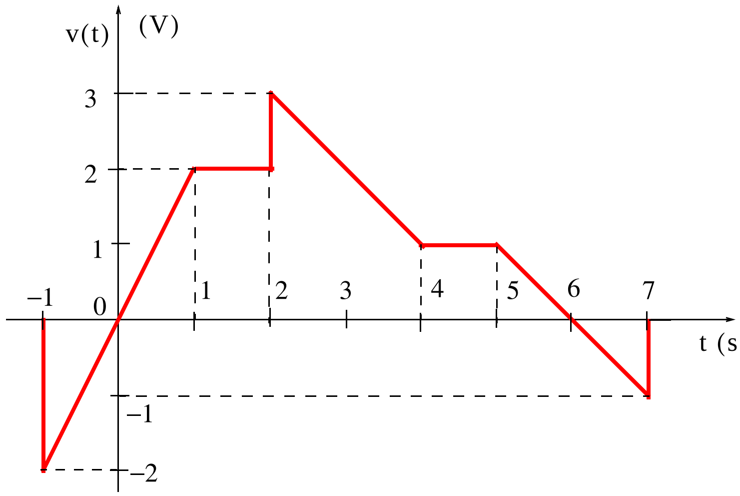


Figure 3.21. Waveform for Example 3.9

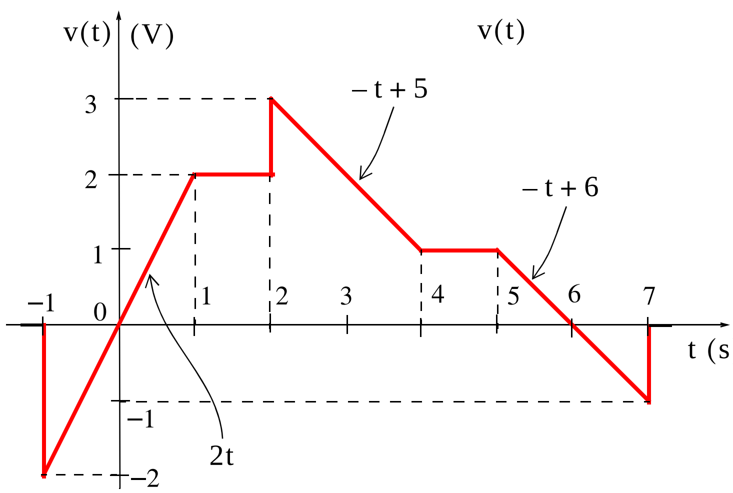


Figure 3.22. Equations for the linear segments of Figure 3.21

Multiplying and collecting like terms in (3.52), we obtain

$$
\begin{aligned}
v(t) = {} & 2tu_0(t+1) - 2tu_0(t-1) - 2u_0(t-1) - 2u_0(t-2) - tu_0(t-2) \\
& + 5u_0(t-2) + tu_0(t-4) - 5u_0(t-4) + u_0(t-4) - u_0(t-5) \\
& - tu_0(t-5) + 6u_0(t-5) + tu_0(t-7) - 6u_0(t-7)
\end{aligned}
$$

or

$$
\begin{aligned}
v(t) = {} & 2tu_0(t+1) + (-2t+2)u_0(t-1) + (-t+3)u_0(t-2) \\
& + (t-4)u_0(t-4) + (-t+5)u_0(t-5) + (t-6)u_0(t-7)
\end{aligned}
$$

b. The derivative of $v(t)$ is

$$\frac{dv}{dt} = 2u_0(t+1) + 2t\delta(t+1) - 2u_0(t-1) + (-2t+2)\delta(t-1)$$
$$- u_0(t-2) + (-t+3)\delta(t-2) + u_0(t-4) + (t-4)\delta(t-4) \qquad (3.53)$$
$$- u_0(t-5) + (-t+5)\delta(t-5) + u_0(t-7) + (t-6)\delta(t-7)$$

From the given waveform, we observe that discontinuities occur only at $t = -1$, $t = 2$, and $t = 7$. Therefore, $\delta(t-1) = 0$, $\delta(t-4) = 0$, and $\delta(t-5) = 0$, and the terms that contain these delta functions vanish. Also, by application of the sampling property,

$$2t\delta(t+1) = \{2t|_{t=-1}\}\delta(t+1) = -2\delta(t+1)$$
$$(-t+3)\delta(t-2) = \{(-t+3)|_{t=2}\}\delta(t-2) = \delta(t-2)$$
$$(t-6)\delta(t-7) = \{(t-6)|_{t=7}\}\delta(t-7) = \delta(t-7)$$

and by substitution into (3.53), we obtain

$$\frac{dv}{dt} = 2u_0(t+1) - 2\delta(t+1) - 2u_0(t-1) - u_0(t-2) \qquad (3.54)$$
$$+ \delta(t-2) + u_0(t-4) - u_0(t-5) + u_0(t-7) + \delta(t-7)$$

The plot of dv/dt is shown in Figure 3.23.

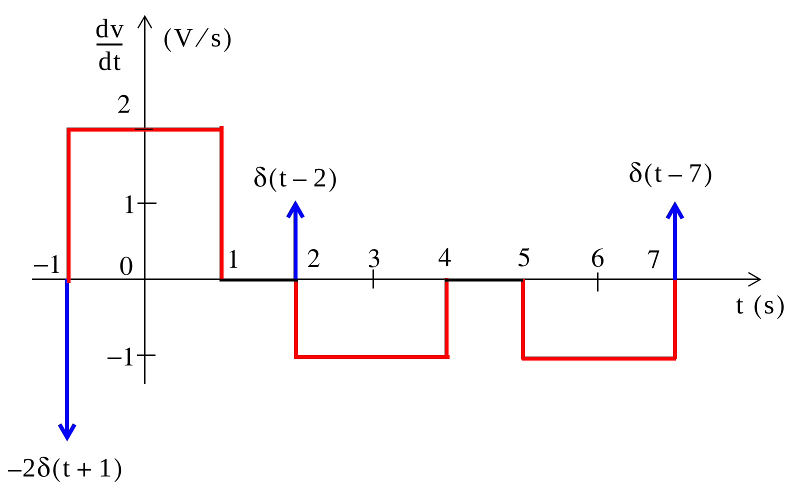


Figure 3.23. Plot of the derivative of the waveform of Figure 3.21

We observe that a negative spike of magnitude 2 occurs at $t = -1$, and two positive spikes of magnitude 1 occur at $t = 2$, and $t = 7$. These spikes occur because of the discontinuities at these points.

It would be interesting to observe the given signal and its derivative on the Scope block of the Simulink®* model of Figure 3.24. They are shown in Figure 3.25.

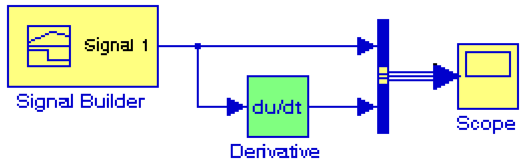


Figure 3.24. Simulink model for Example 3.9

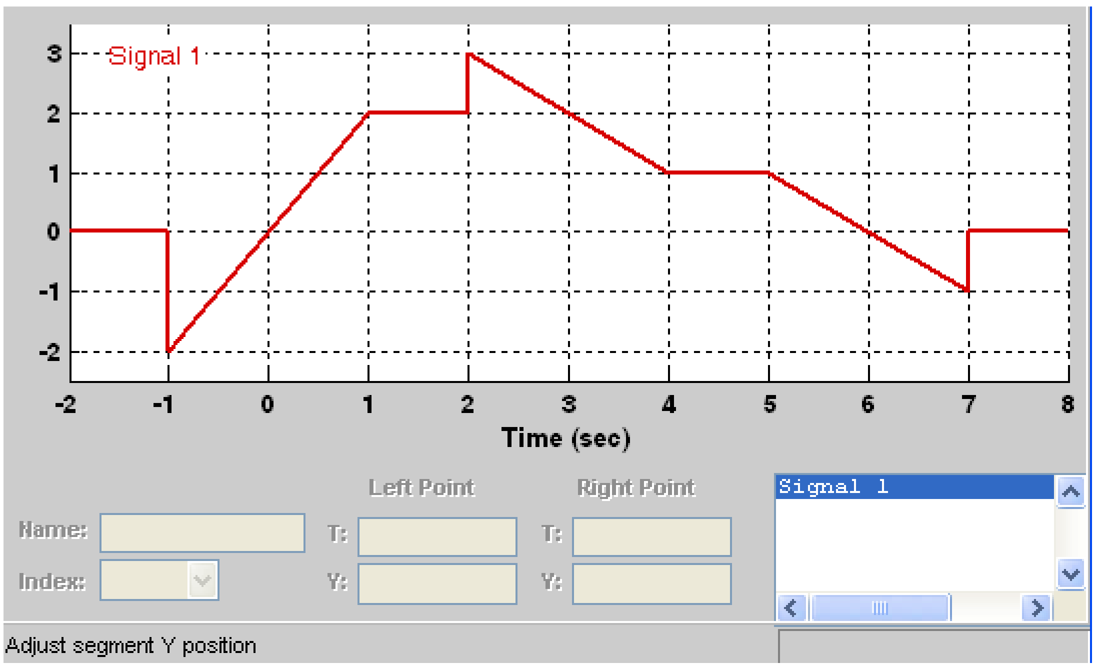


Figure 3.25. Piece−wise linear waveform for the Signal Builder block in Figure 3.24

The waveform in Figure 3.25 is created with the following procedure:

1. We open a new model by clicking the new model icon shown as a blank page on the left corner of the top menu bar. Initially, the name **Untitled** appears on the top of this new model. We save it with the name Figure_3.25 and Simulink appends the .mdl extension to it.

2. From the **Sources** library, we drag the **Signal Builder** block into this new model. We also drag the **Derivative** block from the **Continuous** library, the **Bus Creator** block from the **Commonly Used Blocks** library, and the **Scope** block into this model, and we interconnect these blocks as shown in Figure 3.24.

* *A brief introduction to Simulink is presented in Appendix B. For a detailed procedure for generating piece-wise linear functions with Simulink's Signal Builder block, please refer to Introduction to Simulink with Engineering Applications, ISBN 978−1−934404−09−6*

3. We double–click the Signal Builder block in Figure 3.24, and on the plot which appears as a square pulse, we click the y–axis and we enter Minimum: −2.5, and Maximum: 3.5. Likewise we right–click anywhere on the plot and we specify the Change Time Range at Min time: −2, and Max time: 8.

4. To select a particular point, we position the mouse cursor over that point and we left–click. A circle is drawn around that point to indicate that it is selected.

5. To select a line segment, we left–click on that segment. That line segment is now shown as a thick line indicating that it is selected. To deselect it, we press the Esc key.

6. To drag a line segment to a new position, we place the mouse cursor over that line segment and the cursor shape shows the position in which we can drag the segment.

7. To drag a point along the y–axis, we move the mouse cursor over that point, and the cursor changes to a circle indicating that we can drag that point. Then, we can move that point in a direction parallel to the x–axis.

8. To drag a point along the x–axis, we select that point, and we hold down the Shift key while dragging that point.

9. When we select a line segment on the time axis (x–axis) we observe that at the lower end of the waveform display window the **Left Point** and **Right Point** fields become visible. We can then reshape the given waveform by specifying the Time (**T**) and Amplitude (**Y**) points.

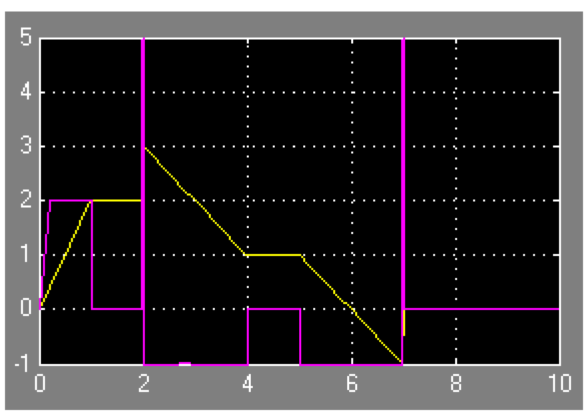


Figure 3.26. Waveforms for the Simulink model in Figure 3.24

The two positive spikes that occur at $t = 2$, and $t = 7$, are clearly shown in Figure 3.26.

MATLAB* has built-in functions for the unit step, and the delta functions. These are denoted by the names of the mathematicians who used them in their work. The unit step function $u_0(t)$ is referred to as **Heaviside(t)**, and the delta function $\delta(t)$ is referred to as **Dirac(t)**.

* *An introduction to MATLAB® is given in Appendix A.*

3.6 Summary

- The unit step function $u_0(t)$ is defined as

$$u_0(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

- The unit step function offers a convenient method of describing the sudden application of a voltage or current source.
- The unit ramp function, denoted as $u_1(t)$, is defined as

$$u_1(t) = \int_{-\infty}^{t} u_0(\tau)d\tau$$

- The unit impulse or delta function, denoted as $\delta(t)$, is the derivative of the unit step $u_0(t)$. It is also defined as

$$\int_{-\infty}^{t} \delta(\tau)d\tau = u_0(t)$$

and

$$\delta(t) = 0 \;\; \text{for all} \;\; t \neq 0$$

- The sampling property of the delta function states that

$$f(t)\delta(t-a) = f(a)\delta(t)$$

or, when $a = 0$,

$$f(t)\delta(t) = f(0)\delta(t)$$

- The sifting property of the delta function states that

$$\int_{-\infty}^{\infty} f(t)\delta(t-\alpha)dt = f(\alpha)$$

- The sampling property of the doublet function $\delta'(t)$ states that

$$f(t)\delta'(t-a) = f(a)\delta'(t-a) - f'(a)\delta(t-a)$$

3.7 Exercises

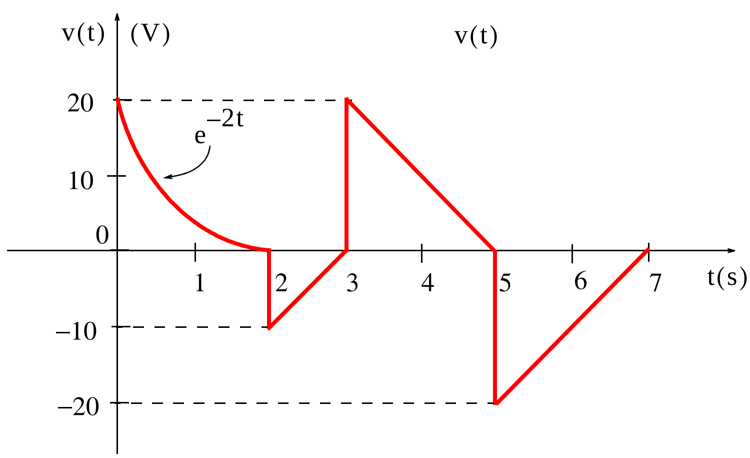
1. Evaluate the following functions:

a. $\sin t\delta\left(t-\frac{\pi}{6}\right)$ b. $\cos 2t\delta\left(t-\frac{\pi}{4}\right)$ c. $\cos^2 t\delta\left(t-\frac{\pi}{2}\right)$

d. $\tan 2t\delta\left(t-\frac{\pi}{8}\right)$ e. $\int_{-\infty}^{\infty} t^2 e^{-t}\delta(t-2)dt$ f. $\sin^2 t\delta^1\left(t-\frac{\pi}{2}\right)$

2.

a. Express the voltage waveform $v(t)$ shown below as a sum of unit step functions for the time interval $0 < t < 7$ s.



b. Using the result of part (a), compute the derivative of $v(t)$, and sketch its waveform. This waveform cannot be used with Sinulink's **Function Builder** block because it contains the decaying exponential segment which is a non–linear function.

3.8 Solutions to End–of–Chapter Exercises

1. We apply the sampling property of the $\delta(t)$ function for all expressions except (e) where we apply the sifting property. For part (f) we apply the sampling property of the doublet.

We recall that the sampling property states that $f(t)\delta(t-a) = f(a)\delta(t)$. Thus,

a. $\sin t\,\delta\left(t-\frac{\pi}{6}\right) = \sin t\big|_{t=\pi/6}\delta(t) = \sin\frac{\pi}{6}\delta(t) = 0.5\delta(t)$

b. $\cos 2t\,\delta\left(t-\frac{\pi}{4}\right) = \cos 2t\big|_{t=\pi/4}\delta(t) = \cos\frac{\pi}{2}\delta(t) = 0$

c. $\cos^2 t\,\delta\left(t-\frac{\pi}{2}\right) = \frac{1}{2}(1+\cos 2t)\Big|_{t=\pi/2}\delta(t) = \frac{1}{2}(1+\cos\pi)\delta(t) = \frac{1}{2}(1-1)\delta(t) = 0$

d. $\tan 2t\,\delta\left(t-\frac{\pi}{8}\right) = \tan 2t\big|_{t=\pi/8}\delta(t) = \tan\frac{\pi}{4}\delta(t) = \delta(t)$

We recall that the sampling property states that $\int_{-\infty}^{\infty} f(t)\delta(t-\alpha)dt = f(\alpha)$. Thus,

e. $\int_{-\infty}^{\infty} t^2 e^{-t}\delta(t-2)dt = t^2 e^{-t}\Big|_{t=2} = 4e^{-2} = 0.54$

f. We recall that the sampling property for the doublet states that

$$f(t)\delta'(t-a) = f(a)\delta'(t-a) - f'(a)\delta(t-a)$$

Thus,

$$\sin^2 t\,\delta'\left(t-\frac{\pi}{2}\right) = \sin^2 t\big|_{t=\pi/2}\delta'\left(t-\frac{\pi}{2}\right) - \frac{d}{dt}\sin^2 t\big|_{t=\pi/2}\delta\left(t-\frac{\pi}{2}\right)$$

$$= \frac{1}{2}(1-\cos 2t)\big|_{t=\pi/2}\delta'\left(t-\frac{\pi}{2}\right) - \sin 2t\big|_{t=\pi/2}\delta\left(t-\frac{\pi}{2}\right)$$

$$= \frac{1}{2}(1+1)\delta'\left(t-\frac{\pi}{2}\right) - \sin\pi\,\delta\left(t-\frac{\pi}{2}\right) = \delta'\left(t-\frac{\pi}{2}\right)$$

2.

a.
$$v(t) = e^{-2t}[u_0(t)-u_0(t-2)] + (10t-30)[u_0(t-2)-u_0(t-3)]$$
$$+(-10t+50)[u_0(t-3)-u_0(t-5)] + (10t-70)[u_0(t-5)-u_0(t-7)]$$

$$v(t) = e^{-2t}u_0(t) - e^{-2t}u_0(t-2) + 10tu_0(t-2) - 30u_0(t-2) - 10tu_0(t-3) + 30u_0(t-3)$$
$$- 10tu_0(t-3) + 50u_0(t-3) + 10tu_0(t-5) - 50u_0(t-5) + 10tu_0(t-5)$$
$$-70u_0(t-5) - 10tu_0(t-7) + 70u_0(t-7)$$

$$v(t) = e^{-2t}u_0(t) + (-e^{-2t} + 10t - 30)u_0(t-2) + (-20t + 80)u_0(t-3) + (20t - 120)u_0(t-5)$$
$$+(-10t + 70)u_0(t-7)$$

b.

$$\frac{dv}{dt} = -2e^{-2t}u_0(t) + e^{-2t}\delta(t) + (2e^{-2t} + 10)u_0(t-2) + (-e^{-2t} + 10t - 30)\delta(t-2)$$
$$-20u_0(t-3) + (-20t + 80)\delta(t-3) + 20u_0(t-5) + (20t - 120)\delta(t-5) \qquad (1)$$
$$-10u_0(t-7) + (-10t + 70)\delta(t-7)$$

Referring to the given waveform we observe that discontinuities occur only at $t = 2$, $t = 3$, and $t = 5$. Therefore, $\delta(t) = 0$ and $\delta(t-7) = 0$. Also, by the sampling property of the delta function

$$(-e^{-2t} + 10t - 30)\delta(t-2) = (-e^{-2t} + 10t - 30)\big|_{t=2}\delta(t-2) \approx -10\delta(t-2)$$

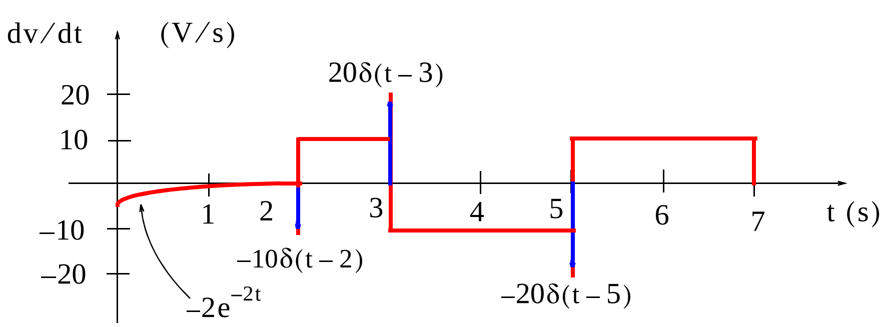
$$(-20t + 80)\delta(t-3) = (-20t + 80)\big|_{t=3}\delta(t-3) = 20\delta(t-3)$$

$$(20t - 120)\delta(t-5) = (20t - 120)\big|_{t=5}\delta(t-5) = -20\delta(t-5)$$

and with these simplifications (1) above reduces to

$$dv/dt = -2e^{-2t}u_0(t) + 2e^{-2t}u_0(t-2) + 10u_0(t-2) - 10\delta(t-2)$$
$$-20u_0(t-3) + 20\delta(t-3) + 20u_0(t-5) - 20\delta(t-5) - 10u_0(t-7)$$
$$= -2e^{-2t}[u_0(t) - u_0(t-2)] - 10\delta(t-2) + 10[u_0(t-2) - u_0(t-3)] + 20\delta(t-3)$$
$$- 10[u_0(t-3) - u_0(t-5)] - 20\delta(t-5) + 10[u_0(t-5) - u_0(t-7)]$$

The waveform for dv/dt is shown below.



Chapter 4

The Laplace Transformation

This chapter begins with an introduction to the Laplace transformation, definitions, and properties of the Laplace transformation. The initial value and final value theorems are also discussed and proved. It continues with the derivation of the Laplace transform of common functions of time, and concludes with the derivation of the Laplace transforms of common waveforms.

4.1 Definition of the Laplace Transformation

The *two–sided* or *bilateral* Laplace Transform pair is defined as

$$\mathcal{L}\{f(t)\} = F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt \tag{4.1}$$

$$\mathcal{L}^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi j}\int_{\sigma - j\omega}^{\sigma + j\omega} F(s)e^{st}ds \tag{4.2}$$

where $\mathcal{L}\{f(t)\}$ denotes the Laplace transform of the time function $f(t)$, $\mathcal{L}^{-1}\{F(s)\}$ denotes the Inverse Laplace transform, and s is a complex variable whose real part is σ, and imaginary part ω, that is, $s = \sigma + j\omega$.

In most problems, we are concerned with values of time t greater than some reference time, say $t = t_0 = 0$, and since the initial conditions are generally known, the two–sided Laplace transform pair of (4.1) and (4.2) simplifies to the *unilateral* or *one–sided Laplace transform* defined as

$$\mathcal{L}\{f(t)\} = F(s) = \int_{t_0}^{\infty} f(t)e^{-st}dt = \int_{0}^{\infty} f(t)e^{-st}dt \tag{4.3}$$

$$\mathcal{L}^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi j}\int_{\sigma - j\omega}^{\sigma + j\omega} F(s)e^{st}ds \tag{4.4}$$

The Laplace Transform of (4.3) has meaning only if the integral converges (reaches a limit), that is, if

$$\left|\int_{0}^{\infty} f(t)e^{-st}dt\right| < \infty \tag{4.5}$$

To determine the conditions that will ensure us that the integral of (4.3) converges, we rewrite (4.5) as

$$\left|\int_0^{\infty} f(t)e^{-\sigma t}e^{-j\omega t}dt\right| < \infty \tag{4.6}$$

The term $e^{-j\omega t}$ in the integral of (4.6) has magnitude of unity, i.e., $|e^{-j\omega t}| = 1$, and thus the condition for convergence becomes

$$\left|\int_0^{\infty} f(t)e^{-\sigma t}dt\right| < \infty \tag{4.7}$$

Fortunately, in most engineering applications the functions $f(t)$ are of *exponential order**. Then, we can express (4.7) as,

$$\left|\int_0^{\infty} f(t)e^{-\sigma t}dt\right| < \left|\int_0^{\infty} ke^{\sigma_0 t}e^{-\sigma t}dt\right| \tag{4.8}$$

and we see that the integral on the right side of the inequality sign in (4.8), converges if $\sigma > \sigma_0$. Therefore, we conclude that if $f(t)$ is of exponential order, $\mathcal{L}\{f(t)\}$ exists if

$$Re\{s\} = \sigma > \sigma_0 \tag{4.9}$$

where $Re\{s\}$ denotes the real part of the complex variable s.

Evaluation of the integral of (4.4) involves contour integration in the complex plane, and thus, it will not be attempted in this chapter. We will see in the next chapter that many Laplace transforms can be inverted with the use of a few standard pairs, and thus there is no need to use (4.4) to obtain the Inverse Laplace transform.

In our subsequent discussion, we will denote transformation from the time domain to the complex frequency domain, and vice versa, as

$$f(t) \Leftrightarrow F(s) \tag{4.10}$$

4.2 Properties and Theorems of the Laplace Transform

The most common properties and theorems of the Laplace transform are presented in Subsections 4.2.1 through 4.2.13 below.

4.2.1 Linearity Property

The *linearity property* states that if the functions

$$f_1(t),\ f_2(t),\ \ldots,\ f_n(t)$$

* *A function* $f(t)$ *is said to be of exponential order if* $|f(t)| < ke^{\sigma_0 t}$ for all $t \ge 0$.

have Laplace transforms

$$F_1(s), F_2(s), \ldots, F_n(s)$$

respectively, and

$$c_1, c_2, \ldots, c_n$$

are arbitrary constants, then,

$$\boxed{c_1 f_1(t) + c_2 f_2(t) + \ldots + c_n f_n(t) \Leftrightarrow c_1 F_1(s) + c_2 F_2(s) + \ldots + c_n F_n(s)} \tag{4.11}$$

Proof:

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t) + \ldots + c_n f_n(t)\} = \int_{t_0}^{\infty} [c_1 f_1(t) + c_2 f_2(t) + \ldots + c_n f_n(t)]dt$$

$$= c_1 \int_{t_0}^{\infty} f_1(t)e^{-st}dt + c_2 \int_{t_0}^{\infty} f_2(t)e^{-st}dt + \ldots + c_n \int_{t_0}^{\infty} f_n(t)e^{-st}dt$$

$$= c_1 F_1(s) + c_2 F_2(s) + \ldots + c_n F_n(s)$$

Note 1:

It is desirable to multiply $f(t)$ by the unit step function $u_0(t)$ to eliminate any unwanted non–zero values of $f(t)$ for $t < 0$.

4.2.2 Time Shifting Property

The *time shifting property* states that a right shift in the time domain by a units, corresponds to multiplication by e^{-as} in the complex frequency domain. Thus,

$$\boxed{f(t-a)u_0(t-a) \Leftrightarrow e^{-as}F(s)} \tag{4.12}$$

Proof:

$$\mathcal{L}\{f(t-a)u_0(t-a)\} = \int_0^a 0e^{-st}dt + \int_a^{\infty} f(t-a)e^{-st}dt \tag{4.13}$$

Now, we let $t - a = \tau$; then, $t = \tau + a$ and $dt = d\tau$. With these substitutions and with $a \to 0$, the second integral on the right side of (4.13) is expressed as

$$\int_0^{\infty} f(\tau)e^{-s(\tau + a)}d\tau = e^{-as}\int_0^{\infty} f(\tau)e^{-s\tau}d\tau = e^{-as}F(s)$$

4.2.3 Frequency Shifting Property

The *frequency shifting property* states that if we multiply a time domain function $f(t)$ by an exponential function e^{-at} where a is an arbitrary positive constant, this multiplication will produce a shift of the s variable in the complex frequency domain by a units. Thus,

$$\boxed{e^{-at}f(t) \Leftrightarrow F(s+a)} \qquad (4.14)$$

Proof:

$$\mathcal{L}\{e^{-at}f(t)\} = \int_0^\infty e^{-at}f(t)e^{-st}dt = \int_0^\infty f(t)e^{-(s+a)t}dt = F(s+a)$$

Note 2:

A change of scale is represented by multiplication of the time variable t by a positive scaling factor a. Thus, the function $f(t)$ after scaling the time axis, becomes $f(at)$.

4.2.4 Scaling Property

Let a be an arbitrary positive constant; then, the *scaling property* states that

$$\boxed{f(at) \Leftrightarrow \frac{1}{a}F\left(\frac{s}{a}\right)} \qquad (4.15)$$

Proof:

$$\mathcal{L}\{f(at)\} = \int_0^\infty f(at)e^{-st}dt$$

and letting $t = \tau/a$, we obtain

$$\mathcal{L}\{f(at)\} = \int_0^\infty f(\tau)e^{-s(\tau/a)}d\left(\frac{\tau}{a}\right) = \frac{1}{a}\int_0^\infty f(\tau)e^{-(s/a)\tau}d(\tau) = \frac{1}{a}F\left(\frac{s}{a}\right)$$

Note 3:

Generally, the initial value of $f(t)$ is taken at $t = 0^-$ to include any discontinuity that may be present at $t = 0$. If it is known that no such discontinuity exists at $t = 0^-$, we simply interpret $f(0^-)$ as $f(0)$.

4.2.5 Differentiation in Time Domain Property

The *differentiation in time domain* property states that differentiation in the time domain corresponds to multiplication by s in the complex frequency domain, minus the initial value of $f(t)$ at $t = 0^-$. Thus,

$$\boxed{f'(t) = \frac{d}{dt}f(t) \Leftrightarrow sF(s) - f(0^-)} \qquad (4.16)$$

Proof:

$$\mathcal{L}\{f'(t)\} = \int_0^\infty f'(t)e^{-st}dt$$

Using integration by parts where

$$\int v\,du = uv - \int u\,dv \tag{4.17}$$

we let $du = f'(t)$ and $v = e^{-st}$. Then, $u = f(t)$, $dv = -se^{-st}$, and thus

$$\mathscr{L}\{f'(t)\} = f(t)e^{-st}\Big|_{0^-}^{\infty} + s\int_{0^-}^{\infty} f(t)e^{-st}dt = \lim_{a\to\infty}\left[f(t)e^{-st}\Big|_{0^-}^{a}\right] + sF(s)$$

$$= \lim_{a\to\infty}[e^{-sa}f(a) - f(0^-)] + sF(s) = 0 - f(0^-) + sF(s)$$

The time differentiation property can be extended to show that

$$\boxed{\frac{d^2}{dt^2}f(t) \Leftrightarrow s^2F(s) - sf(0^-) - f'(0^-)} \tag{4.18}$$

$$\boxed{\frac{d^3}{dt^3}f(t) \Leftrightarrow s^3F(s) - s^2f(0^-) - sf'(0^-) - f''(0^-)} \tag{4.19}$$

and in general

$$\boxed{\frac{d^n}{dt^n}f(t) \Leftrightarrow s^nF(s) - s^{n-1}f(0^-) - s^{n-2}f'(0^-) - \dots - f^{n-1}(0^-)} \tag{4.20}$$

To prove (4.18), we let

$$g(t) = f'(t) = \frac{d}{dt}f(t)$$

and as we found above,

$$\mathscr{L}\{g'(t)\} = s\mathscr{L}\{g(t)\} - g(0^-)$$

Then,

$$\mathscr{L}\{f''(t)\} = s\mathscr{L}\{f'(t)\} - f'(0^-) = s[s\mathscr{L}[f(t)] - f(0^-)] - f'(0^-)$$

$$= s^2F(s) - sf(0^-) - f'(0^-)$$

Relations (4.19) and (4.20) can be proved by similar procedures.

We must remember that the terms $f(0^-)$, $f'(0^-)$, $f''(0^-)$, and so on, represent the initial conditions. Therefore, when all initial conditions are zero, and we differentiate a time function $f(t)$ n times, this corresponds to $F(s)$ multiplied by s to the nth power.

4.2.6 Differentiation in Complex Frequency Domain Property

This property states that *differentiation in complex frequency domain* and multiplication by minus one, corresponds to multiplication of $f(t)$ by t in the time domain. In other words,

$$\boxed{tf(t) \Leftrightarrow -\frac{d}{ds}F(s)} \qquad (4.21)$$

Proof:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} f(t)e^{-st}dt$$

Differentiating with respect to s and applying *Leibnitz's rule** *for differentiation under the integral*, we obtain

$$\frac{d}{ds}F(s) = \frac{d}{ds}\int_0^{\infty} f(t)e^{-st}dt = \int_0^{\infty} \frac{\partial}{\partial s}e^{-st}f(t)dt = \int_0^{\infty} -te^{-st}f(t)dt = -\int_0^{\infty} [tf(t)]e^{-st}dt = -\mathcal{L}[tf(t)]$$

In general,

$$\boxed{t^n f(t) \Leftrightarrow (-1)^n \frac{d^n}{ds^n}F(s)} \qquad (4.22)$$

The proof for $n \ge 2$ follows by taking the second and higher–order derivatives of $F(s)$ with respect to s.

4.2.7 Integration in Time Domain Property

This property states that *integration in time domain* corresponds to $F(s)$ divided by s plus the initial value of $f(t)$ at $t = 0^-$, also divided by s. That is,

$$\boxed{\int_{-\infty}^{t} f(\tau)d\tau \Leftrightarrow \frac{F(s)}{s} + \frac{f(0^-)}{s}} \qquad (4.23)$$

Proof:

We begin by expressing the integral on the left side of (4.23) as two integrals, that is,

$$\int_{-\infty}^{t} f(\tau)d\tau = \int_{-\infty}^{0} f(\tau)d\tau + \int_0^{t} f(\tau)d\tau \qquad (4.24)$$

The first integral on the right side of (4.24), represents a constant value since neither the upper, nor the lower limits of integration are functions of time, and this constant is an initial condition denoted as $f(0^-)$. We will find the Laplace transform of this constant, the transform of the sec-

* *This rule states that if a function of a parameter* α *is defined by the equation* $F(\alpha) = \int_a^b f(x, \alpha)dx$ *where f is some known function of integration x and the parameter* α, *a and b are constants independent of x and* α, *and the partial derivative* $\partial f/\partial\alpha$ *exists and it is continuous, then* $\frac{dF}{d\alpha} = \int_a^b \frac{\partial(x, \alpha)}{\partial(\alpha)}dx$.

ond integral on the right side of (4.24), and will prove (4.23) by the linearity property. Thus,

$$\mathcal{L}\{f(0^-)\} = \int_0^\infty f(0^-)e^{-st}dt = f(0^-)\int_0^\infty e^{-st}dt = f(0^-)\frac{e^{-st}}{-s}\bigg|_0^\infty$$
$$= f(0^-)\times 0 - \left(-\frac{f(0^-)}{s}\right) = \frac{f(0^-)}{s} \tag{4.25}$$

This is the value of the first integral in (4.24). Next, we will show that

$$\int_0^t f(\tau)d\tau \Leftrightarrow \frac{F(s)}{s}$$

We let

$$g(t) = \int_0^t f(\tau)d\tau$$

then,

$$g'(t) = f(\tau)$$

and

$$g(0) = \int_0^0 f(\tau)d\tau = 0$$

Now,

$$\mathcal{L}\{g'(t)\} = G(s) = s\mathcal{L}\{g(t)\} - g(0^-) = G(s) - 0$$

$$s\mathcal{L}\{g(t)\} = G(s)$$

$$\mathcal{L}\{g(t)\} = \frac{G(s)}{s}$$

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s} \tag{4.26}$$

and the proof of (4.23) follows from (4.25) and (4.26).

4.2.8 Integration in Complex Frequency Domain Property

This property states that *integration in complex frequency domain* with respect to s corresponds to division of a time function $f(t)$ by the variable t, provided that the limit $\lim_{t \to 0} \frac{f(t)}{t}$ exists. Thus,

$$\boxed{\frac{f(t)}{t} \Leftrightarrow \int_s^\infty F(s)ds} \tag{4.27}$$

Proof:

$$F(s) = \int_0^\infty f(t)e^{-st}dt$$

Integrating both sides from s to ∞, we obtain

$$\int_s^{\infty} F(s)ds = \int_s^{\infty}\left[\int_0^{\infty} f(t)e^{-st}dt\right]ds$$

Next, we interchange the order of integration, i.e.,

$$\int_s^{\infty} F(s)ds = \int_0^{\infty}\left[\int_s^{\infty} e^{-st}ds\right] f(t)dt$$

and performing the inner integration on the right side integral with respect to s, we obtain

$$\int_s^{\infty} F(s)ds = \int_0^{\infty}\left[-\frac{1}{t}e^{-st}\Big|_s^{\infty}\right] f(t)dt = \int_0^{\infty}\frac{f(t)}{t}e^{-st}dt = \mathcal{L}\left\{\frac{f(t)}{t}\right\}$$

4.2.9 Time Periodicity Property

The *time periodicity* property states that a periodic function of time with period T corresponds to the integral $\int_0^T f(t)e^{-st}dt$ divided by $(1-e^{-sT})$ in the complex frequency domain. Thus, if we let $f(t)$ be a periodic function with period T, that is, $f(t) = f(t+nT)$, for $n = 1, 2, 3, \ldots$ we obtain the transform pair

$$f(t+nT) \Leftrightarrow \frac{\int_0^T f(t)e^{-st}dt}{1-e^{-sT}} \tag{4.28}$$

Proof:

The Laplace transform of a periodic function can be expressed as

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)e^{-st}dt = \int_0^{T} f(t)e^{-st}dt + \int_T^{2T} f(t)e^{-st}dt + \int_{2T}^{3T} f(t)e^{-st}dt + \ldots$$

In the first integral of the right side, we let $t = \tau$, in the second $t = \tau + T$, in the third $t = \tau + 2T$, and so on. The areas under each period of $f(t)$ are equal, and thus the upper and lower limits of integration are the same for each integral. Then,

$$\mathcal{L}\{f(t)\} = \int_0^T f(\tau)e^{-s\tau}d\tau + \int_0^T f(\tau+T)e^{-s(\tau+T)}d\tau + \int_0^T f(\tau+2T)e^{-s(\tau+2T)}d\tau + \ldots \tag{4.29}$$

Since the function is periodic, i.e., $f(\tau) = f(\tau+T) = f(\tau+2T) = \ldots = f(\tau+nT)$, we can express (4.29) as

$$\mathcal{L}\{f(\tau)\} = (1 + e^{-sT} + e^{-2sT} + \dots)\int_0^T f(\tau)e^{-s\tau}d\tau \tag{4.30}$$

By application of the binomial theorem, that is,

$$1 + a + a^2 + a^3 + \dots = \frac{1}{1-a} \tag{4.31}$$

we find that expression (4.30) reduces to

$$\mathcal{L}\{f(\tau)\} = \frac{\int_0^T f(\tau)e^{-s\tau}d\tau}{1 - e^{-sT}}$$

4.2.10 Initial Value Theorem

The *initial value theorem* states that the initial value $f(0^-)$ of the time function $f(t)$ can be found from its Laplace transform multiplied by s and letting $s \to \infty$. That is,

$$\boxed{\lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s) = f(0^-)} \tag{4.32}$$

Proof:

From the time domain differentiation property,

$$\frac{d}{dt} f(t) \Leftrightarrow sF(s) - f(0^-)$$

or

$$\mathcal{L}\left\{\frac{d}{dt} f(t)\right\} = sF(s) - f(0^-) = \int_0^\infty \frac{d}{dt} f(t)e^{-st}dt$$

Taking the limit of both sides by letting $s \to \infty$, we obtain

$$\lim_{s \to \infty} [sF(s) - f(0^-)] = \lim_{s \to \infty} \left[\lim_{\substack{T \to \infty \\ \varepsilon \to 0}} \int_\varepsilon^T \frac{d}{dt} f(t)e^{-st}dt \right]$$

Interchanging the limiting process, we obtain

$$\lim_{s \to \infty} [sF(s) - f(0^-)] = \lim_{\substack{T \to \infty \\ \varepsilon \to 0}} \int_\varepsilon^T \frac{d}{dt} f(t)\left[\lim_{s \to \infty} e^{-st}\right]dt$$

and since

$$\lim_{s \to \infty} e^{-st} = 0$$

the above expression reduces to

$$\lim_{s \to \infty} [sF(s) - f(0^-)] = 0$$

or

$$\lim_{s \to \infty} sF(s) = f(0^-)$$

4.2.11 Final Value Theorem

The *final value theorem* states that the final value $f(\infty)$ of the time function $f(t)$ can be found from its Laplace transform multiplied by s, then, letting $s \to 0$. That is,

$$\boxed{\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s) = f(\infty)} \qquad (4.33)$$

Proof:

From the time domain differentiation property,

$$\frac{d}{dt} f(t) \Leftrightarrow sF(s) - f(0^-)$$

or

$$\mathcal{L}\left\{\frac{d}{dt} f(t)\right\} = sF(s) - f(0^-) = \int_0^{\infty} \frac{d}{dt} f(t) e^{-st} dt$$

Taking the limit of both sides by letting $s \to 0$, we obtain

$$\lim_{s \to 0} [sF(s) - f(0^-)] = \lim_{s \to 0} \left[\lim_{\substack{T \to \infty \\ \varepsilon \to 0}} \int_{\varepsilon}^{T} \frac{d}{dt} f(t) e^{-st} dt \right]$$

and by interchanging the limiting process, the expression above is written as

$$\lim_{s \to 0} [sF(s) - f(0^-)] = \lim_{\substack{T \to \infty \\ \varepsilon \to 0}} \int_{\varepsilon}^{T} \frac{d}{dt} f(t) \left[\lim_{s \to 0} e^{-st} \right] dt$$

Also, since

$$\lim_{s \to 0} e^{-st} = 1$$

it reduces to

$$\lim_{s \to 0} [sF(s) - f(0^-)] = \lim_{\substack{T \to \infty \\ \varepsilon \to 0}} \int_{\varepsilon}^{T} \frac{d}{dt} f(t) dt = \lim_{\substack{T \to \infty \\ \varepsilon \to 0}} \int_{\varepsilon}^{T} f(t) = \lim_{\substack{T \to \infty \\ \varepsilon \to 0}} [f(T) - f(\varepsilon)] = f(\infty) - f(0^-)$$

Therefore,

$$\lim_{s \to 0} sF(s) = f(\infty)$$

4.2.12 Convolution in Time Domain Property

*Convolution** *in the time domain* corresponds to multiplication in the complex frequency domain, that is,

$$f_1(t)*f_2(t) \Leftrightarrow F_1(s)F_2(s) \tag{4.34}$$

Proof:

$$\begin{aligned}\mathcal{L}\{f_1(t)*f_2(t)\} &= \mathcal{L}\left[\int_{-\infty}^{\infty} f_1(\tau)f_2(t-\tau)d\tau\right] = \int_0^{\infty}\left[\int_0^{\infty} f_1(\tau)f_2(t-\tau)d\tau\right]e^{-st}dt \\ &= \int_0^{\infty} f_1(\tau)\left[\int_0^{\infty} f_2(t-\tau)e^{-st}dt\right]d\tau\end{aligned} \tag{4.35}$$

We let $t-\tau = \lambda$; then, $t = \lambda+\tau$, and $dt = d\lambda$. Then, by substitution into (4.35),

$$\begin{aligned}\mathcal{L}\{f_1(t)*f_2(t)\} &= \int_0^{\infty} f_1(\tau)\left[\int_0^{\infty} f_2(\lambda)e^{-s(\lambda+\tau)}d\lambda\right]d\tau = \int_0^{\infty} f_1(\tau)e^{-s\tau}d\tau\int_0^{\infty} f_2(\lambda)e^{-s\lambda}d\lambda \\ &= F_1(s)F_2(s)\end{aligned}$$

4.2.13 Convolution in Complex Frequency Domain Property

Convolution in the complex frequency domain divided by $1/2\pi j$, corresponds to multiplication in the time domain. That is,

$$f_1(t)f_2(t) \Leftrightarrow \frac{1}{2\pi j}F_1(s)*F_2(s) \tag{4.36}$$

Proof:

$$\mathcal{L}\{f_1(t)f_2(t)\} = \int_0^{\infty} f_1(t)f_2(t)e^{-st}dt \tag{4.37}$$

and recalling that the Inverse Laplace transform from (4.2) is

$$f_1(t) = \frac{1}{2\pi j}\int_{\sigma-j\omega}^{\sigma+j\omega} F_1(\mu)e^{\mu t}d\mu$$

* *Convolution is the process of overlapping two time functions* $f_1(t)$ *and* $f_2(t)$. *The convolution integral indicates the amount of overlap of one function as it is shifted over another function The convolution of two time functions* $f_1(t)$ *and* $f_2(t)$ *is denoted as* $f_1(t)*f_2(t)$, *and by definition,* $f_1(t)*f_2(t) = \int_{-\infty}^{\infty} f_1(\tau)f_2(t-\tau)d\tau$ *where* τ *is a dummy variable. Convolution is discussed in Signals and Systems with MATLAB Computing and Simulink Modeling, ISBN 978−1−934404−11−9.*

by substitution into (4.37), we obtain

$$\mathcal{L}\{f_1(t)f_2(t)\} = \int_0^{\infty}\left[\frac{1}{2\pi j}\int_{\sigma-j\omega}^{\sigma+j\omega}F_1(\mu)e^{\mu t}d\mu\right]f_2(t)e^{-st}dt = \frac{1}{2\pi j}\int_{\sigma-j\omega}^{\sigma+j\omega}F_1(\mu)\left[\int_0^{\infty}f_2(t)e^{-(s-\mu)t}dt\right]d\mu$$

We observe that the bracketed integral is $F_2(s-\mu)$; therefore,

$$\mathcal{L}\{f_1(t)f_2(t)\} = \frac{1}{2\pi j}\int_{\sigma-j\omega}^{\sigma+j\omega}F_1(\mu)F_2(s-\mu)d\mu = \frac{1}{2\pi j}F_1(s)*F_2(s)$$

For easy reference, the Laplace transform pairs and theorems are summarized in Table 4.1.

4.3 Laplace Transform of Common Functions of Time

In this section, we will derive the Laplace transform of common functions of time. They are presented in Subsections 4.3.1 through 4.3.11 below.

4.3.1 Laplace Transform of the Unit Step Function $u_0(t)$

We begin with the definition of the Laplace transform, that is,

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty}f(t)e^{-st}dt$$

or

$$\mathcal{L}\{u_0(t)\} = \int_0^{\infty}1e^{-st}dt = \left.\frac{-e^{st}}{s}\right|_0^{\infty} = 0-\left(-\frac{1}{s}\right) = \frac{1}{s}$$

Thus, we have obtained the transform pair

$$\boxed{u_0(t) \Leftrightarrow \frac{1}{s}} \tag{4.38}$$

for $\mathrm{Re}\{s\} = \sigma > 0$.*

4.3.2 Laplace Transform of the Ramp Function $u_1(t)$

We apply the definition

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty}f(t)e^{-st}dt$$

* *This condition was established in relation (4.9), Page 4−2.*

TABLE 4.1 Summary of Laplace Transform Properties and Theorems

	Property/Theorem	Time Domain	Complex Frequency Domain
1	Linearity	$c_1 f_1(t) + c_2 f_2(t) + \ldots + c_n f_n(t)$	$c_1 F_1(s) + c_2 F_2(s) + \ldots + c_n F_n(s)$
2	Time Shifting	$f(t-a)u_0(t-a)$	$e^{-as}F(s)$
3	Frequency Shifting	$e^{-as}f(t)$	$F(s+a)$
4	Time Scaling	$f(at)$	$\frac{1}{a}F\left(\frac{s}{a}\right)$
5	Time Differentiation See also (4.18) through (4.20)	$\frac{d}{dt}f(t)$	$sF(s) - f(0^-)$
6	Frequency Differentiation See also (4.22)	$tf(t)$	$-\frac{d}{ds}F(s)$
7	Time Integration	$\int_{-\infty}^{t} f(\tau)d\tau$	$\frac{F(s)}{s} + \frac{f(0^-)}{s}$
8	Frequency Integration	$\frac{f(t)}{t}$	$\int_{s}^{\infty} F(s)ds$
9	Time Periodicity	$f(t+nT)$	$\frac{\int_0^T f(t)e^{-st}dt}{1-e^{-sT}}$
10	Initial Value Theorem	$\lim_{t \to 0} f(t)$	$\lim_{s \to \infty} sF(s) = f(0^-)$
11	Final Value Theorem	$\lim_{t \to \infty} f(t)$	$\lim_{s \to 0} sF(s) = f(\infty)$
12	Time Convolution	$f_1(t)*f_2(t)$	$F_1(s)F_2(s)$
13	Frequency Convolution	$f_1(t)f_2(t)$	$\frac{1}{2\pi j} F_1(s)*F_2(s)$

or

$$\mathcal{L}\{u_1(t)\} = \mathcal{L}\{t\} = \int_0^{\infty} te^{-st}dt$$

We will perform integration by parts by recalling that

$$\int u dv = uv - \int v du \tag{4.39}$$

We let

$$u = t \text{ and } dv = e^{-st}$$

then,

$$du = 1 \text{ and } v = \frac{-e^{-st}}{s}$$

By substitution into (4.39),

$$\mathcal{L}\{t\} = \left.\frac{-te^{-st}}{s}\right|_0^\infty - \int_0^\infty \frac{-e^{-st}}{s}dt = \left.\left[\frac{-te^{-st}}{s} - \frac{e^{-st}}{s^2}\right]\right|_0^\infty \tag{4.40}$$

Since the upper limit of integration in (4.40) produces an indeterminate form, we apply *L' Hôpital's rule**, that is,

$$\lim_{t \to \infty} te^{-st} = \lim_{t \to \infty} \frac{t}{e^{st}} = \lim_{t \to \infty} \frac{\frac{d}{dt}(t)}{\frac{d}{dt}(e^{st})} = \lim_{t \to \infty} \frac{1}{se^{st}} = 0$$

Evaluating the second term of (4.40), we obtain $\mathcal{L}\{t\} = \frac{1}{s^2}$

Thus, we have obtained the transform pair

$$\boxed{t \Leftrightarrow \frac{1}{s^2}} \tag{4.41}$$

for $\sigma > 0$.

4.3.3 Laplace Transform of $t^n u_0(t)$

Before deriving the Laplace transform of this function, we digress to review the *gamma* or *general-*

* *Often, the ratio of two functions, such as* $\frac{f(x)}{g(x)}$, *for some value of x, say a, results in an indeterminate form. To work around this problem, we consider the limit* $\lim_{x \to a} \frac{f(x)}{g(x)}$, *and we wish to find this limit, if it exists. To find this limit, we use L'Hôpital's rule which states that if* $f(a) = g(a) = 0$, *and if the limit* $\frac{d}{dx}f(x)/\frac{d}{dx}g(x)$ *as x approaches a exists, then,* $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \left(\frac{d}{dx}f(x)/\frac{d}{dx}g(x)\right)$

ized factorial function $\Gamma(n)$ which is an *improper integral** but converges (approaches a limit) for all $n > 0$. It is defined as

$$\Gamma(n) = \int_0^{\infty} x^{n-1}e^{-x}dx \tag{4.42}$$

We will now derive the basic properties of the gamma function, and its relation to the well known factorial function

$$n! = n(n-1)(n-2)\cdot\ \ \cdot 3\cdot 2\cdot 1$$

The integral of (4.42) can be evaluated by performing integration by parts. Thus, in (4.42) we let

$$u = e^{-x} \quad \text{and} \quad dv = x^{n-1}$$

Then,

$$du = -e^{-x}dx \quad \text{and} \quad v = \frac{x^n}{n}$$

and (4.42) is written as

$$\Gamma(n) = \left.\frac{x^n e^{-x}}{n}\right|_{x=0}^{\infty} + \frac{1}{n}\int_0^{\infty} x^n e^{-x}dx \tag{4.43}$$

With the condition that $n > 0$, the first term on the right side of (4.43) vanishes at the lower limit $x = 0$. It also vanishes at the upper limit as $x \to \infty$. This can be proved with L' Hôpital's rule by differentiating both numerator and denominator m times, where $m \ge n$. Then,

$$\lim_{x\to\infty}\frac{x^n e^{-x}}{n} = \lim_{x\to\infty}\frac{x^n}{ne^x} = \lim_{x\to\infty}\frac{\frac{d^m}{dx^m}x^n}{\frac{d^m}{dx^m}ne^x} = \lim_{x\to\infty}\frac{\frac{d^{m-1}}{dx^{m-1}}nx^{n-1}}{\frac{d^{m-1}}{dx^{m-1}}ne^x} = \ldots$$

$$= \lim_{x\to\infty}\frac{n(n-1)(n-2)\ldots(n-m+1)x^{n-m}}{ne^x} = \lim_{x\to\infty}\frac{(n-1)(n-2)\ldots(n-m+1)}{x^{m-n}e^x} = 0$$

Therefore, (4.43) reduces to

$$\Gamma(n) = \frac{1}{n}\int_0^{\infty} x^n e^{-x}dx$$

and with (4.42), we have

---

* *Improper integrals are two types and these are:*

a. $\int_a^b f(x)dx$ *where the limits of integration a or b or both are infinite*

b. $\int_a^b f(x)dx$ *where f(x) becomes infinite at a value x between the lower and upper limits of integration inclusive.*

$$\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx = \frac{1}{n}\int_0^\infty x^n e^{-x}dx \tag{4.44}$$

By comparing the integrals in (4.44), we observe that

$$\boxed{\Gamma(n) = \frac{\Gamma(n+1)}{n}} \tag{4.45}$$

or

$$\boxed{n\Gamma(n) = \Gamma(n+1)} \tag{4.46}$$

It is convenient to use (4.45) for $n<0$, and (4.46) for $n>0$. From (4.45), we see that $\Gamma(n)$ becomes infinite as $n \to 0$.

For $n = 1$, (4.42) yields

$$\Gamma(1) = \int_0^\infty e^{-x}dx = -e^{-x}\Big|_0^\infty = 1 \tag{4.47}$$

and thus we have obtained the important relation,

$$\Gamma(1) = 1 \tag{4.48}$$

From the recurring relation of (4.46), we obtain

$$\begin{aligned}\Gamma(2) &= 1\cdot\Gamma(1) = 1\\ \Gamma(3) &= 2\cdot\Gamma(2) = 2\cdot 1 = 2!\\ \Gamma(4) &= 3\cdot\Gamma(3) = 3\cdot 2 = 3!\end{aligned} \tag{4.49}$$

and in general

$$\boxed{\Gamma(n+1) = n!} \tag{4.50}$$

for $n = 1, 2, 3, \ldots$

The formula of (4.50) is a noteworthy relation; it establishes the relationship between the $\Gamma(n)$ function and the factorial $n!$

We now return to the problem of finding the Laplace transform pair for $t^n u_0 t$, that is,

$$\mathcal{L}\{t^n u_0 t\} = \int_0^\infty t^n e^{-st}dt \tag{4.51}$$

To make this integral resemble the integral of the gamma function, we let $st = y$, or $t = y/s$, and thus $dt = dy/s$. Now, we rewrite (4.51) as

$$\mathcal{L}\{t^n u_0 t\} = \int_0^\infty \left(\frac{y}{s}\right)^n e^{-y} d\left(\frac{y}{s}\right) = \frac{1}{s^{n+1}}\int_0^\infty y^n e^{-y}dy = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}}$$

Therefore, we have obtained the transform pair

$$\boxed{t^n u_0(t) \Leftrightarrow \frac{n!}{s^{n+1}}} \tag{4.52}$$

for positive integers of n and $\sigma > 0$.

4.3.4 Laplace Transform of the Delta Function $\delta(t)$

We apply the definition

$$\mathcal{L}\{\delta(t)\} = \int_0^\infty \delta(t) e^{-st} dt$$

and using the sifting property of the delta function,* we obtain

$$\mathcal{L}\{\delta(t)\} = \int_0^\infty \delta(t) e^{-st} dt = e^{-s(0)} = 1$$

Thus, we have the transform pair

$$\boxed{\delta(t) \Leftrightarrow 1} \tag{4.53}$$

for all σ.

4.3.5 Laplace Transform of the Delayed Delta Function $\delta(t-a)$

We apply the definition

$$\mathcal{L}\{\delta(t-a)\} = \int_0^\infty \delta(t-a) e^{-st} dt$$

and again, using the sifting property of the delta function, we obtain

$$\mathcal{L}\{\delta(t-a)\} = \int_0^\infty \delta(t-a) e^{-st} dt = e^{-as}$$

Thus, we have the transform pair

$$\boxed{\delta(t-a) \Leftrightarrow e^{-as}} \tag{4.54}$$

for $\sigma > 0$.

* *The sifting property of the $\delta(t)$ is described in Subsection 3.4.2, Chapter 3.*

4.3.6 Laplace Transform of $e^{-at}u_0(t)$

We apply the definition

$$\mathcal{L}\{e^{-at}u_0(t)\} = \int_0^{\infty} e^{-at}e^{-st}dt = \int_0^{\infty} e^{-(s+a)t}dt = \left(-\frac{1}{s+a}\right)e^{-(s+a)t}\Bigg|_0^{\infty} = \frac{1}{s+a}$$

Thus, we have the transform pair

$$\boxed{e^{-at}u_0(t) \Leftrightarrow \frac{1}{s+a}} \qquad (4.55)$$

for $\sigma > -a$.

4.3.7 Laplace Transform of $t^n e^{-at}u_0(t)$

For this derivation, we will use the transform pair of (4.52), i.e.,

$$t^n u_0(t) \Leftrightarrow \frac{n!}{s^{n+1}} \qquad (4.56)$$

and the frequency shifting property of (4.14), that is,

$$e^{-at}f(t) \Leftrightarrow F(s+a) \qquad (4.57)$$

Then, replacing s with $s+a$ in (4.56), we obtain the transform pair

$$\boxed{t^n e^{-at}u_0(t) \Leftrightarrow \frac{n!}{(s+a)^{n+1}}} \qquad (4.58)$$

where n is a positive integer, and $\sigma > -a$. Thus, for $n = 1$, we obtain the transform pair

$$\boxed{te^{-at}u_0(t) \Leftrightarrow \frac{1}{(s+a)^2}} \qquad (4.59)$$

for $\sigma > -a$.

For $n = 2$, we obtain the transform

$$\boxed{t^2 e^{-at}u_0(t) \Leftrightarrow \frac{2!}{(s+a)^3}} \qquad (4.60)$$

and in general,

$$\boxed{t^n e^{-at}u_0(t) \Leftrightarrow \frac{n!}{(s+a)^{n+1}}} \qquad (4.61)$$

for $\sigma > -a$.

4.3.8 Laplace Transform of $\sin\omega t\, u_0(t)$

We apply the definition

$$\mathcal{L}\{\sin\omega t\; u_0(t)\} = \int_0^{\infty}(\sin\omega t)e^{-st}dt = \lim_{a\to\infty}\int_0^{a}(\sin\omega t)e^{-st}dt$$

and from tables of integrals*

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}$$

Then,

$$\mathcal{L}\{\sin\omega t\; u_0(t)\} = \lim_{a\to\infty}\left.\frac{e^{-st}(-s\sin\omega t - \omega\cos\omega t)}{s^2+\omega^2}\right|_0^a$$

$$= \lim_{a\to\infty}\left[\frac{e^{-as}(-s\sin\omega a - \omega\cos\omega a)}{s^2+\omega^2} + \frac{\omega}{s^2+\omega^2}\right] = \frac{\omega}{s^2+\omega^2}$$

Thus, we have obtained the transform pair

$$\boxed{\sin\omega t\; u_0 t \Leftrightarrow \frac{\omega}{s^2+\omega^2}} \qquad (4.62)$$

for $\sigma > 0$.

4.3.9 Laplace Transform of $\cos\omega t\, u_0(t)$

We apply the definition

$$\mathcal{L}\{\cos\omega t\; u_0(t)\} = \int_0^{\infty}(\cos\omega t)e^{-st}dt = \lim_{a\to\infty}\int_0^{a}(\cos\omega t)e^{-st}dt$$

and from tables of integrals†

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$$

Then,

---

* *This can also be derived from* $\sin\omega t = \frac{1}{j2}(e^{j\omega t} - e^{-j\omega t})$, *and the use of (4.55) where* $e^{-at}u_0(t) \Leftrightarrow \frac{1}{s+a}$. *By the linearity property, the sum of these terms corresponds to the sum of their Laplace transforms. Therefore,* $\mathcal{L}[\sin\omega t u_0(t)] = \frac{1}{j2}\left(\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right) = \frac{\omega}{s^2+\omega^2}$

$$\mathcal{L}\{\cos\omega t\, u_0(t)\} = \lim_{a \to \infty} \left. \frac{e^{-st}(-s\cos\omega t + \omega\sin\omega t)}{s^2+\omega^2} \right|_0^a$$

$$= \lim_{a \to \infty} \left[\frac{e^{-as}(-s\cos\omega a + \omega\sin\omega a)}{s^2+\omega^2} + \frac{s}{s^2+\omega^2} \right] = \frac{s}{s^2+\omega^2}$$

Thus, we have the fransform pair

$$\boxed{\cos\omega t\, u_0 t \Leftrightarrow \frac{s}{s^2+\omega^2}} \tag{4.63}$$

for $\sigma > 0$.

4.3.10 Laplace Transform of $e^{-at}\sin\omega t\, u_0(t)$

From (4.62),

$$\sin\omega t u_0 t \Leftrightarrow \frac{\omega}{s^2+\omega^2}$$

Using the frequency shifting property of (4.14), that is,

$$e^{-at} f(t) \Leftrightarrow F(s+a) \tag{4.64}$$

we replace s with $s+a$, and we obtain

$$\boxed{e^{-at}\sin\omega t\, u_0(t) \Leftrightarrow \frac{\omega}{(s+a)^2+\omega^2}} \tag{4.65}$$

for $\sigma > 0$ and $a > 0$.

4.3.11 Laplace Transform of $e^{-at}\cos\omega t\, u_0(t)$

From (4.63),

$$\cos\omega t\, u_0(t) \Leftrightarrow \frac{s}{s^2+\omega^2}$$

† *We can use the relation* $\cos\omega t = \frac{1}{2}(e^{j\omega t} + e^{-j\omega t})$ *and the linearity property, as in the derivation of the transform of* $\sin\omega t$ *on the footnote of the previous page. We can also use the transform pair* $\frac{d}{dt} f(t) \Leftrightarrow sF(s) - f(0^-)$ *; this is the time differentiation property of (4.16). Applying this transform pair for this derivation, we obtain*
$\mathcal{L}[\cos\omega t u_0(t)] = L\left[\frac{1}{\omega}\frac{d}{dt}\sin\omega t u_0(t)\right] = \frac{1}{\omega}\mathcal{L}\left[\frac{d}{dt}\sin\omega t u_0(t)\right] = \frac{1}{\omega}s\frac{\omega}{s^2+\omega^2} = \frac{s}{s^2+\omega^2}$

and using the frequency shifting property of (4.14), we replace s with $s+a$, and we obtain

$$e^{-at}\cos\omega t\, u_0(t) \Leftrightarrow \frac{s+a}{(s+a)^2+\omega^2} \tag{4.66}$$

for $\sigma > 0$ and $a > 0$.

For easy reference, we have summarized the above derivations in Table 4.2.

TABLE 4.2 Laplace Transform Pairs for Common Functions

	$f(t)$	$F(s)$
1	$u_0(t)$	$1/s$
2	$tu_0(t)$	$1/s^2$
3	$t^n u_0(t)$	$\frac{n!}{s^{n+1}}$
4	$\delta(t)$	1
5	$\delta(t-a)$	e^{-as}
6	$e^{-at}u_0(t)$	$\frac{1}{s+a}$
7	$t^n e^{-at} u_0(t)$	$\frac{n!}{(s+a)^{n+1}}$
8	$\sin\omega t\, u_0(t)$	$\frac{\omega}{s^2+\omega^2}$
9	$\cos\omega t\, u_0(t)$	$\frac{s}{s^2+\omega^2}$
10	$e^{-at}\sin\omega t\, u_0(t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
11	$e^{-at}\cos\omega t\, u_0(t)$	$\frac{s+a}{(s+a)^2+\omega^2}$

4.4 Laplace Transform of Common Waveforms

In this section, we will present procedures for deriving the Laplace transform of common waveforms using the transform pairs of Tables 4.1 and 4.2. The derivations are described in Subsections 4.4.1 through 4.4.5 below.

4.4.1 Laplace Transform of a Pulse

The waveform of a pulse, denoted as $f_P(t)$, is shown in Figure 4.1.

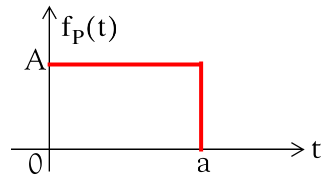


Figure 4.1. Waveform for a pulse

We first express the given waveform as a sum of unit step functions as we've learned in Chapter 3. Then,

$$f_P(t) = A[u_0(t) - u_0(t-a)] \tag{4.67}$$

From Table 4.1, Page 4–13,

$$f(t-a)u_0(t-a) \Leftrightarrow e^{-as}F(s)$$

and from Table 4.2, Page 4–22

$$u_0(t) \Leftrightarrow 1/s$$

Thus,

$$Au_0(t) \Leftrightarrow A/s$$

and

$$Au_0(t-a) \Leftrightarrow e^{-as}\frac{A}{s}$$

Then, in accordance with the linearity property, the Laplace transform of the pulse of Figure 4.1 is

$$\boxed{A[u_0(t) - u_0(t-a)] \Leftrightarrow \frac{A}{s} - e^{-as}\frac{A}{s} = \frac{A}{s}(1 - e^{-as})}$$

4.4.2 Laplace Transform of a Linear Segment

The waveform of a linear segment, denoted as $f_L(t)$, is shown in Figure 4.2.

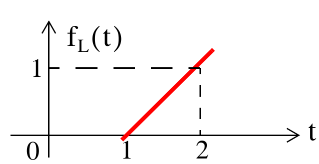


Figure 4.2. Waveform for a linear segment

We must first derive the equation of the linear segment. This is shown in Figure 4.3.

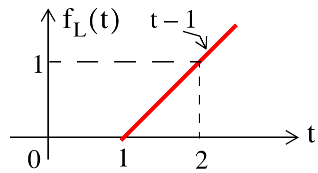


Figure 4.3. Waveform for a linear segment with the equation that describes it

Next, we express the given waveform in terms of the unit step function as follows:

$$f_L(t) = (t-1)u_0(t-1)$$

From Table 4.1, Page 4−13,

$$f(t-a)u_0(t-a) \Leftrightarrow e^{-as}F(s)$$

and from Table 4.2, Page 4−22,

$$tu_0(t) \Leftrightarrow \frac{1}{s^2}$$

Therefore, the Laplace transform of the linear segment of Figure 4.2 is

$$\boxed{(t-1)u_0(t-1) \Leftrightarrow e^{-s}\frac{1}{s^2}} \qquad (4.68)$$

4.4.3 Laplace Transform of a Triangular Waveform

The waveform of a triangular waveform, denoted as $f_T(t)$, is shown in Figure 4.4.

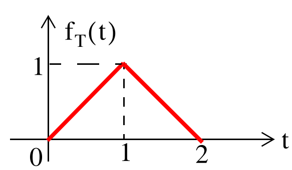


Figure 4.4. Triangular waveform

The equations of the linear segments are shown in Figure 4.5.

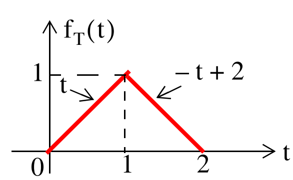


Figure 4.5. Triangular waveform with the equations of the linear segments

Next, we express the given waveform in terms of the unit step function.

$$\begin{aligned} f_T(t) &= t[u_0(t) - u_0(t-1)] + (-t+2)[u_0(t-1) - u_0(t-2)] \\ &= tu_0(t) - tu_0(t-1) - tu_0(t-1) + 2u_0(t-1) + tu_0(t-2) - 2u_0(t-2) \end{aligned}$$

Collecting like terms, we obtain

$$f_T(t) = tu_0(t) - 2(t-1)u_0(t-1) + (t-2)u_0(t-2)$$

From Table 4.1, Page 4−13,

$$f(t-a)u_0(t-a) \Leftrightarrow e^{-as}F(s)$$

and from Table 4.2, Page 4−22,

$$tu_0(t) \Leftrightarrow \frac{1}{s^2}$$

Then,

$$tu_0(t) - 2(t-1)u_0(t-1) + (t-2)u_0(t-2) \Leftrightarrow \frac{1}{s^2} - 2e^{-s}\frac{1}{s^2} + e^{-2s}\frac{1}{s^2}$$

or

$$tu_0(t) - 2(t-1)u_0(t-1) + (t-2)u_0(t-2) \Leftrightarrow \frac{1}{s^2}(1 - 2e^{-s} + e^{-2s})$$

Therefore, the Laplace transform of the triangular waveform of Figure 4.4 is

$$\boxed{f_T(t) \Leftrightarrow \frac{1}{s^2}(1-e^{-s})^2} \tag{4.69}$$

4.4.4 Laplace Transform of a Rectangular Periodic Waveform

The waveform of a rectangular periodic waveform, denoted as $f_R(t)$, is shown in Figure 4.6. This is a periodic waveform with period $T = 2a$, and we can apply the time periodicity property

$$\mathcal{L}\{f(\tau)\} = \frac{\int_0^T f(\tau)e^{-s\tau}d\tau}{1-e^{-sT}}$$

where the denominator represents the periodicity of $f(t)$.

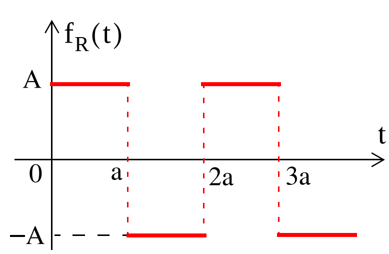


Figure 4.6. Rectangular periodic waveform

For this waveform,

$$\mathcal{L}\{f_R(t)\} = \frac{1}{1-e^{-2as}}\int_0^{2a} f_R(t)e^{-st}dt = \frac{1}{1-e^{-2as}}\left[\int_0^a Ae^{-st}dt + \int_a^{2a}(-A)e^{-st}dt\right]$$

$$= \frac{A}{1-e^{-2as}}\left[\frac{-e^{-st}}{s}\bigg|_0^a + \frac{e^{-st}}{s}\bigg|_a^{2a}\right]$$

$$\mathcal{L}\{f_R(t)\} = \frac{A}{s(1-e^{-2as})}(-e^{-as}+1+e^{-2as}-e^{-as})$$

$$= \frac{A}{s(1-e^{-2as})}(1-2e^{-as}+e^{-2as}) = \frac{A(1-e^{-as})^2}{s(1+e^{-as})(1-e^{-as})}$$

$$= \frac{A}{s}\frac{(1-e^{-as})}{(1+e^{-as})} = \frac{A}{s}\left(\frac{e^{as/2}e^{-as/2}-e^{-as/2}e^{-as/2}}{e^{as/2}e^{-as/2}+e^{-as/2}e^{-as/2}}\right)$$

$$= \frac{A}{s}\frac{e^{-as/2}}{e^{-as/2}}\left(\frac{e^{as/2}-e^{-as/2}}{e^{as/2}+e^{-as/2}}\right) = \frac{A}{s}\frac{\sinh(as/2)}{\cosh(as/2)}$$

$$\boxed{f_R(t) \Leftrightarrow \frac{A}{s}\tanh\left(\frac{as}{2}\right)} \tag{4.70}$$

4.4.5 Laplace Transform of a Half-Rectified Sine Waveform

The waveform of a half–rectified sine waveform, denoted as $f_{HW}(t)$, is shown in Figure 4.7. This is a periodic waveform with period $T = 2a$, and we can apply the time periodicity property

$$\mathcal{L}\{f(\tau)\} = \frac{\int_0^T f(\tau)e^{-s\tau}d\tau}{1-e^{-sT}}$$

where the denominator represents the periodicity of $f(t)$.

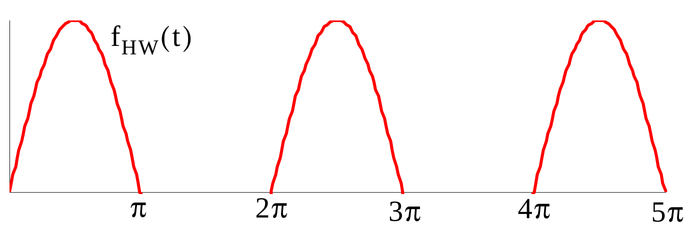


Figure 4.7. Half–rectified sine waveform*

For this waveform,

$$\mathcal{L}\{f_{HW}(t)\} = \frac{1}{1-e^{-2\pi s}}\int_0^{2\pi} f(t)e^{-st}dt = \frac{1}{1-e^{-2\pi s}}\int_0^{\pi} \sin t e^{-st}dt$$

$$= \frac{1}{1-e^{-2\pi s}}\left[\frac{e^{-st}(s\sin t - \cos t)}{s^2+1}\right]\Bigg|_0^{\pi} = \frac{1}{(s^2+1)}\frac{(1+e^{-\pi s})}{(1-e^{-2\pi s})}$$

$$\mathcal{L}\{f_{HW}(t)\} = \frac{1}{(s^2+1)}\frac{(1+e^{-\pi s})}{(1+e^{-\pi s})(1-e^{-\pi s})}$$

$$\boxed{f_{HW}(t) \Leftrightarrow \frac{1}{(s^2+1)(1-e^{-\pi s})}} \qquad (4.71)$$

4.5 Using MATLAB for Finding the Laplace Transforms of Time Functions

We can use the MATLAB function laplace to find the Laplace transform of a time function. For examples, please type

```
help laplace
```

in MATLAB's Command prompt.

We will be using this function extensively in the subsequent chapters of this book.

---

* *This waveform was produced with the following MATLAB script:*

```
t=0:pi/64:5*pi; x=sin(t); y=sin(t–2*pi); z=sin(t–4*pi); plot(t,x,t,y,t,z); axis([0 5*pi 0 1])
```

4.6 Summary

- The two–sided or bilateral Laplace Transform pair is defined as

$$\mathcal{L}\{f(t)\} = F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt$$

$$\mathcal{L}^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi j}\int_{\sigma - j\omega}^{\sigma + j\omega} F(s)e^{st}ds$$

where $\mathcal{L}\{f(t)\}$ denotes the Laplace transform of the time function $f(t)$, $\mathcal{L}^{-1}\{F(s)\}$ denotes the Inverse Laplace transform, and s is a complex variable whose real part is σ, and imaginary part ω, that is, $s = \sigma + j\omega$.

- The unilateral or one–sided Laplace transform defined as

$$\mathcal{L}\{f(t)\} = F(s) = \int_{t_0}^{\infty} f(t)e^{-st}dt = \int_{0}^{\infty} f(t)e^{-st}dt$$

- We denote transformation from the time domain to the complex frequency domain, and vice versa, as

$$f(t) \Leftrightarrow F(s)$$

- The linearity property states that

$$c_1 f_1(t) + c_2 f_2(t) + \ldots + c_n f_n(t) \Leftrightarrow c_1 F_1(s) + c_2 F_2(s) + \ldots + c_n F_n(s)$$

- The time shifting property states that

$$f(t-a)u_0(t-a) \Leftrightarrow e^{-as}F(s)$$

- The frequency shifting property states that

$$e^{-at}f(t) \Leftrightarrow F(s+a)$$

- The scaling property states that

$$f(at) \Leftrightarrow \frac{1}{a}F\left(\frac{s}{a}\right)$$

- The differentiation in time domain property states that

$$f'(t) = \frac{d}{dt}f(t) \Leftrightarrow sF(s) - f(0^-)$$

Also,

$$\frac{d^2}{dt^2}f(t) \Leftrightarrow s^2F(s) - sf(0^-) - f'(0^-)$$

$$\frac{d^3}{dt^3}f(t) \Leftrightarrow s^3F(s) - s^2f(0^-) - sf'(0^-) - f''(0^-)$$

and in general

$$\frac{d^n}{dt^n}f(t) \Leftrightarrow s^nF(s) - s^{n-1}f(0^-) - s^{n-2}f'(0^-) - \ldots - f^{n-1}(0^-)$$

where the terms $f(0^-)$, $f'(0^-)$, $f''(0^-)$, and so on, represent the initial conditions.

- The differentiation in complex frequency domain property states that

$$tf(t) \Leftrightarrow -\frac{d}{ds}F(s)$$

and in general,

$$t^nf(t) \Leftrightarrow (-1)^n\frac{d^n}{ds^n}F(s)$$

- The integration in time domain property states that

$$\int_{-\infty}^{t} f(\tau)d\tau \Leftrightarrow \frac{F(s)}{s} + \frac{f(0^-)}{s}$$

- The integration in complex frequency domain property states that

$$\frac{f(t)}{t} \Leftrightarrow \int_{s}^{\infty} F(s)ds$$

provided that the limit $\lim_{t \to 0} \frac{f(t)}{t}$ exists.

- The time periodicity property states that

$$f(t+nT) \Leftrightarrow \frac{\int_0^T f(t)e^{-st}dt}{1-e^{-sT}}$$

- The initial value theorem states that

$$\lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s) = f(0^-)$$

- The final value theorem states that

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s) = f(\infty)$$

- Convolution in the time domain corresponds to multiplication in the complex frequency domain, that is,

$$f_1(t)*f_2(t) \Leftrightarrow F_1(s)F_2(s)$$

- Convolution in the complex frequency domain divided by $1/2\pi j$, corresponds to multiplication in the time domain. That is,

$$f_1(t)f_2(t) \Leftrightarrow \frac{1}{2\pi j} F_1(s)*F_2(s)$$

- The Laplace transforms of some common functions of time are shown in Table 4.1, Page 4–13
- The Laplace transforms of some common waveforms are shown in Table 4.2, Page 4–22
- We can use the MATLAB function **laplace** to find the Laplace transform of a time function

4.7 Exercises

1. Derive the Laplace transform of the following time domain functions:

a. 12 b. $6u_0(t)$ c. $24u_0(t-12)$ d. $5tu_0(t)$ e. $4t^5u_0(t)$

2. Derive the Laplace transform of the following time domain functions:

a. $j8$ b. $j5\angle -90°$ c. $5e^{-5t}u_0(t)$ d. $8t^7e^{-5t}u_0(t)$ e. $15\delta(t-4)$

3. Derive the Laplace transform of the following time domain functions:

a. $(t^3+3t^2+4t+3)u_0(t)$ b. $3(2t-3)\delta(t-3)$

c. $(3\sin 5t)u_0(t)$ d. $(5\cos 3t)u_0(t)$

e. $(2\tan 4t)u_0(t)$ Be careful with this! Comment and you may skip derivation.

4. Derive the Laplace transform of the following time domain functions:

a. $3t(\sin 5t)u_0(t)$ b. $2t^2(\cos 3t)u_0(t)$ c. $2e^{-5t}\sin 5t$

d. $8e^{-3t}\cos 4t$ e. $(\cos t)\delta(t-\pi/4)$

5. Derive the Laplace transform of the following time domain functions:

a. $5tu_0(t-3)$ b. $(2t^2-5t+4)u_0(t-3)$ c. $(t-3)e^{-2t}u_0(t-2)$

d. $(2t-4)e^{2(t-2)}u_0(t-3)$ e. $4te^{-3t}(\cos 2t)u_0(t)$

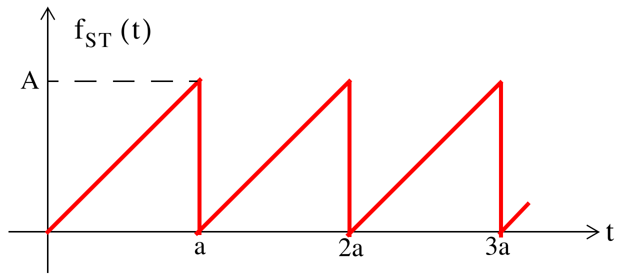
6. Derive the Laplace transform of the following time domain functions:

a. $\frac{d}{dt}(\sin 3t)$ b. $\frac{d}{dt}(3e^{-4t})$ c. $\frac{d}{dt}(t^2\cos 2t)$ d. $\frac{d}{dt}(e^{-2t}\sin 2t)$ e. $\frac{d}{dt}(t^2e^{-2t})$

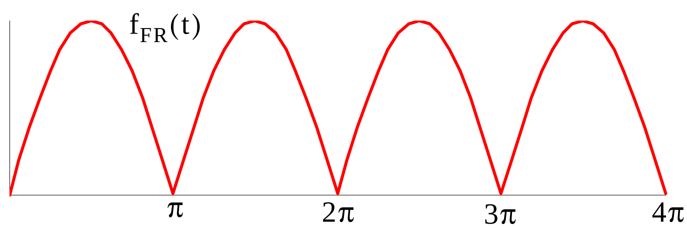
7. Derive the Laplace transform of the following time domain functions:

a. $\frac{\sin t}{t}$ b. $\int_0^t \frac{\sin\tau}{\tau}d\tau$ c. $\frac{\sin at}{t}$ d. $\int_t^\infty \frac{\cos\tau}{\tau}d\tau$ e. $\int_t^\infty \frac{e^{-\tau}}{\tau}d\tau$

8. Derive the Laplace transform for the sawtooth waveform $f_{ST}(t)$ below.



9. Derive the Laplace transform for the full–rectified waveform $f_{FR}(t)$ below.



Write a simple MATLAB script that will produce the waveform above.

4.8 Solutions to End-of-Chapter Exercises

1. From the definition of the Laplace transform or from Table 4.2, Page 4–22, we obtain:

a. $12/s$ b. $6/s$ c. $e^{-12s}\cdot\frac{24}{s}$ d. $5/s^2$ e. $4\cdot\frac{5!}{s^6}$

2. From the definition of the Laplace transform or from Table 4.2, Page 4–22, we obtain:

a. $j8/s$ b. $5/s$ c. $\frac{5}{s+5}$ d. $8\cdot\frac{7!}{(s+5)^8}$ e. $15e^{-4s}$

3.

a. From Table 4.2, Page 4–22, and the linearity property, we obtain $\frac{3!}{s^4}+\frac{3\times 2!}{s^3}+\frac{4}{s^2}+\frac{3}{s}$

b. $3(2t-3)\delta(t-3) = 3(2t-3)|_{t=3}\delta(t-3) = 9\delta(t-3)$ and $9\delta(t-3)\Leftrightarrow 9e^{-3s}$

c. $3\cdot\frac{5}{s^2+5^2}$ d. $5\cdot\frac{s}{s^2+3^2}$ e. $2\tan 4t = 2\cdot\frac{\sin 4t}{\cos 4t}\Leftrightarrow 2\cdot\frac{4/(s^2+2^2)}{s/(s^2+2^2)} = \frac{8}{s}$

This answer for part (e) looks suspicious because $8/s\Leftrightarrow 8u_0(t)$ and the Laplace transform is unilateral, that is, there is one–to–one correspondence between the time domain and the complex frequency domain. The fallacy with this procedure is that we assumed that if $f_1(t)\Leftrightarrow F_1(s)$ and $f_2(t)\Leftrightarrow F_2(s)$, we cannot conclude that $\frac{f_1(t)}{f_2(t)}\Leftrightarrow\frac{F_1(s)}{F_2(s)}$. For this exercise $f_1(t)\cdot f_2(t) = \sin 4t\cdot\frac{1}{\cos 4t}$, and as we've learned, multiplication in the time domain corresponds to convolution in the complex frequency domain. Accordingly, we must use the Laplace transform definition $\int_0^{\infty}(2\tan 4t)e^{-st}dt$ and this requires integration by parts. We skip this analytical derivation. The interested reader may try to find the answer with the MATLAB script

```
syms s t; 2*laplace(sin(4*t)/cos(4*t))
```

4. From (4.22), Page 4–6,

$$t^n f(t)\Leftrightarrow(-1)^n\frac{d^n}{ds^n}F(s)$$

a.

$$3(-1)^1\frac{d}{ds}\left(\frac{5}{s^2+5^2}\right) = -3\left[\frac{-5\cdot(2s)}{(s^2+25)^2}\right] = \frac{30s}{(s^2+25)^2}$$

b.

$$2(-1)^2\frac{d^2}{ds^2}\left(\frac{s}{s^2+3^2}\right) = 2\frac{d}{ds}\left[\frac{s^2+3^2-s(2s)}{(s^2+9)^2}\right] = 2\frac{d}{ds}\left(\frac{-s^2+9}{(s^2+9)^2}\right)$$

$$= 2\cdot\frac{(s^2+9)^2(-2s)-2(s^2+9)(2s)(-s^2+9)}{(s^2+9)^4}$$

$$= 2\cdot\frac{(s^2+9)(-2s)-4s(-s^2+9)}{(s^2+9)^3} = 2\cdot\frac{-2s^3-18s+4s^3-36s}{(s^2+9)^3}$$

$$= 2\cdot\frac{2s^3-54s}{(s^2+9)^3} = 2\cdot\frac{2s(s^2-27)}{(s^2+9)^3} = \frac{4s(s^2-27)}{(s^2+9)^3}$$

c.

$$\frac{2\times 5}{(s+5)^2+5^2} = \frac{10}{(s+5)^2+25}$$

d.

$$\frac{8(s+3)}{(s+3)^2+4^2} = \frac{8(s+3)}{(s+3)^2+16}$$

e.

$$\cos t|_{\pi/4}\delta(t-\pi/4) = (\sqrt{2}/2)\delta(t-\pi/4) \text{ and } (\sqrt{2}/2)\delta(t-\pi/4) \Leftrightarrow (\sqrt{2}/2)e^{-(\pi/4)s}$$

5.

a.

$$5tu_0(t-3) = [5(t-3)+15]u_0(t-3) \Leftrightarrow e^{-3s}\left(\frac{5}{s^2}+\frac{15}{s}\right) = \frac{5}{s}e^{-3s}\left(\frac{1}{s}+3\right)$$

b.

$$(2t^2-5t+4)u_0(t-3) = [2(t-3)^2+12t-18-5t+4]u_0(t-3)$$

$$= [2(t-3)^2+7t-14]u_0(t-3)$$

$$= [2(t-3)^2+7(t-3)+21-14]u_0(t-3)$$

$$= [2(t-3)^2+7(t-3)+7]u_0(t-3) \Leftrightarrow e^{-3s}\left(\frac{2\times 2!}{s^3}+\frac{7}{s^2}+\frac{7}{s}\right)$$

c.

$$(t-3)e^{-2t}u_0(t-2) = [(t-2)-1]e^{-2(t-2)}\cdot e^{-4}u_0(t-2)$$

$$\Leftrightarrow e^{-4}\cdot e^{-2s}\left[\frac{1}{(s+2)^2}-\frac{1}{(s+2)}\right] = e^{-4}\cdot e^{-2s}\left[\frac{-(s+1)}{(s+2)^2}\right]$$

d.

$$(2t-4)e^{2(t-2)}u_0(t-3) = [2(t-3)+6-4]e^{-2(t-3)}\cdot e^{-2}u_0(t-3)$$

$$\Leftrightarrow e^{-2}\cdot e^{-3s}\left[\frac{2}{(s+3)^2}+\frac{2}{(s+3)}\right] = 2e^{-2}\cdot e^{-3s}\left[\frac{s+4}{(s+3)^2}\right]$$

e.

$$4te^{-3t}(\cos 2t)u_0(t) \Leftrightarrow 4(-1)^1\frac{d}{ds}\left[\frac{s+3}{(s+3)^2+2^2}\right] = -4\frac{d}{ds}\left[\frac{s+3}{s^2+6s+9+4}\right]$$

$$\Leftrightarrow -4\frac{d}{ds}\left[\frac{s+3}{s^2+6s+13}\right] = -4\left[\frac{s^2+6s+13-(s+3)(2s+6)}{(s^2+6s+13)^2}\right]$$

$$\Leftrightarrow -4\left[\frac{s^2+6s+13-2s^2-6s-6s-18}{(s^2+6s+13)^2}\right] = \frac{4(s^2+6s+5)}{(s^2+6s+13)^2}$$

6. a.

$$\sin 3t \Leftrightarrow \frac{3}{s^2+3^2} \qquad \frac{d}{dt}f(t) \Leftrightarrow sF(s)-f(0^-) \qquad f(0^-) = \sin 3t|_{t=0} = 0$$

$$\frac{d}{dt}(\sin 3t) \Leftrightarrow s\frac{3}{s^2+3^2} - 0 = \frac{3s}{s^2+9}$$

b.

$$3e^{-4t} \Leftrightarrow \frac{3}{s+4} \qquad \frac{d}{dt}f(t) \Leftrightarrow sF(s)-f(0^-) \qquad f(0^-) = 3e^{-4t}|_{t=0} = 3$$

$$\frac{d}{dt}(3e^{-4t}) \Leftrightarrow s\frac{3}{s+4} - 3 = \frac{3s}{s+4} - \frac{3(s+4)}{s+4} = \frac{-12}{s+4}$$

c.

$$\cos 2t \Leftrightarrow \frac{s}{s^2+2^2} \qquad t^2\cos 2t \Leftrightarrow (-1)^2\frac{d^2}{ds^2}\left[\frac{s}{s^2+4}\right]$$

$$\frac{d}{ds}\left[\frac{s^2+4-s(2s)}{(s^2+4)^2}\right] = \frac{d}{ds}\left[\frac{-s^2+4}{(s^2+4)^2}\right] = \frac{(s^2+4)^2(-2s)-(-s^2+4)(s^2+4)2(2s)}{(s^2+4)^4}$$

$$= \frac{(s^2+4)(-2s)-(-s^2+4)(4s)}{(s^2+4)^3} = \frac{-2s^3-8s+4s^3-16s}{(s^2+4)^3} = \frac{2s(s^2-12)}{(s^2+4)^3}$$

Thus,

$$t^2\cos 2t \Leftrightarrow \frac{2s(s^2-12)}{(s^2+4)^3}$$

and

$$\frac{d}{dt}(t^2\cos 2t) \Leftrightarrow sF(s)-f(0^-) \quad \Leftrightarrow \quad s\frac{2s(s^2-12)}{(s^2+4)^3} - 0 = \frac{2s^2(s^2-12)}{(s^2+4)^3}$$

d.

$$\sin 2t \Leftrightarrow \frac{2}{s^2+2^2} \qquad e^{-2t}\sin 2t \Leftrightarrow \frac{2}{(s+2)^2+4} \qquad \frac{d}{dt}f(t) \Leftrightarrow sF(s)-f(0^-)$$

$$\frac{d}{dt}(e^{-2t}\sin 2t) \Leftrightarrow s\frac{2}{(s+2)^2+4} - 0 = \frac{2s}{(s+2)^2+4}$$

e.

$$t^2 \Leftrightarrow \frac{2!}{s^3} \qquad t^2e^{-2t} \Leftrightarrow \frac{2!}{(s+2)^3} \qquad \frac{d}{dt}f(t) \Leftrightarrow sF(s) - f(0^-)$$

$$\frac{d}{dt}(t^2e^{-2t}) \Leftrightarrow s\frac{2!}{(s+2)^3} - 0 = \frac{2s}{(s+2)^3}$$

7.

a.

$\sin t \Leftrightarrow \frac{1}{s^2+1}$ but to find $\mathscr{L}\left\{\frac{\sin t}{t}\right\}$ we must first show that the limit $\lim_{t \to 0} \frac{\sin t}{t}$ exists. Since $\lim_{x \to 0} \frac{\sin x}{x} = 1$, this condition is satisfied and thus $\frac{\sin t}{t} \Leftrightarrow \int_s^{\infty} \frac{1}{s^2+1} ds$. From tables of integrals, $\int \frac{1}{x^2+a^2} dx = \frac{1}{a}\tan^{-1}(x/a) + C$. Then, $\int \frac{1}{s^2+1} ds = \tan^{-1}(1/s) + C$ and the constant of integration C is evaluated from the final value theorem. Thus,

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s) = \lim_{s \to 0} s[\tan^{-1}(1/s) + C] = 0 \text{ and } \frac{\sin t}{t} \Leftrightarrow \tan^{-1}(1/s)$$

b.

From (a) above, $\frac{\sin t}{t} \Leftrightarrow \tan^{-1}(1/s)$ and since $\int_{-\infty}^{t} f(\tau)d\tau \Leftrightarrow \frac{F(s)}{s} + \frac{f(0^-)}{s}$, it follows that

$$\int_0^t \frac{\sin \tau}{\tau} d\tau \Leftrightarrow \frac{1}{s}\tan^{-1}(1/s)$$

c.

From (a) above $\frac{\sin t}{t} \Leftrightarrow \tan^{-1}(1/s)$ and since $f(at) \Leftrightarrow \frac{1}{a}F\left(\frac{s}{a}\right)$, it follows that

$$\frac{\sin at}{at} \Leftrightarrow \frac{1}{a}\tan^{-1}\left(\frac{1/s}{a}\right) \text{ or } \frac{\sin at}{t} \Leftrightarrow \tan^{-1}(a/s)$$

d.

$\cos t \Leftrightarrow \frac{s}{s^2+1}$, $\frac{\cos t}{t} \Leftrightarrow \int_s^{\infty} \frac{s}{s^2+1} ds$, and from tables of integrals,

$\int \frac{x}{x^2+a^2} dx = \frac{1}{2}\ln(x^2+a^2) + C$. Then, $\int \frac{s}{s^2+1} ds = \frac{1}{2}\ln(s^2+1) + C$ and the constant of integration C is evaluated from the final value theorem. Thus,

$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s) = \lim_{s \to 0} s\left[\frac{1}{2}\ln(s^2+1) + C\right] = 0$ and using $\int_{-\infty}^{t} f(\tau)d\tau \Leftrightarrow \frac{F(s)}{s} + \frac{f(0^-)}{s}$ we obtain

$$\int_t^{\infty}\frac{\cos\tau}{\tau}d\tau \Leftrightarrow \frac{1}{2s}\ln(s^2+1)$$

e.

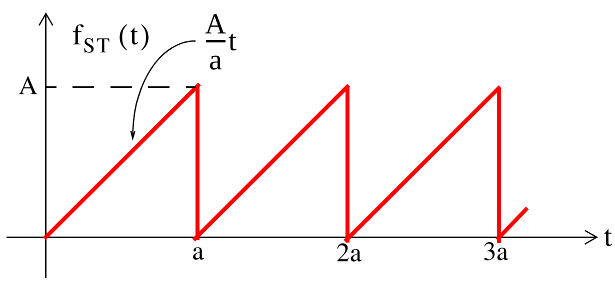
$e^{-t}\Leftrightarrow\frac{1}{s+1}$, $\frac{e^{-t}}{t}\Leftrightarrow\int_s^{\infty}\frac{1}{s+1}ds$, and from tables of integrals $\int\frac{1}{ax+b}dx = \frac{1}{2}\ln(ax+b)$. Then, $\int\frac{1}{s+1}ds = \ln(s+1)+C$ and the constant of integration C is evaluated from the final value theorem. Thus,

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) = \lim_{s\to 0} s[\ln(s+1)+C] = 0$$

and using $\int_{-\infty}^{t} f(\tau)d\tau \Leftrightarrow \frac{F(s)}{s}+\frac{f(0^-)}{s}$, we obtain

$$\int_t^{\infty}\frac{e^{-\tau}}{\tau}d\tau \Leftrightarrow \frac{1}{s}\ln(s+1)$$

8.



This is a periodic waveform with period $T = a$, and its Laplace transform is

$$F(s) = \frac{1}{1-e^{-as}}\int_0^a \frac{A}{a}te^{-st}dt = \frac{A}{a(1-e^{-as})}\int_0^a te^{-st}dt \quad (1)$$

and from (4.41), Page 4–14, and limits of integration 0 to a, we obtain

$$\mathscr{L}\{t\}\big|_0^a = \int_0^a te^{-st}dt = \left[-\frac{te^{-st}}{s}-\frac{e^{-st}}{s^2}\right]\Bigg|_0^a = \left[\frac{te^{-st}}{s}+\frac{e^{-st}}{s^2}\right]\Bigg|_a^0$$

$$= \left[\frac{1}{s^2}-\frac{ae^{-as}}{s}-\frac{e^{-as}}{s^2}\right] = \frac{1}{s^2}[1-(1+as)e^{-as}]$$

Adding and subtracting as in the last expression above, we obtain

$$\mathcal{L}\{t\}\Big|_0^a = \frac{1}{s^2}[(1+as)-(1+as)e^{-as}-as] = \frac{1}{s^2}[(1+as)(1-e^{-as})-as]$$

By substitution into (1) we obtain

$$F(s) = \frac{A}{a(1-e^{-as})}\cdot\frac{1}{s^2}[(1+as)(1-e^{-as})-as] = \frac{A}{as^2(1-e^{-as})}\cdot[(1+as)(1-e^{-as})-as]$$

$$= \frac{A(1+as)}{as^2} - \frac{Aa}{as(1-e^{-as})} = \frac{A}{as}\left[\frac{(1+as)}{s} - \frac{a}{(1-e^{-as})}\right]$$

9.

This is a periodic waveform with period $T = a = \pi$ and its Laplace transform is

$$F(s) = \frac{1}{1-e^{-sT}}\int_0^T f(t)e^{-st}dt = \frac{1}{(1-e^{-\pi s})}\int_0^{\pi} \sin t e^{-st}dt$$

From tables of integrals,

$$\int \sin bx e^{ax}dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}$$

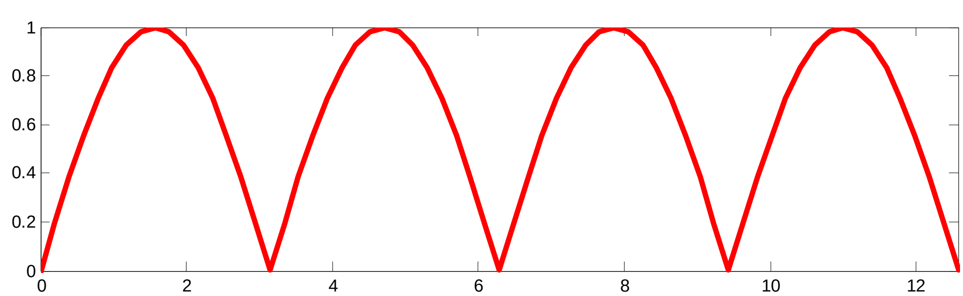
Then,

$$F(s) = \frac{1}{1-e^{-\pi s}}\cdot\frac{e^{-st}(s\sin t - \cos t)}{s^2+1}\Bigg|_0^{\pi} = \frac{1}{1-e^{-\pi s}}\cdot\frac{1+e^{-\pi s}}{s^2+1}$$

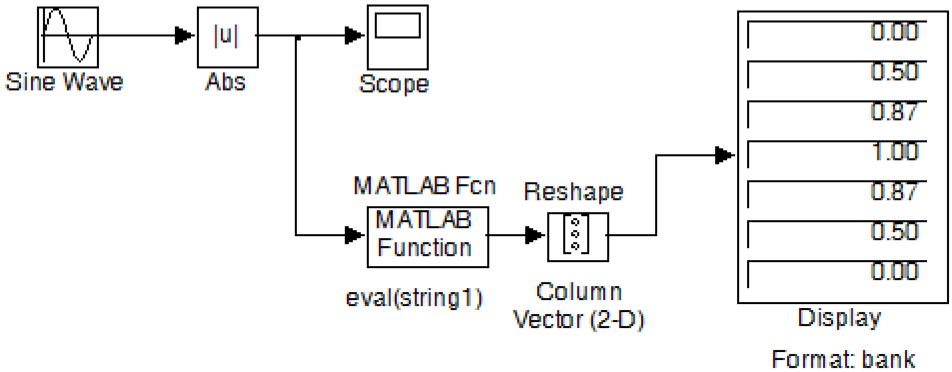
$$= \frac{1}{s^2+1}\cdot\frac{1+e^{-\pi s}}{1-e^{-\pi s}} = \frac{1}{s^2+1}\coth\left(\frac{\pi s}{2}\right)$$

The full–rectified waveform can be produced with the MATLAB script below.

```
t=0:pi/16:4*pi; x=sin(t); plot(t,abs(x)); axis([0 4*pi 0 1])
```



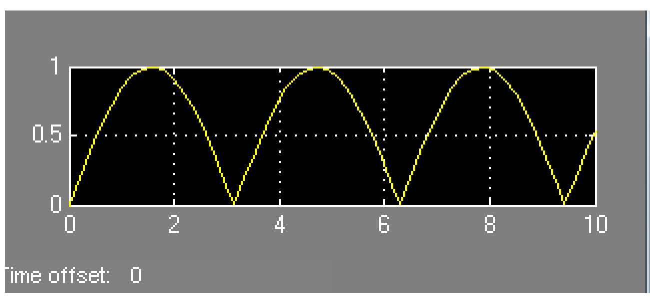
The full–rectified waveform can also be produced with the Simulink model below. The **Sine Wave**, **Abs**, and **Reshape** blocks are in the **Math Operations** library, the **MATLAB Function** block is in the **User–Defined Functions** library, and the **Scope** and **Display** blocks are found in the **Sinks** library.



Before simulation execution, the following script must be entered at the MATLAB command prompt:

```
x=[0 pi/6 pi/3 pi/2 2*pi/3 5*pi/6 pi]; string1='abs(sin(x))';
```

The **Scope** block displays the waveform shown below.



We can use MATLAB polyfit(x,y,n) and polyval(p,x) functions to find a suitable polynomial* that approximates the full–rectifier waveform.

* *For an example with a step–by–step procedure, please refer to Numerical Analysis Using MATLAB and Excel, ISBN 978–1–934404–03–4, Chapter 8, Example 8.8.*

Chapter 5

The Inverse Laplace Transformation

This chapter is a continuation to the Laplace transformation topic of the previous chapter and presents several methods of finding the Inverse Laplace Transformation. The partial fraction expansion method is explained thoroughly and it is illustrated with several examples.

5.1 The Inverse Laplace Transform Integral

The Inverse Laplace Transform Integral was stated in the previous chapter; it is repeated here for convenience.

$$\mathcal{L}^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi j}\int_{\sigma - j\omega}^{\sigma + j\omega} F(s)e^{st}ds \tag{5.1}$$

This integral is difficult to evaluate because it requires contour integration using complex variables theory. Fortunately, for most engineering problems we can refer to Tables of Properties, and Common Laplace transform pairs to lookup the Inverse Laplace transform.

5.2 Partial Fraction Expansion

Quite often the Laplace transform expressions are not in recognizable form, but in most cases appear in a rational form of s, that is,

$$F(s) = \frac{N(s)}{D(s)} \tag{5.2}$$

where $N(s)$ and $D(s)$ are polynomials, and thus (5.2) can be expressed as

$$F(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + b_{m-2}s^{m-2} + \ldots + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \ldots + a_1 s + a_0} \tag{5.3}$$

The coefficients a_k and b_k are real numbers for $k = 1, 2, \ldots, n$, and if the highest power m of $N(s)$ is less than the highest power n of $D(s)$, i.e., $m < n$, $F(s)$ is said to be expressed as a *proper rational function*. If $m \ge n$, $F(s)$ is an *improper rational function.*

In a proper rational function, the roots of $N(s)$ in (5.3) are found by setting $N(s) = 0$; these are called the *zeros* of $F(s)$. The roots of $D(s)$, found by setting $D(s) = 0$, are called the *poles* of $F(s)$. We assume that $F(s)$ in (5.3) is a proper rational function. Then, it is customary and very convenient to make the coefficient of s^n unity; thus, we rewrite $F(s)$ as

$$F(s) = \frac{N(s)}{D(s)} = \frac{\frac{1}{a_n}(b_m s^m + b_{m-1}s^{m-1} + b_{m-2}s^{m-2} + \ldots + b_1 s + b_0)}{s^n + \frac{a_{n-1}}{a_n}s^{n-1} + \frac{a_{n-2}}{a_n}s^{n-2} + \ldots + \frac{a_1}{a_n}s + \frac{a_0}{a_n}} \qquad (5.4)$$

The zeros and poles of (5.4) can be real and distinct, repeated, complex conjugates, or combinations of real and complex conjugates. However, we are mostly interested in the nature of the poles, so we will consider each case separately, as indicated in Subsections 5.2.1 through 5.2.3 below.

5.2.1 Distinct Poles

If all the poles $p_1, p_2, p_3, \ldots, p_n$ of $F(s)$ are *distinct* (different from each another), we can factor the denominator of $F(s)$ in the form

$$F(s) = \frac{N(s)}{(s-p_1)\cdot(s-p_2)\cdot(s-p_3)\cdot\ldots\cdot(s-p_n)} \qquad (5.5)$$

where p_k is distinct from all other poles. Next, using the *partial fraction expansion method,** we can express (5.5) as

$$F(s) = \frac{r_1}{(s-p_1)} + \frac{r_2}{(s-p_2)} + \frac{r_3}{(s-p_3)} + \ldots + \frac{r_n}{(s-p_n)} \qquad (5.6)$$

where $r_1, r_2, r_3, \ldots, r_n$ are the *residues*, and $p_1, p_2, p_3, \ldots, p_n$ are the *poles* of $F(s)$.

To evaluate the residue r_k, we multiply both sides of (5.6) by $(s-p_k)$; then, we let $s \to p_k$, that is,

$$r_k = \lim_{s \to p_k}(s-p_k)F(s) = (s-p_k)F(s)\big|_{s=p_k} \qquad (5.7)$$

Example 5.1

Use the partial fraction expansion method to simplify $F_1(s)$ of (5.8) below, and find the time domain function $f_1(t)$ corresponding to $F_1(s)$.

$$F_1(s) = \frac{3s+2}{s^2+3s+2} \qquad (5.8)$$

* *The partial fraction expansion method applies only to proper rational functions. It is used extensively in integration, and in finding the inverses of the Laplace transform, the Fourier transform, and the z-transform. This method allows us to decompose a rational polynomial into smaller rational polynomials with simpler denominators from which we can easily recognize their integrals and inverse transformations. This method is also being taught in intermediate algebra and introductory calculus courses.*

Solution:

Using (5.6), we obtain

$$F_1(s) = \frac{3s+2}{s^2+3s+2} = \frac{3s+2}{(s+1)(s+2)} = \frac{r_1}{(s+1)} + \frac{r_2}{(s+2)} \tag{5.9}$$

The residues are

$$r_1 = \lim_{s \to -1} (s+1)F(s) = \left.\frac{3s+2}{(s+2)}\right|_{s=-1} = -1 \tag{5.10}$$

and

$$r_2 = \lim_{s \to -2} (s+2)F(s) = \left.\frac{3s+2}{(s+1)}\right|_{s=-2} = 4 \tag{5.11}$$

Therefore, we express (5.9) as

$$F_1(s) = \frac{3s+2}{s^2+3s+2} = \frac{-1}{(s+1)} + \frac{4}{(s+2)} \tag{5.12}$$

and from Table 4.2, Chapter 4, Page 4–22, we find that

$$e^{-at}u_0(t) \Leftrightarrow \frac{1}{s+a} \tag{5.13}$$

Therefore,

$$F_1(s) = \frac{-1}{(s+1)} + \frac{4}{(s+2)} \Leftrightarrow (-e^{-t} + 4e^{-2t})\,u_0(t) = f_1(t) \tag{5.14}$$

The residues and poles of a rational function of polynomials such as (5.8), can be found easily using the MATLAB **residue(a,b)** function. For this example, we use the script

```
Ns = [3, 2]; Ds = [1, 3, 2]; [r, p, k] = residue(Ns, Ds)
```

and MATLAB returns the values

```
r =
     4
    -1
p =
    -2
    -1
k =
     []
```

For the MATLAB script above, we defined `Ns` and `Ds` as two vectors that contain the numerator and denominator coefficients of $F(s)$. When this script is executed, MATLAB displays the `r` and `p` vectors that represent the residues and poles respectively. The first value of the vector `r` is associated with the first value of the vector `p`, the second value of `r` is associated with the second value of p, and so on.

The vector k is referred to as the *direct term* and it is always empty (has no value) whenever $F(s)$ is a proper rational function, that is, when the highest degree of the denominator is larger than that of the numerator. For this example, we observe that the highest power of the denominator is s^2, whereas the highest power of the numerator is s and therefore the direct term is empty.

We can also use the MATLAB **ilaplace(f)** function to obtain the time domain function directly from $F(s)$. This is done with the script that follows.

```
syms s t; Fs=(3*s+2)/(s^2+3*s+2); ft=ilaplace(Fs); pretty(ft)
% Must have Symbolic Math Toolbox installed
```

When this script is executed, MATLAB displays the expression

```
4 exp(-2 t)- exp(-t)
```

Example 5.2

Use the partial fraction expansion method to simplify $F_2(s)$ of (5.15) below, and find the time domain function $f_2(t)$ corresponding to $F_2(s)$.

$$F_2(s) = \frac{3s^2+2s+5}{s^3+12s^2+44s+48} \tag{5.15}$$

Solution:

First, we use the MATLAB **factor(s)** symbolic function to express the denominator polynomial of $F_2(s)$ in factored form. For this example,

```
syms s; factor(s^3 + 12*s^2 + 44*s + 48)  % Must have Symbolic Math Toolbox installed
```

```
ans =
(s+2)*(s+4)*(s+6)
```

Then,

$$F_2(s) = \frac{3s^2+2s+5}{s^3+12s^2+44s+48} = \frac{3s^2+2s+5}{(s+2)(s+4)(s+6)} = \frac{r_1}{(s+2)} + \frac{r_2}{(s+4)} + \frac{r_3}{(s+6)} \tag{5.16}$$

The residues are

$$r_1 = \left.\frac{3s^2+2s+5}{(s+4)(s+6)}\right|_{s=-2} = \frac{9}{8} \tag{5.17}$$

$$r_2 = \left.\frac{3s^2+2s+5}{(s+2)(s+6)}\right|_{s=-4} = -\frac{37}{4} \tag{5.18}$$

$$r_3 = \left.\frac{3s^2 + 2s + 5}{(s+2)(s+4)}\right|_{s=-6} = \frac{89}{8} \qquad (5.19)$$

Then, by substitution into (5.16) we obtain

$$F_2(s) = \frac{3s^2 + 2s + 5}{s^3 + 12s^2 + 44s + 48} = \frac{9/8}{(s+2)} + \frac{-37/4}{(s+4)} + \frac{89/8}{(s+6)} \qquad (5.20)$$

From Table 4.2, Chapter 4, Page 4–22,

$$e^{-at}u_0(t) \Leftrightarrow \frac{1}{s+a} \qquad (5.21)$$

Therefore,

$$F_2(s) = \frac{9/8}{(s+2)} + \frac{-37/4}{(s+4)} + \frac{89/8}{(s+6)} \Leftrightarrow \left(\frac{9}{8}e^{-2t} - \frac{37}{4}e^{-4t} + \frac{89}{8}e^{-6t}\right)u_0(t) = f_2(t) \qquad (5.22)$$

Check with MATLAB:

```
syms s t; Fs = (3*s^2 + 4*s + 5) / (s^3 + 12*s^2 + 44*s + 48); ft = ilaplace(Fs)
```

```
ft =
-37/4*exp(-4*t)+9/8*exp(-2*t)+89/8*exp(-6*t)
```

5.2.2 Complex Poles

Quite often, the poles of $F(s)$ are complex,* and since complex poles occur in complex conjugate pairs, the number of complex poles is even. Thus, if p_k is a complex root of $D(s)$, then, its complex conjugate pole, denoted as p_k^*, is also a root of $D(s)$. The partial fraction expansion method can also be used in this case, but it may be necessary to manipulate the terms of the expansion in order to express them in a recognizable form. The procedure is illustrated with the following example.

Example 5.3

Use the partial fraction expansion method to simplify $F_3(s)$ of (5.23) below, and find the time domain function $f_3(t)$ corresponding to $F_3(s)$.

$$F_3(s) = \frac{s+3}{s^3 + 5s^2 + 12s + 8} \qquad (5.23)$$

* *A review of complex numbers is presented in Appendix D*

Solution:

Let us first express the denominator in factored form to identify the poles of $F_3(s)$ using the MATLAB **factor(s)** symbolic function. Then,

```
syms s; factor(s^3 + 5*s^2 + 12*s + 8)
```

```
ans =
(s+1)*(s^2+4*s+8)
```

The **factor(s)** function did not factor the quadratic term. We will use the **roots(p)** function.

```
p=[1 4 8]; roots_p=roots(p)
```

```
roots_p =
  -2.0000 + 2.0000i
  -2.0000 - 2.0000i
```

Then,

$$F_3(s) = \frac{s+3}{s^3+5s^2+12s+8} = \frac{s+3}{(s+1)(s+2+j2)(s+2-j2)}$$

or

$$F_3(s) = \frac{s+3}{s^3+5s^2+12s+8} = \frac{r_1}{(s+1)} + \frac{r_2}{(s+2+j2)} + \frac{r_2{}^*}{(s+2-j2)} \tag{5.24}$$

The residues are

$$r_1 = \left.\frac{s+3}{s^2+4s+8}\right|_{s=-1} = \frac{2}{5} \tag{5.25}$$

$$\begin{aligned} r_2 &= \left.\frac{s+3}{(s+1)(s+2-j2)}\right|_{s=-2-j2} = \frac{1-j2}{(-1-j2)(-j4)} = \frac{1-j2}{-8+j4} \\ &= \frac{(1-j2)}{(-8+j4)}\frac{(-8-j4)}{(-8-j4)} = \frac{-16+j12}{80} = -\frac{1}{5}+\frac{j3}{20} \end{aligned} \tag{5.26}$$

$$r_2{}^* = \left(-\frac{1}{5}+\frac{j3}{20}\right)^* = -\frac{1}{5}-\frac{j3}{20} \tag{5.27}$$

By substitution into (5.24),

$$F_3(s) = \frac{2/5}{(s+1)} + \frac{-1/5+j3/20}{(s+2+j2)} + \frac{-1/5-j3/20}{(s+2-j2)} \tag{5.28}$$

The last two terms on the right side of (5.28), do not resemble any Laplace transform pair that we derived in Chapter 2. Therefore, we will express them in a different form. We combine them into a single term*, and now (5.28) is written as

$$F_3(s) = \frac{2/5}{(s+1)} - \frac{1}{5} \cdot \frac{(2s+1)}{(s^2+4s+8)} \tag{5.29}$$

For convenience, we denote the first term on the right side of (5.29) as $F_{31}(s)$, and the second as $F_{32}(s)$. Then,

$$F_{31}(s) = \frac{2/5}{(s+1)} \Leftrightarrow \frac{2}{5}e^{-t} = f_{31}(t) \tag{5.30}$$

Next, for $F_{32}(s)$

$$F_{32}(s) = -\frac{1}{5} \cdot \frac{(2s+1)}{(s^2+4s+8)} \tag{5.31}$$

From Table 4.2, Chapter 4, Page 4–22,

$$\begin{aligned} e^{-at}\mathit{sin}\,\omega t u_0 t &\Leftrightarrow \frac{\omega}{(s+a)^2+\omega^2} \\ e^{-at}\mathit{cos}\,\omega t u_0 t &\Leftrightarrow \frac{s+a}{(s+a)^2+\omega^2} \end{aligned} \tag{5.32}$$

Accordingly, we express $F_{32}(s)$ as

$$\begin{aligned} F_{32}(s) &= -\frac{2}{5}\left(\frac{s+\frac{1}{2}+\frac{3}{2}-\frac{3}{2}}{(s+2)^2+2^2)}\right) = -\frac{2}{5}\left(\frac{s+2}{(s+2)^2+2^2)} + \frac{-3/2}{(s+2)^2+2^2)}\right) \\ &= -\frac{2}{5}\left(\frac{s+2}{(s+2)^2+2^2)}\right) + \frac{6/10}{2}\left(\frac{2}{(s+2)^2+2^2)}\right) \\ &= -\frac{2}{5}\left(\frac{s+2}{(s+2)^2+2^2)}\right) + \frac{3}{10}\left(\frac{2}{(s+2)^2+2^2)}\right) \end{aligned} \tag{5.33}$$

Addition of (5.30) with (5.33) yields

$$F_3(s) = F_{31}(s) + F_{32}(s) = \frac{2/5}{(s+1)} - \frac{2}{5}\left(\frac{s+2}{(s+2)^2+2^2)}\right) + \frac{3}{10}\left(\frac{2}{(s+2)^2+2^2)}\right)$$

$$\Leftrightarrow \frac{2}{5}e^{-t} - \frac{2}{5}e^{-2t}\cos 2t + \frac{3}{10}e^{-2t}\sin 2t = f_3(t)$$

Check with MATLAB:

```
syms a s t w;  % Define several symbolic variables. Must have Symbolic Math Tootbox installed
Fs=(s + 3)/(s^3 + 5*s^2 + 12*s + 8); ft=ilaplace(Fs)
```

* *Here, we used MATLAB function* simple((−1/5 +3j/20)/(s+2+2j)+(−1/5 −3j/20)/(s+2−2j)). *The* **simple** *function, after several simplification tools that were displayed on the screen, returned* `(-2*s-1)/(5*s^2+20*s+40).`

```
ft =
2/5*exp(-t)-2/5*exp(-2*t)*cos(2*t)+3/10*exp(-2*t)*sin(2*t)
```

5.2.3 Multiple (Repeated) Poles

In this case, $F(s)$ has simple poles, but one of the poles, say p_1, has a multiplicity m. For this condition, we express it as

$$F(s) = \frac{N(s)}{(s-p_1)^m(s-p_2)\ldots(s-p_{n-1})(s-p_n)} \tag{5.34}$$

Denoting the m residues corresponding to multiple pole p_1 as $r_{11}, r_{12}, r_{13}, \ldots, r_{1m}$, the partial fraction expansion of (5.34) is expressed as

$$\begin{aligned} F(s) &= \frac{r_{11}}{(s-p_1)^m} + \frac{r_{12}}{(s-p_1)^{m-1}} + \frac{r_{13}}{(s-p_1)^{m-2}} + \ldots + \frac{r_{1m}}{(s-p_1)} \\ &+ \frac{r_2}{(s-p_2)} + \frac{r_3}{(s-p_3)} + \ldots + \frac{r_n}{(s-p_n)} \end{aligned} \tag{5.35}$$

For the simple poles $p_1, p_2, \ldots, p_n$, we proceed as before, that is, we find the residues from

$$r_k = \lim_{s \to p_k}(s-p_k)F(s) = (s-p_k)F(s)\big|_{s = p_k} \tag{5.36}$$

The residues $r_{11}, r_{12}, r_{13}, \ldots, r_{1m}$ corresponding to the repeated poles, are found by multiplication of both sides of (5.35) by $(s-p)^m$. Then,

$$\begin{aligned} (s-p_1)^m F(s) &= r_{11} + (s-p_1)r_{12} + (s-p_1)^2 r_{13} + \ldots + (s-p_1)^{m-1} r_{1m} \\ &+ (s-p_1)^m\left(\frac{r_2}{(s-p_2)} + \frac{r_3}{(s-p_3)} + \ldots + \frac{r_n}{(s-p_n)}\right) \end{aligned} \tag{5.37}$$

Next, taking the limit as $s \to p_1$ on both sides of (5.37), we obtain

$$\begin{aligned} \lim_{s \to p_1}(s-p_1)^m F(s) &= r_{11} + \lim_{s \to p_1}[(s-p_1)r_{12} + (s-p_1)^2 r_{13} + \ldots + (s-p_1)^{m-1} r_{1m}] \\ &+ \lim_{s \to p_1}\left[(s-p_1)^m\left(\frac{r_2}{(s-p_2)} + \frac{r_3}{(s-p_3)} + \ldots + \frac{r_n}{(s-p_n)}\right)\right] \end{aligned}$$

or

$$r_{11} = \lim_{s \to p_1}(s-p_1)^m F(s) \tag{5.38}$$

and thus (5.38) yields the residue of the first repeated pole.

The residue r_{12} for the second repeated pole p_1, is found by differentiating (5.37) with respect to s and again, we let $s \to p_1$, that is,

$$r_{12} = \lim_{s \to p_1} \frac{d}{ds}[(s-p_1)^m F(s)] \tag{5.39}$$

In general, the residue r_{1k} can be found from

$$(s-p_1)^m F(s) = r_{11} + r_{12}(s-p_1) + r_{13}(s-p_1)^2 + \ldots \tag{5.40}$$

whose $(m-1)$th derivative of both sides is

$$(k-1)!r_{1k} = \lim_{s \to p_1} \frac{1}{(k-1)!} \frac{d^{k-1}}{ds^{k-1}}[(s-p_1)^m F(s)] \tag{5.41}$$

or

$$r_{1k} = \lim_{s \to p_1} \frac{1}{(k-1)!} \frac{d^{k-1}}{ds^{k-1}}[(s-p_1)^m F(s)] \tag{5.42}$$

Example 5.4

Use the partial fraction expansion method to simplify $F_4(s)$ of (5.43) below, and find the time domain function $f_4(t)$ corresponding to $F_4(s)$.

$$F_4(s) = \frac{s+3}{(s+2)(s+1)^2} \tag{5.43}$$

Solution:

We observe that there is a pole of multiplicity 2 at $s = -1$, and thus in partial fraction expansion form, $F_4(s)$ is written as

$$F_4(s) = \frac{s+3}{(s+2)(s+1)^2} = \frac{r_1}{(s+2)} + \frac{r_{21}}{(s+1)^2} + \frac{r_{22}}{(s+1)} \tag{5.44}$$

The residues are

$$r_1 = \left.\frac{s+3}{(s+1)^2}\right|_{s=-2} = 1$$

$$r_{21} = \left.\frac{s+3}{s+2}\right|_{s=-1} = 2$$

$$r_{22} = \left.\frac{d}{ds}\left(\frac{s+3}{s+2}\right)\right|_{s=-1} = \left.\frac{(s+2)-(s+3)}{(s+2)^2}\right|_{s=-1} = -1$$

The value of the residue r_{22} can also be found without differentiation as follows:

Substitution of the already known values of r_1 and r_{21} into (5.44), and letting $s = 0^*$, we obtain

$$\left.\frac{s+3}{(s+1)^2(s+2)}\right|_{s=0} = \left.\frac{1}{(s+2)}\right|_{s=0} + \left.\frac{2}{(s+1)^2}\right|_{s=0} + \left.\frac{r_{22}}{(s+1)}\right|_{s=0}$$

or

$$\frac{3}{2} = \frac{1}{2} + 2 + r_{22}$$

from which $r_{22} = -1$ as before. Finally,

$$F_4(s) = \frac{s+3}{(s+2)(s+1)^2} = \frac{1}{(s+2)} + \frac{2}{(s+1)^2} + \frac{-1}{(s+1)} \Leftrightarrow e^{-2t} + 2te^{-t} - e^{-t} = f_4(t) \tag{5.45}$$

Check with MATLAB:

```
syms s t; Fs=(s+3)/((s+2)*(s+1)^2); ft=ilaplace(Fs) % Must have Symbolic Math Toolbox installed
```

```
ft = exp(-2*t)+2*t*exp(-t)-exp(-t)
```

We can use the following script to check the partial fraction expansion.

```
syms s
Ns = [1  3];            % Coefficients of the numerator N(s) of F(s)
expand((s + 1)^2);      % Expands (s + 1)^2 to s^2 + 2*s + 1;
d1 = [1  2  1];         % Coefficients of (s + 1)^2 = s^2 + 2*s + 1 term in D(s)
d2 = [0  1  2];         % Coefficients of (s + 2) term in D(s)
Ds=conv(d1,d2);         % Multiplies polynomials d1 and d2 to express the
                        % denominator D(s) of F(s) as a polynomial
[r,p,k]=residue(Ns,Ds)
```

```
r =
     1.0000
    -1.0000
     2.0000
```

* *This is permissible since (5.44) is an identity.*

```
p =
   -2.0000
   -1.0000
   -1.0000
k =
     []
```

Example 5.5

Use the partial fraction expansion method to simplify $F_5(s)$ of (5.46) below, and find the time domain function $f_5(t)$ corresponding to the given $F_5(s)$.

$$F_5(s) = \frac{s^2 + 3s + 1}{(s+1)^3(s+2)^2} \tag{5.46}$$

Solution:

We observe that there is a pole of multiplicity 3 at $s = -1$, and a pole of multiplicity 2 at $s = -2$. Then, in partial fraction expansion form, $F_5(s)$ is written as

$$F_5(s) = \frac{r_{11}}{(s+1)^3} + \frac{r_{12}}{(s+1)^2} + \frac{r_{13}}{(s+1)} + \frac{r_{21}}{(s+2)^2} + \frac{r_{22}}{(s+2)} \tag{5.47}$$

The residues are

$$r_{11} = \left.\frac{s^2 + 3s + 1}{(s+2)^2}\right|_{s=-1} = -1$$

$$r_{12} = \left.\frac{d}{ds}\left(\frac{s^2 + 3s + 1}{(s+2)^2}\right)\right|_{s=-1}$$

$$= \left.\frac{(s+2)^2(2s+3) - 2(s+2)(s^2+3s+1)}{(s+2)^4}\right|_{s=-1} = \left.\frac{s+4}{(s+2)^3}\right|_{s=-1} = 3$$

$$r_{13} = \left.\frac{1}{2!}\frac{d^2}{ds^2}\left(\frac{s^2 + 3s + 1}{(s+2)^2}\right)\right|_{s=-1} = \left.\frac{1}{2}\frac{d}{ds}\left[\frac{d}{ds}\left(\frac{s^2 + 3s + 1}{(s+2)^2}\right)\right]\right|_{s=-1} = \left.\frac{1}{2}\frac{d}{ds}\left(\frac{s+4}{(s+2)^3}\right)\right|_{s=-1}$$

$$= \left.\frac{1}{2}\left[\frac{(s+2)^3 - 3(s+2)^2(s+4)}{(s+2)^6}\right]\right|_{s=-1} = \left.\frac{1}{2}\left(\frac{s+2-3s-12}{(s+2)^4}\right)\right|_{s=-1} = \left.\frac{-s-5}{(s+2)^4}\right|_{s=-1} = -4$$

Next, for the pole at $s = -2$,

$$r_{21} = \left.\frac{s^2+3s+1}{(s+1)^3}\right|_{s=-2} = 1$$

and

$$r_{22} = \left.\frac{d}{ds}\left(\frac{s^2+3s+1}{(s+1)^3}\right)\right|_{s=-2} = \left.\frac{(s+1)^3(2s+3)-3(s+1)^2(s^2+3s+1)}{(s+1)^6}\right|_{s=-2}$$

$$= \left.\frac{(s+1)(2s+3)-3(s^2+3s+1)}{(s+1)^4}\right|_{s=-2} = \left.\frac{-s^2-4s}{(s+1)^4}\right|_{s=-2} = 4$$

By substitution of the residues into (5.47), we obtain

$$F_5(s) = \frac{-1}{(s+1)^3}+\frac{3}{(s+1)^2}+\frac{-4}{(s+1)}+\frac{1}{(s+2)^2}+\frac{4}{(s+2)} \tag{5.48}$$

We will check the values of these residues with the MATLAB script below.

```
syms s;    % The function collect(s) below multiplies (s+1)^3 by (s+2)^2
           % and we use it to express the denominator D(s) as a polynomial so that we can
           % use the coefficients of the resulting polynomial with the residue function
Ds=collect(((s+1)^3)*((s+2)^2))
Ds =
s^5+7*s^4+19*s^3+25*s^2+16*s+4

Ns=[1 3 1];  Ds=[1 7 19 25 16 4];  [r,p,k]=residue(Ns,Ds)

r =
    4.0000
    1.0000
   -4.0000
    3.0000
   -1.0000
p =
   -2.0000
   -2.0000
   -1.0000
   -1.0000
   -1.0000
k =
     []
```

From Table 4.2, Chapter 4,

$$e^{-at} \Leftrightarrow \frac{1}{s+a} \qquad te^{-at} \Leftrightarrow \frac{1}{(s+a)^2} \qquad t^{n-1}e^{-at} \Leftrightarrow \frac{(n-1)!}{(s+a)^n}$$

and with these, we derive $f_5(t)$ from (5.48) as

$$f_5(t) = -\frac{1}{2}t^2e^{-t} + 3te^{-t} - 4e^{-t} + te^{-2t} + 4e^{-2t} \tag{5.49}$$

We can verify (5.49) with MATLAB as follows:

```
syms s t; Fs=-1/((s+1)^3) + 3/((s+1)^2) - 4/(s+1) + 1/((s+2)^2) + 4/(s+2); ft=ilaplace(Fs)
```

```
ft = -1/2*t^2*exp(-t)+3*t*exp(-t)-4*exp(-t)
          +t*exp(-2*t)+4*exp(-2*t)
```

5.3 Case where F(s) is Improper Rational Function

Our discussion thus far, was based on the condition that $F(s)$ is a proper rational function. However, if $F(s)$ is an improper rational function, that is, if $m \ge n$, we must first divide the numerator $N(s)$ by the denominator $D(s)$ to obtain an expression of the form

$$F(s) = k_0 + k_1s + k_2s^2 + \ldots + k_{m-n}s^{m-n} + \frac{N(s)}{D(s)} \tag{5.50}$$

where $N(s)/D(s)$ is a proper rational function.

Example 5.6

Derive the Inverse Laplace transform $f_6(t)$ of

$$F_6(s) = \frac{s^2 + 2s + 2}{s + 1} \tag{5.51}$$

Solution:

For this example, $F_6(s)$ is an improper rational function. Therefore, we must express it in the form of (5.50) before we use the partial fraction expansion method.

By long division, we obtain

$$F_6(s) = \frac{s^2 + 2s + 2}{s + 1} = \frac{1}{s + 1} + 1 + s$$

Now, we recognize that

$$\frac{1}{s + 1} \Leftrightarrow e^{-t}$$

and

$$1 \Leftrightarrow \delta(t)$$

but

$$s \Leftrightarrow ?$$

To answer that question, we recall that

$$u_0'(t) = \delta(t)$$

and

$$u_0''(t) = \delta'(t)$$

where $\delta'(t)$ is the doublet of the delta function. Also, by the time differentiation property

$$u_0''(t) = \delta'(t) \Leftrightarrow s^2F(s) - sf(0) - f'(0) = s^2F(s) = s^2 \cdot \frac{1}{s} = s$$

Therefore, we have the new transform pair

$$s \Leftrightarrow \delta'(t) \tag{5.52}$$

and thus,

$$F_6(s) = \frac{s^2 + 2s + 2}{s + 1} = \frac{1}{s + 1} + 1 + s \Leftrightarrow e^{-t} + \delta(t) + \delta'(t) = f_6(t) \tag{5.53}$$

In general,

$$\frac{d^n}{dt^n}\delta(t) \Leftrightarrow s^n \tag{5.54}$$

We verify (5.53) with MATLAB as follows:

```
Ns = [1  2  2]; Ds = [1  1]; [r, p, k] = residue(Ns, Ds)
```

```
r =
     1
p =
    -1
k =
     1      1
```

The direct terms `k= [1 1]` above are the coefficients of $\delta(t)$ and $\delta'(t)$ respectively.

5.4 Alternate Method of Partial Fraction Expansion

Partial fraction expansion can also be performed with the *method of clearing the fractions*, that is, making the denominators of both sides the same, then equating the numerators. As before, we assume that $F(s)$ is a proper rational function. If not, we first perform a long division, and then work with the quotient and the remainder as we did in Example 5.6. We also assume that the denominator $D(s)$ can be expressed as a product of real linear and quadratic factors. If these assumptions prevail, we let $(s-a)$ be a linear factor of $D(s)$, and we assume that $(s-a)^m$ is the highest power of $(s-a)$ that divides $D(s)$. Then, we can express $F(s)$ as

$$F(s) = \frac{N(s)}{D(s)} = \frac{r_1}{s-a} + \frac{r_2}{(s-a)^2} + \dots \frac{r_m}{(s-a)^m} \tag{5.55}$$

Let $s^2 + \alpha s + \beta$ be a quadratic factor of $D(s)$, and suppose that $(s^2 + \alpha s + \beta)^n$ is the highest power of this factor that divides $D(s)$. Now, we perform the following steps:

1. To this factor, we assign the sum of *n* partial fractions, that is,

$$\frac{r_1 s + k_1}{s^2 + \alpha s + \beta} + \frac{r_2 s + k_2}{(s^2 + \alpha s + \beta)^2} + \dots + \frac{r_n s + k_n}{(s^2 + \alpha s + \beta)^n}$$

2. We repeat step 1 for each of the distinct linear and quadratic factors of $D(s)$
3. We set the given $F(s)$ equal to the sum of these partial fractions
4. We clear the resulting expression of fractions and arrange the terms in decreasing powers of s
5. We equate the coefficients of corresponding powers of s
6. We solve the resulting equations for the residues

Example 5.7

Express $F_7(s)$ of (5.56) below as a sum of partial fractions using the method of clearing the fractions.

$$F_7(s) = \frac{-2s + 4}{(s^2 + 1)(s-1)^2} \tag{5.56}$$

Solution:

Using Steps 1 through 3 above, we obtain

$$F_7(s) = \frac{-2s + 4}{(s^2 + 1)(s-1)^2} = \frac{r_1 s + A}{(s^2 + 1)} + \frac{r_{21}}{(s-1)^2} + \frac{r_{22}}{(s-1)} \tag{5.57}$$

With Step 4,

$$-2s + 4 = (r_1 s + A)(s-1)^2 + r_{21}(s^2 + 1) + r_{22}(s-1)(s^2 + 1) \tag{5.58}$$

and with Step 5,

$$\begin{aligned} -2s + 4 = {} & (r_1 + r_{22})s^3 + (-2r_1 + A - r_{22} + r_{21})s^2 \\ & + (r_1 - 2A + r_{22})\, s + (A - r_{22} + r_{21}) \end{aligned} \tag{5.59}$$

Relation (5.59) will be an identity is s if each power of s is the same on both sides of this relation. Therefore, we equate like powers of s and we obtain

$$\begin{aligned} 0 &= r_1 + r_{22} \\ 0 &= -2r_1 + A - r_{22} + r_{21} \\ -2 &= r_1 - 2A + r_{22} \\ 4 &= A - r_{22} + r_{21} \end{aligned} \tag{5.60}$$

Subtracting the second equation of (5.60) from the fourth, we obtain

$$4 = 2r_1$$

or

$$r_1 = 2 \tag{5.61}$$

By substitution of (5.61) into the first equation of (5.60), we obtain

$$0 = 2 + r_{22}$$

or

$$r_{22} = -2 \tag{5.62}$$

Next, substitution of (5.61) and (5.62) into the third equation of (5.60) yields

$$-2 = 2 - 2A - 2$$

or

$$A = 1 \tag{5.63}$$

Finally by substitution of (5.61), (5.62), and (5.63) into the fourth equation of (5.60), we obtain

$$4 = 1 + 2 + r_{21}$$

or

$$r_{21} = 1 \tag{5.64}$$

Substitution of these values into (5.57) yields

$$F_7(s) = \frac{-2s + 4}{(s^2 + 1)(s - 1)^2} = \frac{2s + 1}{(s^2 + 1)} + \frac{1}{(s - 1)^2} - \frac{2}{(s - 1)} \tag{5.65}$$

Example 5.8

Use partial fraction expansion to simplify $F_8(s)$ of (5.66) below, and find the time domain function $f_8(t)$ corresponding to $F_8(s)$.

$$F_8(s) = \frac{s + 3}{s^3 + 5s^2 + 12s + 8} \tag{5.66}$$

Solution:

This is the same transform as in Example 5.3, Page 5–6, where we found that the denominator

$D(s)$ can be expressed in factored form of a linear term and a quadratic. Thus, we write $F_8(s)$ as

$$F_8(s) = \frac{s+3}{(s+1)(s^2+4s+8)} \tag{5.67}$$

and using the method of clearing the fractions, we express (5.67) as

$$F_8(s) = \frac{s+3}{(s+1)(s^2+4s+8)} = \frac{r_1}{s+1} + \frac{r_2 s + r_3}{s^2+4s+8} \tag{5.68}$$

As in Example 5.3,

$$r_1 = \left.\frac{s+3}{s^2+4s+8}\right|_{s=-1} = \frac{2}{5} \tag{5.69}$$

Next, to compute r_2 and r_3, we follow the procedure of this section and we obtain

$$(s+3) = r_1(s^2+4s+8) + (r_2 s + r_3)(s+1) \tag{5.70}$$

Since r_1 is already known, we only need two equations in r_2 and r_3. Equating the coefficient of s^2 on the left side, which is zero, with the coefficients of s^2 on the right side of (5.70), we obtain

$$0 = r_1 + r_2 \tag{5.71}$$

and since $r_1 = 2/5$, it follows that $r_2 = -2/5$.

To obtain the third residue r_3, we equate the constant terms of (5.70). Then, $3 = 8r_1 + r_3$ or $3 = 8 \times 2/5 + r_3$, or $r_3 = -1/5$. Then, by substitution into (5.68), we obtain

$$F_8(s) = \frac{2/5}{(s+1)} - \frac{1}{5} \cdot \frac{(2s+1)}{(s^2+4s+8)} \tag{5.72}$$

as before.

The remaining steps are the same as in Example 5.3, and thus $f_8(t)$ is the same as $f_3(t)$, that is,

$$f_8(t) = f_3(t) = \left(\frac{2}{5}e^{-t} - \frac{2}{5}e^{-2t}\cos 2t + \frac{3}{10}e^{-2t}\sin 2t\right)u_0(t)$$

5.5 Summary

- The Inverse Laplace Transform Integral defined as

$$\mathcal{L}^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi j}\int_{\sigma - j\omega}^{\sigma + j\omega} F(s)e^{st}ds$$

 is difficult to evaluate because it requires contour integration using complex variables theory.

- For most engineering problems we can refer to Tables of Properties, and Common Laplace transform pairs to lookup the Inverse Laplace transform. The partial fraction expansion method offers a convenient means of expressing Laplace transforms in a recognizable form from which we can obtain the equivalent time–domain functions. The partial fraction expansion method can be applied whether the poles of $F(s)$ are distinct, complex conjugates, repeated, or a combination of these. The method of clearing the fractions is an alternate method of partial fraction expansion.

- If the highest power m of the numerator $N(s)$ is less than the highest power n of the denominator $D(s)$, i.e., $m < n$, $F(s)$ is said to be expressed as a proper rational function. If $m \ge n$, $F(s)$ is an improper rational function. The Laplace transform $F(s)$ must be expressed as a proper rational function before applying the partial fraction expansion. If $F(s)$ is an improper rational function, that is, if $m \ge n$, we must first divide the numerator $N(s)$ by the denominator $D(s)$ to obtain an expression of the form

$$F(s) = k_0 + k_1 s + k_2 s^2 + \ldots + k_{m-n}s^{m-n} + \frac{N(s)}{D(s)}$$

- In a proper rational function, the roots of numerator $N(s)$ are called the zeros of $F(s)$ and the roots of the denominator $D(s)$ are called the poles of $F(s)$.

- When $F(s)$ is expressed as

$$F(s) = \frac{r_1}{(s-p_1)} + \frac{r_2}{(s-p_2)} + \frac{r_3}{(s-p_3)} + \ldots + \frac{r_n}{(s-p_n)}$$

 $r_1, r_2, r_3, \ldots, r_n$ are called the residues and $p_1, p_2, p_3, \ldots, p_n$ are the poles of $F(s)$.

- The residues and poles of a rational function of polynomials can be found easily using the MATLAB **residue(a,b)** function. The direct term is always empty (has no value) whenever $F(s)$ is a proper rational function. We can use the MATLAB **factor(s)** symbolic function to convert the denominator polynomial form of $F_2(s)$ into a factored form. We can also use the MATLAB **collect(s)** and **expand(s)** symbolic functions to convert the denominator factored form of $F_2(s)$ into a polynomial form. In this chapter we introduced the new transform pair $s \Leftrightarrow \delta'(t)$ and in general, $\frac{d^n}{dt^n}\delta(t) \Leftrightarrow s^n$

5.6 Exercises

1. Find the Inverse Laplace transform of the following:

a. $\dfrac{4}{s+3}$ b. $\dfrac{4}{(s+3)^2}$ c. $\dfrac{4}{(s+3)^4}$ d. $\dfrac{3s+4}{(s+3)^5}$ e. $\dfrac{s^2+6s+3}{(s+3)^5}$

2. Find the Inverse Laplace transform of the following:

a. $\dfrac{3s+4}{s^2+4s+85}$ b. $\dfrac{4s+5}{s^2+5s+18.5}$ c. $\dfrac{s^2+3s+2}{s^3+5s^2+10.5s+9}$

d. $\dfrac{s^2-16}{s^3+8s^2+24s+32}$ e. $\dfrac{s+1}{s^3+6s^2+11s+6}$

3. Find the Inverse Laplace transform of the following:

a. $\dfrac{3s+2}{s^2+25}$ b. $\dfrac{5s^2+3}{(s^2+4)^2}$ Hint: $\left\{\begin{array}{l} \dfrac{1}{2\alpha}(\sin\alpha t+\alpha t\cos\alpha t) \Leftrightarrow \dfrac{s^2}{(s^2+\alpha^2)^2} \\ \dfrac{1}{2\alpha^3}(\sin\alpha t-\alpha t\cos\alpha t) \Leftrightarrow \dfrac{1}{(s^2+\alpha^2)^2} \end{array}\right\}$

c. $\dfrac{2s+3}{s^2+4.25s+1}$ d. $\dfrac{s^3+8s^2+24s+32}{s^2+6s+8}$ e. $e^{-2s}\dfrac{3}{(2s+3)^3}$

4. Use the Initial Value Theorem to find $f(0)$ given that the Laplace transform of $f(t)$ is

$$\frac{2s+3}{s^2+4.25s+1}$$

Compare your answer with that of Exercise 3(c).

5. It is known that the Laplace transform $F(s)$ has two distinct poles, one at $s = 0$, the other at $s = -1$. It also has a single zero at $s = 1$, and we know that $\lim_{t\to\infty} f(t) = 10$. Find $F(s)$ and $f(t)$.

5.7 Solutions to End-of-Chapter Exercises

1.

a. $\frac{4}{s+3} \Leftrightarrow 4e^{-3t}$ b. $\frac{4}{(s+3)^2} \Leftrightarrow 4te^{-3t}$ c. $\frac{4}{(s+3)^4} \Leftrightarrow \frac{4}{3!}t^3e^{-3t} = \frac{2}{3}t^3e^{-3t}$

d.
$$\frac{3s+4}{(s+3)^5} = \frac{3(s+4/3+5/3-5/3)}{(s+3)^5} = 3\cdot\frac{(s+3)-5/3}{(s+3)^5} = 3\cdot\frac{1}{(s+3)^4} - 5\cdot\frac{1}{(s+3)^5}$$
$$\Leftrightarrow \frac{3}{3!}t^3e^{-3t} - \frac{5}{4!}t^4e^{-3t} = \frac{1}{2}\left(t^3e^{-3t} - \frac{5}{12}t^4e^{-3t}\right)$$

e.
$$\frac{s^2+6s+3}{(s+3)^5} = \frac{s^2+6s+9-6}{(s+3)^5} = \frac{(s+3)^2}{(s+3)^5} - \frac{6}{(s+3)^5} = \frac{1}{(s+3)^3} - 6\cdot\frac{1}{(s+3)^5}$$
$$\Leftrightarrow \frac{1}{2!}t^2e^{-3t} - \frac{6}{4!}t^4e^{-3t} = \frac{1}{2}\left(t^2e^{-3t} - \frac{1}{2}t^4e^{-3t}\right)$$

2.

a.

$$\frac{3s+4}{s^2+4s+85} = \frac{3(s+4/3+2/3-2/3)}{(s+2)^2+81} = 3\cdot\frac{(s+2)-2/3}{(s+2)^2+9^2} = 3\cdot\frac{(s+2)}{(s+2)^2+9^2} - \frac{1}{9}\cdot\frac{2\times 9}{(s+2)^2+9^2}$$
$$= 3\cdot\frac{(s+2)}{(s+2)^2+9^2} - \frac{2}{9}\cdot\frac{9}{(s+2)^2+9^2} \Leftrightarrow 3e^{-2t}\cos 9t - \frac{2}{9}e^{-2t}\sin 9t$$

b.

$$\frac{4s+5}{s^2+5s+18.5} = \frac{4s+5}{s^2+5s+6.25+12.25} = \frac{4s+5}{(s+2.5)^2+3.5^2} = 4\cdot\frac{s+5/4}{(s+2.5)^2+3.5^2}$$
$$= 4\cdot\frac{s+10/4-10/4+5/4}{(s+2.5)^2+3.5^2} = 4\cdot\frac{s+2.5}{(s+2.5)^2+3.5^2} - \frac{1}{3.5}\cdot\frac{5\times 3.5}{(s+2.5)^2+3.5^2}$$
$$= 4\cdot\frac{(s+2.5)}{(s+2.5)^2+3.5^2} - \frac{10}{7}\cdot\frac{3.5}{(s+2.5)^2+3.5^2} \Leftrightarrow 4e^{-2.5t}\cos 3.5t - \frac{10}{7}e^{-2.5t}\sin 3.5t$$

c. Using the MATLAB **factor(s)** function we obtain:

```
syms s; factor(s^2+3*s+2), factor(s^3+5*s^2+10.5*s+9)
% Must have Symbolic Math Toolbox installed
```

```
ans = (s+2)*(s+1)
ans = 1/2*(s+2)*(2*s^2+6*s+9)
```

Then,

$$\frac{s^2+3s+2}{s^3+5s^2+10.5s+9} = \frac{(s+1)(s+2)}{(s+2)(s^2+3s+4.5)} = \frac{(s+1)}{(s^2+3s+4.5)} = \frac{s+1}{s^2+3s+2.25-2.25+4.5}$$

$$= \frac{s+1.5-1.5+1}{(s+1.5)^2+(1.5)^2} = \frac{s+1.5}{(s+1.5)^2+(1.5)^2} - \frac{1}{1.5}\cdot\frac{0.5\times 1.5}{(s+1.5)^2+(1.5)^2}$$

$$= \frac{s+1.5}{(s+1.5)^2+(1.5)^2} - \frac{1}{3}\cdot\frac{1.5}{(s+2.5)^2+3.5^2} \Leftrightarrow e^{-1.5t}\cos 1.5t - \frac{1}{3}e^{-1.5t}\sin 1.5t$$

d.

$$\frac{s^2-16}{s^3+8s^2+24s+32} = \frac{(s+4)(s-4)}{(s+4)(s^2+4s+8)} = \frac{(s-4)}{(s+2)^2+2^2} = \frac{s+2-2-4}{(s+2)^2+2^2}$$

$$= \frac{s+2}{(s+2)^2+2^2} - \frac{1}{2}\cdot\frac{6\times 2}{(s+2)^2+2^2}$$

$$= \frac{s+2}{(s+2)^2+2^2} - 3\cdot\frac{2}{(s+2)^2+2^2} \Leftrightarrow e^{-2t}\cos 2t - 3e^{-2t}\sin 2t$$

e.

$$\frac{s+1}{s^3+6s^2+11s+6} = \frac{(s+1)}{(s+1)(s+2)(s+3)} = \frac{1}{(s+2)(s+3)}$$

$$= \frac{1}{(s+2)(s+3)} = \frac{r_1}{s+2} + \frac{r_2}{s+3} \qquad r_1 = \left.\frac{1}{s+3}\right|_{s=-2} = 1 \qquad r_2 = \left.\frac{1}{s+2}\right|_{s=-3} = -1$$

$$= \frac{1}{(s+2)(s+3)} = \left[\frac{1}{s+2} - \frac{1}{s+3}\right] \Leftrightarrow e^{-2t} - e^{-3t}$$

3.

a. $$\frac{3s+2}{s^2+25} = \frac{3s}{s^2+5^2} + \frac{1}{5}\cdot\frac{2\times 5}{s^2+5^2} = 3\cdot\frac{s}{s^2+5^2} + \frac{2}{5}\cdot\frac{5}{s^2+5^2} \Leftrightarrow 3\cos 5t + \frac{2}{5}\sin 5t$$

b. $$\frac{5s^2+3}{(s^2+4)^2} = \frac{5s^2}{(s^2+2^2)^2} + \frac{3}{(s^2+2^2)^2} \Leftrightarrow 5\cdot\frac{1}{2\times 2}(\sin 2t + 2t\cos 2t) + 3\cdot\frac{1}{2\times 8}(\sin 2t - 2t\cos 2t)$$

$$\Leftrightarrow \left(\frac{5}{4}+\frac{3}{16}\right)\sin 2t + \left(\frac{5}{4}-\frac{3}{16}\right)2t\cos 2t = \frac{23}{16}\sin 2t + \frac{17}{8}t\cos 2t$$

c. $$\frac{2s+3}{s^2+4.25s+1} = \frac{2s+3}{(s+4)(s+1/4)} = \frac{r_1}{s+4} + \frac{r_2}{s+1/4}$$

$$r_1 = \left.\frac{2s+3}{s+1/4}\right|_{s=-4} = \frac{-5}{-15/4} = \frac{4}{3} \qquad r_2 = \left.\frac{2s+3}{s+4}\right|_{s=-1/4} = \frac{5/2}{15/4} = \frac{2}{3}$$

$$\frac{4/3}{s+4}+\frac{2/3}{s+1/4}\Leftrightarrow\frac{2}{3}(2e^{-4t}+e^{-t/4})$$

d.

$$\frac{s^3+8s^2+24s+32}{s^2+6s+8}=\frac{(s+4)(s^2+4s+8)}{(s+2)(s+4)}=\frac{(s^2+4s+8)}{(s+2)}$$ and by long division

$$\frac{s^2+4s+8}{s+2}=s+2+\frac{4}{s+2}\Leftrightarrow\delta'(t)+2\delta(t)+4e^{-2t}$$

e.

$$e^{-2s}\frac{3}{(2s+3)^3}\qquad e^{-2s}F(s)\Leftrightarrow f(t-2)u_0(t-2)$$

$$F(s)=\frac{3}{(2s+3)^3}=\frac{3/2^3}{(2s+3)^3/2^3}=\frac{3/8}{[(2s+3)/2]^3}=\frac{3/8}{(s+3/2)^3}\Leftrightarrow\frac{3}{8}\left(\frac{1}{2!}t^2e^{-(3/2)t}\right)=\frac{3}{16}t^2e^{-(3/2)t}$$

$$e^{-2s}F(s)=e^{-2s}\frac{3}{(2s+3)^3}\Leftrightarrow\frac{3}{16}(t-2)^2e^{-(3/2)(t-2)}u_0(t-2)$$

4. The initial value theorem states that $\lim_{t\to 0} f(t)=\lim_{s\to\infty} sF(s)$. Then,

$$f(0)=\lim_{s\to\infty} s\frac{2s+3}{s^2+4.25s+1}=\lim_{s\to\infty}\frac{2s^2+3s}{s^2+4.25s+1}$$

$$=\lim_{s\to\infty}\frac{2s^2/s^2+3s/s^2}{s^2/s^2+4.25s/s^2+1/s^2}=\lim_{s\to\infty}\frac{2+3/s}{1+4.25/s+1/s^2}=2$$

The value $f(0)=2$ is the same as in the time domain expression found in Exercise 3(c).

5. We are given that $F(s)=\frac{A(s-1)}{s(s+1)}$ and $\lim_{t\to\infty} f(t)=\lim_{s\to 0} sF(s)=10$. Then,

$$\lim_{s\to 0} s\frac{A(s-1)}{s(s+1)}=A\lim_{s\to 0}\frac{(s-1)}{(s+1)}=-A=10$$

Therefore,

$$F(s)=\frac{-10(s-1)}{s(s+1)}=\frac{r_1}{s}+\frac{r_2}{s+1}=\frac{10}{s}-\frac{20}{s+1}\Leftrightarrow(10-20e^{-t})u_0(t)$$

that is,

$$f(t)=(10-20e^{-t})u_0(t)$$

and we observe that

$$\lim_{t\to\infty} f(t)=10$$

Chapter 6

Circuit Analysis with Laplace Transforms

This chapter presents applications of the Laplace transform. Several examples are presented to illustrate how the Laplace transformation is applied to circuit analysis. Complex impedance, complex admittance, and transfer functions are also defined.

6.1 Circuit Transformation from Time to Complex Frequency

In this section we will show the voltage–current relationships for the three elementary circuit networks, i.e., resistive, inductive, and capacitive in the time and complex frequency domains. They are described in Subsections 6.1.1 through 6.1.3 below.

6.1.1 Resistive Network Transformation

The time and complex frequency domains for purely resistive networks are shown in Figure 6.1.

Figure 6.1. Resistive network in time domain and complex frequency domain

6.1.2 Inductive Network Transformation

The time and complex frequency domains for purely inductive networks are shown in Figure 6.2.

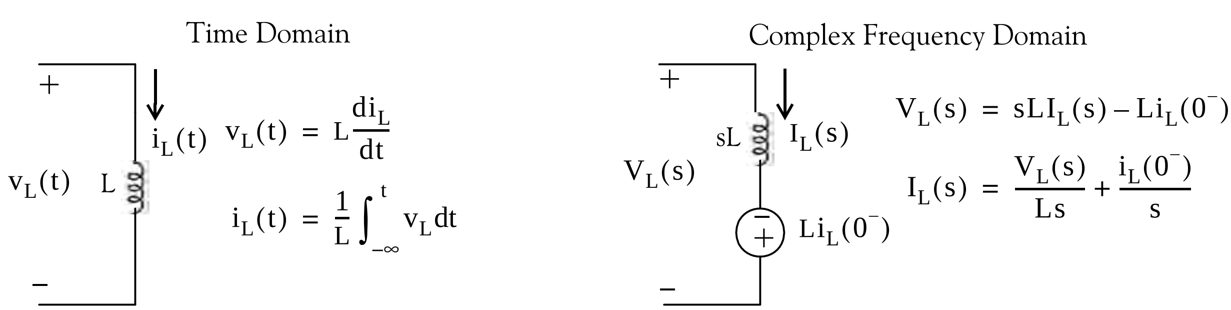
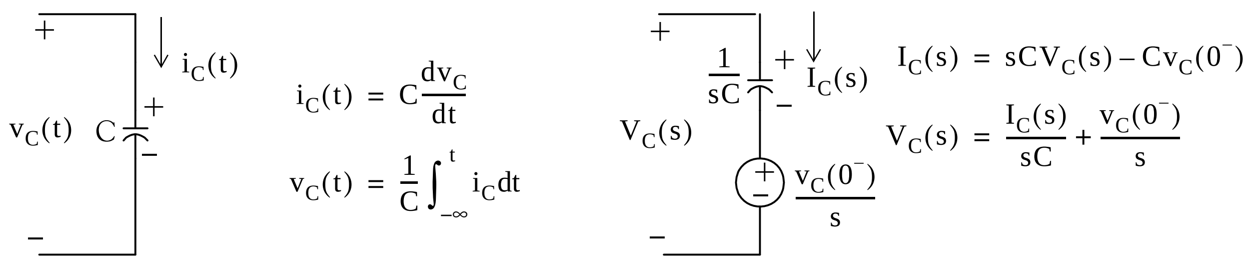


Figure 6.2. Inductive network in time domain and complex frequency domain

6.1.3 Capacitive Network Transformation

The time and complex frequency domains for purely capacitive networks are shown in Figure 6.3.



$$i_C(t) = C\frac{dv_C}{dt}$$

$$v_C(t) = \frac{1}{C}\int_{-\infty}^{t} i_C dt$$

$$I_C(s) = sCV_C(s) - Cv_C(0^-)$$

$$V_C(s) = \frac{I_C(s)}{sC} + \frac{v_C(0^-)}{s}$$

Figure 6.3. Capacitive circuit in time domain and complex frequency domain

Note:

In the complex frequency domain, the terms sL and $1/sC$ are referred to as *complex inductive impedance*, and *complex capacitive impedance* respectively. Likewise, the terms and sC and $1/sL$ are called *complex capacitive admittance* and *complex inductive admittance* respectively.

Example 6.1

Use the Laplace transform method and apply Kirchoff's Current Law (KCL) to find the voltage $v_C(t)$ across the capacitor for the circuit of Figure 6.4, given that $v_C(0^-) = 6\ V$.

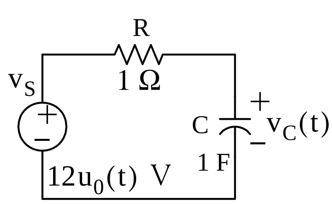


Figure 6.4. Circuit for Example 6.1

Solution:

We apply KCL at node A as shown in Figure 6.5.

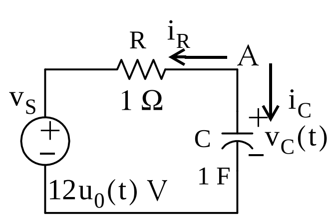


Figure 6.5. Application of KCL for the circuit of Example 6.1

Then,

$$i_R + i_C = 0$$

or

$$\frac{v_C(t) - 12u_0(t)}{1} + 1 \cdot \frac{dv_C}{dt} = 0$$

$$\frac{dv_C}{dt} + v_C(t) = 12u_0(t) \qquad (6.1)$$

The Laplace transform of (6.1) is

$$sV_C(s) - v_C(0^-) + V_C(s) = \frac{12}{s}$$

$$(s+1)V_C(s) = \frac{12}{s} + 6$$

$$V_C(s) = \frac{6s+12}{s(s+1)}$$

By partial fraction expansion,

$$V_C(s) = \frac{6s+12}{s(s+1)} = \frac{r_1}{s} + \frac{r_2}{(s+1)}$$

$$r_1 = \left.\frac{6s+12}{(s+1)}\right|_{s=0} = 12$$

$$r_2 = \left.\frac{6s+12}{s}\right|_{s=-1} = -6$$

Therefore,

$$V_C(s) = \frac{12}{s} - \frac{6}{s+1} \Leftrightarrow 12 - 6e^{-t} = (12 - 6e^{-t})u_0(t) = v_C(t)$$

Example 6.2

Use the Laplace transform method and apply Kirchoff's Voltage Law (KVL) to find the voltage $v_C(t)$ across the capacitor for the circuit of Figure 6.6, given that $v_C(0^-) = 6\ V$.

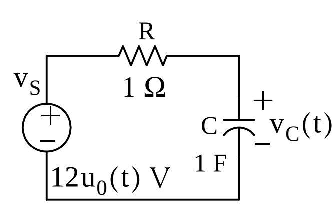


Figure 6.6. Circuit for Example 6.2

Solution:

This is the same circuit as in Example 6.1. We apply KVL for the loop shown in Figure 6.7.

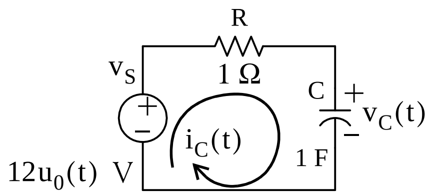


Figure 6.7. Application of KVL for the circuit of Example 6.2

$$Ri_C(t) + \frac{1}{C}\int_{-\infty}^{t} i_C(t)dt = 12u_0(t)$$

and with $R = 1$ and $C = 1$, we obtain

$$i_C(t) + \int_{-\infty}^{t} i_C(t)dt = 12u_0(t) \tag{6.2}$$

Next, taking the Laplace transform of both sides of (6.2), we obtain

$$I_C(s) + \frac{I_C(s)}{s} + \frac{v_C(0^-)}{s} = \frac{12}{s}$$

$$\left(1 + \frac{1}{s}\right)I_C(s) = \frac{12}{s} - \frac{6}{s} = \frac{6}{s}$$

$$\left(\frac{s+1}{s}\right)I_C(s) = \frac{6}{s}$$

or

$$I_C(s) = \frac{6}{s+1} \Leftrightarrow i_C(t) = 6e^{-t}u_0(t)$$

Check: From Example 6.1,

$$v_C(t) = (12 - 6e^{-t})u_0(t)$$

Then,

$$i_C(t) = C\frac{dv_C}{dt} = \frac{dv_C}{dt} = \frac{d}{dt}(12 - 6e^{-t})u_0(t) = 6e^{-t}u_0(t) + 6\delta(t) \tag{6.3}$$

The presence of the delta function in (6.3) is a result of the unit step that is applied at $t = 0$.

Example 6.3

In the circuit of Figure 6.8, switch S_1 closes at $t = 0$, while at the same time, switch S_2 opens. Use the Laplace transform method to find $v_{out}(t)$ for $t > 0$.

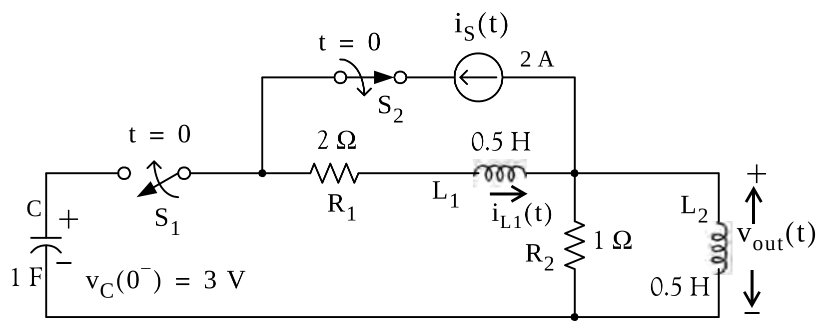


Figure 6.8. Circuit for Example 6.3

Solution:

Since the circuit contains a capacitor and an inductor, we must consider two initial conditions One is given as $v_C(0^-) = 3\ V$. The other initial condition is obtained by observing that there is an initial current of $2\ A$ in inductor L_1; this is provided by the $2\ A$ current source just before switch S_2 opens. Therefore, our second initial condition is $i_{L1}(0^-) = 2\ A$.

For $t > 0$, we transform the circuit of Figure 6.8 into its *s–domain** equivalent shown in Figure 6.9.

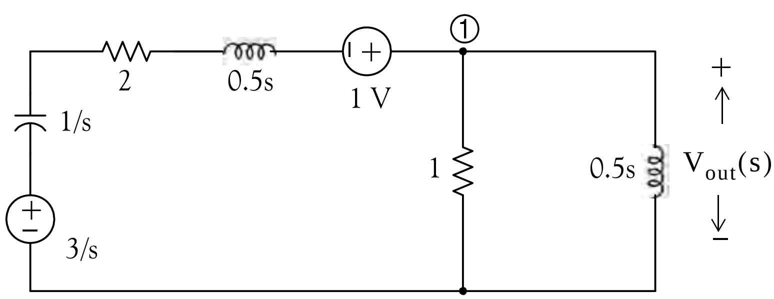


Figure 6.9. Transformed circuit of Example 6.3

In Figure 6.9 the current in inductor L_1 has been replaced by a voltage source of $1\ V$. This is found from the relation

$$L_1 i_{L1}(0^-) = \frac{1}{2} \times 2 = 1\ V \tag{6.4}$$

* *Henceforth, for convenience, we will refer the time domain as t–domain and the complex frequency domain as s–domain.*

The polarity of this voltage source is as shown in Figure 6.9 so that it is consistent with the direction of the current $i_{L1}(t)$ in the circuit of Figure 6.8 just before switch S_2 opens. The initial capacitor voltage is replaced by a voltage source equal to $3/s$.

Applying KCL at node ① we obtain

$$\frac{V_{out}(s) - 1 - 3/s}{1/s + 2 + s/2} + \frac{V_{out}(s)}{1} + \frac{V_{out}(s)}{s/2} = 0 \qquad (6.5)$$

and after simplification,

$$V_{out}(s) = \frac{2s(s+3)}{s^3 + 8s^2 + 10s + 4} \qquad (6.6)$$

We will use MATLAB to factor the denominator $D(s)$ of (6.6) into a linear and a quadratic factor.

```
p=[1  8  10  4]; r=roots(p)                          % Find the roots of D(s)
```

```
r =
  -6.5708
  -0.7146 + 0.3132i
  -0.7146 - 0.3132i
```

```
y=expand((s + 0.7146 – 0.3132j)*(s + 0.7146 + 0.3132j))   % Find quadratic form
```

```
y =
s^2+3573/2500*s+3043737/5000000
```

```
3573/2500                                            % Simplify coefficient of s
```

```
ans =
    1.4292
```

```
3043737/5000000                                      % Simplify constant term
```

```
ans =
    0.6087
```

Therefore,

$$V_{out}(s) = \frac{2s(s+3)}{s^3 + 8s^2 + 10s + 4} = \frac{2s(s+3)}{(s + 6.57)(s^2 + 1.43s + 0.61)} \qquad (6.7)$$

Next, we perform partial fraction expansion.

$$V_{out}(s) = \frac{2s(s+3)}{(s + 6.57)(s^2 + 1.43s + 0.61)} = \frac{r_1}{s + 6.57} + \frac{r_2 s + r_3}{s^2 + 1.43s + 0.61} \qquad (6.8)$$

$$r_1 = \left.\frac{2s(s+3)}{s^2+1.43s+0.61}\right|_{s=-6.57} = 1.36 \tag{6.9}$$

The residues r_2 and r_3 are found from the equality

$$2s(s+3) = r_1(s^2+1.43s+0.61)+(r_2 s+r_3)(s+6.57) \tag{6.10}$$

Equating constant terms of (6.10), we obtain

$$0 = 0.61r_1 + 6.57r_3$$

and by substitution of the known value of r_1 from (6.9), we obtain

$$r_3 = -0.12$$

Similarly, equating coefficients of s^2, we obtain

$$2 = r_1 + r_2$$

and using the known value of r_1, we obtain

$$r_2 = 0.64 \tag{6.11}$$

By substitution into (6.8),

$$V_{out}(s) = \frac{1.36}{s+6.57}+\frac{0.64s-0.12}{s^2+1.43s+0.61} = \frac{1.36}{s+6.57}+\frac{0.64s+0.46-0.58}{s^2+1.43s+0.51+0.1}\ ^{*}$$

or

$$V_{out}(s) = \frac{1.36}{s+6.57}+(0.64)\frac{s+0.715-0.91}{(s+0.715)^2+(0.316)^2}$$

$$= \frac{1.36}{s+6.57}+\frac{0.64(s+0.715)}{(s+0.715)^2+(0.316)^2}-\frac{0.58}{(s+0.715)^2+(0.316)^2} \tag{6.12}$$

$$= \frac{1.36}{s+6.57}+\frac{0.64(s+0.715)}{(s+0.715)^2+(0.316)^2}-\frac{1.84\times 0.316}{(s+0.715)^2+(0.316)^2}$$

Taking the Inverse Laplace of (6.12), we obtain

$$v_{out}(t) = (1.36e^{-6.57t}+0.64e^{-0.715t}\cos 0.316t - 1.84e^{-0.715t}\sin 0.316t)u_0(t) \tag{6.13}$$

---

* *We perform these steps to express the term* $\frac{0.64s-0.12}{s^2+1.43s+0.61}$ *in a form that resembles the transform pairs* $e^{-at}\cos\omega t u_0(t) \Leftrightarrow \frac{s+a}{(s+a)^2+\omega^2}$ *and* $e^{-at}\sin\omega t u_0(t) \Leftrightarrow \frac{\omega}{(s+a)^2+\omega^2}$. *The remaining steps are carried out in (6.12).*

From (6.13), we observe that as $t \to \infty$, $v_{out}(t) \to 0$. This is to be expected because $v_{out}(t)$ is the voltage across the inductor as we can see from the circuit of Figure 6.9. The MATLAB script below will plot the relation (6.13) above.

```
t=0:0.01:10;...
Vout=1.36.*exp(–6.57.*t)+0.64.*exp(–0.715.*t).*cos(0.316.*t)–1.84.*exp(–0.715.*t).*sin(0.316.*t);...
plot(t,Vout); grid
```

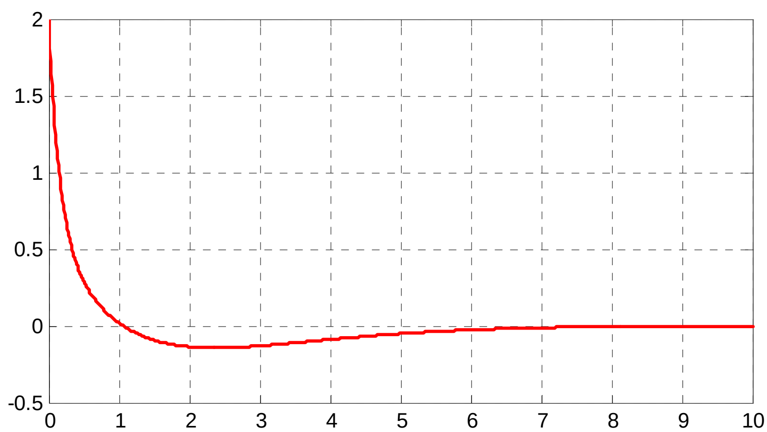


Figure 6.10. Plot of $v_{out}(t)$ for the circuit of Example 6.3

Figure 6.11 shows the Simulink/SimPower Systems model for the circuit in Figure 6.8.

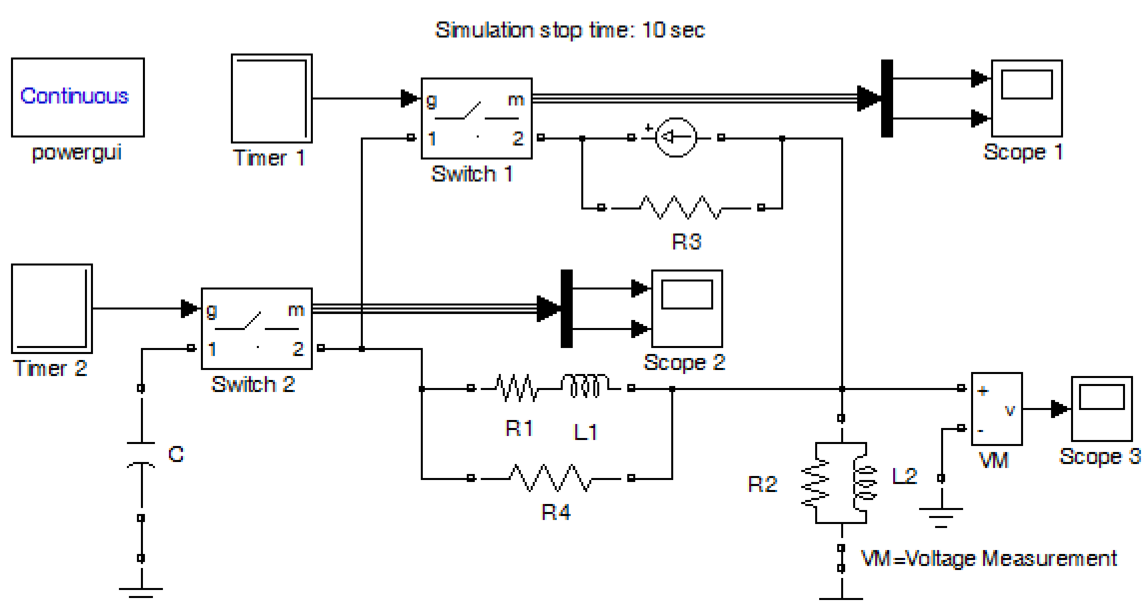


Figure 6.11. The Simulink/SimPowerSystems model for the circuit in Figure 6.8

In the model in Figure 6.11, the **Switch 1** and **Switch 2** blocks are modeled as current sources and unless a snubber* circuit is present, cannot be connected in series with a current source or in series with an inductor. The **Current Source** block and the series **RL** block in Figure 6.11 do not include snubbers and in this case, the **Resistor** blocks R3 and R4, both set as 1 MΩ, are connected in parallel with the **Current Source** block and the series **RL** block to act as snubbers.

The Block Parameters for the Simulink/SimPowerSystems blocks in Figure 6.11 are set as follows:

On the model in Figure 6.11 window click **Simulation>Configuration Parameters**, and select:

Type: Variable Step, **Solver**: ode23. Leave unlisted parameters in their default states.

Timer 1 and **Timer 2** blocks – **Time(s)**: [0 3/60]

Amplitude – Timer 1: [1 0] (Closed, then Open after 3/60 s)

Timer 2: [0 1] (Open, then Closed after 3/60 s)

Switch 1 block – as shown in Figure 6.12

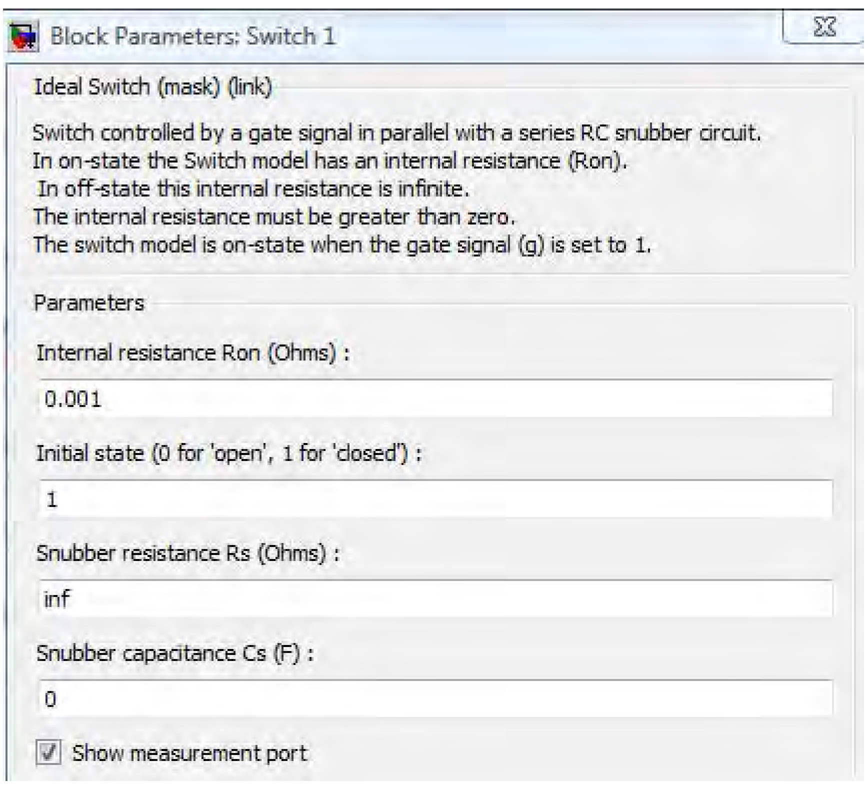


Figure 6.12. Block parameters for Switch 1 block

Switch 2 block – as shown in Figure 6.12, except **Initial state** 0.

* *A snubber is a device used to suppress transients such as voltage in electrical systems, force in mechanical systems, and pressure in fluid mechanics.*

Current Source block – **Peak Amplitude**: 2, **Phase**: 90, Frequency: 0, **Measurement**: Current
With these settings the **Current Source** block behaves as a 2 Amp DC current source.

R1 L1 block - As shown in Figure 6.13.

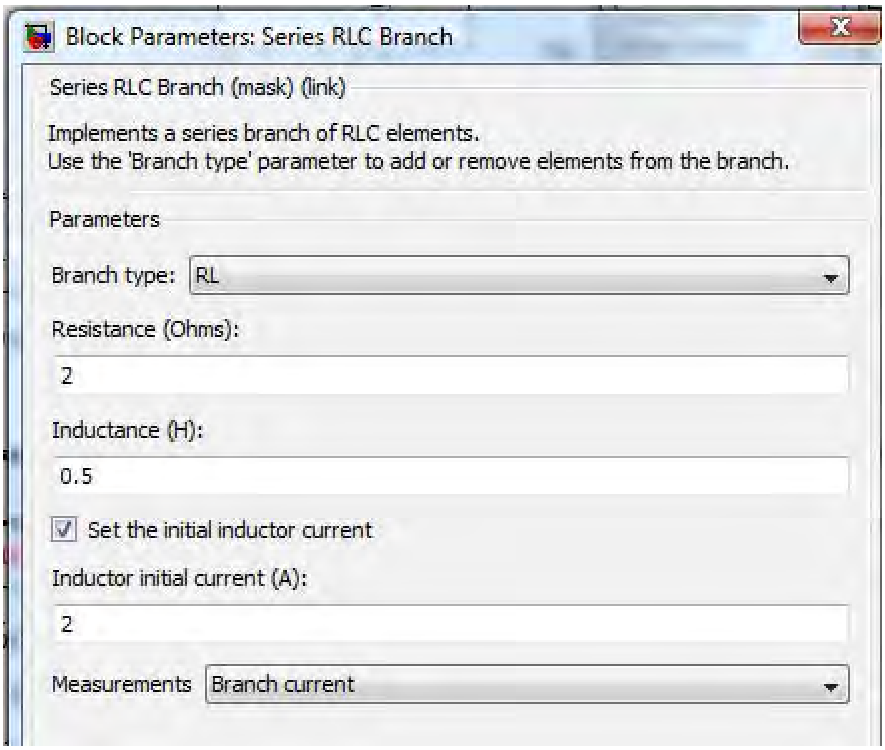


Figure 6.13. Block parameters for R1 L1 branch

The waveform for the voltage $v_{out}(t)$ in expression 6.13 is displayed by the Scope 3 block in Figure 6.11 is shown in Figure 6.14 and it compares favorably with the waveform produced with MATLAB in Figure 6.10.

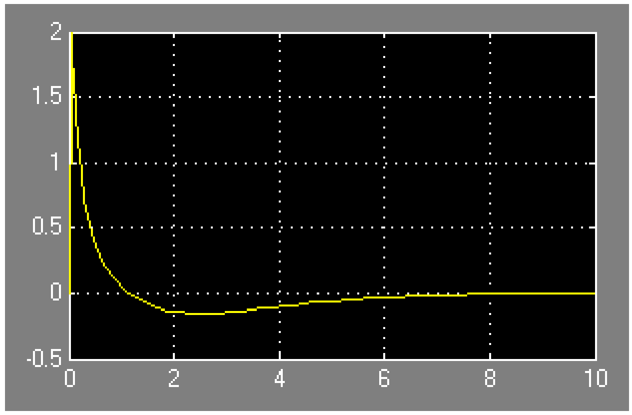


Figure 6.14. Waveform displayed by the Scope 3 block in Figure 6.11.

6.2 Complex Impedance Z(s)

Consider the s – domain RLC series circuit of Figure 6.11, where the initial conditions are assumed to be zero.

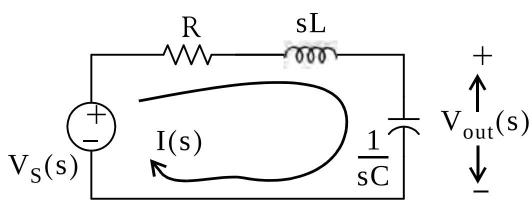


Figure 6.15. Series RLC circuit in s–domain

For this circuit, the sum $R + sL + \frac{1}{sC}$ represents the total opposition to current flow. Then,

$$I(s) = \frac{V_S(s)}{R + sL + 1/sC} \tag{6.14}$$

and defining the ratio $V_s(s)/I(s)$ as $Z(s)$, we obtain

$$\boxed{Z(s) \equiv \frac{V_S(s)}{I(s)} = R + sL + \frac{1}{sC}} \tag{6.15}$$

and thus, the s – domain current $I(s)$ can be found from the relation (6.16) below.

$$\boxed{I(s) = \frac{V_S(s)}{Z(s)}} \tag{6.16}$$

where

$$Z(s) = R + sL + \frac{1}{sC} \tag{6.17}$$

We recall that $s = \sigma + j\omega$. Therefore, $Z(s)$ is a complex quantity, and it is referred to as the *complex input impedance* of an $s-\mathrm{domain}$ RLC series circuit. In other words, $Z(s)$ is the ratio of the voltage excitation $V_s(s)$ to the current response $I(s)$ under *zero state* (zero initial conditions).

Example 6.4

For the network of Figure 6.16, all values are in Ω (ohms). Find $Z(s)$ using:

a. nodal analysis

b. successive combinations of series and parallel impedances

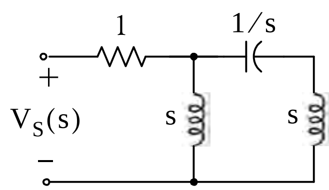


Figure 6.16. Circuit for Example 6.4

Solution:

a.

We will first find $I(s)$, and we will compute $Z(s)$ using (6.15). We assign the voltage $V_A(s)$ at node A as shown in Figure 6.17.

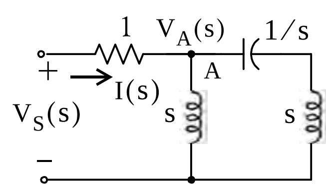


Figure 6.17. Network for finding $I(s)$ in Example 6.4

By nodal analysis,

$$\frac{V_A(s) - V_S(s)}{1} + \frac{V_A(s)}{s} + \frac{V_A(s)}{s + 1/s} = 0$$

$$\left(1 + \frac{1}{s} + \frac{1}{s + 1/s}\right)V_A(s) = V_S(s)$$

$$V_A(s) = \frac{s^3+1}{s^3+2s^2+s+1}\cdot V_S(s)$$

The current $I(s)$ is now found as

$$I(s) = \frac{V_S(s)-V_A(s)}{1} = \left(1-\frac{s^3+1}{s^3+2s^2+s+1}\right)V_S(s) = \frac{2s^2+1}{s^3+2s^2+s+1}\cdot V_S(s)$$

and thus,

$$Z(s) = \frac{V_S(s)}{I(s)} = \frac{s^3+2s^2+s+1}{2s^2+1} \qquad (6.18)$$

b.

The impedance $Z(s)$ can also be found by successive combinations of series and parallel impedances, as it is done with series and parallel resistances. For convenience, we denote the network devices as Z_1, Z_2, Z_3 and Z_4 shown in Figure 6.16.

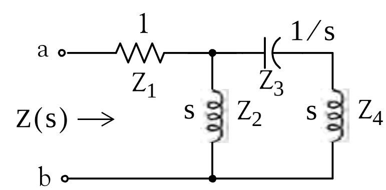


Figure 6.18. Computation of the impedance of Example 6.4 by series − parallel combinations

To find the equivalent impedance $Z(s)$, looking to the right of terminals a and b, we begin on the right side of the network and we proceed to the left combining impedances as we would combine resistances where the symbol $||$ denotes parallel combination. Then,

$$Z(s) = [(Z_3+Z_4)\ ||\ Z_2] + Z_1$$

$$Z(s) = \frac{s(s+1/s)}{s+s+1/s}+1 = \frac{s^2+1}{(2s^2+1)/s}+1 = \frac{s^3+s}{2s^2+1}+1 = \frac{s^3+2s^2+s+1}{2s^2+1} \qquad (6.19)$$

We observe that (6.19) is the same as (6.18).

6.3 Complex Admittance Y(s)

Consider the s – domain GLC parallel circuit of Figure 6.19 where the initial conditions are zero.

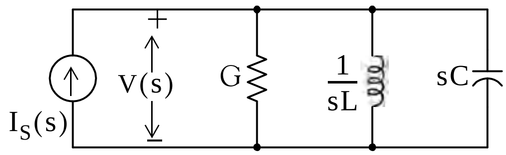


Figure 6.19. Parallel GLC circuit in s-domain

For the circuit of Figure 6.19,

$$GV(s) + \frac{1}{sL}V(s) + sCV(s) = I(s)$$

$$\left(G + \frac{1}{sL} + sC\right)(V(s)) = I(s)$$

Defining the ratio $I_S(s)/V(s)$ as $Y(s)$, we obtain

$$Y(s) \equiv \frac{I(s)}{V(s)} = G + \frac{1}{sL} + sC = \frac{1}{Z(s)} \qquad (6.20)$$

and thus the $s-\mathrm{domain}$ voltage $V(s)$ can be found from

$$\boxed{V(s) = \frac{I_S(s)}{Y(s)}} \qquad (6.21)$$

where

$$\boxed{Y(s) = G + \frac{1}{sL} + sC} \qquad (6.22)$$

We recall that $s = \sigma + j\omega$. Therefore, $Y(s)$ is a complex quantity, and it is referred to as the *complex input admittance* of an $s-\mathrm{domain}$ GLC parallel circuit. In other words, $Y(s)$ is the ratio of the current excitation $I_S(s)$ to the voltage response $V(s)$ under *zero state* (zero initial conditions).

Example 6.5

Compute $Z(s)$ and $Y(s)$ for the circuit of Figure 6.20. All values are in Ω (ohms). Verify your answers with MATLAB.

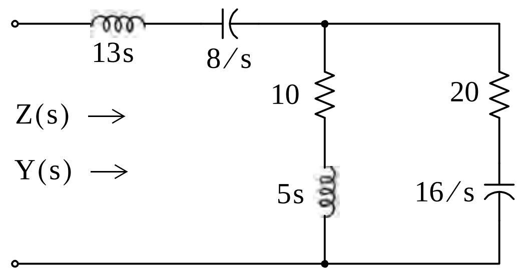


Figure 6.20. Circuit for Example 6.5

Solution:

It is convenient to represent the given circuit as shown in Figure 6.17.

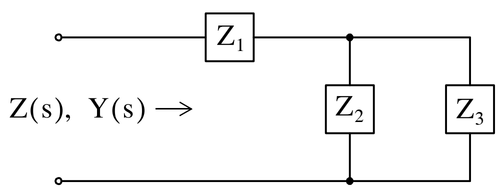


Figure 6.21. Simplified circuit for Example 6.5

where

$$Z_1 = 13s + \frac{8}{s} = \frac{13s^2 + 8}{s}$$

$$Z_2 = 10 + 5s$$

$$Z_3 = 20 + \frac{16}{s} = \frac{4(5s+4)}{s}$$

Then,

$$Z(s) = Z_1 + \frac{Z_2 Z_3}{Z_2 + Z_3} = \frac{13s^2+8}{s} + \frac{(10+5s)\left(\frac{4(5s+4)}{s}\right)}{10+5s+\frac{4(5s+4)}{s}} = \frac{13s^2+8}{s} + \frac{(10+5s)\left(\frac{4(5s+4)}{s}\right)}{\frac{5s^2+10s+4(5s+4)}{s}}$$

$$= \frac{13s^2+8}{s} + \frac{20(5s^2+14s+8)}{5s^2+30s+16} = \frac{65s^4+490s^3+528s^2+400s+128}{s(5s^2+30s+16)}$$

Check with MATLAB:

```
syms s;                % Define symbolic variable s. Must have Symbolic Math Toolbox installed
z1 = 13*s + 8/s; z2 = 5*s + 10; z3 = 20 + 16/s; z = z1 + z2 * z3 / (z2+z3)

z =
13*s+8/s+(5*s+10)*(20+16/s)/(5*s+30+16/s)

z10 = simplify(z)
```

```
z10 =
(65*s^4+490*s^3+528*s^2+400*s+128)/s/(5*s^2+30*s+16)
```

pretty(z10)

```
         4         3         2
   65 s  + 490 s  + 528 s  + 400 s + 128
   -------------------------------------
                      2
             s (5 s  + 30 s + 16)
```

The complex input admittance $Y(s)$ is found by taking the reciprocal of $Z(s)$, that is,

$$Y(s) = \frac{1}{Z(s)} = \frac{s(5s^2+30s+16)}{65s^4+490s^3+528s^2+400s+128} \quad (6.23)$$

6.4 Transfer Functions

In an $s-$**domain** circuit, the ratio of the output voltage $V_{out}(s)$ to the input voltage $V_{in}(s)$ *under zero state conditions*, is of great interest* in network analysis. This ratio is referred to as the *voltage transfer function* and it is denoted as $G_v(s)$, that is,

$$G_v(s) \equiv \frac{V_{out}(s)}{V_{in}(s)} \quad (6.24)$$

Similarly, the ratio of the output current $I_{out}(s)$ to the input current $I_{in}(s)$ *under zero state conditions*, is called the *current transfer function* denoted as $G_i(s)$, that is,

$$G_i(s) \equiv \frac{I_{out}(s)}{I_{in}(s)} \quad (6.25)$$

The current transfer function of (6.25) is rarely used; therefore, from now on, the transfer function will have the meaning of the voltage transfer function, i.e.,

* *To appreciate the usefulness of the transfer function, let us express relation (6.24) as* $V_{out}(s) = G_v(s) \cdot V_{in}(s)$. *This relation indicates that if we know the transfer function of a network, we can compute its output by multiplication of the transfer function by its input. We should also remember that the transfer function concept exists only in the complex frequency domain. In the time domain this concept is known as the* ***impulse response****, and it is discussed in Signals and Systems with MATLAB Computing and Simulink Modeling, ISBN 978–1–934404–11–9.*

$$\boxed{G(s) \equiv \frac{V_{out}(s)}{V_{in}(s)}} \tag{6.26}$$

Example 6.6

Derive an expression for the transfer function $G(s)$ for the circuit of Figure 6.22, where R_g represents the internal resistance of the applied (source) voltage V_S, and R_L represents the resistance of the load that consists of R_L, L, and C.

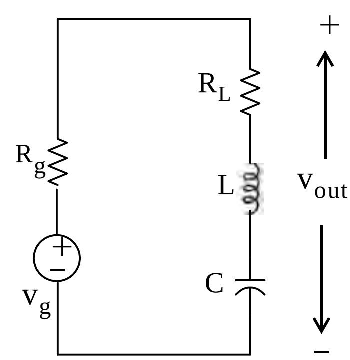


Figure 6.22. Circuit for Example 6.6

Solution:

No initial conditions are given, and even if they were, we would disregard them since the transfer function was defined as the ratio of the output voltage $V_{out}(s)$ to the input voltage $V_{in}(s) = V_g(s)$ under zero initial conditions. The $s-\text{domain}$ circuit is shown in Figure 6.23.

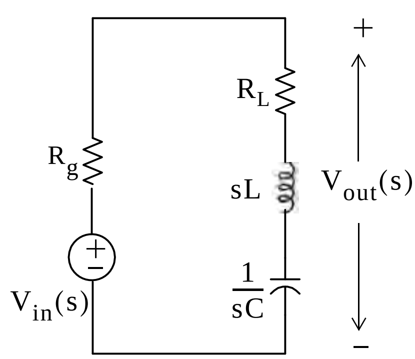


Figure 6.23. The s–domain circuit for Example 6.6

The transfer function $G(s)$ is readily found by application of the voltage division expression of the $s-\text{domain}$ circuit of Figure 6.23. Thus,

$$V_{out}(s) = \frac{R_L + sL + 1/sC}{R_g + R_L + sL + 1/sC} V_{in}(s)$$

Therefore,

$$G(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{R_L + Ls + 1/sC}{R_g + R_L + Ls + 1/sC} \tag{6.27}$$

Example 6.7

Compute the transfer function $G(s)$ for the circuit of Figure 6.24 in terms of the circuit constants R_1, R_2, R_3, C_1, and C_2 Then, replace the complex variable s with $j\omega$, and the circuit constants with their numerical values and plot the magnitude $|G(s)| = V_{out}(s)/V_{in}(s)$ versus radian frequency ω.

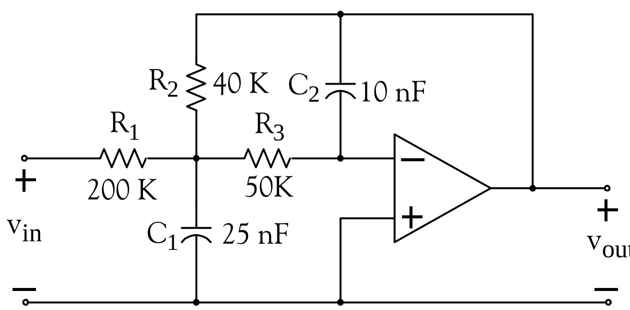


Figure 6.24. Circuit for Example 6.7

Solution:

The complex frequency domain equivalent circuit is shown in Figure 6.25.

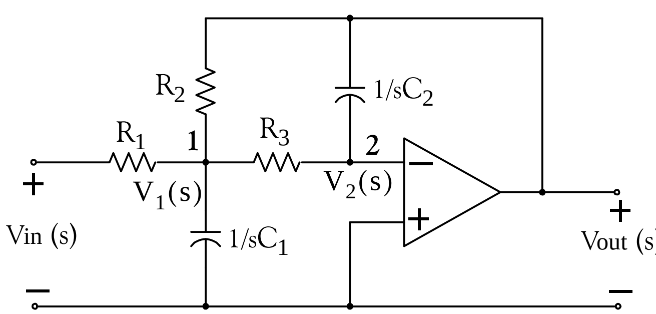


Figure 6.25. The s–domain circuit for Example 6.7

Next, we write nodal equations at nodes **1** and **2**. At node **1**,

$$\frac{V_1(s) - V_{in}(s)}{R_1} + \frac{V_1}{1/sC_1} + \frac{V_1(s) - V_{out}(s)}{R_2} + \frac{V_1(s) - V_2(s)}{R_3} = 0 \tag{6.28}$$

At node **2**,

$$\frac{V_2(s) - V_1(s)}{R_3} = \frac{V_{out}(s)}{1/sC_2} \tag{6.29}$$

Since $V_2(s) = 0$ (virtual ground), we express (6.29) as

$$V_1(s) = (-sR_3C_2)V_{out}(s) \tag{6.30}$$

and by substitution of (6.30) into (6.28), rearranging, and collecting like terms, we obtain:

$$\left[\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + sC_1\right)(-sR_3C_2) - \frac{1}{R_2}\right]V_{out}(s) = \frac{1}{R_1}V_{in}(s)$$

or

$$G(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{-1}{R_1[(1/R_1 + 1/R_2 + 1/R_3 + sC_1)(sR_3C_2) + 1/R_2]} \tag{6.31}$$

To simplify the denominator of (6.31), we use the MATLAB script below with the given values of the resistors and the capacitors.

```
syms s;                % Define symbolic variable s
R1=2*10^5; R2=4*10^4; R3=5*10^4; C1=25*10^(-9); C2=10*10^(-9);...
DEN=R1*((1/R1+1/R2+1/R3+s*C1)*(s*R3*C2)+1/R2); simplify(DEN)
```

```
ans =
1/200*s+188894659314785825/75557863725914323419136*s^2+5
```

```
188894659314785825/75557863725914323419136     % Simplify coefficient of s^2
```

```
ans =
  2.5000e-006
```

```
1/200                                          % Simplify coefficient of s^2
```

```
ans =
    0.0050
```

Therefore,

$$G(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{-1}{2.5 \times 10^{-6}s^2 + 5 \times 10^{-3}s + 5}$$

By substitution of s with $j\omega$ we obtain

$$G(j\omega) = \frac{V_{out}(j\omega)}{V_{in}(j\omega)} = \frac{-1}{2.5 \times 10^{-6}\omega^2 - j5 \times 10^{-3}\omega + 5} \tag{6.32}$$

We use MATLAB to plot the magnitude of (6.32) on a semilog scale with the following script:

```
w=1:10:10000; Gs=–1./(2.5.*10.^(–6).*w.^2–5.*j.*10.^(–3).*w+5);...
semilogx(w,abs(Gs)); xlabel('Radian Frequency w'); ylabel('|Vout/Vin|');...
title('Magnitude Vout/Vin vs. Radian Frequency'); grid
```

The plot is shown in Figure 6.22. We observe that the given op amp circuit is a second order low–pass filter whose cutoff frequency $(-3\ \text{dB})$ occurs at about $700\ r/s$.

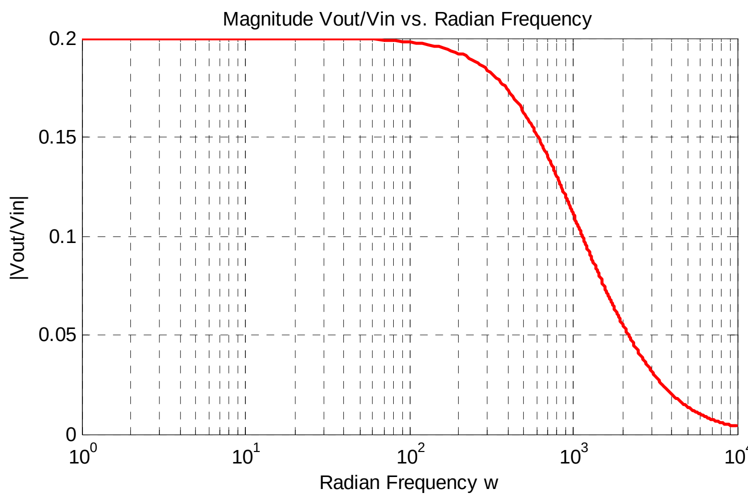
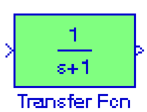


Figure 6.26. $|G(j\omega)|$ versus ω for the circuit of Example 6.7

6.5 Using the Simulink Transfer Fcn Block



The Simulink **Transfer Fcn** block implements a transfer function where the input $V_{IN}(s)$ and the output $V_{OUT}(s)$ can be expressed in transfer function form as

$$G(s) = \frac{V_{OUT}(s)}{V_{IN}(s)} \tag{6.33}$$

Example 6.8

Let us reconsider the active low–pass filter op amp circuit of Figure 6.24, Page 6-18 where we found that the transfer function is

$$G(s)=\frac{V_{out}(s)}{V_{in}(s)}=\frac{-1}{R_1[(1/R_1+1/R_2+1/R_3+sC_1)(sR_3C_2)+1/R_2]} \qquad (6.34)$$

and for simplicity, let $R_1 = R_2 = R_3 = 1\ \Omega$, and $C_1 = C_2 = 1\ F$. By substitution into (6.34) we obtain

$$G(s)=\frac{V_{out}(s)}{V_{in}(s)}=\frac{-1}{s^2+3s+1} \qquad (6.35)$$

Next, we let the input be the unit step function $u_0(t)$, and as we know from Chapter 4, $u_0(t) \Leftrightarrow 1/s$. Therefore,

$$V_{out}(s) = G(s)\cdot V_{in}(s) = \frac{1}{s}\cdot\frac{-1}{s^2+3s+1} = \frac{-1}{s^3+3s^2+s} \qquad (6.36)$$

To find $v_{out}(t)$, we perform partial fraction expansion, and for convenience, we use the MATLAB **residue** function as follows:

```
num=–1; den=[1 3 1 0];[r p k]=residue(num,den)
```

```
r =
   -0.1708
    1.1708
   -1.0000
p =
   -2.6180
   -0.3820
         0
k =
     []
```

Therefore,

$$\left(\frac{1}{s}\cdot\frac{-1}{s^2+3s+1}\right) = -\frac{1}{s}+\frac{1.171}{s+0.382}-\frac{0.171}{s+2.618} \Leftrightarrow -1+1.171e^{-0.382t}-0.171e^{-2.618t} = v_{out}(t) \qquad (6.37)$$

The plot for $v_{out}(t)$ is obtained with the following MATLAB script, and it is shown in Figure 6.27.

```
t=0:0.01:10; ft=–1+1.171.*exp(–0.382.*t)–0.171.*exp(–2.618.*t); plot(t,ft); grid
```

The same plot can be obtained using the Simulink model of Figure 6.29, where in the **Function Block Parameters** dialog box for the **Transfer Fcn** block we enter -1 for the numerator, and

[1 3 1] for the denominator. After the simulation command is executed, the **Scope** block displays the waveform of Figure 6.29.

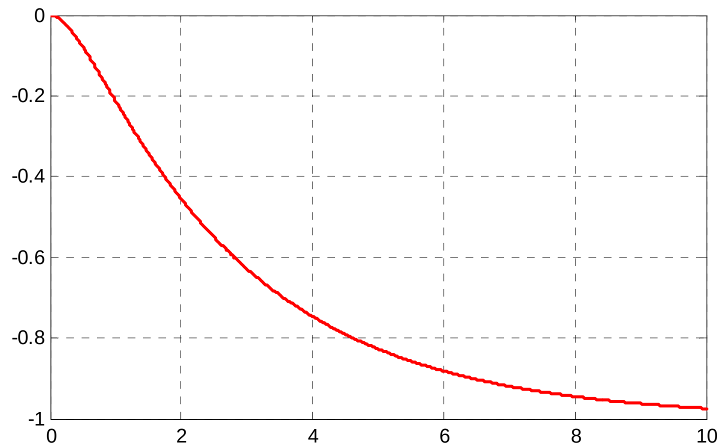


Figure 6.27. Plot of $v_{out}(t)$ *for Example 6.8.*

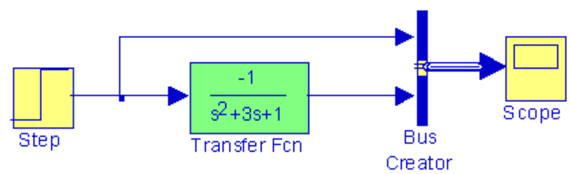


Figure 6.28. Simulink model for Example 6.8

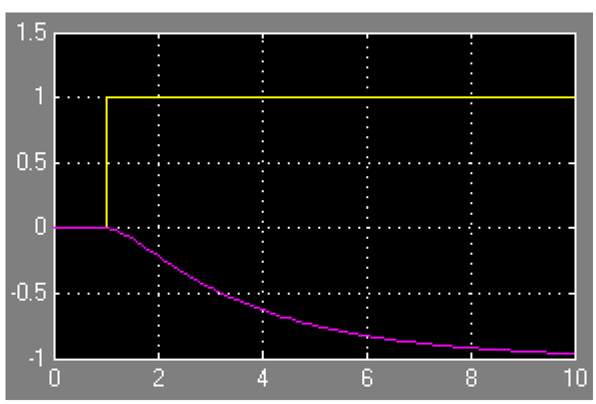


Figure 6.29. Waveform for the Simulink model of Figure 6.28

6.6 Summary

- The Laplace transformation provides a convenient method of analyzing electric circuits since integrodifferential equations in the $t-\text{domain}$ are transformed to algebraic equations in the $s-\text{domain}$.

- In the $s-\text{domain}$ the terms sL and $1/sC$ are called complex inductive impedance, and complex capacitive impedance respectively. Likewise, the terms and sC and $1/sL$ are called complex capacitive admittance and complex inductive admittance respectively.

- The expression

$$Z(s) = R + sL + \frac{1}{sC}$$

is a complex quantity, and it is referred to as the complex input impedance of an $s-\text{domain}$ RLC series circuit.

- In the $s-\text{domain}$ the current $I(s)$ can be found from

$$I(s) = \frac{V_S(s)}{Z(s)}$$

- The expression

$$Y(s) = G + \frac{1}{sL} + sC$$

is a complex quantity, and it is referred to as the complex input admittance of an $s-\text{domain}$ GLC parallel circuit.

- In the $s-\text{domain}$ the voltage $V(s)$ can be found from

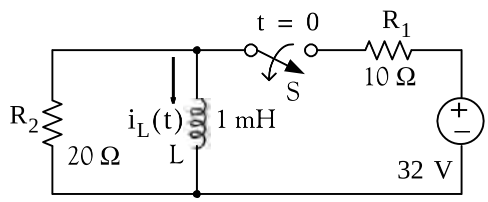
$$V(s) = \frac{I_S(s)}{Y(s)}$$

- In an $s-\text{domain}$ circuit, the ratio of the output voltage $V_{out}(s)$ to the input voltage $V_{in}(s)$ under zero state conditions is referred to as the voltage transfer function and it is denoted as $G(s)$, that is,

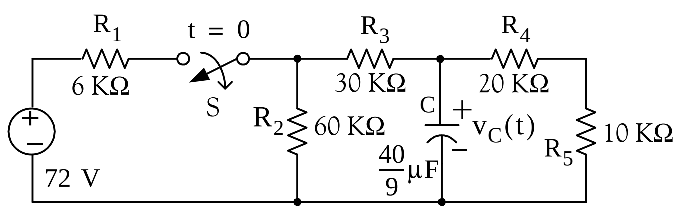
$$G(s) \equiv \frac{V_{out}(s)}{V_{in}(s)}$$

6.7 Exercises

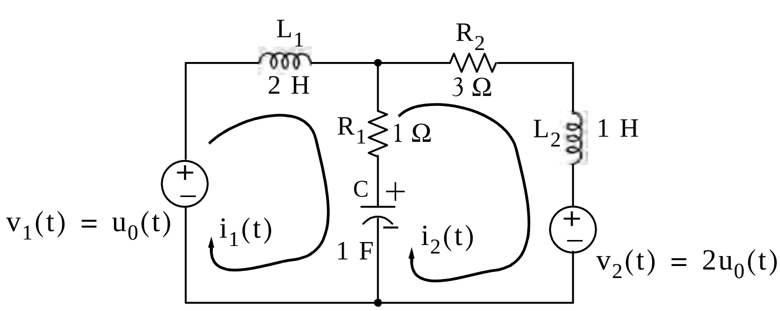
1. In the circuit below, switch S has been closed for a long time, and opens at $t = 0$. Use the Laplace transform method to compute $i_L(t)$ for $t > 0$.



2. In the circuit below, switch S has been closed for a long time, and opens at $t = 0$. Use the Laplace transform method to compute $v_c(t)$ for $t > 0$.



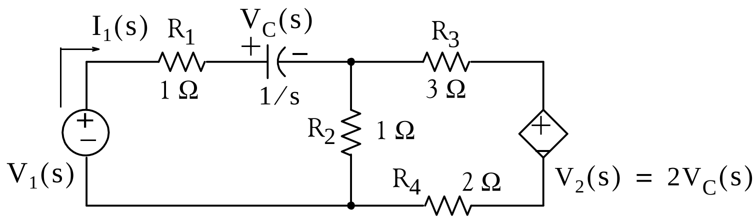
3. Use mesh analysis and the Laplace transform method, to compute $i_1(t)$ and $i_2(t)$ for the circuit below, given that $i_L(0^-) = 0$ and $v_C(0^-) = 0$.



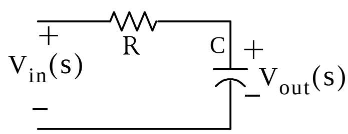
4. For the $s-\text{domain}$ circuit below,

   a. compute the admittance $Y(s) = I_1(s)/V_1(s)$

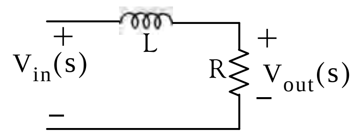
   b. compute the $t-\text{domain}$ value of $i_1(t)$ when $v_1(t) = u_0(t)$, and all initial conditions are zero.



5. Derive the transfer functions for the networks (a) and (b) below.

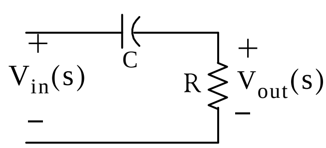


(a)

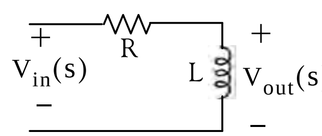


(b)

6. Derive the transfer functions for the networks (a) and (b) below.

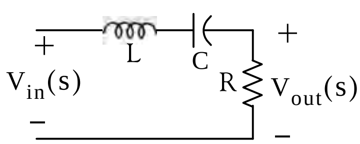


(a)

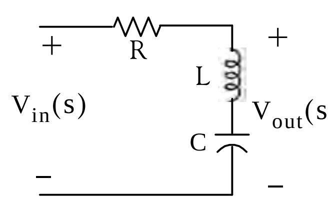


(b)

7. Derive the transfer functions for the networks (a) and (b) below.

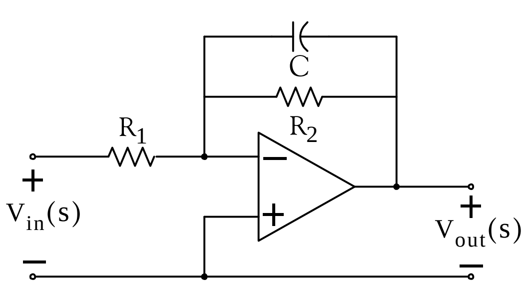


(a)

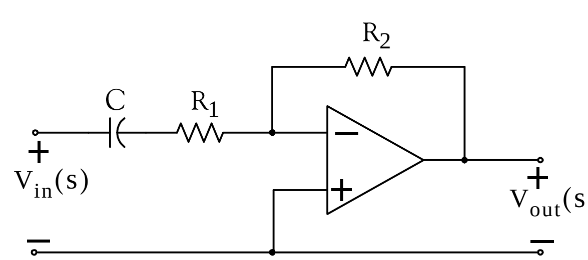


(b)

8. Derive the transfer function for the networks (a) and (b) below.

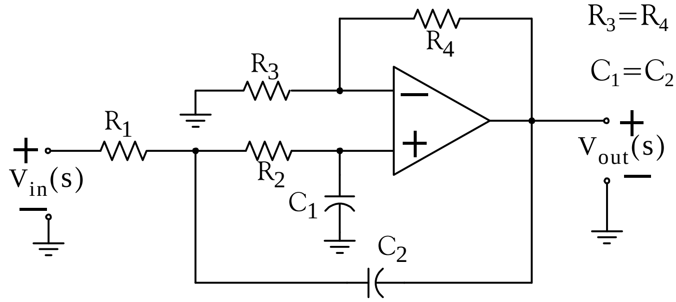


(a)



(b)

9. Derive the transfer function for the network below. Using MATLAB, plot $|G(s)|$ versus frequency in Hertz, on a semilog scale.



$R_1 = 11.3\ k\Omega$

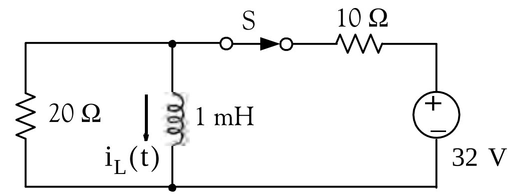
$R_2 = 22.6\ k\Omega$

$R_3 = R_4 = 68.1\ k\Omega$

$C_1 = C_2 = 0.01\ \mu F$

6.8 Solutions to End–of–Chapter Exercises

1. At $t = 0^-$, the switch is closed, and the $t-\text{domain}$ circuit is as shown below where the $20\ \Omega$ resistor is shorted out by the inductor.

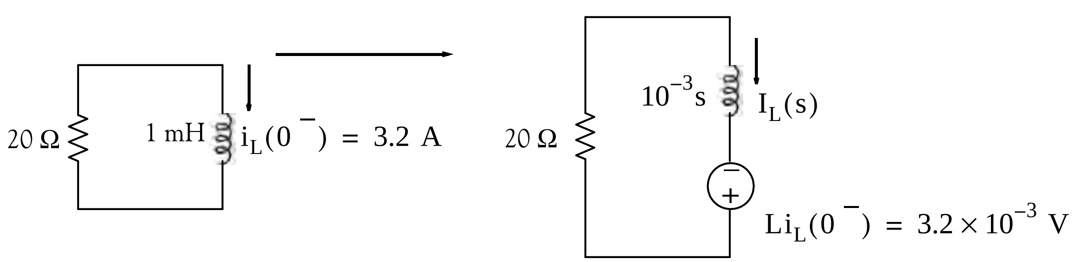


Then,

$$i_L(t)\Big|_{t=0^-} = \frac{32}{10} = 3.2\ \text{A}$$

and thus the initial condition has been established as $i_L(0^-) = 3.2\ \text{A}$

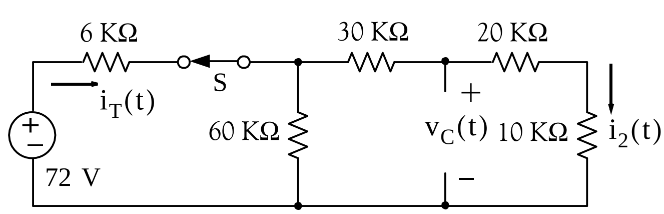
For all $t > 0$ the $t-\text{domain}$ and $s-\text{domain}$ circuits are as shown below.



From the $s-\text{domain}$ circuit on the right side above we obtain

$$I_L(s) = \frac{3.2 \times 10^{-3}}{20 + 10^{-3}s} = \frac{3.2}{s + 20000} \Leftrightarrow 3.2e^{-20000t}u_0(t) = i_L(t)$$

2. At $t = 0^-$, the switch is closed and the $t-\text{domain}$ circuit is as shown below.



Then,

$$i_T(0^-) = \frac{72\ V}{6\ K\Omega + 60\ K\Omega \parallel 60\ K\Omega} = \frac{72\ V}{6\ K\Omega + 30\ K\Omega} = \frac{72\ V}{36\ K\Omega} = 2\ mA$$

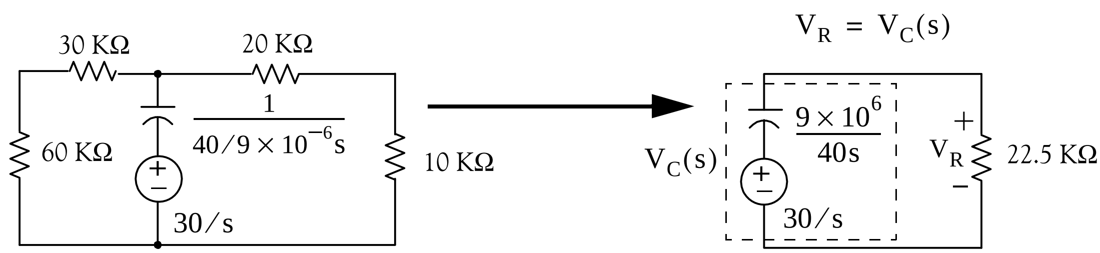
and

$$i_2(0^-) = \frac{1}{2} i_T(0^-) = 1\ mA$$

Therefore, the initial condition is

$$v_C(0^-) = (20\ K\Omega + 10\ K\Omega) \cdot i_2(0^-) = (30\ K\Omega) \cdot (1\ mA) = 30\ V$$

For all $t > 0$, the $s-\mathrm{domain}$ circuit is as shown below.



$$(60\ K\Omega + 30\ K\Omega) \parallel (20\ K\Omega + 10\ K\Omega) = 22.5\ K\Omega$$

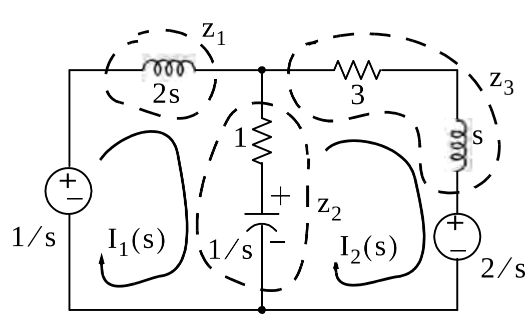
$$V_C(s) = V_R = \frac{22.5 \times 10^3}{9 \times 10^6 / 40s + 22.5 \times 10^3} \cdot \frac{30}{s} = \frac{30 \times 22.5 \times 10^3}{9 \times 10^6 / 40 + 22.5 \times 10^3 s}$$

$$= \frac{(30 \times 22.5 \times 10^3)/(22.5 \times 10^3)}{9 \times 10^6 / (40 \times 22.5 \times 10^3) + s} = \frac{30}{9 \times 10^6 / 90 \times 10^4 + s} = \frac{30}{10 + s}$$

Then,

$$V_C(s) = \frac{30}{s+10} \Leftrightarrow 30e^{-10t} u_0(t)\ V = v_C(t)$$

3. The $s-\mathrm{domain}$ circuit is shown below where $z_1 = 2s$, $z_2 = 1 + 1/s$, and $z_3 = s + 3$



Then,

$$(z_1 + z_2)I_1(s) - z_2I_2(s) = 1/s$$
$$-z_2I_1(s) + (z_2 + z_3)I_2(s) = -2/s$$

and in matrix form

$$\begin{bmatrix} (z_1 + z_2) & -z_2 \\ -z_2 & (z_2 + z_3) \end{bmatrix} \cdot \begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} 1/s \\ -2/s \end{bmatrix}$$

We use the MATLAB script below we obtain the values of the currents.

```
syms s; z1=2*s; z2=1+1/s; z3=s+3;  % Must have Symbolic Math Toolbox installed
Z=[z1+z2  –z2; –z2  z2+z3]; Vs=[1/s  –2/s]'; Is=Z\Vs; fprintf(' \n');...
disp('Is1 = '); pretty(Is(1)); disp('Is2 = '); pretty(Is(2))
```

```
Is1 =
                                    2
                        2 s - 1 + s
                 -------------------------------
                                2      3
                 (6 s + 3 + 9 s   + 2 s )
Is2 =
                                2
                         4 s   + s + 1
                - -------------------------------
                                2      3
                  (6 s + 3 + 9 s   + 2 s ) conj(s)
```

Therefore,

$$I_1(s) = \frac{s^2 + 2s - 1}{2s^3 + 9s^2 + 6s + 3} \quad (1)$$

$$I_2(s) = -\frac{4s^2 + s + 1}{2s^3 + 9s^2 + 6s + 3} \quad (2)$$

We use MATLAB to express the denominators of (1) and (2) as a product of a linear and a quadratic term.

```
p=[2  9  6  3]; r=roots(p); fprintf(' \n'); disp('root1 ='); disp(r(1));...
disp('root2 ='); disp(r(2)); disp('root3 ='); disp(r(3)); disp('root2 + root3 ='); disp(r(2)+r(3));...
disp('root2 * root3 ='); disp(r(2)*r(3))
```

```
root1 =
   -3.8170

root2 =
  -0.3415 + 0.5257i
```

```
root3 =
 -0.3415 - 0.5257i

root2 + root3 =
   -0.6830

root2 * root3 =
    0.3930
```

and with these values (1) is written as

$$I_1(s) = \frac{s^2 + 2s - 1}{(s + 3.817) \cdot (s^2 + 0.683s + 0.393)} = \frac{r_1}{(s + 3.817)} + \frac{r_2 s + r_3}{(s^2 + 0.683s + 0.393)} \quad (3)$$

Multiplying every term by the denominator and equating numerators we obtain

$$s^2 + 2s - 1 = r_1(s^2 + 0.683s + 0.393) + (r_2 s + r_3)(s + 3.817)$$

Equating s^2, s, and constant terms we obtain

$$\begin{aligned} r_1 + r_2 &= 1 \\ 0.683r_1 + 3.817r_2 + r_3 &= 2 \\ 0.393r_1 + 3.817r_3 &= -1 \end{aligned}$$

We will use MATLAB to find these residues.

```
A=[1  1  0; 0.683  3.817  1; 0.393  0  3.817]; B=[1  2  -1]'; r=A\B; fprintf(' \n');...
fprintf('r1 = %5.2f \t',r(1)); fprintf('r2 = %5.2f \t',r(2)); fprintf('r3 = %5.2f',r(3))
```

```
r1 = 0.48    r2 = 0.52    r3 = -0.31
```

By substitution of these values into (3) we obtain

$$I_1(s) = \frac{r_1}{(s + 3.817)} + \frac{r_2 s + r_3}{(s^2 + 0.683s + 0.393)} = \frac{0.48}{(s + 3.817)} + \frac{0.52s - 0.31}{(s^2 + 0.683s + 0.393)} \quad (4)$$

By inspection, the Inverse Laplace of first term on the right side of (4) is

$$\frac{0.48}{(s + 3.82)} \Leftrightarrow 0.48e^{-3.82t} \quad (5)$$

The second term on the right side of (4) requires some manipulation. Therefore, we will use the MATLAB **ilaplace(s)** function to find the Inverse Laplace as shown below.

```
syms s t  % Must have Symbolic Math Toolbox installed
IL=ilaplace((0.52*s-0.31)/(s^2+0.68*s+0.39));
pretty(IL)
```

```
     1217        17       1/2              1/2
   - ---- exp(- -- t) 14    sin(7/50 14    t)
     4900        50
          13        17                1/2
        + -- exp(- -- t) cos(7/50 14    t)
          25        50
```

Thus,

$$i_1(t) = 0.48e^{-3.82t} - 0.93e^{-0.34t}\sin 0.53t + 0.52e^{-0.34t}\cos 0.53t$$

Next, we will find $I_2(s)$. We found earlier that

$$I_2(s) = -\frac{4s^2+s+1}{2s^3+9s^2+6s+3}$$

and following the same procedure we obtain

$$I_2(s) = \frac{-4s^2-s-1}{(s+3.817)\cdot(s^2+0.683s+0.393)} = \frac{r_1}{(s+3.817)} + \frac{r_2s+r_3}{(s^2+0.683s+0.393)} \quad (6)$$

Multiplying every term by the denominator and equating numerators we obtain

$$-4s^2-s-1 = r_1(s^2+0.683s+0.393)+(r_2s+r_3)(s+3.817)$$

Equating s^2, s, and constant terms, we obtain

$$r_1+r_2 = -4$$
$$0.683r_1+3.817r_2+r_3 = -1$$
$$0.393r_1+3.817r_3 = -1$$

We will use MATLAB to find these residues.

```
A=[1  1  0; 0.683  3.817  1; 0.393  0  3.817]; B=[−4  −1  −1]'; r=A\B; fprintf(' \n');...
fprintf('r1 = %5.2f \t',r(1)); fprintf('r2 = %5.2f \t',r(2)); fprintf('r3 = %5.2f',r(3))
```

```
r1 = -4.49   r2 = 0.49   r3 = 0.20
```

By substitution of these values into (6) we obtain

$$I_1(s) = \frac{r_1}{(s+3.817)} + \frac{r_2s+r_3}{(s^2+0.683s+0.393)} = \frac{-4.49}{(s+3.817)} + \frac{0.49s+0.20}{(s^2+0.683s+0.393)} \quad (7)$$

By inspection, the Inverse Laplace of first term on the right side of (7) is

$$\frac{0.48}{(s+3.82)} \Leftrightarrow -4.47e^{-3.82t} \quad (8)$$

The second term on the right side of (7) requires some manipulation. Therefore, we will use the MATLAB **ilaplace(s)** function to find the Inverse Laplace as shown below.

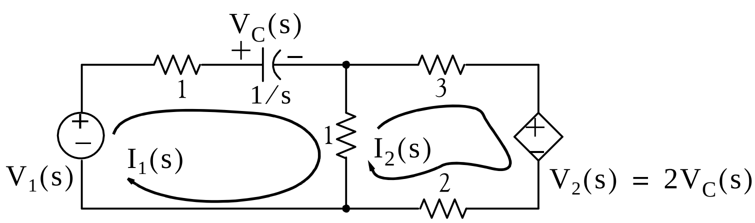
```
syms s t  % Must have Symbolic Math Toolbox installed
IL=ilaplace((0.49*s+0.20)/(s^2+0.68*s+0.39)); pretty(IL)
```

```
  167         17     1/2               1/2
 ---- exp(- -- t) 14    sin(7/50 14    t)
  9800        50

    49        17                 1/2
 + --- exp(- -- t) cos(7/50 14    t)
   100        50
```

Thus,

$$i_2(t) = -4.47e^{-3.82t} + 0.06e^{-0.34t}\sin 0.53t + 0.49e^{-0.34t}\cos 0.53t$$

4.



a. Mesh 1:

$$(2 + 1/s) \cdot I_1(s) - I_2(s) = V_1(s)$$

or

$$6(2 + 1/s) \cdot I_1(s) - 6I_2(s) = 6V_1(s) \quad (1)$$

Mesh 2:

$$-I_1(s) + 6I_2(s) = -V_2(s) = -(2/s)I_1(s) \quad (2)$$

Addition of (1) and (2) yields

$$(12 + 6/s) \cdot I_1(s) + (2/s - 1) \cdot I_1(s) = 6V_1(s)$$

or

$$(11 + 8/s) \cdot I_1(s) = 6V_1(s)$$

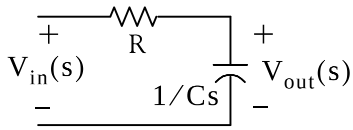
and thus

$$Y(s) = \frac{I_1(s)}{V_1(s)} = \frac{6}{11 + 8/s} = \frac{6s}{11s + 8}$$

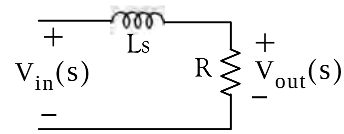
b. With $V_1(s) = 1/s$ we obtain

$$I_1(s) = Y(s) \cdot V_1(s) = \frac{6s}{11s+8} \cdot \frac{1}{s} = \frac{6}{11s+8} = \frac{6/11}{s + 8/11} \Leftrightarrow \frac{6}{11}e^{-(8/11)t} = i_1(t)$$

5.



(a)



(b)

Network (a):

$$V_{out}(s) = \frac{1/Cs}{R + 1/Cs} \cdot V_{in}(s)$$

and thus

$$G(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{1/Cs}{R + 1/Cs} = \frac{1/Cs}{(RCs + 1)/(Cs)} = \frac{1}{RCs + 1} = \frac{1/RC}{s + 1/RC}$$

Network (b):

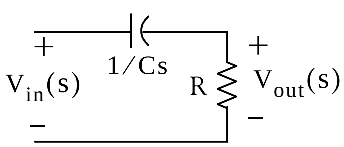
$$V_{out}(s) = \frac{R}{Ls + R} \cdot V_{in}(s)$$

and thus

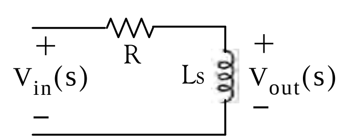
$$G(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{R}{Ls + R} = \frac{R/L}{s + R/L}$$

Both of these networks are first–order low–pass filters.

6.



(a)



(b)

Network (a):

$$V_{out}(s) = \frac{R}{1/Cs + R} \cdot V_{in}(s)$$

and

$$G(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{R}{1/Cs + R} = \frac{RCs}{(RCs + 1)} = \frac{s}{s + 1/RC}$$

Network (b):

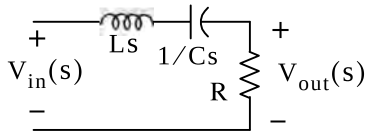
$$V_{out}(s) = \frac{Ls}{R + Ls} \cdot V_{in}(s)$$

and

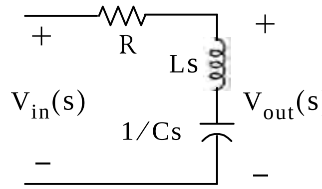
$$G(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{Ls}{R + Ls} = \frac{s}{s + R/L}$$

Both of these networks are first–order high–pass filters.

7.



(a)



(b)

Network (a):

$$V_{out}(s) = \frac{R}{Ls + 1/Cs + R} \cdot V_{in}(s)$$

and thus

$$G(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{R}{Ls + 1/Cs + R} = \frac{RCs}{LCs^2 + 1 + RCs} = \frac{(R/L)s}{s^2 + (R/L)s + 1/LC}$$

This network is a second–order band–pass filter.

Network (b):

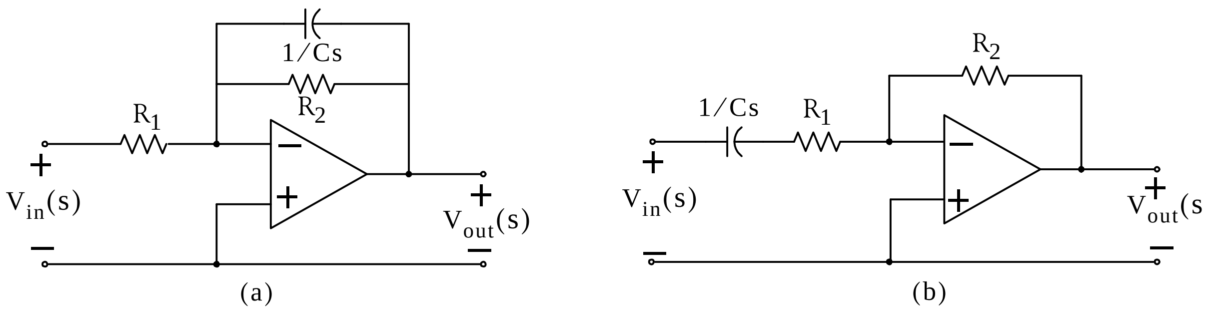
$$V_{out}(s) = \frac{Ls + 1/Cs}{R + Ls + 1/Cs} \cdot V_{in}(s)$$

and

$$G(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{Ls + 1/Cs}{R + Ls + 1/Cs} = \frac{LCs^2 + 1}{LCs^2 + RCs + 1} = \frac{s^2 + 1/LC}{s^2 + (R/L)s + 1/LC}$$

This network is a second–order band–elimination (band–reject) filter.

8.



Network (a):

Let $z_1 = R_1$ and $z_2 = R_2 \| 1/Cs = \dfrac{R_2 \times 1/Cs}{R_2 + 1/Cs}$. For inverting op amps $\dfrac{V_{out}(s)}{V_{in}(s)} = -\dfrac{z_2}{z_1}$, and thus

$$G(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{-[(R_2 \times 1/Cs)/(R_2 + 1/Cs)]}{R_1} = \frac{-(R_2 \times 1/Cs)}{R_1 \cdot (R_2 + 1/Cs)} = \frac{-R_1 C}{s + 1/R_2 C}$$

This network is a first–order active low–pass filter.

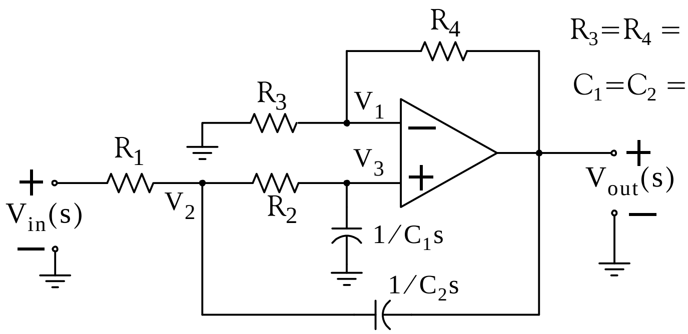
Network (b):

Let $z_1 = R_1 + 1/Cs$ and $z_2 = R_2$. For inverting op-amps $\dfrac{V_{out}(s)}{V_{in}(s)} = -\dfrac{z_2}{z_1}$, and thus

$$G(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{-R_2}{R_1 + 1/Cs} = \frac{-(R_2/R_1)s}{s + 1/R_1 C}$$

This network is a first–order active high–pass filter.

9.



At Node V_1:

$$\frac{V_1(s)}{R_3}+\frac{V_1(s)-V_{out}(s)}{R_4} = 0$$

$$\left(\frac{1}{R_3}+\frac{1}{R_4}\right)V_1(s) = \frac{1}{R_4}V_{out}(s) \quad (1)$$

At Node V_3:

$$\frac{V_3(s)-V_2(s)}{R_2}+\frac{V_3(s)}{1/C_1s} = 0$$

and since $V_3(s) \approx V_1(s)$, we express the last relation above as

$$\frac{V_1(s)-V_2(s)}{R_2}+C_1sV_1(s) = 0$$

$$\left(\frac{1}{R_2}+C_1s\right)V_1(s) = \frac{1}{R_2}V_2(s) \quad (2)$$

At Node V_2:

$$\frac{V_2(s)-V_{in}(s)}{R_1}+\frac{V_2(s)-V_1(s)}{R_2}+\frac{V_2(s)-V_{out}(s)}{1/C_2s} = 0$$

$$\left(\frac{1}{R_1}+\frac{1}{R_2}+C_2s\right)V_2(s) = \frac{V_{in}(s)}{R_1}+\frac{V_1(s)}{R_2}+C_2sV_{out}(s) \quad (3)$$

From (1)

$$V_1(s) = \frac{(1/R_4)}{(R_3+R_4)/R_3R_4}V_{out}(s) = \frac{R_3}{(R_3+R_4)}V_{out}(s) \quad (4)$$

From (2)

$$V_2(s) = R_2\left(\frac{1}{R_2}+C_1s\right)V_1(s) = (1+R_2C_1s)V_1(s)$$

and with (4)

$$V_2(s) = \frac{R_3(1+R_2C_1s)}{(R_3+R_4)}V_{out}(s) \quad (5)$$

By substitution of (4) and (5) into (3) we obtain

$$\left(\frac{1}{R_1}+\frac{1}{R_2}+C_2s\right)\frac{R_3(1+R_2C_1s)}{(R_3+R_4)}V_{out}(s) = \frac{V_{in}(s)}{R_1}+\frac{1}{R_2}\frac{R_3}{(R_3+R_4)}V_{out}(s)+C_2sV_{out}(s)$$

$$\left[\left(\frac{1}{R_1}+\frac{1}{R_2}+C_2s\right)\frac{R_3(1+R_2C_1s)}{(R_3+R_4)}-\frac{1}{R_2}\frac{R_3}{(R_3+R_4)}-C_2s\right]V_{out}(s) = \frac{1}{R_1}V_{in}(s)$$

and thus

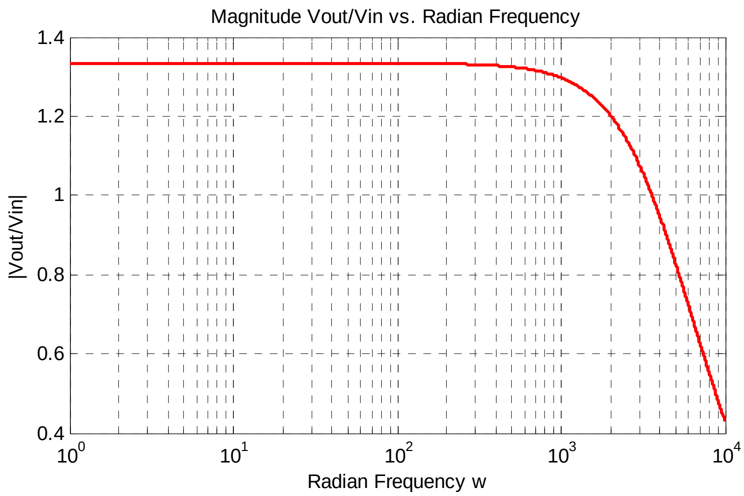
$$G(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{R_1\left[\left(\frac{1}{R_1}+\frac{1}{R_2}+C_2s\right)\frac{R_3(1+R_2C_1s)}{(R_3+R_4)}-\frac{1}{R_2}\frac{R_3}{(R_3+R_4)}-C_2s\right]}$$

By substitution of the given values and after simplification we obtain

$$G(s) = \frac{7.83\times10^7}{s^2+1.77\times10^4s+5.87\times10^7}$$

We use the MATLAB script below to plot this function.

```
w=1:10:10000; s=j.*w; Gs=7.83.*10.^7./(s.^2+1.77.*10.^4.*s+5.87.*10.^7);...
semilogx(w,abs(Gs)); xlabel('Radian Frequency w'); ylabel('|Vout/Vin|');...
title('Magnitude Vout/Vin vs. Radian Frequency'); grid
```



The plot above indicates that this circuit is a second–order low–pass filter.

Chapter 7

State Variables and State Equations

This chapter is an introduction to state variables and state equations as they apply in circuit analysis. The state transition matrix is defined, and the state–space to transfer function equivalence is presented. Several examples are presented to illustrate their application.

7.1 Expressing Differential Equations in State Equation Form

As we know, when we apply Kirchoff's Current Law (KCL) or Kirchoff's Voltage Law (KVL) in networks that contain energy–storing devices, we obtain integro–differential equations. Also, when a network contains just one such device (capacitor or inductor), it is said to be a *first–order circuit*. If it contains two such devices, it is said to be *second–order circuit*, and so on. Thus, a first order linear, time–invariant circuit can be described by a differential equation of the form

$$a_1 \frac{dy}{dt} + a_0\, y(t) = x(t) \tag{7.1}$$

A second order circuit can be described by a second–order differential equation of the same form as (7.1) where the highest order is a second derivative.

An *nth–order* differential equation can be resolved to n first–order simultaneous differential equations with a set of auxiliary variables called *state variables*. The resulting first–order differential equations are called *state–space equations*, or simply *state equations*. These equations can be obtained either from the nth–order differential equation, or directly from the network, provided that the state variables are chosen appropriately. The state variable method offers the advantage that it can also be used with non–linear and time–varying devices. However, our discussion will be limited to linear, time–invariant circuits.

State equations can also be solved with numerical methods such as Taylor series and Runge–Kutta methods, but these will not be discussed in this text*. The state variable method is best illustrated with several examples presented in this chapter.

Example 7.1

A series RLC circuit with excitation

$$v_S(t) = e^{j\omega t} \tag{7.2}$$

* *These are discussed in Numerical Analysis using MATLAB and Excel, ISBN 978–1–934404–03–4.*

is described by the integro–differential equation

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{t} i\,dt = e^{j\omega t} \tag{7.3}$$

Differentiating both sides and dividing by L we obtain

$$\frac{d^2t}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i = \frac{1}{L}j\omega e^{j\omega t} \tag{7.4}$$

or

$$\frac{d^2t}{dt^2} = -\frac{R}{L}\frac{di}{dt} - \frac{1}{LC}i + \frac{1}{L}j\omega e^{j\omega t} \tag{7.5}$$

Next, we define two state variables x_1 and x_2 such that

$$x_1 = i \tag{7.6}$$

and

$$x_2 = \frac{di}{dt} = \frac{dx_1}{dt} = \dot{x}_1 \tag{7.7}$$

Then,

$$\dot{x}_2 = d^2i/dt^2 \tag{7.8}$$

where $\dot{x}_k$ denotes the derivative of the state variable x_k. From (7.5) through (7.8), we obtain the state equations

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{R}{L}x_2 - \frac{1}{LC}x_1 + \frac{1}{L}j\omega e^{j\omega t} \end{aligned} \tag{7.9}$$

It is convenient and customary to express the state equations in matrix* form. Thus, we write the state equations of (7.9) as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L}j\omega e^{j\omega t} \end{bmatrix} u \tag{7.10}$$

We usually express (7.10) in a compact form as

$$\dot{x} = Ax + bu \tag{7.11}$$

where u† is any input

* *For a review of matrix theory, please refer to Appendix E.*

† *In this text, and in all Orchard Publications texts, the unit step function is denoted as* u_0.

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ \frac{1}{L} j\omega e^{j\omega t} \end{bmatrix}, \text{ and } u = \text{any input} \tag{7.12}$$

The output $y(t)$ is expressed by the state equation

$$y = Cx + du \tag{7.13}$$

where C is another matrix, and d is a column vector.

In general, the state representation of a network can be described by the pair of the of the state–space equations

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{Ax + bu} \\ \mathbf{y} &= \mathbf{Cx + du} \end{aligned} \tag{7.14}$$

The state space equations of (7.14) can be realized with the block diagram of Figure 7.1.

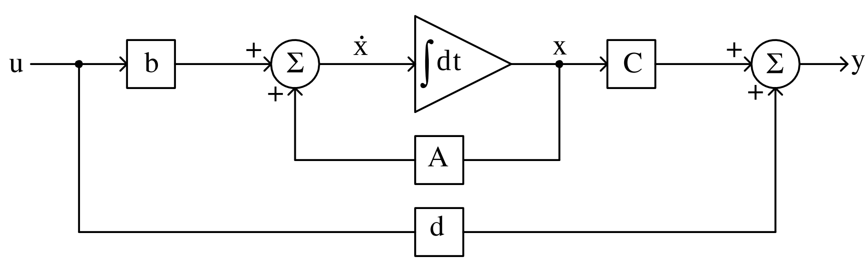


Figure 7.1. Block diagram for the realization of the state equations of (7.14)

We will learn how to solve the matrix equations of (7.14) in the subsequent sections.

Example 7.2

A fourth–order network is described by the differential equation

$$\frac{d^4y}{dt^4} + a_3\frac{d^3y}{dt^3} + a_2\frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_0\,y(t) = u(t) \tag{7.15}$$

where $y(t)$ is the output representing the voltage or current of the network, and $u(t)$ is any input. Express (7.15) as a set of state equations.

Solution:

The differential equation of (7.15) is of fourth–order; therefore, we must define four state variables which will be used with the resulting four first–order state equations.

We denote the state variables as x_1, x_2, x_3, and x_4, and we relate them to the terms of the given differential equation as

$$x_1 = y(t) \qquad x_2 = \frac{dy}{dt} \qquad x_3 = \frac{d^2y}{dt^2} \qquad x_4 = \frac{d^3y}{dt^3} \tag{7.16}$$

We observe that

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= x_4 \\ \frac{d^4y}{dt^4} = \dot{x}_4 &= -a_0x_1 - a_1x_2 - a_2x_3 - a_3x_4 + u(t) \end{aligned} \tag{7.17}$$

and in matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t) \backslash \tag{7.18}$$

In compact form, (7.18) is written as

$$\dot{x} = Ax + bu \tag{7.19}$$

where

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \text{and } u = u(t)$$

We can also obtain the state equations directly from given circuits. We choose the state variables to represent inductor currents and capacitor voltages. In other words, we assign state variables to energy storing devices. The examples below illustrate the procedure.

Example 7.3

Write state equation(s) for the circuit of Figure 7.2, given that $v_C(0^-) = 0$, and $u_0(t)$ is the unit step function.

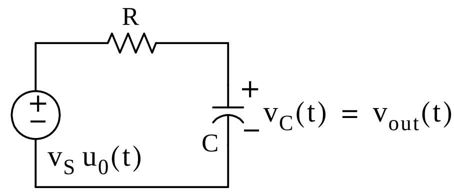


Figure 7.2. Circuit for Example 7.3

Solution:

This circuit contains only one energy–storing device, the capacitor. Therefore, we need only one state variable. We choose the state variable to denote the voltage across the capacitor as shown in Figure 7.3. For this example, the output is defined as the voltage across the capacitor.

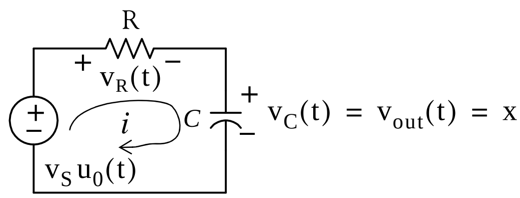


Figure 7.3. Circuit for Example 7.3 with state variable x assigned to it

For this circuit,

$$i_R = i = i_C = C\frac{dv_C}{dt} = C\dot{x}$$

and

$$v_R(t) = Ri = RC\dot{x}$$

By KVL,

$$v_R(t) + v_C(t) = v_S u_0(t)$$

or

$$RC\dot{x} + x = v_S u_0(t)$$

Therefore, the state equations are

$$\begin{aligned} \dot{x} &= -\frac{1}{RC}x + v_S u_0(t) \\ y &= x \end{aligned} \tag{7.20}$$

Example 7.4

Write state equation(s) for the circuit of Figure 7.4 assuming $i_L(0^-) = 0$, and the output y is defined as $y = i(t)$.

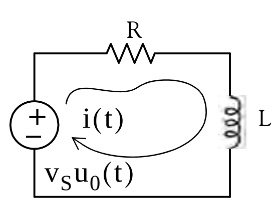


Figure 7.4. Circuit for Example 7.4

Solution:

This circuit contains only one energy–storing device, the inductor; therefore, we need only one state variable. We choose the state variable to denote the current through the inductor as shown in Figure 7.5.

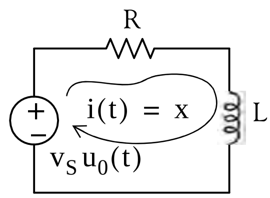


Figure 7.5. Circuit for Example 7.4 with assigned state variable x

By KVL,

$$v_R + v_L = v_S u_0(t)$$

or

$$Ri + L\frac{di}{dt} = v_S u_0(t)$$

or

$$Rx + L\dot{x} = v_S u_0(t)$$

Therefore, the state equations are

$$\begin{aligned} \dot{x} &= -\frac{R}{L}x + \frac{1}{L}v_S u_0(t) \\ y &= x \end{aligned} \tag{7.21}$$

7.2 Solution of Single State Equations

If a circuit contains only one energy–storing device, the state equations are written as

$$\begin{aligned} \dot{x} &= \alpha x + \beta u \\ y &= k_1 x + k_2 u \end{aligned} \tag{7.22}$$

where α, β, k_1, and k_2 are scalar constants, and the initial condition, if non–zero, is denoted as

$$x_0 = x(t_0) \tag{7.23}$$

We will now prove that the solution of the first state equation in (7.22) is

$$x(t) = e^{\alpha(t-t_0)}x_0 + e^{\alpha t}\int_{t_0}^{t} e^{-\alpha\tau}\beta u(\tau)d\tau \tag{7.24}$$

Proof:

First, we must show that (7.24) satisfies the initial condition of (7.23). This is done by substitution of $t = t_0$ in (7.24). Then,

$$x(t_0) = e^{\alpha(t_0 - t_0)}x_0 + e^{\alpha t}\int_{t_0}^{t_0} e^{-\alpha\tau}\beta u(\tau)d\tau \tag{7.25}$$

The first term in the right side of (7.25) reduces to x_0 since

$$e^{\alpha(t_0 - t_0)}x_0 = e^0 x_0 = x_0 \tag{7.26}$$

The second term of (7.25) is zero since the upper and lower limits of integration are the same. Therefore, (7.25) reduces to $x(t_0) = x_0$ and thus the initial condition is satisfied.

Next, we must prove that (7.24) satisfies also the first equation in (7.22). To prove this, we differentiate (7.24) with respect to t and we obtain

$$\dot{x}(t) = \frac{d}{dt}(e^{\alpha(t - t_0)}x_0) + \frac{d}{dt}\left\{e^{\alpha t}\int_{t_0}^{t} e^{-\alpha\tau}\beta u(\tau)d\tau\right\}$$

or

$$\dot{x}(t) = \alpha e^{\alpha(t - t_0)}x_0 + \alpha e^{\alpha t}\int_{t_0}^{t} e^{-\alpha\tau}\beta u(\tau)d\tau + e^{\alpha t}[e^{-\alpha\tau}\beta u(\tau)]\Big|_{\tau = t}$$

$$= \alpha\left[e^{\alpha(t - t_0)}x_0 + e^{\alpha t}\int_{t_0}^{t} e^{-\alpha\tau}\beta u(\tau)d\tau\right] + e^{\alpha t}e^{-\alpha t}\beta u(t)$$

or

$$\dot{x}(t) = \alpha\left[e^{\alpha(t - t_0)}x_0 + \int_{t_0}^{t} e^{\alpha(t - \tau)}\beta u(\tau)d\tau\right] + \beta u(t) \tag{7.27}$$

We observe that the bracketed terms of (7.27) are the same as the right side of the assumed solution of (7.24). Therefore,

$$\dot{x} = \alpha x + \beta u$$

and this is the same as the first equation of (7.22).

In summary, if α and β are scalar constants, the solution of

$$\dot{x} = \alpha x + \beta u \tag{7.28}$$

with initial condition

$$x_0 = x(t_0) \tag{7.29}$$

is obtained from the relation

$$x(t) = e^{\alpha(t - t_0)}x_0 + e^{\alpha t}\int_{t_0}^{t} e^{-\alpha\tau}\beta u(\tau)d\tau \tag{7.30}$$

Example 7.5

Use (7.28) through (7.30) to find the capacitor voltage $v_C(t)$ of the circuit of Figure 7.6 for $t > 0$, given that the initial condition is $v_C(0^-) = 1\ V$

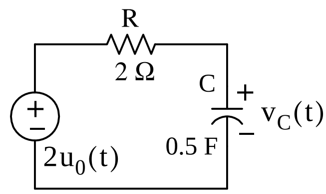


Figure 7.6. Circuit for Example 7.5

Solution:

From (7.20) of Example 7.3, Page 7–5,

$$\dot{x} = -\frac{1}{RC}x + v_S u_0(t)$$

and by comparison with (7.28),

$$\alpha = -\frac{1}{RC} = \frac{-1}{2 \times 0.5} = -1$$

and

$$\beta = 2$$

Then, from (7.30),

$$x(t) = e^{\alpha(t-t_0)}x_0 + e^{\alpha t}\int_{t_0}^{t} e^{-\alpha\tau}\beta u(\tau)d\tau = e^{-1(t-0)}1 + e^{-t}\int_0^t e^{\tau}2u(\tau)d\tau$$

$$= e^{-t} + 2e^{-t}\int_0^t e^{\tau}d\tau = e^{-t} + 2e^{-t}[e^{\tau}]\Big|_0^t = e^{-t} + 2e^{-t}(e^t - 1)$$

or

$$v_C(t) = x(t) = (2 - e^{-t})u_0(t) \tag{7.31}$$

Assuming that the output y is the capacitor voltage, the output state equation is

$$y(t) = x(t) = (2 - e^{-t})u_0(t) \tag{7.32}$$

7.3 The State Transition Matrix

In Section 7.1, relation (7.14), we defined the state equations pair

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= Cx + du \end{aligned} \tag{7.33}$$

where for two or more simultaneous differential equations, A and C are 2×2 or higher order matrices, and b and d are column vectors with two or more rows. In this section we will introduce the *state transition matrix* e^{At}, and we will prove that the solution of the matrix differential equation

$$\dot{x} = Ax + bu \tag{7.34}$$

with initial conditions

$$x(t_0) = x_0 \tag{7.35}$$

is obtained from the relation

$$x(t) = e^{A(t-t_0)}x_0 + e^{At}\int_{t_0}^{t} e^{-A\tau}bu(\tau)d\tau \tag{7.36}$$

Proof:

Let A be any $n \times n$ matrix whose elements are constants. Then, another $n \times n$ matrix denoted as $\varphi(t)$, is said to be the state transition matrix of (7.34), if it is related to the matrix A as the matrix power series

$$\boxed{\varphi(t) \equiv e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \ldots + \frac{1}{n!}A^nt^n} \tag{7.37}$$

where I is the $n \times n$ identity matrix.

From (7.37), we find that

$$\varphi(0) = e^{A0} = I + A0 + \ldots = I \tag{7.38}$$

Differentiation of (7.37) with respect to t yields

$$\varphi'(t) = \frac{d}{dt}e^{At} = 0 + A \cdot 1 + A^2t + \ldots = A + A^2t + \ldots \tag{7.39}$$

and by comparison with (7.37) we obtain

$$\frac{d}{dt}e^{At} = Ae^{At} \tag{7.40}$$

To prove that (7.36) is the solution of (7.34), we must prove that it satisfies both the initial condition and the matrix differential equation. The initial condition is satisfied from the relation

$$x(t_0) = e^{A(t_0-t_0)}x_0 + e^{At_0}\int_{t_0}^{t_0} e^{-A\tau}bu(\tau)d\tau = e^{A0}x_0 + 0 = Ix_0 = x_0 \tag{7.41}$$

where we have used (7.38) for the initial condition. The integral is zero since the upper and lower limits of integration are the same.

To prove that (7.34) is also satisfied, we differentiate the assumed solution

$$x(t) = e^{A(t-t_0)}x_0 + e^{At}\int_{t_0}^{t} e^{-A\tau}bu(\tau)d\tau$$

with respect to t and we use (7.40), that is,

$$\frac{d}{dt}e^{At} = Ae^{At}$$

Then,

$$\dot{x}(t) = Ae^{A(t-t_0)}x_0 + Ae^{At}\int_{t_0}^{t} e^{-A\tau}bu(\tau)d\tau + e^{At}e^{-At}bu(t)$$

or

$$\dot{x}(t) = A\left[e^{A(t-t_0)}x_0 + e^{At}\int_{t_0}^{t} e^{-A\tau}bu(\tau)d\tau\right] + e^{At}e^{-At}bu(t) \quad (7.42)$$

We recognize the bracketed terms in (7.42) as $x(t)$, and the last term as $bu(t)$. Thus, the expression (7.42) reduces to

$$\dot{x}(t) = Ax + bu$$

In summary, if A is an $n \times n$ matrix whose elements are constants, $n \ge 2$, and b is a column vector with *n* elements, the solution of

$$\dot{x}(t) = Ax + bu \quad (7.43)$$

with initial condition

$$x_0 = x(t_0) \quad (7.44)$$

is

$$\boxed{x(t) = e^{A(t-t_0)}x_0 + e^{At}\int_{t_0}^{t} e^{-A\tau}bu(\tau)d\tau} \quad (7.45)$$

Therefore, the solution of second or higher order circuits using the state variable method, entails the computation of the state transition matrix e^{At}, and integration of (7.45).

7.4 Computation of the State Transition Matrix e^{At}

Let A be an $n \times n$ matrix, and I be the $n \times n$ identity matrix. By definition, the *eigenvalues* λ_i, $i = 1, 2, \ldots, n$ of A are the roots of the nth order polynomial

$$\boxed{\det[A - \lambda I] = 0} \quad (7.46)$$

We recall that expansion of a determinant produces a polynomial. The roots of the polynomial of (7.46) can be real (unequal or equal), or complex numbers.

Evaluation of the state transition matrix e^{At} is based on the *Cayley–Hamilton theorem*. This theorem states that a matrix can be expressed as an $(n-1)$th degree polynomial in terms of the matrix A as

$$\boxed{e^{At} = a_0 I + a_1 A + a_2 A^2 + \ldots + a_{n-1} A^{n-1}} \tag{7.47}$$

where the coefficients a_i are functions of the eigenvalues λ.

We accept (7.47) without proving it. The proof can be found in Linear Algebra and Matrix Theory textbooks.

Since the coefficients a_i are functions of the eigenvalues λ, we must consider the two cases discussed in Subsections 7.4.1 and 7.4.2 below.

7.4.1 Distinct Eigenvalues (Real of Complex)

If $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \ldots \neq \lambda_n$, that is, if all eigenvalues of a given matrix A are distinct, the coefficients a_i are found from the simultaneous solution of the following system of equations:

$$\boxed{\begin{aligned} a_0 + a_1\lambda_1 + a_2\lambda_1^2 + \ldots + a_{n-1}\lambda_1^{n-1} &= e^{\lambda_1 t} \\ a_0 + a_1\lambda_2 + a_2\lambda_2^2 + \ldots + a_{n-1}\lambda_2^{n-1} &= e^{\lambda_2 t} \\ &\ldots \\ a_0 + a_1\lambda_n + a_2\lambda_n^2 + \ldots + a_{n-1}\lambda_n^{n-1} &= e^{\lambda_n t} \end{aligned}} \tag{7.48}$$

Example 7.6

Compute the state transition matrix e^{At} given that

$$A = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}$$

Solution:

We must first find the eigenvalues λ of the given matrix A. These are found from the expansion of

$$\det[A - \lambda I] = 0$$

For this example,

$$\det[A-\lambda I] = \det\left\{\begin{bmatrix}-2 & 1\\ 0 & -1\end{bmatrix} - \lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right\} = \det\begin{bmatrix}-2-\lambda & 1\\ 0 & -1-\lambda\end{bmatrix} = 0$$

$$= (-2-\lambda)(-1-\lambda) = 0$$

or

$$(\lambda+1)(\lambda+2) = 0$$

Therefore,

$$\lambda_1 = -1 \text{ and } \lambda_2 = -2 \tag{7.49}$$

Next, we must find the coefficients a_i of (7.47). Since A is a 2×2 matrix, we only need to consider the first two terms of that relation, that is,

$$e^{At} = a_0 I + a_1 A \tag{7.50}$$

The coefficients a_0 and a_1 are found from (7.48). For this example,

$$a_0 + a_1\lambda_1 = e^{\lambda_1 t}$$

$$a_0 + a_1\lambda_2 = e^{\lambda_2 t}$$

or

$$\begin{aligned} a_0 + a_1(-1) &= e^{-t} \\ a_0 + a_1(-2) &= e^{-2t} \end{aligned} \tag{7.51}$$

Simultaneous solution of (7.51) yields

$$\begin{aligned} a_0 &= 2e^{-t} - e^{-2t} \\ a_1 &= e^{-t} - e^{-2t} \end{aligned} \tag{7.52}$$

and by substitution into (7.50),

$$e^{At} = (2e^{-t} - e^{-2t})\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + (e^{-t} - e^{-2t})\begin{bmatrix}-2 & 1\\ 0 & -1\end{bmatrix}$$

or

$$e^{At} = \begin{bmatrix}e^{-2t} & e^{-t} - e^{-2t}\\ 0 & e^{-t}\end{bmatrix} \tag{7.53}$$

In summary, we compute the state transition matrix e^{At} for a given matrix A using the following procedure:

1. We find the eigenvalues λ from $\det[A-\lambda I] = 0$. We can write $[A-\lambda I]$ at once by subtracting λ from each of the main diagonal elements of A. If the dimension of A is a 2×2 matrix, it will yield two eigenvalues; if it is a 3×3 matrix, it will yield three eigenvalues, and so on. If the eigenvalues are distinct, we perform steps 2 through 4; otherwise we refer to Subsection 7.4.2 below.

2. If the dimension of A is a 2×2 matrix, we use only the first *2* terms of the right side of the state transition matrix

$$e^{At} = a_0 I + a_1 A + a_2 A^2 + \dots + a_{n-1}A^{n-1} \tag{7.54}$$

   If A matrix is a 3×3 matrix, we use the first 3 terms of (7.54), and so on.

3. We obtain the a_i coefficients from

$$\begin{aligned} a_0 + a_1\lambda_1 + a_2\lambda_1^2 + \dots + a_{n-1}\lambda_1^{n-1} &= e^{\lambda_1 t} \\ a_0 + a_1\lambda_2 + a_2\lambda_2^2 + \dots + a_{n-1}\lambda_2^{n-1} &= e^{\lambda_2 t} \\ &\dots \\ a_0 + a_1\lambda_n + a_2\lambda_n^2 + \dots + a_{n-1}\lambda_n^{n-1} &= e^{\lambda_n t} \end{aligned}$$

   We use as many equations as the number of the eigenvalues, and we solve for the coefficients a_i.

4. We substitute the a_i coefficients into the state transition matrix of (7.54), and we simplify.

Example 7.7

Compute the state transition matrix e^{At} given that

$$A = \begin{bmatrix} 5 & 7 & -5 \\ 0 & 4 & -1 \\ 2 & 8 & -3 \end{bmatrix} \tag{7.55}$$

Solution:

1. We first compute the eigenvalues from $\det[A-\lambda I] = 0$. We obtain $[A-\lambda I]$ at once, by subtracting λ from each of the main diagonal elements of A. Then,

$$\det[A-\lambda I] = \det\begin{bmatrix} 5-\lambda & 7 & -5 \\ 0 & 4-\lambda & -1 \\ 2 & 8 & -3-\lambda \end{bmatrix} = 0 \quad (7.56)$$

and expansion of this determinant yields the polynomial

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0 \quad (7.57)$$

We will use MATLAB **roots(p)** function to obtain the roots of (7.57).

```
p=[1 -6 11 -6]; r=roots(p); fprintf(' \n'); fprintf('lambda1 = %5.2f \t', r(1));...
fprintf('lambda2 = %5.2f \t', r(2)); fprintf('lambda3 = %5.2f', r(3))
```

```
lambda1 = 3.00    lambda2 = 2.00    lambda3 = 1.00
```

and thus the eigenvalues are

$$\lambda_1 = 1 \qquad \lambda_2 = 2 \qquad \lambda_3 = 3 \quad (7.58)$$

2. Since A is a 3×3 matrix, we use the first 3 terms of (7.54), that is,

$$e^{At} = a_0 I + a_1 A + a_2 A^2 \quad (7.59)$$

3. We obtain the coefficients a_0, a_1, and a_2 from

$$a_0 + a_1\lambda_1 + a_2\lambda_1^2 = e^{\lambda_1 t}$$
$$a_0 + a_1\lambda_2 + a_2\lambda_2^2 = e^{\lambda_2 t}$$
$$a_0 + a_1\lambda_3 + a_2\lambda_3^2 = e^{\lambda_3 t}$$

or

$$\begin{aligned} a_0 + a_1 + a_2 &= e^t \\ a_0 + 2a_1 + 4a_2 &= e^{2t} \\ a_0 + 3a_1 + 9a_2 &= e^{3t} \end{aligned} \quad (7.60)$$

We will use the following MATLAB script for the solution of (7.60).

```
B=sym('[1 1 1; 1 2 4; 1 3 9]'); b=sym('[exp(t); exp(2*t); exp(3*t)]'); a=B\b; fprintf(' \n');...
disp('a0 = '); disp(a(1)); disp('a1 = '); disp(a(2)); disp('a2 = '); disp(a(3))
```

```
a0 =
3*exp(t)-3*exp(2*t)+exp(3*t)
a1 =
-5/2*exp(t)+4*exp(2*t)-3/2*exp(3*t)
a2 =
```

```
1/2*exp(t)-exp(2*t)+1/2*exp(3*t)
```

Thus,

$$a_0 = 3e^t - 3e^{2t} + e^{3t}$$
$$a_1 = -\frac{5}{2}e^t + 4e^{2t} - \frac{3}{2}e^{3t} \qquad (7.61)$$
$$a_2 = \frac{1}{2}e^t - e^{2t} + \frac{1}{2}e^{3t}$$

4. We also use MATLAB to perform the substitution into the state transition matrix, and to perform the matrix multiplications. The script is shown below.

```
syms t; a0 = 3*exp(t)+exp(3*t)–3*exp(2*t); a1 = –5/2*exp(t)–3/2*exp(3*t)+4*exp(2*t);...
a2 = 1/2*exp(t)+1/2*exp(3*t)–exp(2*t);...
A = [5 7 –5; 0 4 –1; 2 8 -3]; eAt=a0*eye(3)+a1*A+a2*A^2
```

```
eAt =
[-2*exp(t)+2*exp(2*t)+exp(3*t),  -6*exp(t)+5*exp(2*t)+exp(3*t),
4*exp(t)-3*exp(2*t)-exp(3*t)]
[-exp(t)+2*exp(2*t)-exp(3*t),  -3*exp(t)+5*exp(2*t)-exp(3*t),
2*exp(t)-3*exp(2*t)+exp(3*t)]
[-3*exp(t)+4*exp(2*t)-exp(3*t), -9*exp(t)+10*exp(2*t)-exp(3*t),
6*exp(t)-6*exp(2*t)+exp(3*t)]
```

Thus,

$$e^{At} = \begin{bmatrix} -2e^t + 2e^{2t} + e^{3t} & -6e^t + 5e^{2t} + e^{3t} & 4e^t - 3e^{2t} - e^{3t} \\ -e^t + 2e^{2t} - e^{3t} & -3e^t + 5e^{2t} - e^{3t} & 2e^t - 3e^{2t} + e^{3t} \\ -3e^t + 4e^{2t} - e^{3t} & -9e^t + 10e^{2t} - e^{3t} & 6e^t - 6e^{2t} + e^{3t} \end{bmatrix}$$

7.4.2 Multiple (Repeated) Eigenvalues

In this case, we will assume that the polynomial of

$$\det[A - \lambda I] = 0 \qquad (7.62)$$

has n roots, and m of these roots are equal. In other words, the roots are

$$\lambda_1 = \lambda_2 = \lambda_3 \ldots = \lambda_m,\ \lambda_{m+1},\ \lambda_n \qquad (7.63)$$

The coefficients a_i of the state transition matrix

$$e^{At} = a_0 I + a_1 A + a_2 A^2 + \ldots + a_{n-1} A^{n-1} \qquad (7.64)$$

are found from the simultaneous solution of the system of equations of (7.65) below.

$$
\begin{aligned}
a_0 + a_1\lambda_1 + a_2\lambda_1^2 + \ldots + a_{n-1}\lambda_1^{n-1} &= e^{\lambda_1 t} \\
\frac{d}{d\lambda_1}(a_0 + a_1\lambda_1 + a_2\lambda_1^2 + \ldots + a_{n-1}\lambda_1^{n-1}) &= \frac{d}{d\lambda_1}e^{\lambda_1 t} \\
\frac{d^2}{d\lambda_1^2}(a_0 + a_1\lambda_1 + a_2\lambda_1^2 + \ldots + a_{n-1}\lambda_1^{n-1}) &= \frac{d^2}{d\lambda_1^2}e^{\lambda_1 t} \\
&\ldots \\
\frac{d^{m-1}}{d\lambda_1^{m-1}}(a_0 + a_1\lambda_1 + a_2\lambda_1^2 + \ldots + a_{n-1}\lambda_1^{n-1}) &= \frac{d^{m-1}}{d\lambda_1^{m-1}}e^{\lambda_1 t} \\
a_0 + a_1\lambda_{m+1} + a_2\lambda_{m+1}^2 + \ldots + a_{n-1}\lambda_{m+1}^{n-1} &= e^{\lambda_{m+1} t} \\
&\ldots \\
a_0 + a_1\lambda_n + a_2\lambda_n^2 + \ldots + a_{n-1}\lambda_n^{n-1} &= e^{\lambda_n t}
\end{aligned}
\tag{7.65}
$$

Example 7.8

Compute the state transition matrix e^{At} given that

$$A = \begin{bmatrix} -1 & 0 \\ 2 & -1 \end{bmatrix}$$

Solution:

1. We first find the eigenvalues λ of the matrix A and these are found from the polynomial of $\det[A - \lambda I] = 0$. For this example,

$$\det[A - \lambda I] = \det\begin{bmatrix} -1-\lambda & 0 \\ 2 & -1-\lambda \end{bmatrix} = 0 \qquad (-1-\lambda)(-1-\lambda) = 0 \qquad (\lambda+1)^2 = 0$$

and thus,

$$\lambda_1 = \lambda_2 = -1$$

2. Since A is a 2×2 matrix, we only need the first two terms of the state transition matrix, that is,

$$e^{At} = a_0 I + a_1 A \tag{7.66}$$

3. We find a_0 and a_1 from (7.65). For this example,

$$a_0 + a_1\lambda_1 = e^{\lambda_1 t}$$

$$\frac{d}{d\lambda_1}(a_0 + a_1\lambda_1) = \frac{d}{d\lambda_1}e^{\lambda_1 t}$$

or

$$a_0 + a_1\lambda_1 = e^{\lambda_1 t}$$

$$a_1 = te^{\lambda_1 t}$$

and by substitution with $\lambda_1 = \lambda_2 = -1$, we obtain

$$a_0 - a_1 = e^{-t}$$

$$a_1 = te^{-t}$$

Simultaneous solution of the last two equations yields

$$\begin{aligned} a_0 &= e^{-t} + te^{-t} \\ a_1 &= te^{-t} \end{aligned} \qquad (7.67)$$

4. By substitution of (7.67) into (7.66), we obtain

$$e^{At} = (e^{-t} + te^{-t})\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + te^{-t}\begin{bmatrix} -1 & 0 \\ 2 & -1 \end{bmatrix}$$

or

$$e^{At} = \begin{bmatrix} e^{-t} & 0 \\ 2te^{-t} & e^{-t} \end{bmatrix} \qquad (7.68)$$

We can use the MATLAB **eig(x)** function to find the eigenvalues of an $n \times n$ matrix. To find out how it is used, we invoke the **help eig** command.

We will first use MATLAB to verify the values of the eigenvalues found in Examples 7.6 through 7.8, and we will briefly discuss eigenvectors in the next section.

Example 7.6:

A= [−2 1; 0 −1]; lambda=eig(A)

```
lambda =
    -2
    -1
```

Example 7.7:

B = [5 7 –5; 0 4 –1; 2 8 –3]; lambda=eig(B)

```
lambda =
    1.0000
    3.0000
    2.0000
```

Example 7.8:

C = [–1 0; 2 –1]; lambda=eig(C)

```
lambda =
    -1
    -1
```

7.5 Eigenvectors

Consider the relation

$$AX = \lambda X \tag{7.69}$$

where A is an $n \times n$ matrix, X is a column vector, and λ is a scalar number. We can express this relation in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} \tag{7.70}$$

We express (7.70) as

$$(A-\lambda I)X = 0 \tag{7.71}$$

Then, (7.71) can be written as

$$\begin{bmatrix} (a_{11}-\lambda)x_1 & a_{12}x_2 & \dots & a_{1n}x_n \\ a_{21}x_1 & (a_{22}-\lambda)x_2 & \dots & a_{2n}x_n \\ \dots & \dots & \dots & \dots \\ a_{n1}x_1 & a_{n2}x_2 & \dots & (a_{nn}-\lambda)x_n \end{bmatrix} = 0 \tag{7.72}$$

The equations of (7.72) will have non–trivial solutions if and only if its determinant is zero*, that is, if

* *This is because we want the vector X in (7.71) to be a non-zero vector and the product* $(A-\lambda I)X$ *to be zero.*

$$\det\begin{bmatrix} (a_{11}-\lambda) & a_{12} & \dots & a_{1n} \\ a_{21} & (a_{22}-\lambda) & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & (a_{nn}-\lambda) \end{bmatrix} = 0 \qquad (7.73)$$

Expansion of the determinant of (7.73) results in a polynomial equation of degree n in λ, and it is called the *characteristic equation*.

We can express (7.73) in a compact form as

$$\det(A-\lambda I) = 0 \qquad (7.74)$$

As we know, the roots λ of the characteristic equation are the eigenvalues of the matrix A, and corresponding to each eigenvalue λ, there is a non-trivial solution of the column vector X, i.e., $X \neq 0$. This vector X is called *eigenvector*. Obviously, there is a different eigenvector for each eigenvalue. Eigenvectors are generally expressed as *unit eigenvectors*, that is, they are normalized to unit length. This is done by dividing each component of the eigenvector by the square root of the sum of the squares of their components, so that the sum of the squares of their components is equal to unity.

In many engineering applications the unit eigenvectors are chosen such that $X \cdot X^T = I$ where X^T is the transpose of the eigenvector X, and I is the identity matrix.

Two vectors X and Y are said to be *orthogonal* if their inner (dot) product is zero. A set of eigenvectors constitutes an *orthonormal basis* if the set is normalized (expressed as unit eigenvectors) and these vector are mutually orthogonal. An orthonormal basis can be formed with the *Gram-Schmidt Orthogonalization Procedure*; it is beyond the scope of this chapter to discuss this procedure, and therefore it will not be discussed in this text. It can be found in Linear Algebra and Matrix Theory textbooks.

The example below illustrates the relationships between a matrix A, its eigenvalues, and eigenvectors.

Example 7.9

Given the matrix

$$A = \begin{bmatrix} 5 & 7 & -5 \\ 0 & 4 & -1 \\ 2 & 8 & -3 \end{bmatrix}$$

a. Find the eigenvalues of A

b. Find eigenvectors corresponding to each eigenvalue of A

c. Form a set of unit eigenvectors using the eigenvectors of part (b).

Solution:

a. This is the same matrix as in Example 7.7, relation (7.55), Page 7−14, where we found the eigenvalues to be

$$\lambda_1 = 1 \qquad \lambda_2 = 2 \qquad \lambda_3 = 3$$

b. We begin with

$$AX = \lambda X$$

and we let

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Then,

$$\begin{bmatrix} 5 & 7 & -5 \\ 0 & 4 & -1 \\ 2 & 8 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lambda\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{7.75}$$

or

$$\begin{bmatrix} 5x_1 & 7x_2 & -5x_3 \\ 0 & 4x_2 & -x_3 \\ 2x_1 & 8x_2 & -3x_3 \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \\ \lambda x_3 \end{bmatrix} \tag{7.76}$$

Equating corresponding rows and rearranging, we obtain

$$\begin{bmatrix} (5-\lambda)x_1 & 7x_2 & -5x_3 \\ 0 & (4-\lambda)x_2 & -x_3 \\ 2x_1 & 8x_2 & -(3-\lambda)x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{7.77}$$

For $\lambda = 1$, (7.77) reduces to

$$\begin{aligned} 4x_1 + 7x_2 - 5x_3 &= 0 \\ 3x_2 - x_3 &= 0 \\ 2x_1 + 8x_2 - 4x_3 &= 0 \end{aligned} \tag{7.78}$$

By Crame’s rule, or MATLAB, we obtain the indeterminate values

$$x_1 = 0/0 \qquad x_2 = 0/0 \qquad x_3 = 0/0 \tag{7.79}$$

Since the unknowns $x_1, x_2,$ and x_3 are scalars, we can assume that one of these, say x_2, is known, and solve x_1 and x_3 in terms of x_2. Then, we obtain $x_1 = 2x_2$, and $x_3 = 3x_2$. Therefore, an eigenvector for $\lambda = 1$ is

$$X_{\lambda=1} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \\ 3x_2 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \tag{7.80}$$

since any eigenvector is a scalar multiple of the last vector in (7.80).

Similarly, for $\lambda = 2$, we obtain $x_1 = x_2$, and $x_3 = 2x_2$. Then, an eigenvector for $\lambda = 2$ is

$$X_{\lambda=2} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \\ 2x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \tag{7.81}$$

Finally, for $\lambda = 3$, we obtain $x_1 = -x_2$, and $x_3 = x_2$. Then, an eigenvector for $\lambda = 3$ is

$$X_{\lambda=3} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \tag{7.82}$$

c. We find the unit eigenvectors by dividing the components of each vector by the square root of the sum of the squares of the components. These are:

$$\sqrt{2^2 + 1^2 + 3^2} = \sqrt{14}$$

$$\sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}$$

$$\sqrt{(-1)^2 + 1^2 + 1^2} = \sqrt{3}$$

The unit eigenvectors are

$$\text{Unit } X_{\lambda=1} = \begin{bmatrix} \frac{2}{\sqrt{14}} \\ \frac{1}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \end{bmatrix} \qquad \text{Unit } X_{\lambda=2} = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix} \qquad \text{Unit } X_{\lambda=3} = \begin{bmatrix} \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \tag{7.83}$$

We observe that for the first unit eigenvector the sum of the squares is unity, that is,

$$\left(\frac{2}{\sqrt{14}}\right)^2+\left(\frac{1}{\sqrt{14}}\right)^2+\left(\frac{3}{\sqrt{14}}\right)^2 = \frac{4}{14}+\frac{1}{14}+\frac{9}{14} = 1 \qquad (7.84)$$

and the same is true for the other two unit eigenvectors in (7.83).

7.6 Circuit Analysis with State Variables

In this section we will present two examples to illustrate how the state variable method is used in circuit analysis.

Example 7.10

For the circuit of Figure 7.7, the initial conditions are $i_L(0^-) = 0$, and $v_C(0^-) = 0.5\ V$. Use the state variable method to compute $i_L(t)$ and $v_C(t)$.

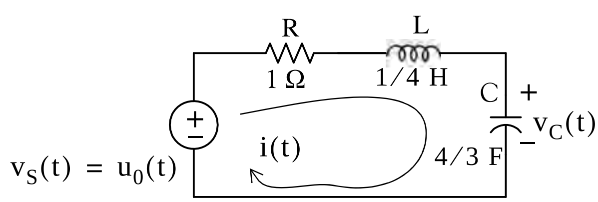


Figure 7.7. Circuit for Example 7.10

Solution:

For this example,

$$i = i_L$$

and

$$Ri_L + L\frac{di_L}{dt} + v_C = u_0(t)$$

Substitution of given values and rearranging, yields

$$\frac{1}{4}\frac{di_L}{dt} = (-1)i_L - v_C + 1$$

or

$$\frac{di_L}{dt} = -4i_L - 4v_C + 4 \qquad (7.85)$$

Next, we define the state variables $x_1 = i_L$ and $x_2 = v_C$. Then,

$$\dot{x}_1 = \frac{di_L}{dt} \qquad (7.86)$$

and

$$\dot{x}_2 = \frac{dv_C}{dt}$$

Also,

$$i_L = C\frac{dv_C}{dt}$$

and thus,

$$x_1 = i_L = C\frac{dv_C}{dt} = C\dot{x}_2 = \frac{4}{3}\dot{x}_2$$

or

$$\dot{x}_2 = \frac{3}{4}x_1 \qquad (7.87)$$

Therefore, from (7.85), (7.86), and (7.87), we obtain the state equations

$$\dot{x}_1 = -4x_1 - 4x_2 + 4$$

$$\dot{x}_2 = \frac{3}{4}x_1$$

and in matrix form,

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-4 & -4\\ 3/4 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}4\\ 0\end{bmatrix}u_0(t) \qquad (7.88)$$

We will compute the solution of (7.88) using

$$x(t) = e^{A(t-t_0)}x_0 + e^{At}\int_{t_0}^{t} e^{-A\tau}bu(\tau)d\tau \qquad (7.89)$$

where

$$A = \begin{bmatrix}-4 & -4\\ 3/4 & 0\end{bmatrix} \quad x_0 = \begin{bmatrix}i_L(0)\\ v_C(0)\end{bmatrix} = \begin{bmatrix}0\\ 1/2\end{bmatrix} \quad b = \begin{bmatrix}4\\ 0\end{bmatrix} \qquad (7.90)$$

First, we compute the state transition matrix e^{At}. We find the eigenvalues from

$$\det[A - \lambda I] = 0$$

Then,

$$\det[A - \lambda I] = \det\begin{bmatrix}-4-\lambda & -4\\ 3/4 & -\lambda\end{bmatrix} = 0 \qquad (-\lambda)(-4-\lambda) + 3 = 0 \qquad \lambda^2 + 4\lambda + 3 = 0$$

Therefore,

$$\lambda_1 = -1 \text{ and } \lambda_2 = -3$$

The next step is to find the coefficients a_i. Since A is a 2×2 matrix, we only need the first two terms of the state transition matrix, that is,

$$e^{At} = a_0 I + a_1 A \qquad (7.91)$$

The constants a_0 and a_1 are found from

$$a_0 + a_1\lambda_1 = e^{\lambda_1 t}$$
$$a_0 + a_1\lambda_2 = e^{\lambda_2 t}$$

and with $\lambda_1 = -1$ and $\lambda_2 = -3$, we obtain

$$\begin{aligned} a_0 - a_1 &= e^{-t} \\ a_0 - 3a_1 &= e^{-3t} \end{aligned} \tag{7.92}$$

Simultaneous solution of (7.92) yields

$$\begin{aligned} a_0 &= 1.5e^{-t} - 0.5e^{-3t} \\ a_1 &= 0.5e^{-t} - 0.5e^{-3t} \end{aligned} \tag{7.93}$$

We now substitute these values into (7.91), and we obtain

$$e^{At} = (1.5e^{-t} - 0.5e^{-3t})\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (0.5e^{-t} - 0.5e^{-2t})\begin{bmatrix} -4 & -4 \\ 3/4 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1.5e^{-t} - 0.5e^{-3t} & 0 \\ 0 & 1.5e^{-t} - 0.5e^{-3t} \end{bmatrix} + \begin{bmatrix} -2e^{-t} + 2e^{-3t} & -2e^{-t} + 2e^{-3t} \\ \frac{3}{8}e^{-t} - \frac{3}{8}e^{-3t} & 0 \end{bmatrix}$$

or

$$e^{At} = \begin{bmatrix} -0.5e^{-t} + 1.5e^{-3t} & -2e^{-t} + 2e^{-3t} \\ \frac{3}{8}e^{-t} - \frac{3}{8}e^{-3t} & 1.5e^{-t} - 0.5e^{-3t} \end{bmatrix}$$

The initial conditions vector is the second vector in (7.90); then, the first term of (7.89) becomes

$$e^{At}x_0 = \begin{bmatrix} -0.5e^{-t} + 1.5e^{-3t} & -2e^{-t} + 2e^{-3t} \\ \frac{3}{8}e^{-t} - \frac{3}{8}e^{-3t} & 1.5e^{-t} - 0.5e^{-3t} \end{bmatrix}\begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$$

or

$$e^{At}x_0 = \begin{bmatrix} -e^{-t} + e^{-3t} \\ 0.75e^{-t} - 0.25e^{-3t} \end{bmatrix} \tag{7.94}$$

We also need to evaluate the integral on the right side of (7.89). From (7.90)

$$b = \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} 4$$

and denoting this integral as Int, we obtain

$$Int = \int_{t_0}^{t} \begin{bmatrix} -0.5e^{-(t-\tau)} + 1.5e^{-3(t-\tau)} & -2e^{-(t-\tau)} + 2e^{-3(t-\tau)} \\ \frac{3}{8}e^{-(t-\tau)} - \frac{3}{8}e^{-3(t-\tau)} & 1.5e^{-(t-\tau)} - 0.5e^{-3(t-\tau)} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} 4d\tau$$

or

$$Int = \int_{t_0}^{t} \begin{bmatrix} -0.5e^{-(t-\tau)} + 1.5e^{-3(t-\tau)} \\ \frac{3}{8}e^{-(t-\tau)} - \frac{3}{8}e^{-3(t-\tau)} \end{bmatrix} 4d\tau \tag{7.95}$$

The integration in (7.95) is with respect to τ; then, integrating the column vector under the integral, we obtain

$$Int = 4 \begin{bmatrix} -0.5e^{-(t-\tau)} + 0.5e^{-3(t-\tau)} \\ 0.375e^{-(t-\tau)} - 0.125e^{-3(t-\tau)} \end{bmatrix} \Bigg|_{\tau=0}^{t}$$

or

$$Int = 4 \begin{bmatrix} -0.5 + 0.5 \\ 0.375 - 0.125 \end{bmatrix} - 4 \begin{bmatrix} -0.5e^{-t} + 0.5e^{-3t} \\ 0.375e^{-t} - 0.125e^{-3t} \end{bmatrix} = 4 \begin{bmatrix} 0.5e^{-t} - 0.5e^{-3t} \\ 0.25 - 0.375e^{-t} + 0.125e^{-3t} \end{bmatrix}$$

By substitution of these values, the solution of

$$x(t) = e^{A(t-t_0)}x_0 + e^{At}\int_{t_0}^{t} e^{-A\tau}bu(\tau)d\tau$$

is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -e^{-t} + e^{-3t} \\ 0.75e^{-t} - 0.25e^{-3t} \end{bmatrix} + 4 \begin{bmatrix} 0.5e^{-t} - 0.5e^{-3t} \\ 0.25 - 0.375e^{-t} + 0.125e^{-3t} \end{bmatrix} = \begin{bmatrix} e^{-t} - e^{-3t} \\ 1 - 0.75e^{-t} + 0.25e^{-3t} \end{bmatrix}$$

Then,

$$x_1 = i_L = e^{-t} - e^{-3t} \tag{7.96}$$

and

$$x_2 = v_C = 1 - 0.75e^{-t} + 0.25e^{-3t} \tag{7.97}$$

Other variables of the circuit can now be computed from (7.96) and (7.97). For example, the voltage across the inductor is

$$v_L = L\frac{di_L}{dt} = \frac{1}{4}\frac{d}{dt}(e^{-t} - e^{-3t}) = -\frac{1}{4}e^{-t} + \frac{3}{4}e^{-3t}$$

We use the MATLAB script below to plot the relation of (7.97).

```
t=0:0.01:10; x2=1–0.75.*exp(–t)+0.25.*exp(–3.*t);...
plot(t,x2); grid
```

The plot is shown in Figure 7.8.

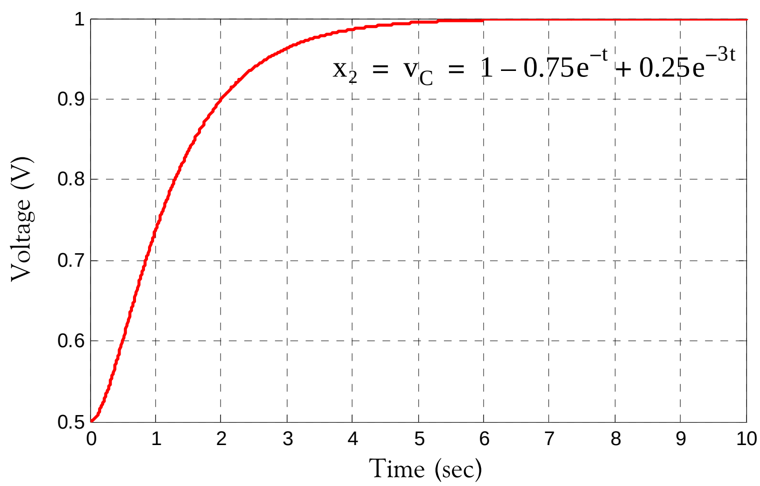


Figure 7.8. Plot for relation (7.97)

We can obtain the plot in Figure 7.8 with the Simulink **State–Space** block with the unit step function as the input using the **Step** block, and the capacitor voltage as the output displayed on the **Scope** block as shown in the model of Figure 7.9 where for the **State–Space** block Function Block Parameters dialog box we have entered:

A: [–4 –4; 3/4 0]
B: [4 0]'
C: [0 1]
D: [0]
Initial conditions: [0 1/2]

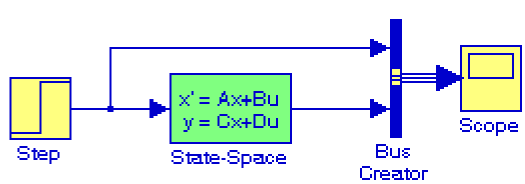


Figure 7.9. Simulink model for Example 7.10

The waveform for the capacitor voltage for the simulation time interval $0 \le t \le 10$ seconds is shown in Figure 7.10 where we observe that the initial condition $v_C(0^-) = 0.5\ V$ is also displayed.

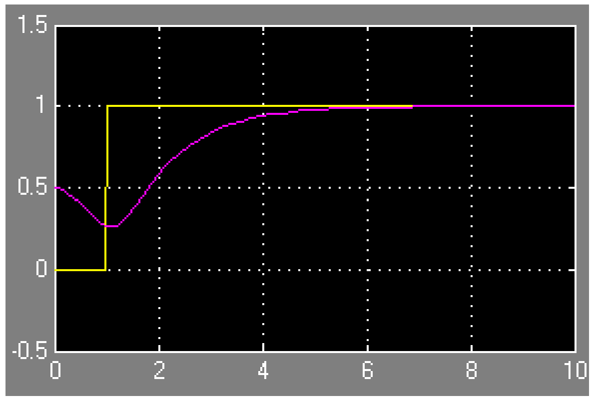


Figure 7.10. Input and output waveforms for the model of Figure 7.9

The SimPowerSystems model for the circuit in Figure 7.7 is shown in Figure 7.11.

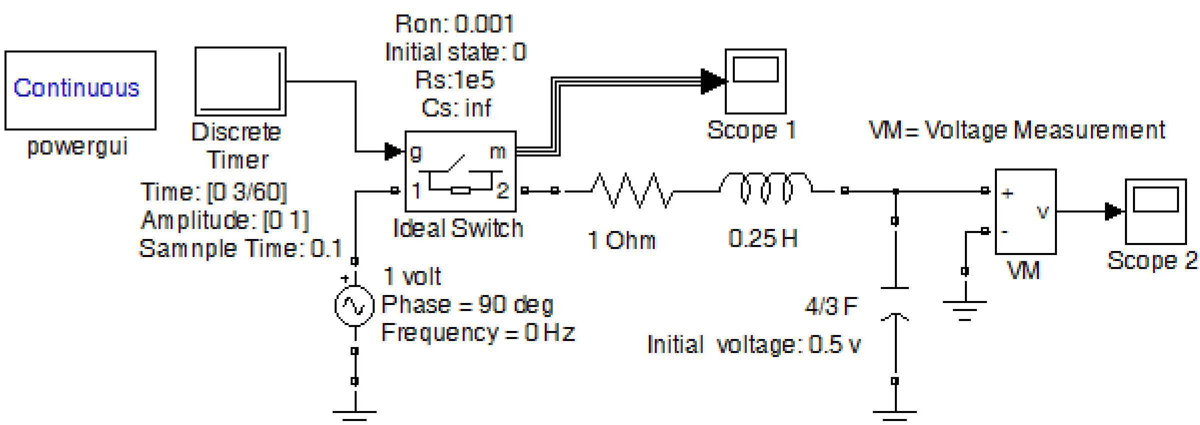


Figure 7.11. Model for the circuit in Figure 7.7. Scope 2 block displays the waveform in Fig.7.8.

Example 7.11

A network is described by the state equation

$$\dot{x} = Ax + bu \tag{7.98}$$

where

$$A = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \qquad x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad b = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \text{and} \quad u = \delta(t) \tag{7.99}$$

Compute the state vector

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Solution:

We compute the eigenvalues from

$$\det[A - \lambda I] = 0$$

For this example,

$$\det[A - \lambda I] = \det\begin{bmatrix} 1-\lambda & 0 \\ 1 & -1-\lambda \end{bmatrix} = 0 \qquad (1-\lambda)(-1-\lambda) = 0$$

Then,

$$\lambda_1 = 1 \text{ and } \lambda_2 = -1$$

Since A is a 2×2 matrix, we only need the first two terms of the state transition matrix to find the coefficients a_i, that is,

$$e^{At} = a_0 I + a_1 A \tag{7.100}$$

The constants a_0 and a_1 are found from

$$\begin{aligned} a_0 + a_1\lambda_1 &= e^{\lambda_1 t} \\ a_0 + a_1\lambda_2 &= e^{\lambda_2 t} \end{aligned} \tag{7.101}$$

and with $\lambda_1 = 1$ and $\lambda_2 = -1$, we obtain

$$\begin{aligned} a_0 + a_1 &= e^{t} \\ a_0 - a_1 &= e^{-t} \end{aligned} \tag{7.102}$$

and simultaneous solution of (7.102) yields

$$a_0 = \frac{e^t + e^{-t}}{2} = \cosh t$$

$$a_1 = \frac{e^t - e^{-t}}{2} = \sinh t$$

By substitution of these values into (7.100), we obtain

$$e^{At} = \cosh t I + \sinh t A = \cosh t\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sinh t\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} \cosh t + \sinh t & 0 \\ \sinh t & \cosh t - \sinh t \end{bmatrix} \tag{7.103}$$

The values of the vector $\mathbf{x}$ are found from

$$x(t) = e^{A(t-t_0)}x_0 + e^{At}\int_{t_0}^{t} e^{-A\tau}bu(\tau)d\tau = e^{At}x_0 + e^{At}\int_{0}^{t} e^{-A\tau}b\delta(\tau)d\tau \tag{7.104}$$

Using the sifting property of the delta function we find that (7.104) reduces to

$$x(t) = e^{At}x_0 + e^{At}b = e^{At}(x_0 + b) = e^{At}\left\{\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}-1\\1\end{bmatrix}\right\} = e^{At}\begin{bmatrix}0\\1\end{bmatrix}$$

$$= \begin{bmatrix}\cosh t + \sinh t & 0\\ \sinh t & \cosh t - \sinh t\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}x_1\\x_2\end{bmatrix}$$

Therefore,

$$x = \begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\ \cosh t - \sinh t\end{bmatrix} = \begin{bmatrix}0\\e^{-t}\end{bmatrix} \tag{7.105}$$

7.7 Relationship between State Equations and Laplace Transform

In this section, we will show that the state transition matrix can be computed from the Inverse Laplace transform. We will also show that the transfer function can be found from the coefficient matrices of the state equations.

Consider the state equation

$$\dot{x} = Ax + bu \tag{7.106}$$

Taking the Laplace of both sides of (7.106), we obtain

$$sX(s) - x(0) = AX(s) + bU(s)$$

or

$$(sI - A)X(s) = x(0) + bU(s) \tag{7.107}$$

Multiplying both sides of (7.107) by $(sI - A)^{-1}$, we obtain

$$X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}bU(s) \tag{7.108}$$

Comparing (7.108) with

$$x(t) = e^{At}x_0 + e^{At}\int_0^t e^{-A\tau}bu(\tau)d\tau \tag{7.109}$$

we observe that the right side of (7.108) is the Laplace transform of (7.109). Therefore, we can compute the state transition matrix e^{At} from the Inverse Laplace of $(sI - A)^{-1}$, that is, we can use the relation

$$\boxed{e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\}} \qquad (7.110)$$

Next, we consider the output state equation

$$y = Cx + du \qquad (7.111)$$

Taking the Laplace of both sides of (7.111), we obtain

$$Y(s) = CX(s) + dU(s) \qquad (7.112)$$

and using (7.108), we obtain

$$Y(s) = C(sI - A)^{-1}x(0) + [C(sI - A)^{-1}b + d]U(s) \qquad (7.113)$$

If the initial condition $x(0) = 0$, (7.113) reduces to

$$Y(s) = [C(sI - A)^{-1}b + d]U(s) \qquad (7.114)$$

In (7.114), $U(s)$ is the Laplace transform of the input $u(t)$; then, division of both sides by $U(s)$ yields the transfer function

$$\boxed{G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}b + d} \qquad (7.115)$$

Example 7.12

In the circuit of Figure 7.12, all initial conditions are zero. Compute the state transition matrix e^{At} using the Inverse Laplace transform method.

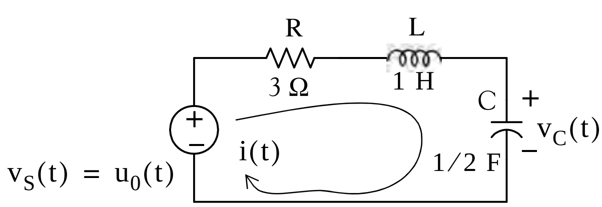


Figure 7.12. Circuit for Example 7.12

Solution:

For this circuit,

$$i = i_L$$

and

$$Ri_L + L\frac{di_L}{dt} + v_C = u_0(t)$$

Substitution of given values and rearranging, yields

$$\frac{di_L}{dt} = -3i_L - v_C + 1 \tag{7.116}$$

Now, we define the state variables

$$x_1 = i_L$$

and

$$x_2 = v_C$$

Then,

$$\dot{x}_1 = \frac{di_L}{dt} = -3i_L - v_C + 1 \tag{7.117}$$

and

$$\dot{x}_2 = \frac{dv_C}{dt}$$

Also,

$$i_L = C\frac{dv_C}{dt} = 0.5\frac{dv_C}{dt} \tag{7.118}$$

and thus,

$$x_1 = i_L = 0.5\frac{dv_C}{dt} = 0.5\dot{x}_2$$

or

$$\dot{x}_2 = 2x_1 \tag{7.119}$$

Therefore, from (7.117) and (7.119) we obtain the state equations

$$\begin{aligned} \dot{x}_1 &= -3x_1 - x_2 + 1 \\ \dot{x}_2 &= 2x_1 \end{aligned} \tag{7.120}$$

and in matrix form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} 1 \tag{7.121}$$

By inspection,

$$A = \begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix} \tag{7.122}$$

Now, we will find the state transition matrix from

$$e^{At} = \mathscr{L}^{-1}\{(sI - A)^{-1}\} \tag{7.123}$$

where

$$(sI - A) = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

Then,

$$(sI - A)^{-1} = \frac{adj(sI - A)}{det(sI - A)} = \frac{1}{s^2 + 3s + 2}\begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix} = \begin{bmatrix} \frac{s}{(s+1)(s+2)} & \frac{-1}{(s+1)(s+2)} \\ \frac{2}{(s+1)(s+2)} & \frac{s+3}{(s+1)(s+2)} \end{bmatrix}$$

We find the Inverse Laplace of each term by partial fraction expansion. Thus,

$$e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\} = \begin{bmatrix} -e^{-t} + 2e^{-2t} & -e^{-t} + e^{-2t} \\ 2e^{-t} - 2e^{-2t} & 2e^{-t} - e^{-2t} \end{bmatrix}$$

Now, we can find the state variables representing the inductor current and the capacitor voltage from

$$x(t) = e^{At}x_0 + e^{At}\int_0^t e^{-A\tau}bu(\tau)d\tau$$

using the procedure of Example 7.11.

MATLAB provides two very useful functions to convert state–space (state equations), to transfer function (s–domain), and vice versa. The function **ss2tf** (state–space to transfer function) converts the state space equations

$$\begin{aligned} \dot{x} &= Ax + Bu\,^{*} \\ y &= Cx + Du \end{aligned} \qquad (7.124)$$

to the rational transfer function form

$$G(s) = \frac{N(s)}{D(s)} \qquad (7.125)$$

This is used with the statement **[num,den]=ss2tf(A,B,C,D,iu)** where **A, B, C, D** are the matrices of (7.124) and **iu** is 1 if there is only one input. The MATLAB **help** command provides the following information:

help ss2tf

```
SS2TF  State-space to transfer function conversion.
   [NUM,DEN] = SS2TF(A,B,C,D,iu) calculates the
    transfer function:
               NUM(s)          -1
      G(s) = -------- = C(sI-A) B + D
               DEN(s)
   of the system:
      x = Ax + Bu
```

* *We have used capital letters for vectors b and c to be consistent with MATLAB's designations.*

```
        y = Cx + Du
from the iu'th input. Vector DEN contains the coefficients of  the
denominator in descending powers of s. The numerator coefficients are
returned in matrix NUM with as many rows as there    are outputs y.

    See also TF2SS
```

The other function, **tf2ss**, converts the transfer function of (7.125) to the state–space equations of (7.124). It is used with the statement **[A,B,C,D]=tf2ss(num,den)** where **A, B, C**, and **D** are the matrices of (7.124), and num, den are N(s) and D(s) of (7.125) respectively. The MATLAB **help** command provides the following information:

help tf2ss

```
 TF2SS  Transfer function to state-space conversion.
    [A,B,C,D] = TF2SS(NUM,DEN) calculates the state-space
    representation:

        x = Ax + Bu
        y = Cx + Du
    of the system:
                 NUM(s)
        G(s) = --------
                 DEN(s)
from a single input. Vector DEN must contain the coefficients of the
denominator in descending powers of s. Matrix NUM must contain the
numerator coefficients with as many rows as there are outputs y. The
A,B,C,D matrices are returned in controller canonical form. This cal-
culation also works for discrete systems. To avoid confusion when using
this function with discrete systems, always use a numerator polynomial
that has been padded with zeros to make it the same length as the
denominator. See the User's guide for more details.

    See also SS2TF.
```

Example 7.13

For the circuit of Figure 7.13, all initial conditions are zero.

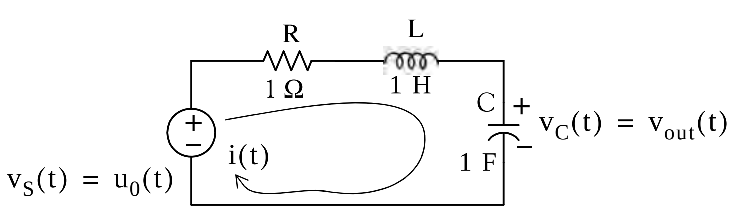


Figure 7.13. Circuit for Example 7.13

a. Derive the state equations and express them in matrix form as

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

b. Derive the transfer function

$$G(s) = \frac{N(s)}{D(s)}$$

c. Verify your answers with MATLAB.

Solution:

a. The differential equation describing the circuit is

$$Ri + L\frac{di}{dt} + v_C = u_0(t)$$

and with the given values,

$$i + \frac{di}{dt} + v_C = u_0(t)$$

or

$$\frac{di}{dt} = -i - v_C + u_0(t)$$

We let

$$x_1 = i_L = i$$

and

$$x_2 = v_C = v_{out}$$

Then,

$$\dot{x}_1 = \frac{di}{dt}$$

and

$$\dot{x}_2 = \frac{dv_c}{dt} = x_1$$

Thus, the state equations are

$$\dot{x}_1 = -x_1 - x_2 + u_0(t)$$
$$\dot{x}_2 = x_1$$
$$y = x_2$$

and in matrix form,

$$\dot{x} = Ax + Bu \leftrightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_0(t)$$
$$y = Cx + Du \leftrightarrow y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} u_0(t) \qquad (7.126)$$

b. The $s-\mathrm{domain}$ circuit is shown in Figure 7.14.

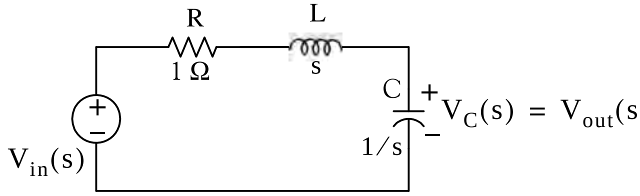


Figure 7.14. Transformed circuit for Example 7.13

By the voltage division expression,

$$V_{out}(s) = \frac{1/s}{1+s+1/s}V_{in}(s)$$

or

$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{s^2+s+1}$$

Therefore,

$$G(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{s^2+s+1} \tag{7.127}$$

c.

```
A = [−1  −1; 1  0]; B = [1  0]';  C = [0  1]; D = [0];       % The matrices of (7.126)
[num, den] = ss2tf(A, B, C, D, 1)                            % Verify coefficients of G(s) in (7.127)

num =
   0      0      1
den =
  1.0000     1.0000     1.0000

num = [0  0  1]; den = [1  1  1];                            % The coefficients of G(s) in (7.127)
[A  B  C  D] = tf2ss(num, den)                               % Verify the matrices of (7.126)

A =
  -1     -1
   1      0
B =
   1
   0
C =
   0      1
D =
   0
```

The equivalence between the state−space equations of (7.126) and the transfer function of (7.127) is also evident from the Simulink models shown in Figure 7.15 where for the **State−Space** block Function Block Parameters dialog box we have entered:

A: [−1 −1; 3/4 0]
B: [1 0]'
C: [0 1]
D: [0]
Initial conditions: [0 0]

For the **Transfer Fcn** block Function Block Parameters dialog box we have entered:

Numerator coefficient: [1]
Denominator coefficient: [1 1 1]

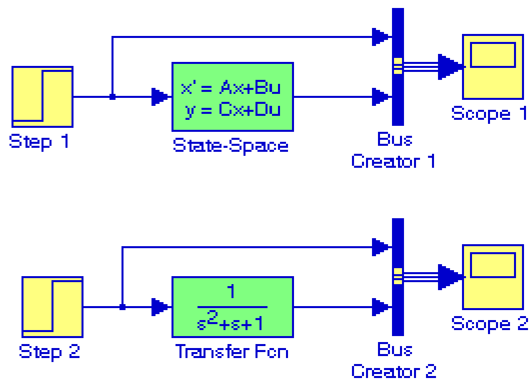


Figure 7.15. Models to show the equivalence between relations (7.126) and (7.127)

After the simulation command is executed, both Scope 1 and Scope 2 blocks display the input and output waveforms shown in Figure 7.15.

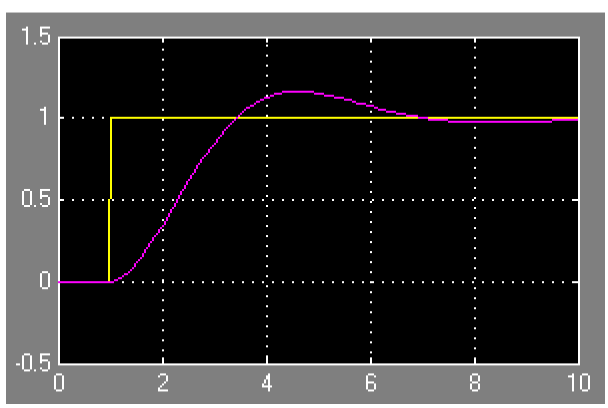


Figure 7.16. Waveforms displayed by Scope 1 and Scope 2 blocks for the models in Figure 7.15

7.8 Summary

- An nth–order differential equation can be resolved to n first–order simultaneous differential equations with a set of auxiliary variables called state variables. The resulting first–order differential equations are called state–space equations, or simply state equations.

- The state–space equations can be obtained either from the nth–order differential equation, or directly from the network, provided that the state variables are chosen appropriately.

- When we obtain the state equations directly from given circuits, we choose the state variables to represent inductor currents and capacitor voltages.

- The state variable method offers the advantage that it can also be used with non–linear and time–varying devices.

- If a circuit contains only one energy–storing device, the state equations are written as

$$\dot{x} = \alpha x + \beta u$$
$$y = k_1 x + k_2 u$$

where α, β, k_1, and k_2 are scalar constants, and the initial condition, if non–zero, is denoted as

$$x_0 = x(t_0)$$

- If α and β are scalar constants, the solution of $\dot{x} = \alpha x + \beta u$ with initial condition $x_0 = x(t_0)$ is obtained from the relation

$$x(t) = e^{\alpha(t-t_0)} x_0 + e^{\alpha t} \int_{t_0}^{t} e^{-\alpha\tau} \beta u(\tau) d\tau$$

- The solution of the state equations pair

$$\dot{x} = Ax + bu$$
$$y = Cx + du$$

where A and C are 2×2 or higher order matrices, and b and d are column vectors with two or more rows, entails the computation of the state transition matrix e^{At}, and integration of

$$x(t) = e^{A(t-t_0)} x_0 + e^{At} \int_{t_0}^{t} e^{-A\tau} bu(\tau) d\tau$$

- The eigenvalues λ_i, where $i = 1, 2, \ldots, n$, of an $n \times n$ matrix A are the roots of the nth order polynomial

$$\det[A - \lambda I] = 0$$

where I is the $n \times n$ identity matrix.

- The Cayley–Hamilton theorem states that a matrix can be expressed as an $(n-1)$th degree polynomial in terms of the matrix A as

$$e^{At} = a_0 I + a_1 A + a_2 A^2 + \ldots + a_{n-1} A^{n-1}$$

where the coefficients a_i are functions of the eigenvalues λ.

- If all eigenvalues of a given matrix A are distinct, that is, if

$$\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \ldots \neq \lambda_n$$

the coefficients a_i are found from the simultaneous solution of the system of equations

$$\begin{aligned}
a_0 + a_1\lambda_1 + a_2\lambda_1^2 + \ldots + a_{n-1}\lambda_1^{n-1} &= e^{\lambda_1 t} \\
a_0 + a_1\lambda_2 + a_2\lambda_2^2 + \ldots + a_{n-1}\lambda_2^{n-1} &= e^{\lambda_2 t} \\
&\ldots \\
a_0 + a_1\lambda_n + a_2\lambda_n^2 + \ldots + a_{n-1}\lambda_n^{n-1} &= e^{\lambda_n t}
\end{aligned}$$

- If some or all eigenvalues of matrix A are repeated, that is, if

$$\lambda_1 = \lambda_2 = \lambda_3 \ldots = \lambda_m, \ \lambda_{m+1}, \ \lambda_n$$

the coefficients a_i of the state transition matrix are found from the simultaneous solution of the system of equations

$$\begin{aligned}
a_0 + a_1\lambda_1 + a_2\lambda_1^2 + \ldots + a_{n-1}\lambda_1^{n-1} &= e^{\lambda_1 t} \\
\frac{d}{d\lambda_1}(a_0 + a_1\lambda_1 + a_2\lambda_1^2 + \ldots + a_{n-1}\lambda_1^{n-1}) &= \frac{d}{d\lambda_1} e^{\lambda_1 t} \\
\frac{d^2}{d\lambda_1^2}(a_0 + a_1\lambda_1 + a_2\lambda_1^2 + \ldots + a_{n-1}\lambda_1^{n-1}) &= \frac{d^2}{d\lambda_1^2} e^{\lambda_1 t} \\
&\ldots \\
\frac{d^{m-1}}{d\lambda_1^{m-1}}(a_0 + a_1\lambda_1 + a_2\lambda_1^2 + \ldots + a_{n-1}\lambda_1^{n-1}) &= \frac{d^{m-1}}{d\lambda_1^{m-1}} e^{\lambda_1 t} \\
a_0 + a_1\lambda_{m+1} + a_2\lambda_{m+1}^2 + \ldots + a_{n-1}\lambda_{m+1}^{n-1} &= e^{\lambda_{m+1} t} \\
&\ldots \\
a_0 + a_1\lambda_n + a_2\lambda_n^2 + \ldots + a_{n-1}\lambda_n^{n-1} &= e^{\lambda_n t}
\end{aligned}$$

- We can use the MATLAB **eig(x)** function to find the eigenvalues of an $n \times n$ matrix.
- A column vector X that satisfies the relation

$$AX = \lambda X$$

where A is an $n \times n$ matrix and λ is a scalar number, is called an eigenvector.

- There is a different eigenvector for each eigenvalue.
- Eigenvectors are generally expressed as unit eigenvectors, that is, they are normalized to unit length. This is done by dividing each component of the eigenvector by the square root of the sum of the squares of their components, so that the sum of the squares of their components is equal to unity.
- Two vectors X and Y are said to be orthogonal if their inner (dot) product is zero.
- A set of eigenvectors constitutes an orthonormal basis if the set is normalized (expressed as unit eigenvectors) and these vector are mutually orthogonal.
- The state transition matrix can be computed from the Inverse Laplace transform using the relation

$$e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\}$$

- If $U(s)$ is the Laplace transform of the input $u(t)$ and $Y(s)$ is the Laplace transform of the output $y(t)$, the transfer function can be computed using the relation

$$G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}b + d$$

- MATLAB provides two very useful functions to convert state–space (state equations), to transfer function (s-domain), and vice versa. The function **ss2tf** (state–space to transfer function) converts the state space equations to the transfer function equivalent, and the function **tf2ss**, converts the transfer function to state–space equations.

7.9 Exercises

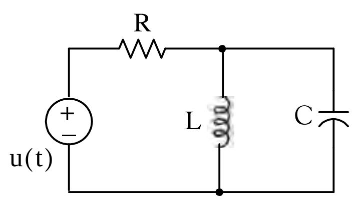
1. Express the integrodifferential equation below as a matrix of state equations where $k_1, k_2,$ and k_3 are constants.

$$\frac{dv^2}{dt^2} + k_3\frac{dv}{dt} + k_2 v + k_1\int_0^t v\,dt = \sin 3t + \cos 3t$$

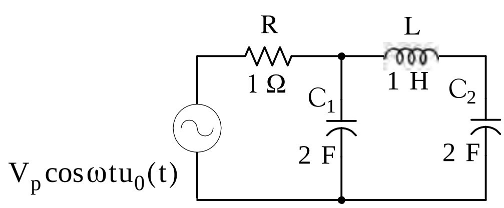
2. Express the matrix of the state equations below as a single differential equation, and let $x(y) = y(t)$.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -2 & -3 & -4 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

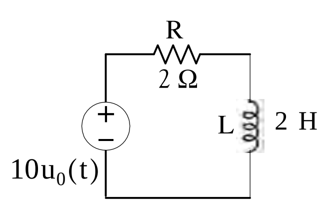
3. For the circuit below, all initial conditions are zero, and $u(t)$ is any input. Write state equations in matrix form.



4. In the circuit below, all initial conditions are zero. Write state equations in matrix form.



5. In the below, $i_L(0^-) = 2\ A$. Use the state variable method to find $i_L(t)$ for $t > 0$.



6. Compute the eigenvalues of the matrices A, B, and C below.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} a & 0 \\ -a & b \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$$

Hint: One of the eigenvalues of matrix C is -1.

7. Compute e^{At} given that

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$$

Observe that this is the same matrix as C of Exercise 6.

8. Find the solution of the matrix state equation $\dot{x} = Ax + bu$ given that

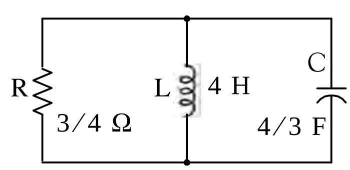
$$A = \begin{bmatrix} 1 & 0 \\ -2 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad x_0 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad u = \delta(t), \quad t_0 = 0$$

9. In the circuit below, $i_L(0^-) = 0$, and $v_C(0^-) = 1\ V$.

   a. Write state equations in matrix form.

   b. Compute e^{At} using the Inverse Laplace transform method.

   c. Find $i_L(t)$ and $v_C(t)$ for $t > 0$.



7.10 Solutions to End–of–Chapter Exercises

1. Differentiating the given integrodifferential equation with respect to t we obtain

$$\frac{dv^3}{dt^3}+k_3\frac{dv^2}{dt^2}+k_2\frac{dv}{dt}+k_1v = 3\cos 3t-3\sin 3t = 3(\cos 3t-\sin 3t)$$

or

$$\frac{dv^3}{dt^3} = -k_3\frac{dv^2}{dt^2}-k_2\frac{dv}{dt}-k_1v+3(\cos 3t-\sin 3t) \quad (1)$$

We let

$$v = x_1 \qquad \frac{dv}{dt} = x_2 = \dot{x}_1 \qquad \frac{dv^2}{dt^2} = x_3 = \dot{x}_2$$

Then,

$$\frac{dv^3}{dt^3} = \dot{x}_3$$

and by substitution into (1)

$$\dot{x}_3 = -k_1x_1-k_2x_2-k_3x_3+3(\cos 3t-\sin 3t)$$

and thus the state equations are

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = -k_1x_1-k_2x_2-k_3x_3+3(\cos 3t-\sin 3t)$$

and in matrix form

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -k_1 & -k_2 & -k_3\end{bmatrix}\cdot\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}\cdot 3(\cos 3t-\sin 3t)$$

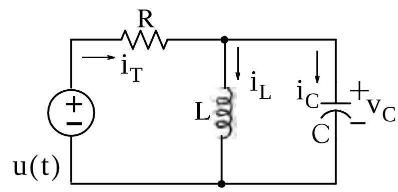
2. Expansion of the given matrix yields

$$\dot{x}_1 = x_2 \qquad \dot{x}_2 = x_3 \qquad \dot{x}_3 = x_2 \qquad \dot{x}_4 = -x_1-2x_2-3x_3-4x_4+u(t)$$

Letting $x = y$ we obtain

$$\frac{dy^4}{dt^4}+4\frac{dy^3}{dt^3}+3\frac{dy^2}{dt^2}+2\frac{dy}{dt}+y = u(t)$$

3.



We let $i_L = x_1$ and $v_C = x_2$. By KCL, $i_T = i_L + i_C$ or

$$\frac{u(t) - v_C}{R} = i_L + C\frac{dv_C}{dt}$$

or

$$\frac{u(t) - x_2}{R} = x_1 + C\dot{x}_2$$

Also,

$$x_2 = L\dot{x}_1$$

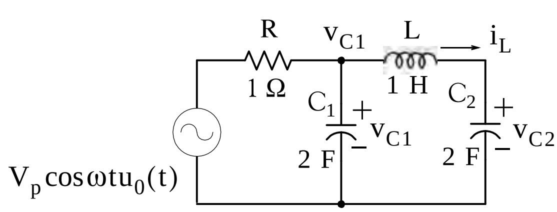
Then,

$$\dot{x}_1 = \frac{1}{L}x_2 \text{ and } \dot{x}_2 = -\frac{1}{C}x_1 - \frac{1}{RC}x_2 + \frac{1}{RC}u(t)$$

and in matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1/L \\ -1/C & -1/RC \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/RC \end{bmatrix} \cdot u(t)$$

4.



We let $i_L = x_1$, $v_{C1} = x_2$, and $v_{C2} = x_3$. By KCL,

$$\frac{v_{C1} - V_p\cos\omega t u_0(t)}{1} + 2\frac{dv_{C1}}{dt} + i_L = 0$$

or

$$x_2 - V_p\cos\omega t u_0(t) + 2\dot{x}_2 + x_1 = 0$$

or

$$\dot{x}_2 = -\frac{1}{2}x_1 - \frac{1}{2}x_2 + \frac{1}{2}V_p\cos\omega t u_0(t) \quad (1)$$

By KVL,

$$v_{C1} = L\frac{di_L}{dt} + v_{C2}$$

or

$$x_2 = 1\dot{x}_1 + x_3$$

or

$$\dot{x}_1 = x_2 - x_3 \quad (2)$$

Also,

$$i_L = C\frac{dv_{C2}}{dt}$$

or

$$x_1 = 2\dot{x}_3$$

or

$$\dot{x}_3 = \frac{1}{2}x_1 \quad (3)$$

Combining (1), (2), and (3) into matrix form we obtain

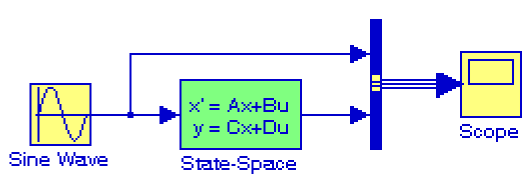
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ -1/2 & -1/2 & 0 \\ 1/2 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/2 \\ 0 \end{bmatrix} \cdot V_p \cos \omega t u_0(t)$$

We will create a Simulink model with $V_p = 1$ and output $y = x_3$. The model is shown below where for the State–Space block Function Block Parameters dialog box we have entered:

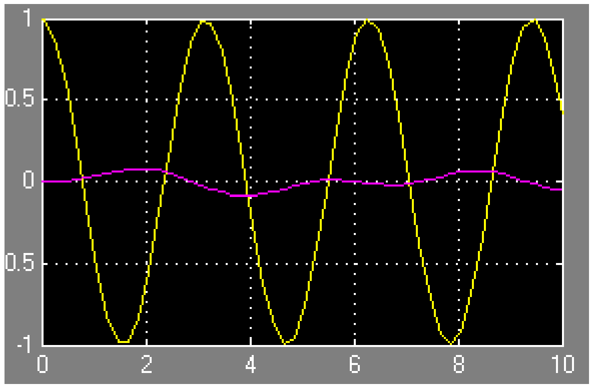
A: [0 1 –1; –1/2 –1/2 0; 1/2 0 0]
B: [0 1/2 0]’
C: [0 0 1]
D: [0]
Initial conditions: [0 0 0]

and for the Sine Wave block Function Block Parameters dialog box we have entered:

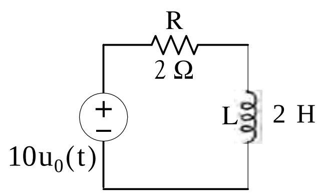
Amplitude: 1
Phase: pi/2



The input and output waveforms are shown below.



5.



From (7.21) of Example 7.4, Page 7–6,

$$\dot{x} = -\frac{R}{L}x + \frac{1}{L}v_S u_0(t)$$

For this exercise, $\alpha = -R/L = -1$ and $b = 10 \times (1/L) = 5$. Then,

$$x(t) = e^{\alpha(t-t_0)}x_0 + e^{\alpha t}\int_{t_0}^{t} e^{-\alpha\tau}\beta u(\tau)d\tau$$

$$= e^{-1(t-0)}2 + e^{-t}\int_0^t e^{\tau}5u_0(\tau)d\tau = 2e^{-t} + 5e^{-t}\int_0^t e^{\tau}d\tau$$

$$= 2e^{-t} + 5e^{-t}(e^{t} - 1) = 2e^{-t} + 5 - 5e^{-t} = (5 - 3e^{-t})u_0(t)$$

and denoting the current i_L as the output y we obtain

$$y(t) = x(t) = (5 - 3e^{-t})u_0(t)$$

6.

a.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \qquad \det(A - \lambda I) = \det\left(\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \det\begin{bmatrix} 1-\lambda & 2 \\ 3 & -1-\lambda \end{bmatrix} = 0$$

$$(1-\lambda)(-1-\lambda) - 6 = 0$$

$$-1-\lambda+\lambda+\lambda^2-6 = 0$$

$$\lambda^2 = 7$$

and thus

$$\lambda_1 = \sqrt{7} \qquad \lambda_2 = -\sqrt{7}$$

b.

$$B = \begin{bmatrix} a & 0 \\ -a & b \end{bmatrix} \qquad \det(B-\lambda I) = \det\left(\begin{bmatrix} a & 0 \\ -a & b \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \det\begin{bmatrix} a-\lambda & 0 \\ -a & b-\lambda \end{bmatrix} = 0$$

$$(a-\lambda)(b-\lambda) = 0$$

and thus

$$\lambda_1 = a \qquad \lambda_2 = b$$

c.

$$C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \qquad \det(C-\lambda I) = \det\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$= \det\begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -6 & -11 & -6-\lambda \end{bmatrix} = 0$$

$$\lambda^2(-6-\lambda)-6-(-11)(-\lambda) = \lambda^3+6\lambda^2+11\lambda+6 = 0$$

and it is given that $\lambda_1 = -1$. Then,

$$\frac{\lambda^3+6\lambda^2+11\lambda+6}{(\lambda+1)} = \lambda^2+5\lambda+6 \Rightarrow (\lambda+1)(\lambda+2)(\lambda+3) = 0$$

and thus

$$\lambda_1 = -1 \qquad \lambda_2 = -2 \qquad \lambda_1 = -3$$

7.

a. Matrix A is the same as Matrix C in Exercise 6. Then,

$$\lambda_1 = -1 \qquad \lambda_2 = -2 \qquad \lambda_1 = -3$$

and since A is a 3×3 matrix the state transition matrix is

$$e^{At} = a_0 I + a_1 A + a_2 A^2 \quad (1)$$

Then,

$$a_0 + a_1\lambda_1 + a_2\lambda_1^2 = e^{\lambda_1 t} \Rightarrow a_0 - a_1 + a_2 = e^{-t}$$

$$a_0 + a_1\lambda_2 + a_2\lambda_2^2 = e^{\lambda_2 t} \Rightarrow a_0 - 2a_1 + 4a_2 = e^{-2t}$$

$$a_0 + a_1\lambda_3 + a_2\lambda_3^2 = e^{\lambda_3 t} \Rightarrow a_0 - 3a_1 + 9a_2 = e^{-3t}$$

```
syms t; A=[1  –1  1; 1  –2  4; 1  –3  9];...
a=sym('[exp(–t); exp(–2*t); exp(–3*t)]'); x=A\a; fprintf(' \n');...
disp('a0 = '); disp(x(1)); disp('a1 = '); disp(x(2)); disp('a2 = '); disp(x(3))
```

```
a0 =
3*exp(-t)-3*exp(-2*t)+exp(-3*t)
a1 =
5/2*exp(-t)-4*exp(-2*t)+3/2*exp(-3*t)
a2 =
1/2*exp(-t)-exp(-2*t)+1/2*exp(-3*t)
```

Thus,

$$\begin{aligned} a_0 &= 3e^{-t} - 3e^{-2t} + 3e^{-3t} \\ a_1 &= 2.5e^{-t} - 4e^{-2t} + 1.5e^{-3t} \\ a_2 &= 0.5e^{-t} - e^{-2t} + 0.5e^{-3t} \end{aligned}$$

Now, we compute e^{At} of (1) with the following MATLAB script:

```
syms t; a0=3*exp(–t)–3*exp(–2*t)+exp(–3*t); a1=5/2*exp(–t)–4*exp(–2*t)+3/2*exp(–3*t);...
a2=1/2*exp(–t)-exp(–2*t)+1/2*exp(–3*t); A=[0 1 0; 0 0 1; –6  –11  –6]; fprintf(' \n');...
eAt=a0*eye(3)+a1*A+a2*A^2
```

```
eAt =
[3*exp(-t)-3*exp(-2*t)+exp(-3*t),   5/2*exp(-t)-4*exp(-2*t)+3/
2*exp(-3*t),     1/2*exp(-t)-exp(-2*t)+1/2*exp(-3*t)]
[-3*exp(-t)+6*exp(-2*t)-3*exp(-3*t),  -5/2*exp(-t)+8*exp(-
2*t)-9/2*exp(-3*t),  -1/2*exp(-t)+2*exp(-2*t)-3/2*exp(-3*t)]
[3*exp(-t)-12*exp(-2*t)+9*exp(-3*t), 5/2*exp(-t)-16*exp(-
2*t)+27/2*exp(-3*t),   1/2*exp(-t)-4*exp(-2*t)+9/2*exp(-3*t)]
```

Thus,

$$e^{At} = \begin{bmatrix} 3e^{-t} - 3e^{-2t} + e^{-3t} & 2.5e^{-t} - 4e^{-2t} + 1.5e^{-3t} & 0.5e^{-t} - e^{-2t} + 0.5e^{-3t} \\ -3e^{-t} + 6e^{-2t} - 3e^{-3t} & -2.5e^{-t} + 8e^{-2t} - 4.5e^{-3t} & -0.5e^{-t} + 2e^{-2t} - 1.5e^{-3t} \\ 3e^{-t} - 12e^{-2t} + 9e^{-3t} & 2.5e^{-t} - 16e^{-2t} + 13.5e^{-3t} & 0.5e^{-t} - 4e^{-2t} + 4.5e^{-3t} \end{bmatrix}$$

8.

$$A = \begin{bmatrix} 1 & 0 \\ -2 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad x_0 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad u = \delta(t), \quad t_0 = 0$$

$$x(t) = e^{A(t-0)}x_0 + e^{At}\int_0^t e^{-A\tau}bu(\tau)d\tau = e^{At}x_0 + e^{At}\int_0^t e^{-A\tau}b\delta(\tau)d\tau$$
$$= e^{At}x_0 + e^{At}b = e^{At}(x_0 + b) = e^{At}\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = e^{At}\begin{bmatrix} 0 \\ 2 \end{bmatrix} \qquad (1)$$

We use the following MATLAB script to find the eigenvalues λ_1 and λ_2.

```
A=[1  0; –2  2]; lambda=eig(A); fprintf(' \n');...
fprintf('lambda1 = %4.2f  \t',lambda(1)); fprintf('lambda2 = %4.2f  \t',lambda(2))
```

```
lambda1 = 2.00   lambda2 = 1.00
```

Next,

$$a_0 + a_1\lambda_1 = e^{\lambda_1 t} \Rightarrow a_0 + a_1 = e^t$$

$$a_0 + a_1\lambda_2 = e^{\lambda_2 t} \Rightarrow a_0 + 2a_1 = e^{2t}$$

Then,

$$a_0 = 2e^t - e^{2t} \qquad a_1 = e^{2t} - e^t$$

and

$$e^{At} = a_0 I + a_1 A = (2e^t - e^{2t})\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (e^{2t} - e^t)\begin{bmatrix} 1 & 0 \\ -2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2e^t - e^{2t} & 0 \\ 0 & 2e^t - e^{2t} \end{bmatrix} + \begin{bmatrix} e^{2t} - e^t & 0 \\ -2e^{2t} + 2e^t & 2e^{2t} - 2e^t \end{bmatrix} = \begin{bmatrix} e^t & 0 \\ 2e^t - 2e^{2t} & e^{2t} \end{bmatrix}$$

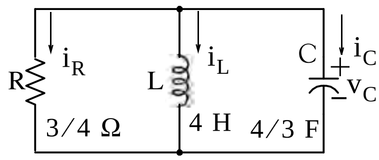
By substitution into (1) we obtain

$$x(t) = e^{At}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} e^t & 0 \\ 2e^t - 2e^{2t} & e^{2t} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2e^{2t} \end{bmatrix}$$

and thus

$$x_1 = 0 \qquad x_2 = 2e^{2t}$$

9.



$i_L(0^-) = 0$

$v_C(0^-) = 1\ V$

We let

$$x_1 = i_L \qquad x_2 = v_C$$

Then,

a.

$$i_R + i_L + i_C = 0$$

$$\frac{v_C}{R} + i_L + C\frac{v_C}{dt} = 0$$

$$\frac{x_2}{3/4} + x_1 + \frac{4}{3}\dot{x}_2 = 0$$

or

$$\dot{x}_2 = -\frac{3}{4}x_1 - x_2 \quad (1)$$

Also,

$$v_L = v_C = L\frac{di_L}{dt} = 4\dot{x}_1 = x_2$$

or

$$\dot{x}_1 = \frac{1}{4}x_2 \quad (2)$$

From (1) and (2)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1/4 \\ -3/4 & -1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

and thus

$$A = \begin{bmatrix} 0 & 1/4 \\ -3/4 & -1 \end{bmatrix}$$

b.

$$e^{At} = \mathcal{L}^{-1}\{[sI - A]^{-1}\}$$

$$[sI - A] = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1/4 \\ -3/4 & -1 \end{bmatrix} = \begin{bmatrix} s & -1/4 \\ 3/4 & s+1 \end{bmatrix}$$

$$\Delta = \det[sI - A] = \det\begin{bmatrix} s & -1/4 \\ 3/4 & s+1 \end{bmatrix} = s^2 + s + 3/16 = (s + 1/4)(s + 3/4)$$

$$\mathrm{adj}[sI - A] = \mathrm{adj}\begin{bmatrix} s & -1/4 \\ 3/4 & s+1 \end{bmatrix} = \begin{bmatrix} s+1 & 1/4 \\ -3/4 & s \end{bmatrix}$$

$$[sI - A]^{-1} = \frac{1}{\Delta}\mathrm{adj}[sI - A] = \frac{1}{(s+1/4)(s+3/4)}\begin{bmatrix} s+1 & 1/4 \\ -3/4 & s \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{s+1}{(s+1/4)(s+3/4)} & \dfrac{1/4}{(s+1/4)(s+3/4)} \\ \dfrac{-3/4}{(s+1/4)(s+3/4)} & \dfrac{s}{(s+1/4)(s+3/4)} \end{bmatrix}$$

We use MATLAB to find $e^{At} = \mathcal{L}^{-1}\{[sI - A]^{-1}\}$ with the script below.

```
syms s t  % Must have Symbolic Math Toolbox installed
Fs1=(s+1)/(s^2+s+3/16); Fs2=(1/4)/(s^2+s+3/16); Fs3=(–3/4)/(s^2+s+3/16);...
Fs4=s/(s^2+s+3/16);...
fprintf(' \n'); disp('a11 = '); disp(simple(ilaplace(Fs1))); disp('a12 = ');...
disp(simple(ilaplace(Fs2)));...
disp('a21 = '); disp(simple(ilaplace(Fs3))); disp('a22 = '); disp(simple(ilaplace(Fs4)))
```

```
a11 =
-1/2*exp(-3/4*t)+3/2*exp(-1/4*t)
a12 =
1/2*exp(-1/4*t)-1/2*exp(-3/4*t)
a21 =
-3/2*exp(-1/4*t)+3/2*exp(-3/4*t)
a22 =
3/2*exp(-3/4*t)-1/2*exp(-1/4*t)
```

Thus,

$$e^{At} = \begin{bmatrix} 1.5e^{-0.25t} - 0.5e^{-0.75t} & 0.5e^{-0.25t} - 0.5e^{-0.75t} \\ -1.5e^{-0.25t} + 1.5e^{-0.75t} & -0.5e^{-0.25t} + 1.5e^{-0.75t} \end{bmatrix}$$

c.

$$x(t) = e^{A(t-0)}x_0 + e^{At}\int_0^t e^{-A\tau}bu(\tau)d\tau = e^{At}x_0 + 0 = e^{At}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 1.5e^{-0.25t} - 0.5e^{-0.75t} & 0.5e^{-0.25t} - 0.5e^{-0.75t} \\ -1.5e^{-0.25t} + 1.5e^{-0.75t} & -0.5e^{-0.25t} + 1.5e^{-0.75t} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5e^{-0.25t} - 0.5e^{-0.75t} \\ -0.5e^{-0.25t} + 1.5e^{-0.75t} \end{bmatrix}$$

and thus for $t > 0$,

$$x_1 = i_L = 0.5e^{-0.25t} - 0.5e^{-0.75t} \qquad x_2 = v_C = -0.5e^{-0.25t} + 1.5e^{-0.75t}$$

Chapter 8

Frequency Response and Bode Plots

This chapter discusses frequency response in terms of both amplitude and phase. This topic will enable us to determine which frequencies are dominant and which frequencies are virtually suppressed. The design of electric filters is based on the study of the frequency response. We will also discuss the Bode method of linear system analysis using two separate plots; one for the magnitude of the transfer function, and the other for the phase, both versus frequency. These plots reveal valuable information about the frequency response behavior.

Note: Throughout this text, the common (base 10) logarithm of a number x will be denoted as $\log(x)$ while its natural (base e) logarithm will be denoted as $\ln(x)$. However, we should remember that in MATLAB the $\log(x)$ function displays the natural logarithm, and the common (base 10) logarithm is defined as $\log 10(x)$.

8.1 Decibel Defined

The ratio of any two values of the same quantity (power, voltage or current) can be expressed in *decibels* (dB). For instance, we say that an amplifier has $10\ dB$ power gain or a transmission line has a power loss of $7\ dB$ (or gain $-7\ dB$). If the gain (or loss) is $0\ dB$, the output is equal to the input. We should remember that a negative voltage or current gain A_V or A_I indicates that there is a $180°$ phase difference between the input and the output waveforms. For instance, if an amplifier has a gain of -100 (dimensionless number), it means that the output is $180°$ out-of-phase with the input. For this reason we use absolute values of power, voltage and current when these are expressed in dB terms to avoid misinterpretation of gain or loss.

By definition,

$$dB = 10\log\left|\frac{P_{out}}{P_{in}}\right| \tag{8.1}$$

Therefore,

$10\ dB$ represents a power ratio of 10

$10n\ dB$ represents a power ratio of 10^n

$20\ dB$ represents a power ratio of 100

$30\ dB$ represents a power ratio of $1,000$

$60\ dB$ represents a power ratio of $1,000,000$

Also,

1 dB represents a power ratio of approximately 1.25
3 dB represents a power ratio of approximately 2
7 dB represents a power ratio of approximately 5

From these, we can estimate other values. For instance, $4\ \text{dB} = 3\ \text{dB} + 1\ \text{dB}$ which is equivalent to a power ratio of approximately $2 \times 1.25 = 2.5$. Likewise, $27\ \text{dB} = 20\ \text{dB} + 7\ \text{dB}$ and this is equivalent to a power ratio of approximately $100 \times 5 = 500$.

Since $y = \log x^2 = 2\log x$ and $P = V^2/R = I^2R$, if we let $R = 1$ the dB values for the voltage and current ratios become:

$$dB_v = 10\log\left|\frac{V_{out}}{V_{in}}\right|^2 = 20\log\left|\frac{V_{out}}{V_{in}}\right| \tag{8.2}$$

and

$$dB_i = 10\log\left|\frac{I_{out}}{I_{in}}\right|^2 = 20\log\left|\frac{I_{out}}{I_{in}}\right| \tag{8.3}$$

Example 8.1

Compute the gain in dB_W for the amplifier shown in Figure 8.1.

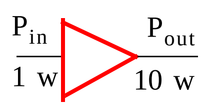


Figure 8.1. Amplifier for Example 8.1

Solution:

$$dB_W = 10\log\frac{P_{out}}{P_{in}} = 10\log\frac{10}{1} = 10\log 10 = 10 \times 1 = 10\ dB_W$$

Example 8.2

Compute the gain in dB_V for the amplifier shown in Figure 8.2 given that $\log 2 = 0.3$.

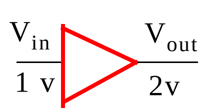


Figure 8.2. Amplifier for Example 8.2.

Solution:

$$dB_V = 20\log\frac{V_{out}}{V_{in}} = 20\log\frac{2}{1} = 20\log 0.3 = 20 \times 0.3 = 6\ dB_V$$

8.2 Bandwidth and Frequency Response

Electric and electronic circuits, such as filters and amplifiers, exhibit a band of frequencies over which the output remains nearly constant. Consider, for example, the magnitude of the output voltage $|V_{out}|$ of an electric or electronic circuit as a function of radian frequency ω as shown in Figure 8.3.

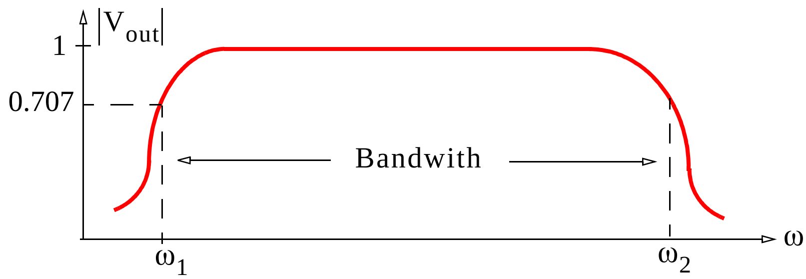


Figure 8.3. Definition of the bandwidth.

As shown in Figure 8.3, the *bandwidth* is $BW = \omega_2 - \omega_1$ where ω_1 and ω_2 are the *lower* and *upper cutoff frequencies* respectively. At these frequencies, $|V_{out}| = \sqrt{2}/2 = 0.707$ and these two points are known as the $3\ dB\ down$ or *half-power points*. They derive their name from the fact that since power $p = v^2/R = i^2R$, for $R = 1$ and for $v = 0.707|V_{out}|$ or $i = 0.707|I_{out}|$ the power is $1/2$, that is, it is "halved". Alternately, we can define the bandwidth as the frequency band between half-power points.

Most amplifiers are used with a feedback path which returns (feeds) some or all its output to the input as shown in Figure 8.4.

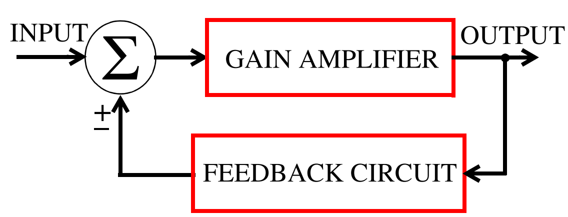


Figure 8.4. Amplifier with partial output feedback

Figure 8.5 shows an amplifier where the entire output is fed back to the input.

Figure 8.5. Amplifier with entire output feedback

The symbol Σ (Greek capital letter sigma) inside the circle indicates the summing point where the output signal, or portion of it, is combined with the input signal. This summing point may be also indicated with a large plus (+) symbol inside the circle. The positive (+) sign below the summing point implies *positive feedback* which means that the output, or portion of it, is added to the input. On the other hand, the negative (–) sign implies *negative feedback* which means that the output, or portion of it, is subtracted from the input. Practically, all amplifiers use used with negative feedback since positive feedback causes circuit instability.

8.3 Octave and Decade

Let us consider two frequencies u_1 and u_2 defining the frequency interval $u_2 - u_1$, and let

$$u_2 - u_1 = log_{10}\omega_2 - log_{10}\omega_1 = log_{10}\frac{\omega_2}{\omega_1} \tag{8.4}$$

If these frequencies are such that $\omega_2 = 2\omega_1$, we say that these frequencies are separated by one *octave* and if $\omega_2 = 10\omega_1$, they are separated by one *decade*.

Let us now consider a transfer function $G(s)$ whose magnitude is evaluated at $s = |j\omega|$, that is,

$$G(s)| = \left.\frac{C}{s^k}\right|_{s = |j\omega|} = |G(\omega)| = \frac{C}{\omega^k} \tag{8.5}$$

Taking the log of both sides of (8.5) and multiplying by 20, we obtain

$$20log_{10}|G(\omega)| = 20log_{10}C - 20log_{10}\omega^k = -20k\,log_{10}\omega + 20log_{10}C$$

or

$$|G(\omega)|_{dB} = -20k\,log_{10}\omega + 20log_{10}C \tag{8.6}$$

Relation (8.6) is an equation of a straight line in a semilog plot with abscissa $log_{10}\omega$ where

$$slope = -20k\ \frac{dB}{decade}$$

and $intercept = C\ dB$ shown in Figure 8.6.

With these concepts in mind, we can now proceed to discuss *Bode Plots* and *Asymptotic Approximations*.

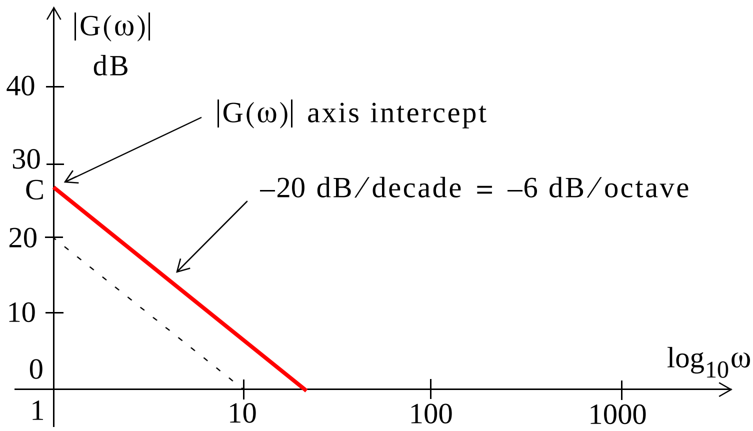


Figure 8.6. Straight line with slope –20 dB/decade = –6 dB/octave

8.4 Bode Plot Scales and Asymptotic Approximations

Bode plots are magnitude and phase plots where the abscissa (frequency axis) is a logarithmic (base 10) scale, and the radian frequency ω is equally spaced between powers of 10 such as 10^{-1}, 10^{0}, 10^{1}, 10^{2} and so on.

The ordinate (dB axis) of the magnitude plot has a scale in dB units, and the ordinate of the phase plot has a scale in degrees as shown in Figure 8.7.

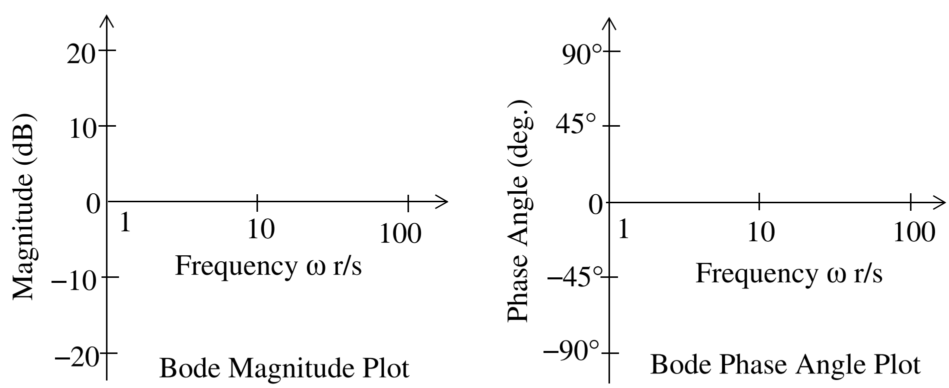


Figure 8.7. Magnitude and phase plots

It is convenient to express the magnitude in dB so that a transfer function $G(s)$, composed of products of terms can be computed by the sum of the dB magnitudes of the individual terms. For example,

$$\frac{20 \cdot \left(1+\dfrac{j\omega}{100}\right)}{1+j\omega} = 20\ \text{dB} + \left(1+\frac{j\omega}{100}\right)\ \text{dB} + \frac{1}{1+j\omega}\ \text{dB}$$

and the Bode plots then can be approximated by straight lines called *asymptotes*.

8.5 Construction of Bode Plots when the Zeros and Poles are Real

Let us consider the transfer function

$$G(s) = \frac{A \cdot (s+z_1) \cdot (s+z_2) \cdot \ldots \cdot (s+z_m)}{s \cdot (s+p_1) \cdot (s+p_2) \cdot (s+p_3) \cdot (s+p_n)} \tag{8.7}$$

where A is a real constant, and the zeros z_i and poles p_i are real numbers. We will consider complex zeros and poles in the next section. Letting $s = j\omega$ in (8.7) we obtain

$$G(j\omega) = \frac{A \cdot (j\omega+z_1) \cdot (j\omega+z_2) \cdot \ldots \cdot (j\omega+z_m)}{j\omega \cdot (j\omega+p_1) \cdot (j\omega+p_2) \cdot (j\omega+p_3) \cdot (j\omega+p_n)} \tag{8.8}$$

Next, we multiply and divide each numerator factor $j\omega + z_i$ by z_i and each denominator factor $j\omega + p_i$ by p_i and we obtain:

$$G(j\omega) = \frac{A \cdot z_1\left(\frac{j\omega}{z_1}+1\right) \cdot z_2\left(\frac{j\omega}{z_2}+1\right) \cdot \ldots \cdot z_m\left(\frac{j\omega}{z_m}+1\right)}{j\omega \cdot p_1\left(\frac{j\omega}{p_1}+1\right) \cdot p_2\left(\frac{j\omega}{p_2}+1\right) \cdot \ldots \cdot p_n\left(\frac{j\omega}{p_n}+1\right)} \tag{8.9}$$

Letting

$$K = \frac{A \cdot z_1 \cdot z_2 \cdot \ldots \cdot z_m}{p_1 \cdot p_2 \cdot \ldots \cdot p_n} = A\frac{\prod_{i=1} z_i}{\prod_{i=1}^{n} p_i} \tag{8.10}$$

we can express (8.9) in dB magnitude and phase form,

$$\begin{aligned}|G(\omega)| = {} & 20\log|K| + 20\log\left(\frac{j\omega}{z_1}+1\right) + 20\log\left(\frac{j\omega}{z_2}+1\right) + \ldots + 20\log\left(\frac{j\omega}{z_m}+1\right) \\ & - 20\log|j\omega| - 20\log\left(\frac{j\omega}{p_1}+1\right) - 20\log\left(\frac{j\omega}{p_2}+1\right) - \ldots - 20\log\left(\frac{j\omega}{p_n}+1\right)\end{aligned} \tag{8.11}$$

$$\begin{aligned}\angle G(\omega) = {} & \angle K + \angle\left(\frac{j\omega}{z_1}+1\right) + \angle\left(\frac{j\omega}{z_2}+1\right) + \ldots + \angle\left(\frac{j\omega}{z_m}+1\right) \\ & - \angle j\omega - \angle\left(\frac{j\omega}{p_1}+1\right) - \angle\left(\frac{j\omega}{p_2}+1\right) - \ldots - \angle\left(\frac{j\omega}{p_n}+1\right)\end{aligned} \tag{8.12}$$

The constant K can be positive or negative. Its magnitude is $|K|$ and its phase angle is $0°$ if $K > 0$, and $-180°$ if $K < 0$. The magnitude and phase plots for the constant K are shown in Figure 8.8.

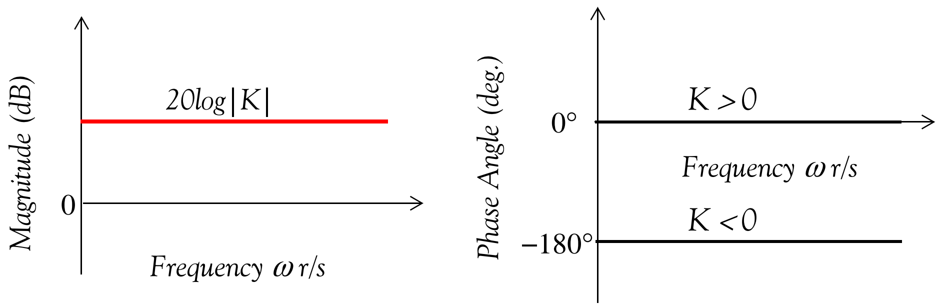


Figure 8.8. Magnitude and phase plots for the constant K

For a zero of order n, that is, $(j\omega)^n$ at the origin, the Bode plots for the magnitude and phase are as shown in Figures 8.9 and 8.10 respectively.

For a pole of order n, that is, $1/(j\omega)^n = (j\omega)^{-n}$ at the origin, the Bode plots are as shown in Figures 8.11 and 8.12 respectively.

Next, we consider the term $G(j\omega) = (a + j\omega)^n$.

The magnitude of this term is

$$|G(j\omega)| = \sqrt{(a^2 + \omega^2)^n} = (a^2 + \omega^2)^{n/2} \tag{8.13}$$

and taking the log of both sides and multiplying by 20 we obtain

$$20\log|G(j\omega)| = 10n\log(a^2 + \omega^2) \tag{8.14}$$

It is convenient to normalize (8.14) by letting

$$u \equiv \omega/a \tag{8.15}$$

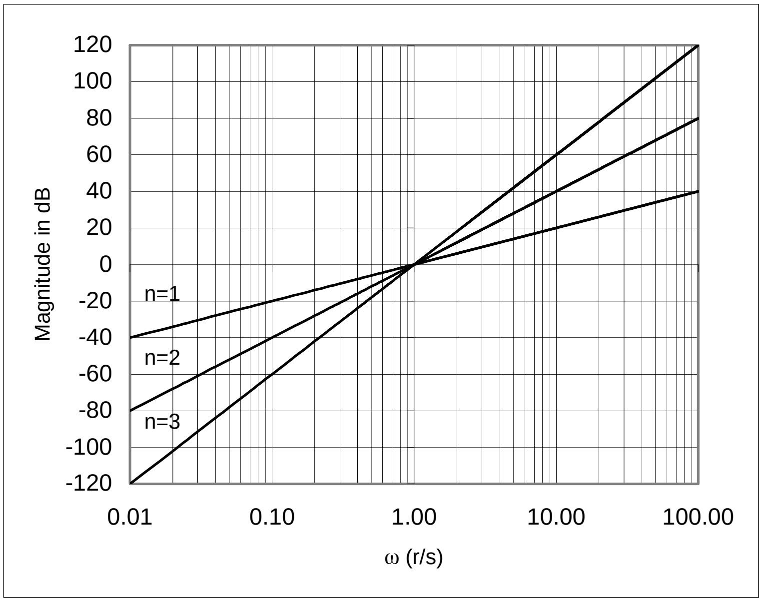


Figure 8.9. Magnitude for zeros of Order n at the origin

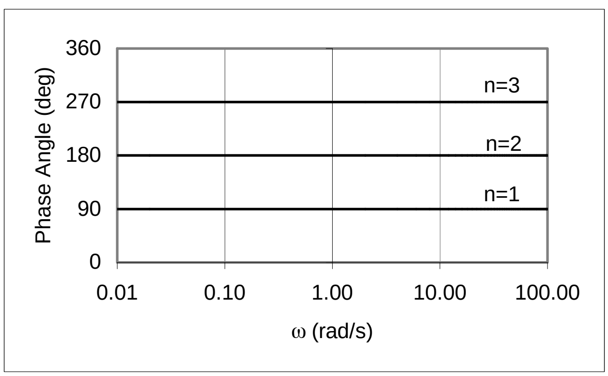


Figure 8.10. Phase for zeros of Order n at the origin

Then, (8.14) becomes

$$20\log|G(ju)| = 10n\log\left(a^2 \cdot \frac{a^2+\omega^2}{a^2}\right) = 10n\log a^2 + 10n\log(1+u^2)$$
$$= 10n\log(1+u^2) + 20n\log a \tag{8.16}$$

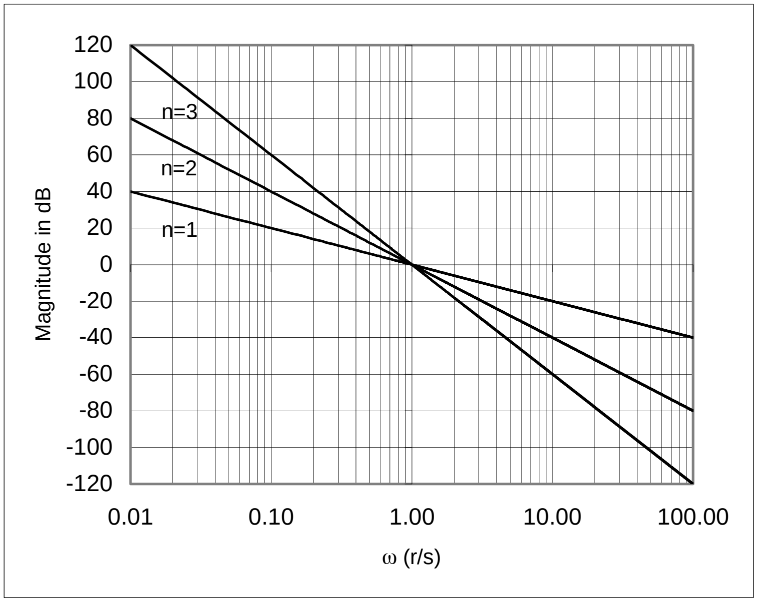


Figure 8.11. Magnitude for poles of Order n at the origin

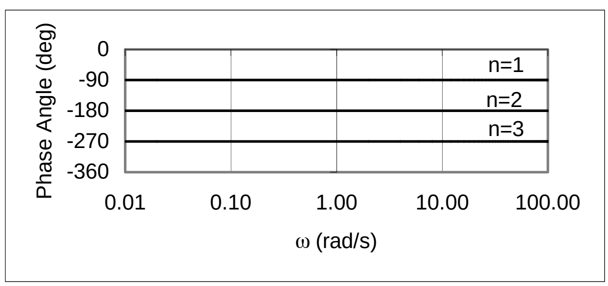


Figure 8.12. Phase for poles of Order n at the origin

For $u \ll 1$ the first term of (8.16) becomes $10n\log 1 = 0\ \mathrm{dB}$. For $u \gg 1$, this term becomes approximately $10n\log u^2 = 20n\log u$ and this has the same form as $G(j\omega) = (j\omega)^n$ which is shown in Figure 8.9 for $n = 1$, $n = 2$, and $n = 3$.

The frequency at which two asymptotes intersect each other forming a corner is referred to as the *corner frequency*. Thus, the two lines defined by the first term of (8.16), one for $u \ll 1$ and the other for $u \gg 1$ intersect at the corner frequency $u = 1$.

The second term of (8.16) represents the ordinate axis intercept defined by this straight line.

The phase response for the term $G(j\omega) = (a + j\omega)^n$ is found as follows:

We let

$$u \equiv \omega / a \tag{8.17}$$

and

$$\phi(u) = \tan^{-1} u \tag{8.18}$$

Then,

$$(a + j\omega)^n = a^n (1 + ju)^n = a^n \left(\sqrt{1 + u^2} \angle \tan^{-1} u \right)^n = a^n (1 + u^2)^{n/2} e^{jn\phi(u)} \tag{8.19}$$

Figure 8.13 shows plots of the magnitude of (8.16) for $a = 10$, $n = 1$, $n = 2$, and $n = 3$.

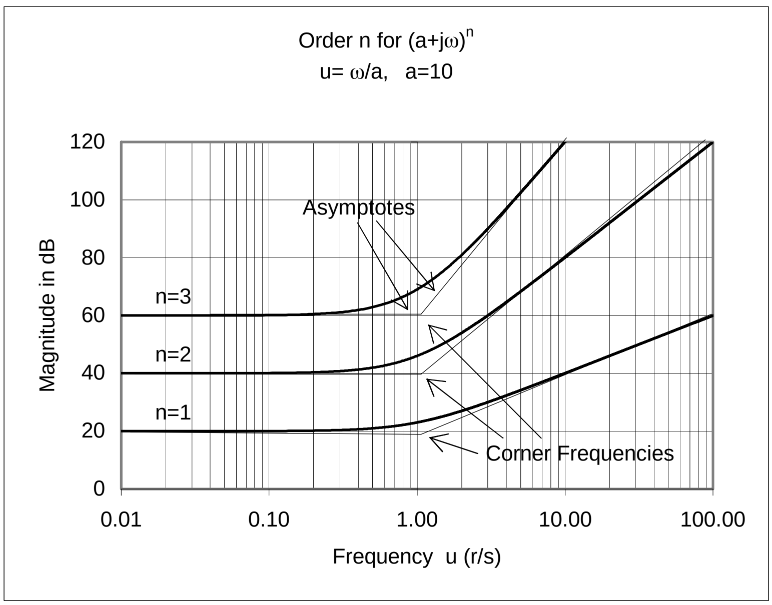


Figure 8.13. Magnitude for zeros of Order n for $(a + j\omega)^n$

As shown in Figure 8.13, a quick sketch can be obtained by drawing the straight line asymptotes given by $10\log 1 = 0$ and $10n\log u^2$ for $u \ll 1$ and $u \gg 1$ respectively.

The phase angle of (8.19) is $n\phi(u)$. Then, with (8.18) and letting

$$n\phi(u) = \theta(u) = n\tan^{-1} u \tag{8.20}$$

we obtain

$$\lim_{u \to 0} \theta(u) = \lim_{u \to 0} n\tan^{-1} u = 0 \tag{8.21}$$

and

$$\lim_{u \to \infty} \theta(u) = \lim_{u \to \infty} n\tan^{-1} u = \frac{n\pi}{2} \tag{8.22}$$

At the corner frequency $u = a$ we obtain $u = 1$ and with (8.20)

$$\theta(1) = n\tan^{-1}1 = \frac{n\pi}{4} \tag{8.23}$$

Figure 8.14 shows the phase angle plot for (8.19).

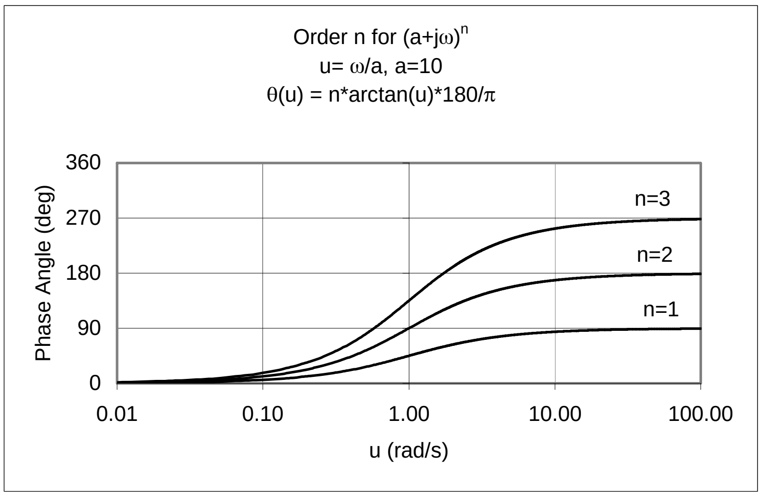


Figure 8.14. Phase for zeros of Order n for $(a + j\omega)^n$

The magnitude and phase plots for $G(j\omega) = 1/(a+j\omega)^n$ are similar to those of $G(j\omega) = (a+j\omega)^n$ except for a minus sign. In this case (8.16) becomes

$$-20\log|G(ju)| = -10n\log(1+u^2) - 20n\log a \tag{8.24}$$

and (8.20) becomes

$$\theta(u) = -n\tan^{-1}u \tag{8.25}$$

The plots for (8.24) and (8.25) are shown in Figures 8.15 and 8.16 respectively.

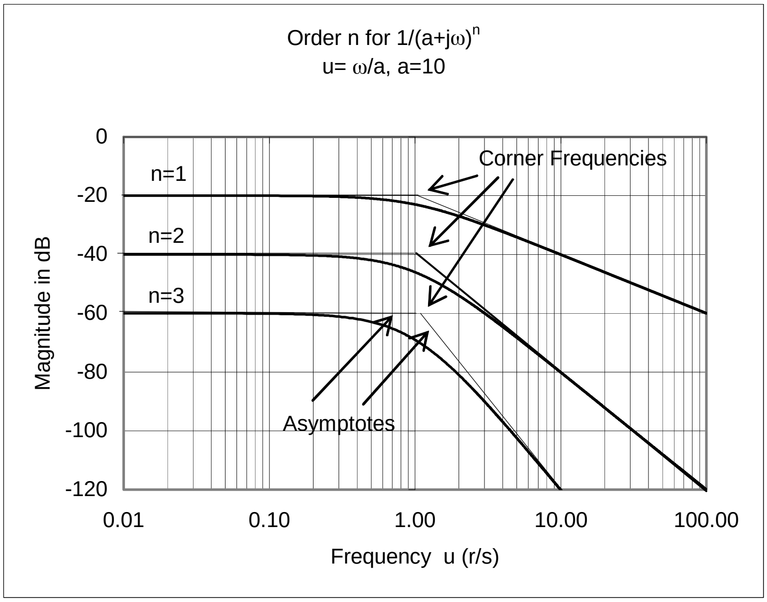


Figure 8.15. Magnitude for poles of Order n for $1/(a+j\omega)^n$

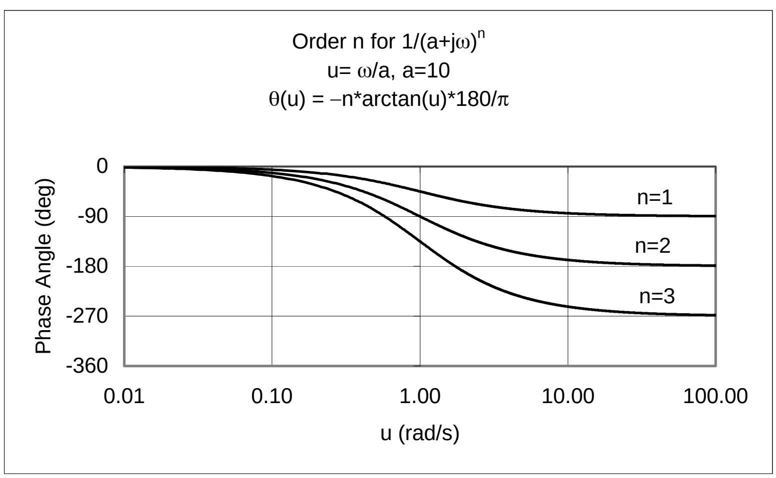


Figure 8.16. Phase for poles of Order n for $1/(a+j\omega)^n$

8.6 Construction of Bode Plots when the Zeros and Poles are Complex

The final type of terms appearing in the transfer function $G(s)$ are quadratic term of the form as^2+bs+c whose roots are complex conjugates. In this case, we express the complex conjugate roots as

$$
\begin{aligned}
(s+\alpha-j\beta)(s+\alpha+j\beta) &= (s+\alpha)^2+\beta^2 \\
&= s^2+2\alpha s+\alpha^2+\beta^2
\end{aligned}
\tag{8.26}
$$

and letting

$$\alpha = \zeta\omega_n \tag{8.27}$$

and

$$\alpha^2+\beta^2 = \omega_n^2 \tag{8.28}$$

by substitution into (8.26) we obtain

$$s^2+2\alpha s+\alpha^2+\beta^2 = s^2+2\zeta\omega_n s+\omega_n^2 \tag{8.29}$$

Next, we let

$$G(s) = s^2+2\zeta\omega_n s+\omega_n^2 \tag{8.30}$$

Then,

$$
\begin{aligned}
G(j\omega) &= (j\omega)^2+j2\omega_n\omega+\omega_n^2 \\
&= (\omega_n^2-\omega^2)+j2\omega_n\omega
\end{aligned}
\tag{8.31}
$$

The magnitude of (8.31) is

$$|G(j\omega)| = \sqrt{(\omega_n^2-\omega^2)^2+4\zeta^2\omega_n^2\omega^2} \tag{8.32}$$

and taking the log of both sides and multiplying by 20 we obtain

$$20\log|G(j\omega)| = 10\log[(\omega_n^2-\omega^2)^2+4\zeta^2\omega_n^2\omega^2] \tag{8.33}$$

As in the previous section, it is convenient to normalize (8.33) by dividing by ω_n^4 to yield a function of the normalized frequency variable u such that

$$u \equiv \omega/\omega_n \tag{8.34}$$

Then, (8.33) is expressed as

$$20\log|G(ju)| = 10\log[(\omega_n^2-\omega^2)^2+4\zeta^2\omega_n^2\omega^2]$$

or

$$
\begin{aligned}
20\log|G(ju)| &= 10\log\left[\omega_n^4\left(\frac{\omega_n^2-\omega^2}{\omega_n^2}\right)^2+4\zeta^2\omega_n^4\frac{\omega^2}{\omega_n^2}\right] = 10\log\left[\omega_n^4\left(\frac{\omega_n^2-\omega^2}{\omega_n^2}\right)^2+4\zeta^2\omega_n^4\frac{\omega^2}{\omega_n^2}\right] \\
&= 10\log\left[\omega_n^4\left\{(1-u^2)^2+4\zeta^2u^2\right\}\right] = 10\log\omega_n^4+10\log[(1-u^2)^2+4\zeta^2u^2]
\end{aligned}
\tag{8.35}
$$

The first term in (8.35) is a constant which represents the ordinate axis intercept defined by this straight line. For the second term, if $u^2 \ll 1$, this term reduces to approximately $10\log 1 = 0\ \text{dB}$

and if $u^2 \gg 1$, this term reduces to approximately $10\log u^4$ and this can be plotted as a straight line increasing at $40\ \mathrm{dB/decade}$. Using these two straight lines as asymptotes for the magnitude curve we see that the asymptotes intersect at the corner frequency $u = 1$. The exact shape of the curve depends on the value of ζ which is called the *damping coefficient*.

A plot of (8.35) for $\zeta = 0.2$, $\zeta = 0.4$, and $\zeta = 0.707$ is shown in Figure 8.17.

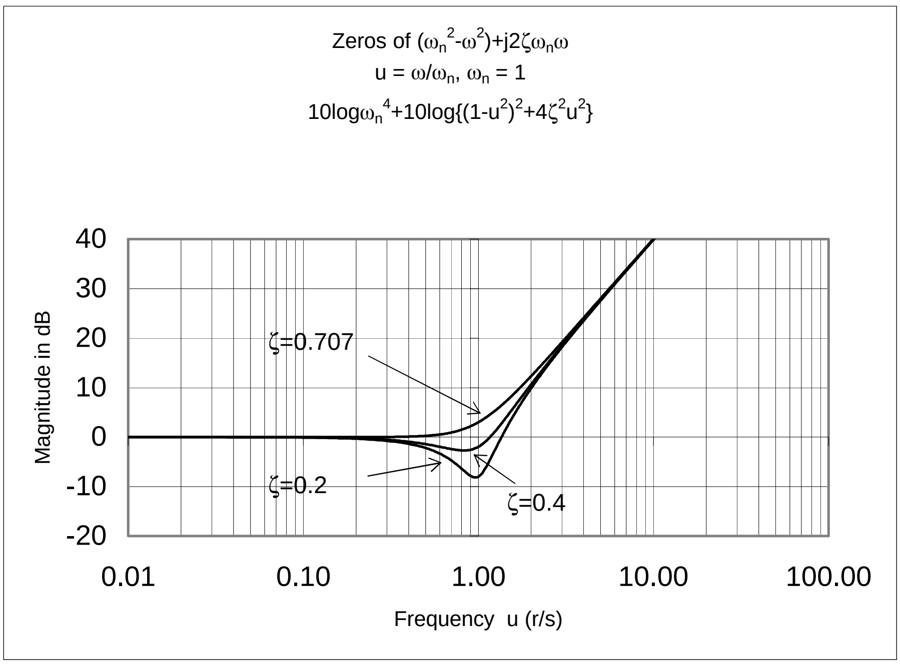


Figure 8.17. Magnitude for zeros of $10\log\omega_n^4 + 10\log[(1-u^2)^2 + 4\zeta^2u^2]$

The phase shift associated with $(\omega_n^2 - \omega^2) + j2\omega_n\omega$ is also simplified by the substitution $u \equiv \omega/\omega_n$ and thus

$$\theta(u) = \tan^{-1}\left(\frac{2\zeta u}{1-u^2}\right) \tag{8.36}$$

The two asymptotic relations of (8.36) are

$$\lim_{u\to 0}\theta(u) = \lim_{u\to 0}\tan^{-1}\left(\frac{2\zeta u}{1-u^2}\right) = 0 \tag{8.37}$$

and

$$\lim_{u \to \infty} \theta(u) = \lim_{u \to \infty} \tan^{-1}\left(\frac{2\zeta u}{1-u^2}\right) = \pi \tag{8.38}$$

At the corner frequency $\omega = \omega_n$, $u = 1$ and

$$\theta(1) = \lim_{u \to 1} \tan^{-1}\left(\frac{2\zeta u}{1-u^2}\right) = \frac{\pi}{2} \tag{8.39}$$

A plot of the phase for $\zeta = 0.2$, $\zeta = 0.4$, and $\zeta = 0.707$ is shown in Figure 8.18.

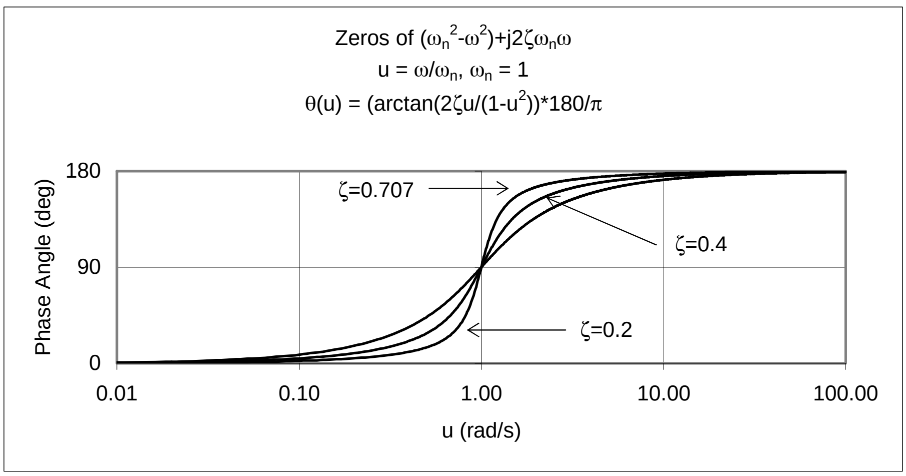


Figure 8.18. Phase for zeros of $10\log\omega_n^4 + 10\log[(1-u^2)^2 + 4\zeta^2u^2]$

The magnitude and phase plots for

$$G(j\omega) = \frac{1}{(\omega_n^2-\omega^2)+j2\omega_n\omega}$$

are similar to those of

$$G(j\omega) = (\omega_n^2-\omega^2)+j2\omega_n\omega$$

except for a minus sign. In this case, (8.35) becomes

$$-10(\log\omega_n^4) - 10\log[(1-u^2)^2 + 4\zeta^2u^2] \tag{8.40}$$

and (8.36) becomes

$$\theta(u) = -\tan^{-1}\left(\frac{2\zeta u}{1-u^2}\right) \tag{8.41}$$

A plot of (8.40) for $\zeta = 0.2$, $\zeta = 0.4$, and $\zeta = 0.707$ is shown in Figure 8.19.

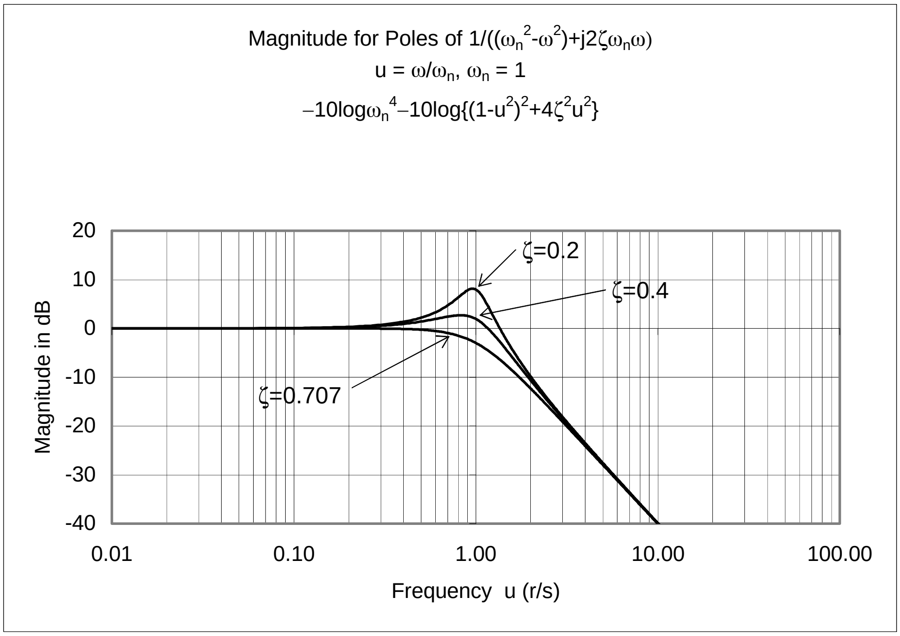


Figure 8.19. Magnitude for poles of $1/10\log\omega_n^4 + 10\log[(1-u^2)^2 + 4\zeta^2u^2]$

A plot of the phase for $\zeta = 0.2$, $\zeta = 0.4$, and $\zeta = 0.707$ is shown in Figure 8.20.

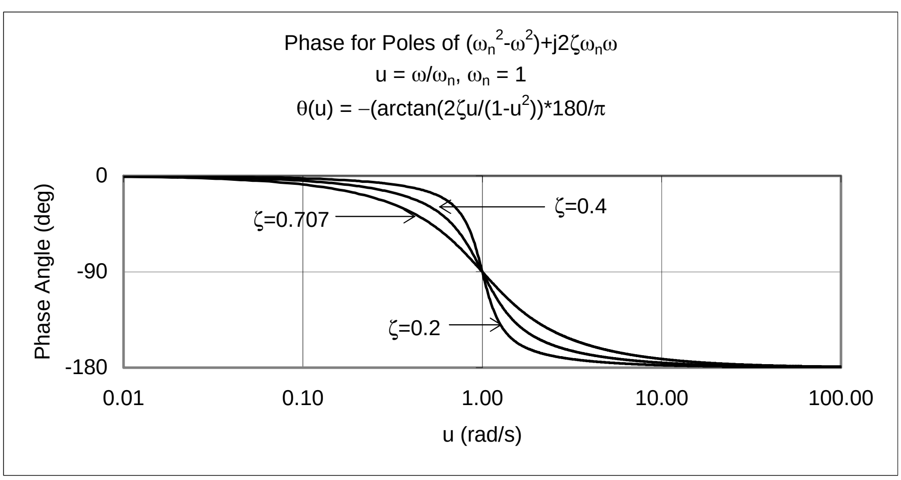


Figure 8.20. Phase for poles of $1/10\log\omega_n^4 + 10\log[(1-u^2)^2 + 4\zeta^2u^2]$

Example 8.3

For the circuit shown in Figure 8.21

a. Compute the transfer function $G(s)$.

b. Construct a straight line approximation for the magnitude of the Bode plot.

c. From the Bode plot obtain the values of $20\log|G(j\omega)|$ at $\omega = 30\ r/s$ and $\omega = 4000\ r/s$. Compare these values with the actual values.

d. If $v_s(t) = 10\cos(5000t + 60°)$, use the Bode plot to compute the output $v_{out}(t)$.

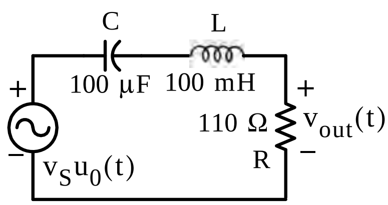


Figure 8.21. Circuit for Example 8.3.

Solution:

a. We transform the given circuit to its equivalent in the s – domain shown in Figure 8.22.

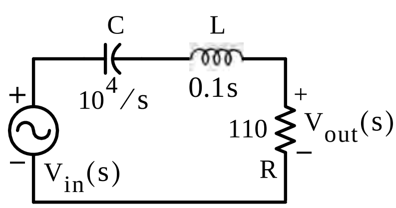


Figure 8.22. Circuit for Example 8.3 in s – domain

By the voltage division expression,

$$V_{out}(s) = \frac{110}{10^4/s + 0.1s + 110} \cdot V_{in}(s)$$

Therefore, the transfer function is

$$G(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{110s}{0.1s^2 + 110s + 10^4} = \frac{1100s}{s^2 + 1100s + 10^5} = \frac{1100s}{(s+100)(s+1000)} \qquad (8.42)$$

b. Letting $s = j\omega$ we obtain

$$G(j\omega) = \frac{1100j\omega}{(j\omega + 100)(j\omega + 1000)}$$

or in standard form

$$G(j\omega) = \frac{0.011j\omega}{(1+j\omega/100)(1+j\omega/1000)} \quad (8.43)$$

Letting the magnitude of (8.43) be denoted as A, and expressing it in decibels we obtain

$$A_{dB} = 20\log|G(j\omega)| = 20\log 0.011 + 20\log|j\omega| - 20\log\left|\left(1+\frac{j\omega}{10}\right)\right| - 20\log\left|\left(1+\frac{j\omega}{100}\right)\right| \quad (8.44)$$

We observe that the first term on the right side of (8.44) is a constant whose value is $20\log 0.011 = -39.17$. The second term is a straight line with slope equal to $20\ dB/decade$. For $\omega < 100\ r/s$ the third term is approximately zero and for $\omega > 100$ it decreases with slope equal to $-20\ dB/decade$. Likewise, for $\omega < 1000\ r/s$ the fourth term is approximately zero and for $\omega > 1000$ it also decreases with slope equal to $-20\ dB/decade$.

For Bode plots we use semilog paper. Instructions to construct semilog paper with Microsoft Excel are provided in Appendix F.

In the Bode plot of Figure 8.23 the individual terms are shown with dotted lines and the sum of these with a solid line.

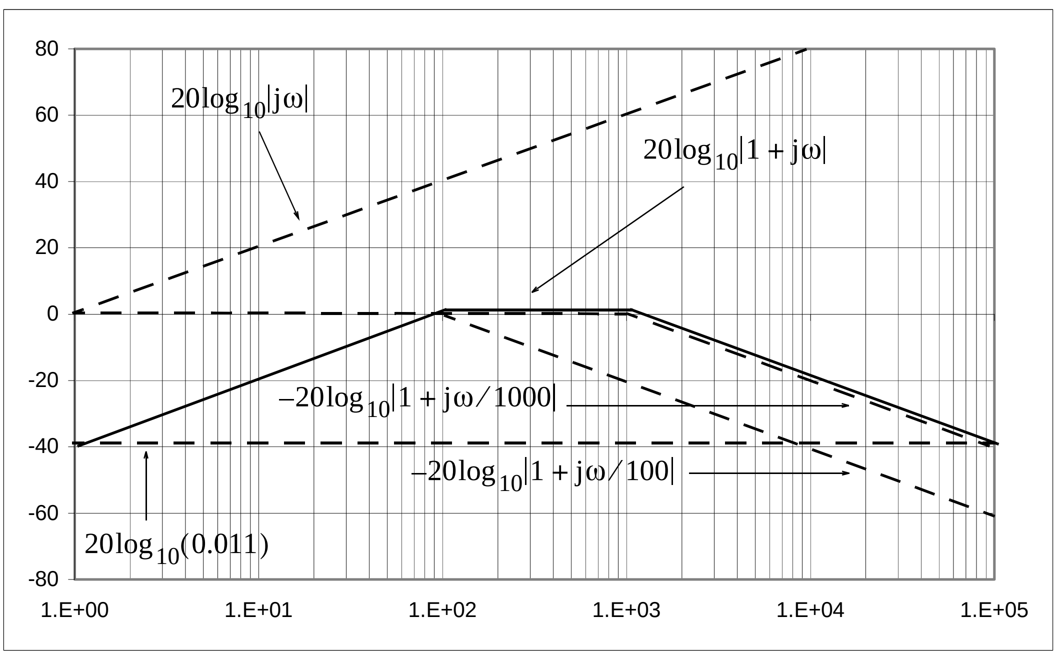


Figure 8.23. Magnitude plot of (8.44)

c. The plot of Figure 8.23 shows that the magnitude of (8.43) at $\omega = 30\ r/s$ is approximately $-9\ dB$ and at $\omega = 4000\ r/s$ is approximately $-10\ dB$. The actual values are found as follows:

At $\omega = 30\ r/s$, (8.43) becomes

$$G(j30) = \frac{0.011 \times j30}{(1 + j0.3)(1 + j0.03)}$$

and using the MATLAB script

```
g30=0.011*30j/((1+0.3j)*(1+0.03j));...
fprintf(' \n'); fprintf('mag = %6.2f \t',abs(g30));...
fprintf('magdB = %6.2f dB',20*log10(abs(g30))); fprintf(' \n'); fprintf(' \n')
```

we obtain

```
mag = 0.32   magdB = -10.01 dB
```

Therefore,

$$|G(j30)| = 0.32$$

and

$$20\log|G(j30)| = 20\log 0.32 \approx -10\ dB$$

Likewise, at $\omega = 4000\ r/s$, (8.43) becomes

$$G(j1000) = \frac{0.11(j4000)}{(1 + j40)(1 + j4)}$$

and using MATLAB script

```
g4000=0.011*4000j/((1+40j)*(1+4j));...
fprintf(' \n'); fprintf('mag = %6.2f \t',abs(g4000));...
fprintf('magdB = %6.2f dB',20*log10(abs(g4000))); fprintf(' \n'); fprintf(' \n')
```

we obtain

```
mag = 0.27    magdB = -11.48 dB
```

Therefore,

$$|G(j4000)| = 0.27$$

and

$$20\log|G(j4000)| = 20\log 0.27 = -11.48\ dB$$

d. From the Bode plot of Figure 8.23, we see that the value of A_{dB} at $\omega = 5000\ r/s$ is approximately $-12\ dB$. Then, since in general $a_{dB} = 20\log b$, and that $y = \log x$ implies $x = 10^y$, we have

$$|A| = 10^{\left(-\frac{12}{20}\right)} = 0.25$$

and therefore

$$V_{out\ max} = |A||V_S| = 0.25 \times 10 = 2.5\ V$$

If we wish to obtain a more accurate value, we substitute $\omega = 5000$ into (8.43) and with the following MATLAB script:

```
g5000=0.011*5000j/((1+50j)*(1+5j));...
fprintf(' \n'); fprintf('mag = %6.2f \t',abs(g5000));...
fprintf('phase = %6.2f deg.',angle(g5000)*180/pi); fprintf(' \n'); fprintf(' \n')
```

and we we obtain

```
mag = 0.22      phase = -77.54 deg.
```

$$G(j5000) = \frac{0.011(j5000)}{(1+j50)(1+j5)} = 0.22\angle{-77.54}$$

Then,

$$V_{out\ max} = |A| \times 10 = 0.22 \times 10 = 2.2 \text{ V}$$

and in the t – domain

$$v_{out}(t) = 2.2\cos(5000t - 77.54°)$$

We can use the MATLAB function **bode(sys)** to draw the Bode plot of a Linear Time Invariant (LTI) System where **sys = tf(num,den)** creates a continuous-time transfer function **sys** with numerator **num** and denominator **den**, and **tf** creates a transfer function. With this function, the frequency range and number of points are chosen automatically. The function **bode(sys,{wmin,wmax})** draws the Bode plot for frequencies between **wmin** and **wmax** (in radians/second) and the function **bode(sys,w)** uses the user-supplied vector **w** of frequencies, in radians/second, at which the Bode response is to be evaluated. To generate logarithmically spaced frequency vectors, we use the command **logspace(first_exponent,last_exponent, number_of_values)**. For example, to generate plots for 100 logarithmically evenly spaced points for the frequency interval $10^{-1} \le \omega \le 10^{2}$ r/s, we use the statement logspace(–1,2,100).

The **bode(sys,w)** function displays both magnitude and phase. If we want to display the magnitude only, we can use the **bodemag(sys,w)** function.

MATLAB requires that we express the numerator and denominator of $G(s)$ as polynomials of s in descending powers.

Let us plot the transfer function of Example 8.3 using MATLAB.

From (8.42),

$$G(s) = \frac{1100s}{s^2 + 1100s + 10^5}$$

and the MATLAB script to generate the magnitude and phase plots is as follows:

```
num=[0 1100 0]; den=[1 1100 10^5]; w=logspace(0,5,100); bode(num,den,w)
```

However, since for this example we are interested in the magnitude only, we will use the script

```
num=[0 1100 0]; den=[1 1100 10^5]; sys=tf(num,den);...
w=logspace(0,5,100); bodemag(sys,w); grid
```

and upon execution, MATLAB displays the plot shown in Figure 8.24.

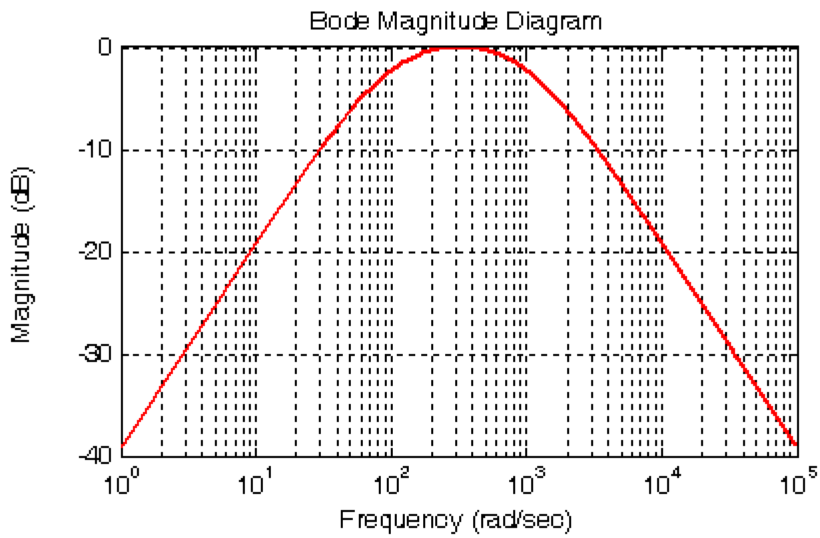


Figure 8.24. Bode plot for Example 8.3.

Example 8.4

For the circuit in Example 8.3

a. Draw a Bode phase plot.

b. Using the Bode phase plot estimate the frequency where the phase is zero degrees.

c. Compute the actual frequency where the phase is zero degrees.

d. Find $v_{out}(t)$ if $v_{in}(t) = 10\cos(\omega t + 60°)$ and ω is the value found in part (c).

Solution:

a. From (8.43) of Example 8.3

$$G(j\omega) = \frac{0.011j\omega}{(1 + j\omega/100)(1 + j\omega/1000)} \tag{8.45}$$

and in magnitude–phase form

$$G(j\omega) = \frac{0.011|j\omega|}{|(1 + j\omega/100)||(1 + j\omega/1000)|}\angle(\alpha - \beta - \gamma)$$

where

$$\angle\alpha = 90° \qquad \angle-\beta = -\tan^{-1}(\omega/100) \qquad \angle-\gamma = -\tan^{-1}(\omega/1000)$$

For $\omega = 100$

$$\angle-\beta = -\tan^{-1}1 = -45°$$

For $\omega = 1000$

$$\angle -\gamma = -\tan^{-1}1 = -45°$$

The straight-line phase angle approximations are shown in Figure 8.25.

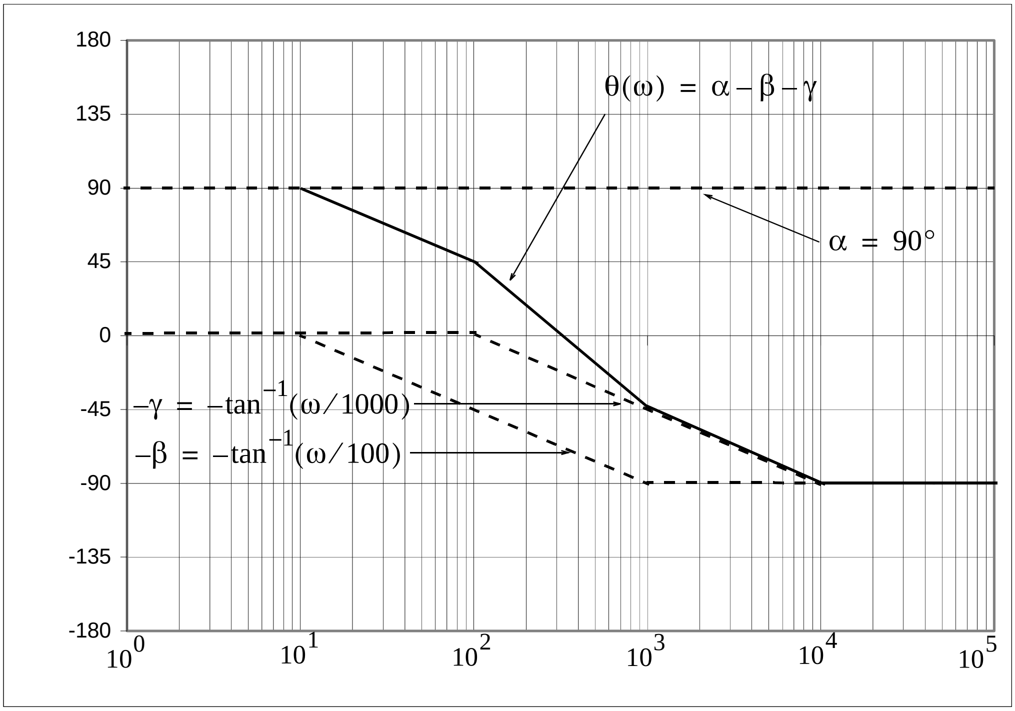


Figure 8.25. Bode plot for Example 8.4.

Figure 8.26 shows the magnitude and phase plots generated with the following MATLAB script:

```
num=[0 1100 0]; den=[1 1100 10^5]; w=logspace(0,5,100); bode(num,den,w)
```

b. From the Bode plot of Figure 8.25 we find that the phase is zero degrees at approximately $\omega = 310\ r/s$

c. From (8.45)

$$G(j\omega) = \frac{0.011j\omega}{(1 + j\omega/100)(1 + j\omega/1000)}$$

and in magnitude–phase form

$$G(j\omega) = \frac{0.011\omega\angle 90°}{|(1 + j\omega/100)|\angle\tan^{-1}(\omega/100)|(1 + j\omega/1000)|\angle\tan^{-1}(\omega/1000)}$$

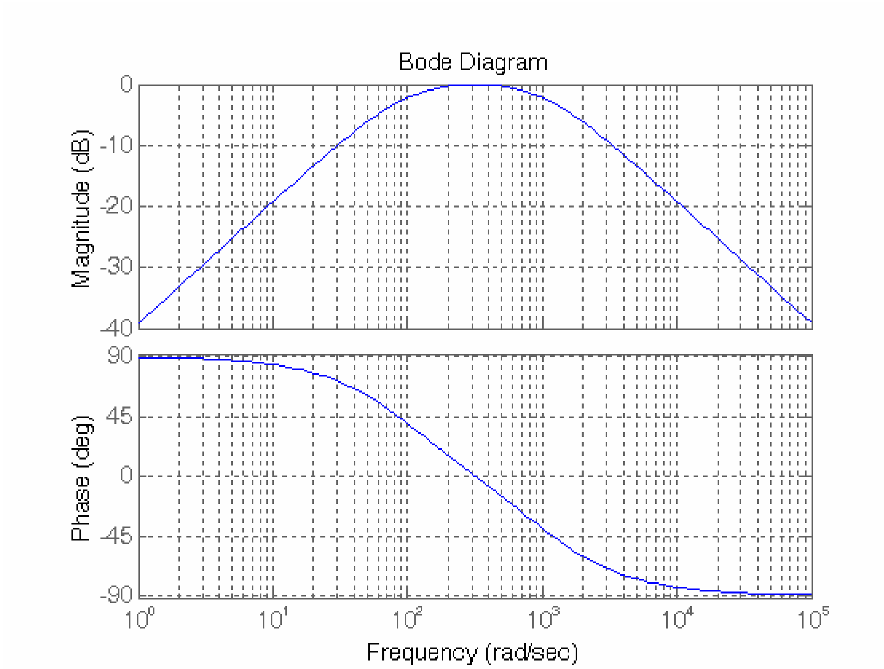


Figure 8.26. Bode plots for Example 8.4 generated with the MATLAB bode *function*

The phase will be zero when

$$\tan^{-1}(\omega/100) + \tan^{-1}(\omega/1000) = 90°$$

This is a trigonometric equation and we will solve it for ω with the **solve(equ)** MATLAB function as follows:

```
syms w; x=solve(atan(w/100)+atan(w/1000)–pi/2)
```

```
ans =
  316.2278
```

Therefore, $\omega = 316.23\ r/s$

d. Evaluating (8.45) at $\omega = 316.23\ r/s$ we obtain:

$$G(j316.23) = \frac{0.011(j316.23)}{(1 + j316.23/100)(1 + j316.23/1000)} \tag{8.46}$$

and with the MATLAB script

```
Gj316=0.011*316.23j/((1+316.23j/100)*(1+316.23j/1000)); fprintf(' \n');...
fprintf('magGj316 = %5.2f \t', abs(Gj316));...
fprintf('phaseGj316 = %5.2f deg.', angle(Gj316)*180/pi)
```

we obtain

```
magGj316 = 1.00  phaseGj316 = -0.00 deg.
```

We are given that $|V_{in}| = 10\ V$ and with $|G(j316.23)| = 1$, we obtain

$$|V_{out}| = |G(j316.23)||V_{in}| = 1 \times 10 = 10\ V$$

The phase angle of the input voltage is given as $\theta_{in} = 60°$ and with $\theta(j316.23) = 0°$ we find that the phase angle of the output voltage is

$$\theta_{out} = \theta_{in} + \theta(j316.23) = 60° + 0° = 60°$$

and thus

$$V_{out} = 10\angle 60°$$

or

$$v_{out}(t) = 10\cos(316.23t + 60°)$$

8.7 Corrected Amplitude Plots

The amplitude plots we have considered thus far are approximate. We can make the straight line more accurate by drawing smooth curves connecting the points at one-half the corner frequency $\omega_n/2$, the corner frequency ω_n and twice the corner frequency $2\omega_n$ as shown in Figure 8.27.

At the corner frequency ω_n, the value of the amplitude A in dB is

$$A_{dB}\big|_{\omega = \omega_n} = \pm 20\log|1 + j| = \pm 20\log\sqrt{2} = \pm 3\ dB \tag{8.47}$$

where the plus (+) sign applies to a first order zero, and the minus (–) to a first order pole. Similarly,

$$A_{dB}\big|_{\omega = \omega_n/2} = \pm 20\log|1 + j/2| = \pm 20\log\sqrt{\frac{5}{4}} = \pm 0.97\ dB \approx \pm 1\ dB \tag{8.48}$$

and

$$A_{dB}\big|_{\omega = 2\omega_n} = \pm 20\log|1 + j2| = \pm 20\log\sqrt{5} = \pm 6.99\ dB \approx \pm 7\ dB \tag{8.49}$$

As we can seen from Figure 8.27, the straight line approximations, shown by dotted lines, yield $0\ dB$ at half the corner frequency and at the corner frequency. At twice the corner frequency, the straight line approximations yield $\pm 6\ dB$ because ω_n and $2\omega_n$ are separated by one octave which is equivalent to $\pm 3\ dB$ per decade. Therefore, the corrections to be made are $\pm 1\ dB$ at half the corner frequency $\omega_n/2$, $\pm 3\ dB$ at the corner frequency ω_n, and $\pm 1\ dB$ at twice the corner frequency $2\omega_n$.

The corrected amplitude plots for a first order zero and first order pole are shown by solid lines in Figure 8.27.

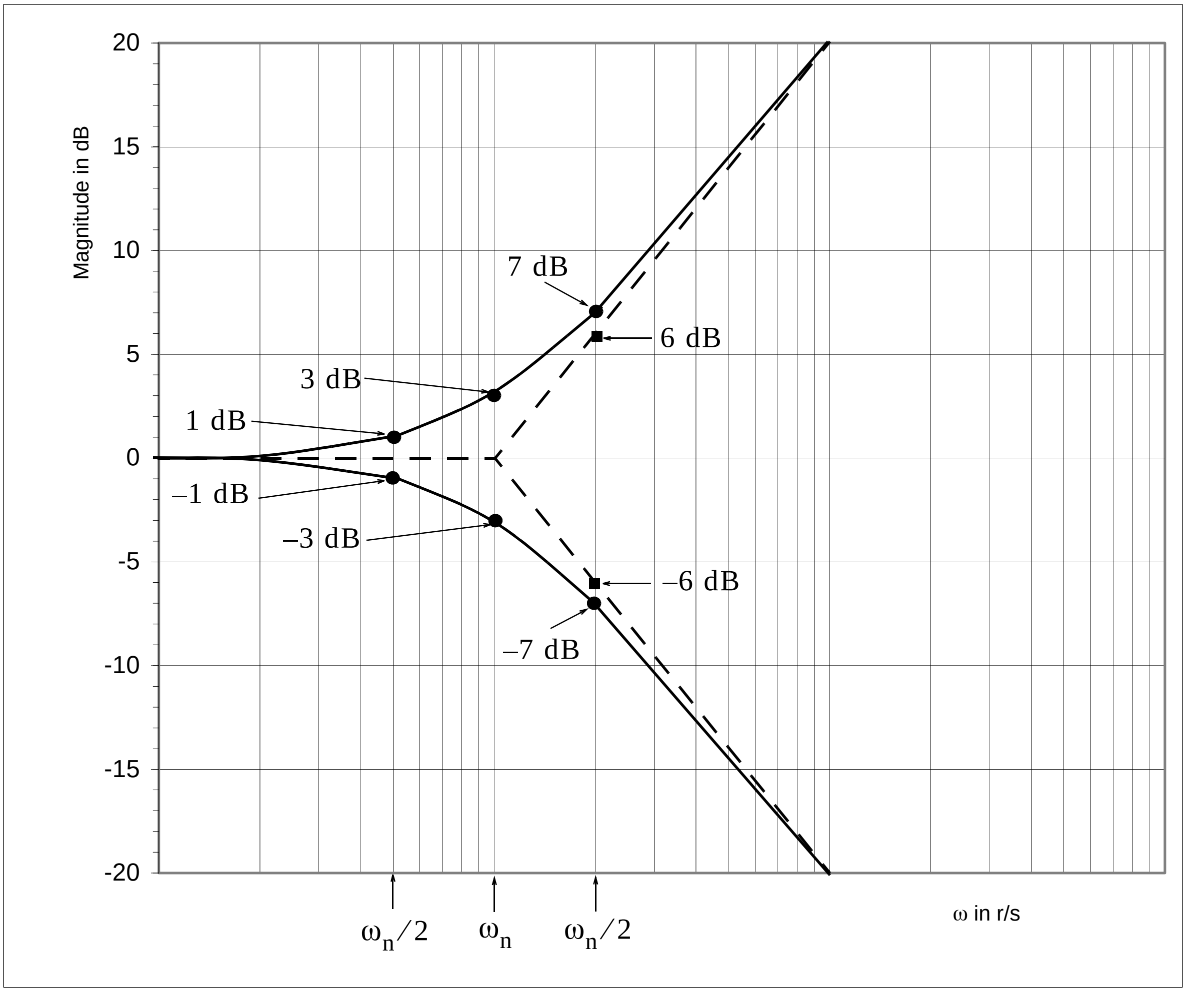


Figure 8.27. Corrections for magnitude Bode plots

The corrections for straight–line amplitude plots when we have complex poles and zeros require different type of correction because they depend on the damping coefficient ζ. Let us refer to the plot in Figure 8.28.

We observe that as the damping coefficient ζ becomes smaller and smaller, larger and larger peaks in the amplitude occur in the vicinity of the corner frequency ω_n. We also observe that when $\zeta \ge 0.707$, the amplitude at the corner frequency ω_n lies below the straight line approximation.

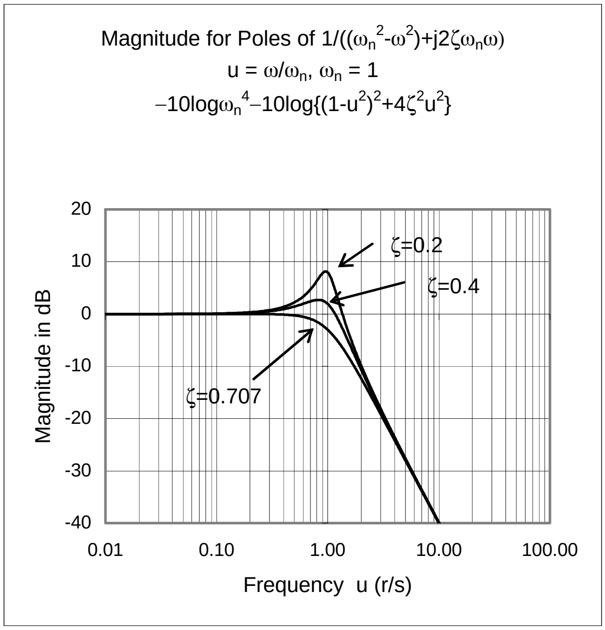


Figure 8.28. Magnitude Bode plots with complex poles

We can obtain a fairly accurate amplitude plot by computing the amplitude at four points near the corner frequency ω_n as shown in Figure 8.28.

The amplitude plot of Figure 8.28 is for complex poles. In analogy with (8.30), i.e., the plot in Figure 8.28 above, we obtain

$$G(s) = s^2 + 2\zeta\omega_n s + \omega_n^2$$

which was derived earlier for complex zeros, the transfer function for complex poles is

$$G(s) = \frac{C}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad (8.50)$$

where C is a constant.

Dividing each term of the denominator of (8.50) by ω_n we obtain

$$G(s) = \frac{C}{\omega_n^2}\,\frac{1}{(s/\omega_n)^2 + 2\zeta(s/\omega_n) + 1}$$

and letting $C/\omega_n^2 = K$ and $s = j\omega$, we obtain

$$G(j\omega) = \frac{K}{1-(\omega/\omega_n)^2 + j2\zeta\omega/\omega_n} \tag{8.51}$$

As before, we let $\omega/\omega_n = u$. Then (8.51) becomes

$$G(ju) = \frac{K}{1-u^2+j2\zeta u} \tag{8.52}$$

and in polar form,

$$G(ju) = \frac{K}{|1-u^2+j2\zeta u|\angle\theta} \tag{8.53}$$

The magnitude of (8.53) in dB is

$$\begin{aligned} A_{dB} &= 20\log|G(ju)| = 20\log K - 20\log|(1-u^2+j2\zeta u)| \\ &= 20\log K - 20\log\sqrt{(1-u^2)^2+4\zeta^2u^2} = 20\log K - 10\log[u^4+2u^2(2\zeta^2-1)+1] \end{aligned} \tag{8.54}$$

and the phase is

$$\theta(u) = -\tan^{-1}\frac{2\zeta u}{1-u^2} \tag{8.55}$$

In (8.54) the term $20\log K$ is constant and thus the amplitude A_{dB}, as a function of frequency, is dependent only the second term on the right side. Also, from this expression, we observe that as $u \to 0$,

$$-10\log[u^4+2u^2(2\zeta^2-1)+1] \to 0 \tag{8.56}$$

and as $u \to \infty$,

$$-10\log[u^4+2u^2(2\zeta^2-1)+1] \to -40\log u \tag{8.57}$$

We are now ready to compute the values of A_{dB} at points 1, 2, 3, and 4 of the plot of Figure 8.29.

At point 1, the corner frequency ω_n corresponds to $u = 1$. Then, from (8.54)

$$\begin{aligned} A_{dB}(\omega_n/2) &= A_{dB}\left(\frac{u}{2}\right) = -10\log[u^4+2u^2(2\zeta^2-1)+1]\Big|_{u=1/2} \\ &= -10\log\left[\frac{1}{16}+2\cdot\frac{1}{4}(2\zeta^2-1)+1\right] = -10\log\left[\frac{1}{16}+\zeta^2-\frac{1}{2}+1\right] \\ &= -10\log(\zeta^2+0.5625) \end{aligned} \tag{8.58}$$

and for $\zeta = 0.4$,

$$A_{dB}(\omega_n/2)\Big|_{point\ 1} = -10\log(0.4^2 + 0.5625) = 1.41\ dB$$

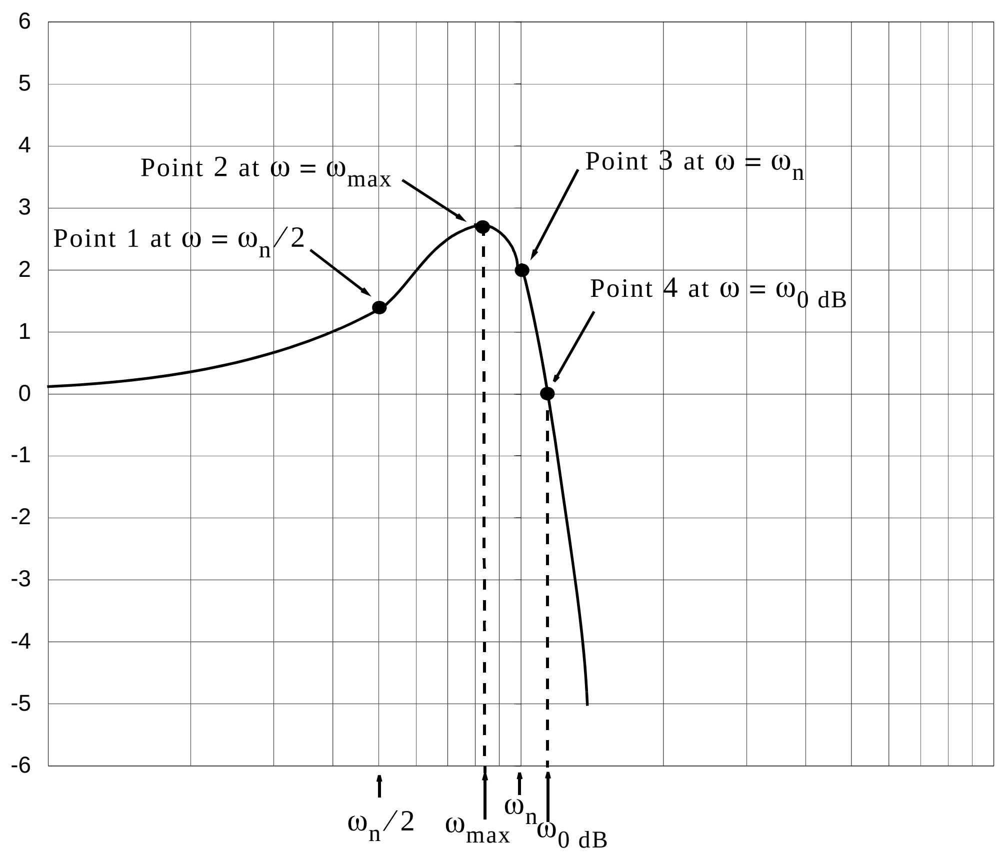


Figure 8.29. Corrections for magnitude Bode plots with complex poles when $\zeta = 0.4$

To find the amplitude at point 2, in (8.54) we let $K = 1$ and we form the magnitude in dB. Then,

$$A_{dB}\Big|_{point\ 2} = 20\log\frac{1}{\left|1-(\omega/\omega_n)^2 + j2\zeta\omega/\omega_n\right|} \tag{8.59}$$

We now recall that the logarithmic function is a monotonically increasing function and therefore (8.59 has a maximum when the absolute magnitude of this expression is maximum. Also, the square of the absolute magnitude is maximum when the absolute magnitude is maximum.

The square of the absolute magnitude is

$$\frac{1}{[1-(\omega/\omega_n)^2]^2+4(\zeta\omega/\omega_n)^2} \tag{8.60}$$

or

$$\frac{1}{1-2\omega^2/\omega_n^2+\omega^4/\omega_n^4+4\zeta^2\omega^2/\omega_n^2} \tag{8.61}$$

To find the maximum, we take the derivative with respect to ω and we set it equal to zero, that is,

$$\frac{4\omega/\omega_n^2-4\omega^3/\omega_n^4-8\zeta^2\omega/\omega_n^2}{\left\{[1-(\omega/\omega_n)^2]^2+4((\zeta\omega)/\omega_n)^2\right\}^2} = 0 \tag{8.62}$$

The expression of (8.62) will be zero when the numerator is set to zero, that is,

$$(\omega/\omega_n^2)(4-4\omega^2/\omega_n^2-8\zeta^2) = 0 \tag{8.63}$$

Of course, we require that the value of ω must be a nonzero value. Then,

$$4-4\omega^2/\omega_n^2-8\zeta^2 = 0$$

or

$$(4\omega^2)/\omega_n^2 = 4-8\zeta^2$$

from which

$$\omega_{max} = \omega = \omega_n\sqrt{1-2\zeta^2} \tag{8.64}$$

provided that $1-2\zeta^2>0$ or $\zeta<1/\sqrt{2}$ or $\zeta<0.707$.

The dB value of the amplitude at point 2 is found by substitution of (8.64) into (8.54), that is,

$$\begin{aligned} A_{dB}(\omega_{max}) &= -10\log[u^4+2u^2(2\zeta^2-1)+1]\Big|_{u=\sqrt{1-2\zeta}} \\ &= -10\log[(1-2\zeta^2)^2+2(1-2\zeta^2)(2\zeta^2-1)+1] \\ &= -10\log[4\zeta^2(1-\zeta^2)] \end{aligned} \tag{8.65}$$

and for $\zeta = 0.4$

$$A_{dB}(\omega_{max}) = -10\log(4\times0.4^2(1-0.4^2)) = 2.69\ dB$$

The dB value of the amplitude at point 3 is found by substitution of $\omega = \omega_n = u = 1$ into (8.54). Then,

$$\begin{aligned} A_{dB}(\omega_n) &= -10\log[u^4 + 2u^2(2\zeta^2 - 1) + 1]\big|_{u=1} \\ &= -10\log[1 + 2(2\zeta^2 - 1) + 1] \\ &= -10\log[4\zeta^2] = -20\log(2\zeta) \end{aligned} \tag{8.66}$$

and for $\zeta = 0.4$

$$A_{dB}(\omega_n) = -20\log(2 \times 0.4) = 1.94\ dB$$

Finally, at point 4, the dB value of the amplitude crosses the 0 dB axis. Therefore, at this point we are interested not in $A_{dB}(\omega_{0\ dB})$ but in the location of $\omega_{0\ dB}$ in relation to the corner frequency ω_n. at this point we must have from (8.57)

$$0\ dB = -10\log[u^4 + 2u^2(2\zeta^2 - 1) + 1]$$

and since $\log 1 = 0$, it follows that

$$u^4 + 2u^2(2\zeta^2 - 1) + 1 = 1$$

$$u^4 + 2u^2(2\zeta^2 - 1) = 0$$

$$u^2(u^2 + 2(2\zeta^2 - 1)) = 0$$

or

$$u^2 + 2(2\zeta^2 - 1) = 0$$

Solving for u and making use of $u = \omega/\omega_n$ we obtain

$$\omega_{0\ dB} = \omega_n\sqrt{2(1 - 2\zeta^2)}$$

From (8.67),

$$\omega_{max} = \omega_n\sqrt{1 - 2\zeta^2}$$

therefore, if we already know the frequency at which the dB amplitude is maximum, we can compute the frequency at point 4 from

$$\omega_{0\ dB} = \sqrt{2}\omega_{max} \tag{8.67}$$

Example 8.5

For the circuit in Figure 8.30,

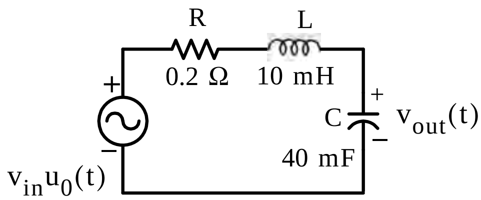


Figure 8.30. Circuit for Example 8.5.

a. Compute the transfer function $G(s)$

b. Find the corner frequency ω_n from $G(s)$.

c. Compute the damping coefficient ζ.

d. Construct a straight line approximation for the magnitude of the Bode plot.

e. Compute the amplitude in dB at one-half the corner frequency $\omega_n/2$, at the frequency ω_{max} at which the amplitude reaches its maximum value, at the corner frequency ω_n, and at the frequency $\omega_{0\ dB}$ where the dB amplitude is zero. Then, draw a smooth curve to connect these four points.

Solution:

a. We transform the given circuit to its equivalent in the $s-\text{domain}$ shown in Figure 8.31 where $R = 1$, $Ls = 0.05s$, and $1/Cs = 25/s$.

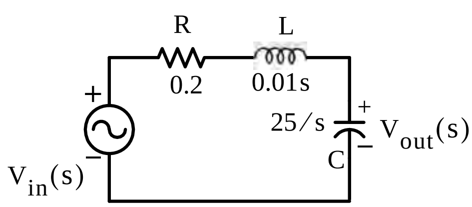


Figure 8.31. Circuit for Example 8.5 in $s-\text{domain}$

and by the voltage division expression,

$$V_{out}(s) = \frac{25/s}{0.2 + 0.01s + 25/s} \cdot V_{in}(s)$$

Therefore, the transfer function is

$$G(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{25}{0.01s^2 + 0.2s + 25} = \frac{2500}{s^2 + 20s + 2500} \qquad (8.68)$$

b. From (8.50),

$$G(s) = \frac{K}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{8.69}$$

and from (8.68) and (8.69) $\omega_n^2 = 2500$ or $\omega_n = 50\ rad/s$

c. From (8.68) and (8.69) $2\zeta\omega_n = 20$. Then, the damping coefficient ζ is

$$\zeta = \frac{20}{2\omega_n} = \frac{20}{2 \times 50} = 0.2 \tag{8.70}$$

d. For $\omega < \omega_n$, the straight line approximation lies along the $0\ dB$ axis, whereas for $\omega > \omega_n$, the straight line approximation has a slope of $-40\ dB$. The corner frequency ω_n was found in part (b) to be $50\ rad/s$ The dB amplitude plot is shown in Figure 8.32.

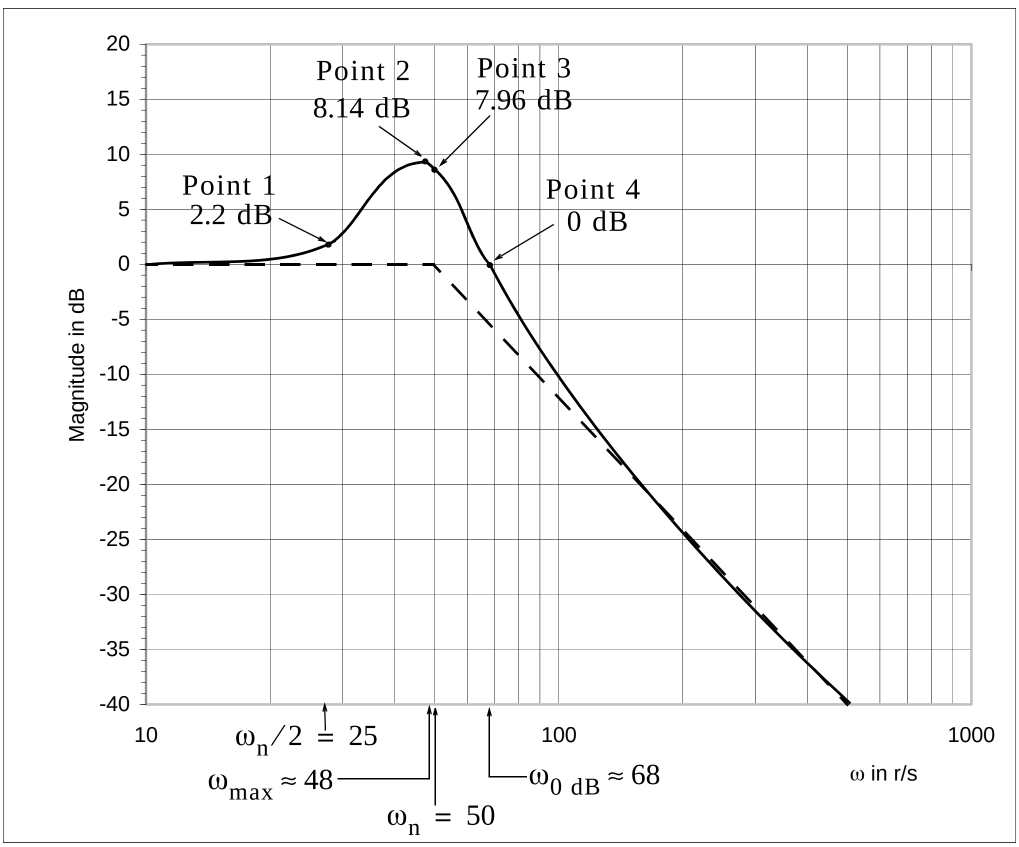


Figure 8.32. Amplitude plot for Example 8.5

e. From (8.61),

$$A_{dB}(\omega_n/2) = -10\log(\zeta^2 + 0.5625)$$

where from (8.74) $\zeta = 0.2$ and thus $\zeta^2 = 0.04$. Then,

$$A_{dB}(\omega_n/2) = -10\log(0.04 + 0.5625) = -10\log(0.6025) = 2.2 \text{ dB}$$

and this value is indicated as Point 1 on the plot of Figure 8.32.

Next, from (8.64)

$$\omega_{max} = \omega_n\sqrt{1-2\zeta^2}$$

Then,

$$\omega_{max} = 50\sqrt{1-2\times 0.04} = 50\sqrt{0.92} = 47.96 \text{ rad/s}$$

Therefore, from (8.65)

$$A_{dB}(\omega_{max}) = -10\log[4\zeta^2(1-\zeta^2)] = -10\log[(0.16)\times(0.96)] = 8.14 \text{ dB}$$

and this value is indicated as Point 2 on the plot of Figure 8.32.

The dB amplitude at the corner frequency is found from (8.66), that is,

$$A_{dB}(\omega_n) = -20\log(2\zeta)$$

Then,

$$A_{dB}(\omega_n) = -20\log(2\times 0.2) = 7.96 \text{ dB}$$

and this value is indicated as Point 3 on the plot of Figure 8.32.

Finally, the frequency at which the amplitude plot crosses the 0 dB axis is found from (8.67), that is,

$$\omega_{0\ dB} = \sqrt{2}\omega_{max}$$

or

$$\omega_{0\ dB} = \sqrt{2}\times 47.96 = 67.83 \text{ rad/s}$$

This frequency is indicated as Point 4 on the plot of Figure 8.32.

The amplitude plot of Figure 8.32 reveals that the given circuit behaves as a low pass filter.

Using the transfer function of (8.68) with the MATLAB script below, we obtain the Bode magnitude plot shown in Figure 8.33.

```
num=[0 0 2500]; den=[1 20 2500]; sys=tf(num,den); w=logspace(0,5,100); bodemag(sys,w)
```

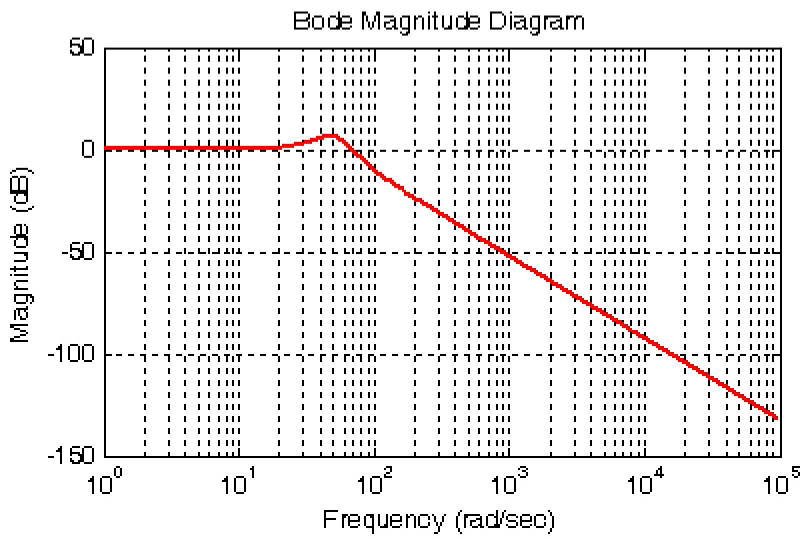


Figure 8.33. Bode plot for Example 8.5 using the MATLAB bodemag *function*

8.8 Summary

- The decibel, denoted as dB, is a unit used to express the ratio between two amounts of power, generally P_{out}/P_{in}. By definition, the number of dB is obtained from $dB_w = 10\log_{10}(P_{out}/P_{in})$. It can also be used to express voltage and current ratios provided that the voltages and currents have identical impedances. Then, for voltages we use the expression $dB_v = 20\log_{10}(V_{out}/V_{in})$, and for currents we use the expression $dB_i = 20\log_{10}(I_{out}/I_{in})$

- The bandwidth, denoted as BW, is a term generally used with electronic amplifiers and filters. For low-pass filters the bandwidth is the band of frequencies from zero frequency to the cutoff frequency where the amplitude fall to 0.707 of its maximum value. For high-pass filters the bandwidth is the band of frequencies from 0.707 of maximum amplitude to infinite frequency. For amplifiers, band-pass, and band-elimination filters the bandwidth is the range of frequencies where the maximum amplitude falls to 0.707 of its maximum value on either side of the frequency response curve.

- If two frequencies ω_1 and ω_2 are such that $\omega_2 = 2\omega_1$, we say that these frequencies are separated by one octave and if $\omega_2 = 10\omega_1$, they are separated by one decade.

- Frequency response is a term used to express the response of an amplifier or filter to input sinusoids of different frequencies. The response of an amplifier or filter to a sinusoid of frequency ω is completely described by the magnitude $|G(j\omega)|$ and phase $\angle G(j\omega)$ of the transfer function.

- Bode plots are frequency response diagrams of magnitude and phase versus frequency ω.

- In Bode plots the 3-dB frequencies, denoted as ω_n, are referred to as the corner frequencies.

- In Bode plots, the transfer function is expressed in linear factors of the form $j\omega + z_i$ for the zero (numerator) linear factors and $j\omega + p_i$ for the pole linear factors. When quadratic factors with complex roots occur in addition to the linear factors, these quadratic factors are expressed in the form $(j\omega)^2 + j2\zeta\omega_n\omega + \omega_n^2$.

- In magnitude Bode plots with quadratic factors the difference between the asymptotic plot and the actual curves depends on the value of the damping factor ζ. But regardless of the value of ζ, the actual curve approaches the asymptotes at both low and high frequencies.

- In Bode plots the corner frequencies ω_n are easily identified by expressing the linear terms as $z_i(j\omega/z_i + 1)$ and $p_i(j\omega/p_i + 1)$ for the zeros and poles respectively. For quadratic factor the corner frequency ω_n appears in the expression $(j\omega)^2 + j2\zeta\omega_n\omega + \omega_n^2$ or $(j\omega/\omega_n)^2 + j2\zeta\omega/\omega_n + 1$
- In both the magnitude and phase Bode plots the frequency (abscissa) scale is logarithmic. The ordinate in the magnitude plot is expressed in dB and in the phase plot is expressed in degrees.
- In magnitude Bode plots, the asymptotes corresponding to the linear terms of the form $(j\omega/z_i + 1)$ and $(j\omega/p_i + 1)$ have a slope $\pm 20\ dB/decade$ where the positive slope applies to zero (numerator) linear factors, and the negative slope applies to pole (denominator) linear factors.
- In magnitude Bode plots, the asymptotes corresponding to the quadratic terms of the form $(j\omega/\omega_n)^2 + j2\zeta\omega/\omega_n + 1$ have a slope $\pm 40\ dB/decade$ where the positive slope applies to zero (numerator) quadratic factors, and the negative slope applies to pole (denominator) quadratic factors.
- In phase Bode plots with linear factors, for frequencies less than one tenth the corner frequency we assume that the phase angle is zero. At the corner frequency the phase angle is $\pm 45°$. For frequencies ten times or greater than the corner frequency, the phase angle is approximately $\pm 90°$ where the positive angle applies to zero (numerator) linear factors, and the negative angle applies to pole (denominator) linear factors.
- In phase Bode plots with quadratic factors, the phase angle is zero for frequencies less than one tenth the corner frequency. At the corner frequency the phase angle is $\pm 90°$. For frequencies ten times or greater than the corner frequency, the phase angle is approximately $\pm 180°$ where the positive angle applies to zero (numerator) quadratic factors, and the negative angle applies to pole (denominator) quadratic factors.
- Bode plots can be easily constructed and verified with the MATLAB function **bode(sys)** function. With this function, the frequency range and number of points are chosen automatically. The function **bode(sys),{wmin,wmax})** draws the Bode plot for frequencies between **wmin** and **wmax** (in radian/second) and the function **bode(sys,w)** uses the user-supplied vector **w** of frequencies, in radian/second, at which the Bode response is to be evaluated. To generate logarithmically spaced frequency vectors, we use the command **logspace(first_exponent,last_exponent, number_of_values)**.

8.9 Exercises

1. For the transfer function

$$G(s) = \frac{10^5(s+5)}{(s+100)(s+5000)}$$

   a. Draw the magnitude Bode plot and find the approximate maximum value of $|G(j\omega)|$ in dB.

   b. Find the value of ω where $|G(j\omega)| = 1$ for $\omega > 5\ r/s$

   c. Check your plot with the plot generated with MATLAB.

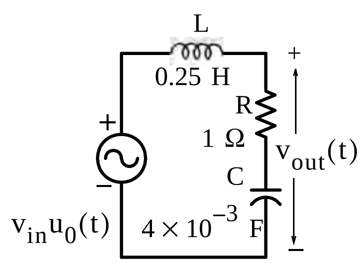
2. For the transfer function of Exercise 1:

   a. Draw a Bode plot for the phase angle and find the approximate phase angle at $\omega = 30\ r/s$, $\omega = 50\ r/s$, $\omega = 100\ r/s$, and $\omega = 5000\ r/s$

   b. Compute the actual values of the phase angle at the frequencies specified in (a).

   c. Check your magnitude plot of Exercise 1 and the phase plot of this exercise with the plots generated with MATLAB.

3. For the circuit below:



   a. Compute the transfer function.
   b. Draw the Bode amplitude plot for $20\log|G(j\omega)|$
   c. From the plot of part (b) determine the type of filter represented by this circuit and estimate the cutoff frequency.

   d. Compute the actual cutoff frequency of this filter.

   e. Draw a straight line phase angle plot of $G(j\omega)$.

   f. Determine the value of $\theta(\omega)$ at the cutoff frequency from the plot of part (c).

   g. Compute the actual value of $\theta(\omega)$ at the cutoff frequency.

8.10 Solutions to End-of-Chapter Exercises

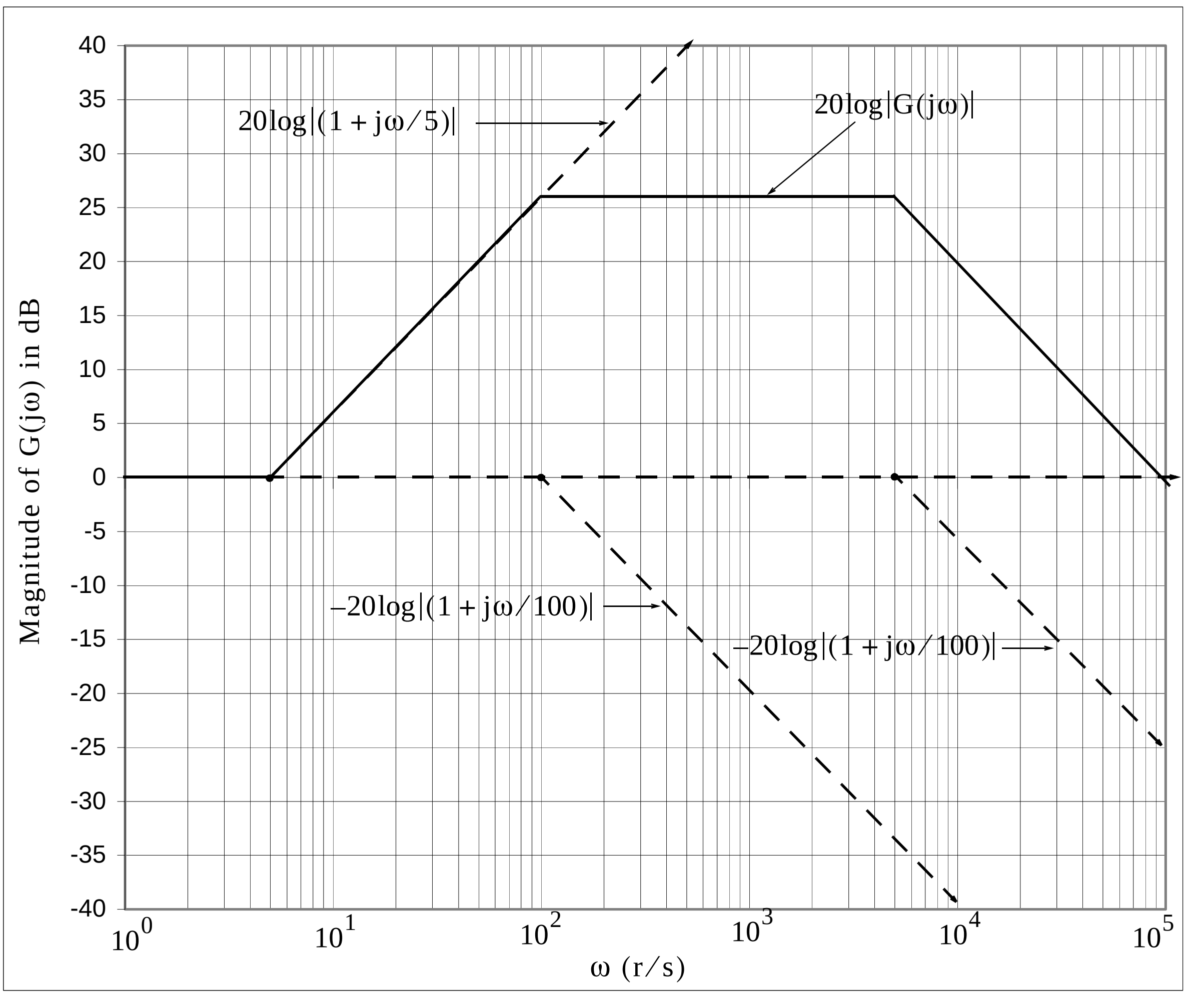
1. a.

$$G(j\omega) = \frac{10^5(j\omega+5)}{(j\omega+100)(j\omega+5000)} = \frac{10^5\times 5\times(1+j\omega/5)}{100\times(1+j\omega/100)\times 5000\times(1+j\omega/5000)}$$

$$= \frac{(1+j\omega/5)}{(1+j\omega/100)\cdot(1+j\omega/5000)}$$

$$20\log|G(j\omega)| = 20\log|1+j\omega/5| - 20\log|1+j\omega/100| - 20\log|1+j\omega/5000|$$

The corner frequencies are at $\omega = 5\ r/s$, $\omega = 100\ r/s$, and $\omega = 5000\ r/s$. The asymptotes are shown as solid lines.



From this plot we observe that $20\log|G(j\omega)|_{max} \approx 26\ dB$ for the interval $10^2 \le \omega \le 5 \times 10^3$

b. By inspection, $20\log|G(j\omega)| = 0\ dB$ at $\omega = 9.85 \times 10^4\ r/s$

2. From the solution of Exercise 1,

$$G(j\omega) = \frac{(1 + j\omega/5)}{(1 + j\omega/100) \cdot (1 + j\omega/5000)}$$

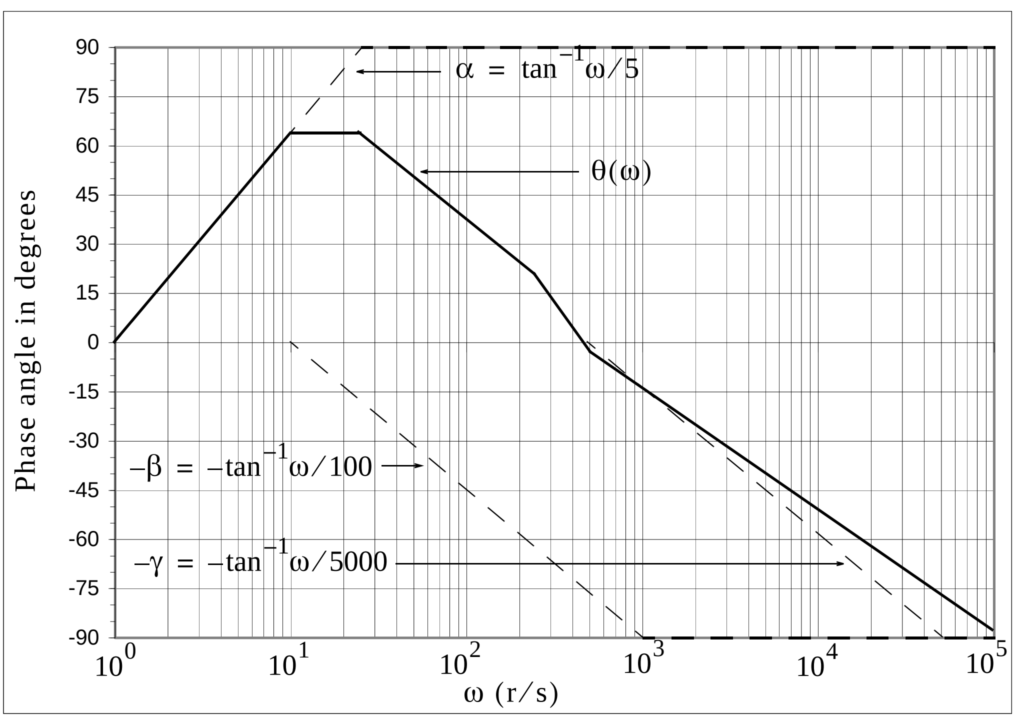
and in magnitude-phase form

$$G(j\omega) = \frac{|(1 + j\omega/5)|}{|(1 + j\omega/100)| \cdot |(1 + j\omega/5000)|}\angle(\alpha - \beta - \gamma)$$

that is, $\theta(\omega) = \alpha - \beta - \gamma$ where $\alpha = \tan^{-1}\omega/5$, $-\beta = -\tan^{-1}\omega/100$, and $-\gamma = -\tan^{-1}\omega/5000$

The corner frequencies are at $\omega = 5\ r/s$, $\omega = 100\ r/s$, and $\omega = 5000\ r/s$ where at those frequencies $\alpha = 45°$, $-\beta = -45°$, and $-\gamma = -45°$ respectively. The asymptotes are shown as solid lines.

From the phase plot we observe that $\theta(30\ r/s) \approx 60°$, $\theta(50\ r/s) \approx 53°$, $\theta(100\ r/s) \approx 38°$, and $\theta(5000\ r/s) \approx -39°$



b. We use the MATLAB script below for the computations.

```
theta_g30=(1+30j/5)/((1+30j/100)*(1+30j/5000));...
theta_g50=(1+50j/5)/((1+50j/100)*(1+50j/5000));...
theta_g100=(1+100j/5)/((1+100j/100)*(1+100j/5000));...
theta_g5000=(1+5000j/5)/((1+5000j/100)*(1+5000j/5000));...
printf(' \n');...
fprintf('theta30r = %5.2f deg. \t', angle(theta_g30)*180/pi);...
fprintf('theta50r = %5.2f deg. ', angle(theta_g50)*180/pi);...
fprintf(' \n');...
fprintf('theta100r = %5.2f deg. \t', angle(theta_g100)*180/pi);...
fprintf('theta5000r = %5.2f deg. ', angle(theta_g5000)*180/pi);...
fprintf(' \n')
```

and we obtain

```
theta30r = 63.49 deg. theta50r = 57.15 deg.
theta100r = 40.99 deg. theta5000r = -43.91 deg.
```

Thus, the actual values are

$$\angle G(j30) = \angle\frac{(1+j30/5)}{(1+j30/100)\cdot(1+j30/5000)} = 63.49°$$

$$\angle G(j50) = \angle\frac{(1+j50/5)}{(1+j50/100)\cdot(1+j50/5000)} = 57.15°$$

$$\angle G(j100) = \angle\frac{(1+j100/5)}{(1+j100/100)\cdot(1+j100/5000)} = 40.99°$$

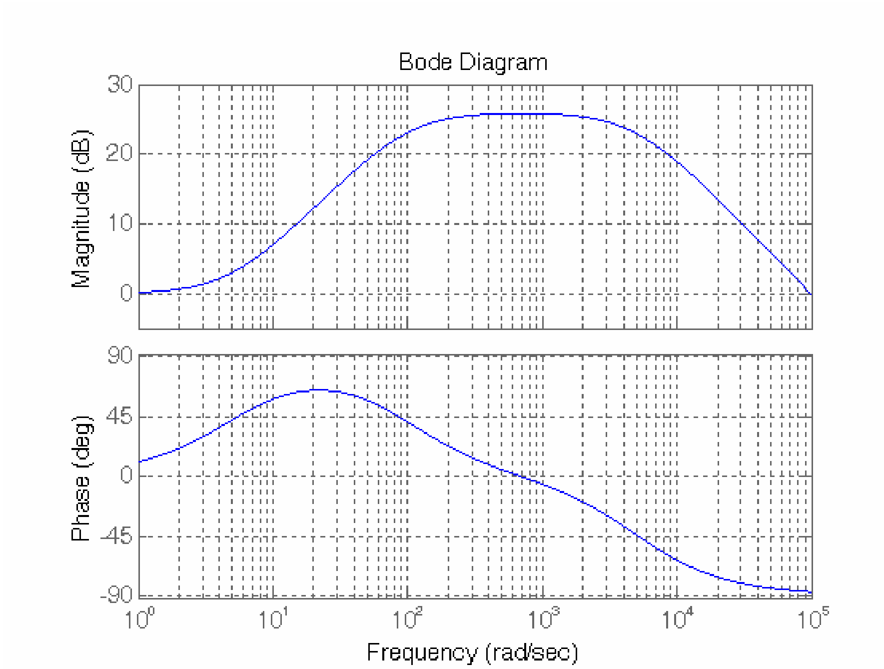
$$\angle G(j5000) = \angle\frac{(1+j5000/5)}{(1+j5000/100)\cdot(1+j5000/5000)} = -43.91°$$

c. The Bode plot generated with MATLAB is shown below.

```
syms s; expand((s+100)*(s+5000))
```

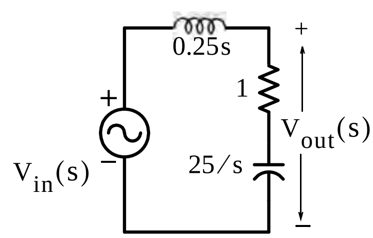
```
ans =
s^2+5100*s+500000
```

```
num=[0  10^5  5*10^5]; den=[1  5.1*10^3  5*10^5];...
w=logspace(0,5,10^4); bode(num,den,w)
```



3. a.

The equivalent s – domain circuit is shown below.



By the voltage division expression,

$$V_{out}(s) = \frac{1 + 25/s}{0.25s + 1 + 25/s} \cdot V_{in}(s)$$

and

$$G(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{s + 25}{0.25s^2 + s + 25} = \frac{4(s + 25)}{s^2 + 4s + 100} \quad (1)$$

b. From (1) with $s = j\omega$,

$$G(j\omega) = \frac{4(j\omega + 25)}{-\omega^2 + 4j\omega + 100} \quad (2)$$

From (8.53),

$$G(s) = \frac{C}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad (3)$$

and from (1) and (3) $\omega_n^2 = 100$, $\omega_n = 10$, and $2\zeta\omega_n = 4$, $\zeta = 0.2$.

Following the procedure of page 8-26 we let $u = \omega/\omega_n = \omega/10$. The numerator of (2) is a linear factor and thus we express it as $100(1 + j\omega/25)$. Then (2) is written as

$$G(j\omega) = \frac{100(1 + j\omega/25)}{100(-\omega^2/100 + 4j\omega/100 + 100/100)} = \frac{(1 + j\omega/25)}{1 - (\omega/10)^2 + j0.4/10}$$

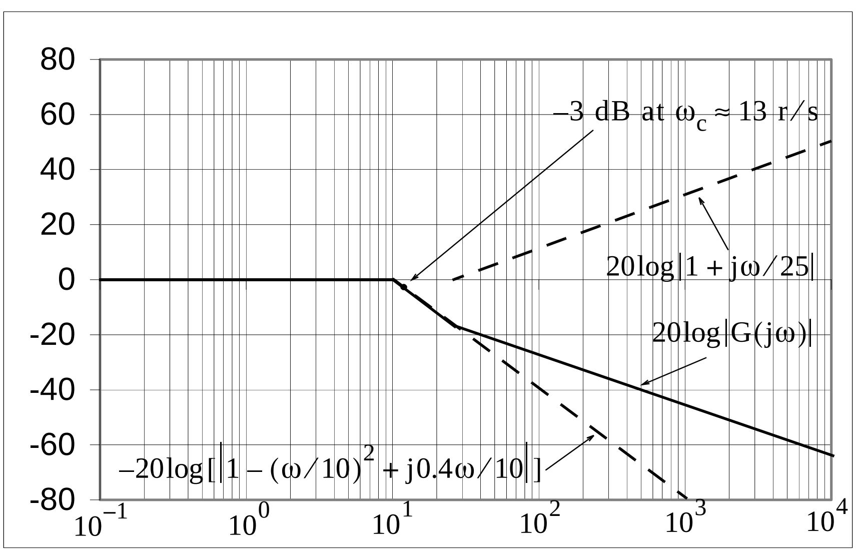
or

$$G(j\omega) = \frac{|1 + j\omega/25|\angle\theta}{\left|1 - (\omega/10)^2 + j0.4\omega/10\right|\angle\phi} \quad (4)$$

The amplitude of $G(j\omega)$ in dB is

$$20\log|G(j\omega)| = 20\log|1 + j\omega/25| - 20\log\left[\left|1 - (\omega/10)^2 + j0.4\omega/10\right|\right] \quad (5)$$

The asymptote of the first term on the right side of (5) has a corner frequency of 25 r/s and rises with slope of 20 dB/decade. The second term has a corner frequency of 10 r/s and rises with slope of -40 dB/decade. The amplitude plot is shown below.



c. The plot above indicates that the circuit is a low-pass filter and the 3 dB cutoff frequency ω_c occurs at approximately 13 r/s.

d. The actual cutoff frequency occurs where

$$|G(j\omega_c)| = |G(j\omega)|_{max}/\sqrt{2} = 1/(\sqrt{2}) = 0.70$$

At this frequency (2) is written as

$$G(j\omega_c) = \frac{100 + 4j\omega_C}{(100-\omega_C^2) + 4j\omega}$$

and considering its magnitude we obtain

$$\frac{\sqrt{100^2 + (4\omega_C)^2}}{\sqrt{(100-\omega_C^2)^2 + (4\omega_C)^2}} = \frac{1}{\sqrt{2}}$$

$$2[100^2 + (4\omega_C)^2] = (100-\omega_C^2)^2 + (4\omega_C)^2$$

$$20000 + 32\omega_C^2 = 10000 - 200\omega_C^2 + \omega_C^4 + 16\omega_C^2$$

$$\omega_C^4 - 216\omega_C^2 - 10000 = 0$$

We will use MATLAB to find the four roots of this equation.

```
syms w; solve(w^4–216*w^2–10000)
```

```
ans =
[  2*(27+1354^(1/2))^(1/2)]  [ -2*(27+1354^(1/2))^(1/2)]
[  2*(27-1354^(1/2))^(1/2)]  [ -2*(27-1354^(1/2))^(1/2)]
```

```
w1=2*(27+1354^(1/2))^(1/2)
```

```
w1 =
  15.9746
```

```
w2=-2*(27+1354^(1/2))^(1/2)
```

```
w2 =
  -15.9746
```

```
w3=2*(27-1354^(1/2))^(1/2)
```

```
w3 =
   0.0000 + 6.2599i
```

```
w4=-2*(27-1354^(1/2))^(1/2)
```

```
w4 =
```

```
-0.0000 - 6.2599i
```

From these four roots we accept only the first, that is, $\omega_C \approx 16\ r/s$

e. From (4)

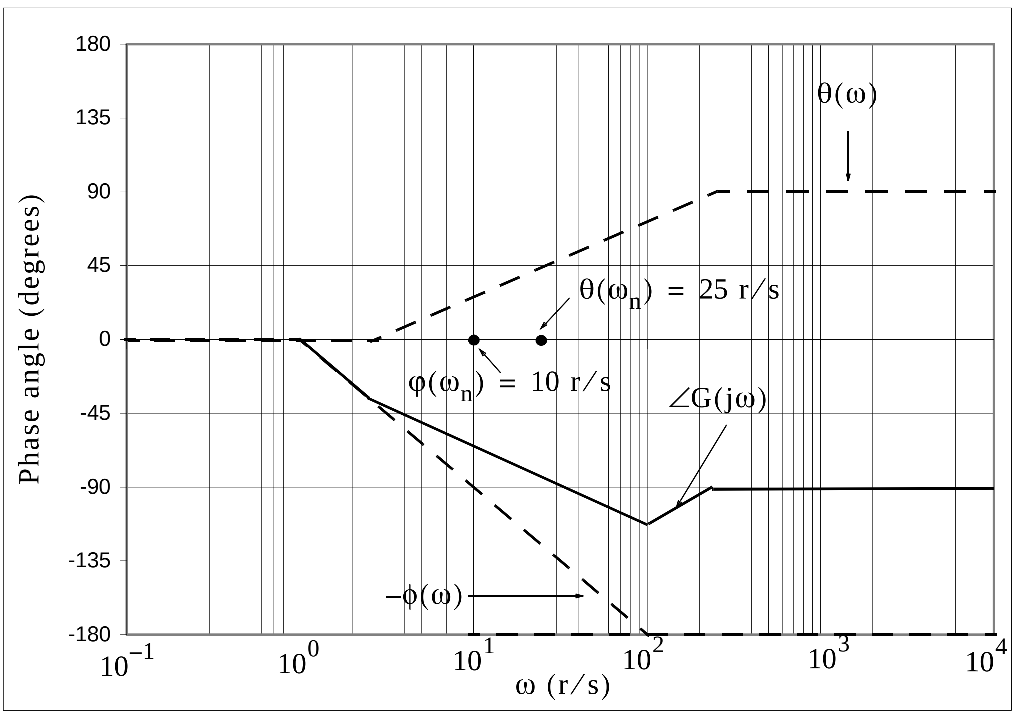
$$\theta = \tan^{-1}(\omega/25)$$

and

$$\phi = \frac{0.4\omega/10}{1-(\omega/10)^2}$$

For a first order zero or pole not at the origin, the straight line phase angle plot approximations are as follows:

I. For frequencies less than one tenth the corner frequency we assume that the phase angle is zero. For this exercise the corner frequency of $\theta(\omega)$ is $\omega_n = 25\ r/s$ and thus for $1 \le \omega \le 2.5\ r/s$ the phase angle is zero as shown on the Bode plot below.



II For frequencies ten times or greater than the corner frequency, the phase angle is approximately $\pm 90°$. The numerator phase angle $\theta(\omega)$ is zero at one tenth the corner frequency, it is $45°$ at the corner frequency, and $90°$ for frequencies ten times or greater the corner frequency. For this exercise, in the interval $2.5 \le \omega \le 250\ r/s$ the phase angle is zero at $2.5\ r/s$ and rises to $90°$ at $250\ r/s$.

III As shown in Figure 8.20, for complex poles the phase angle is zero at zero frequency, $-90°$ at the corner frequency and approaches $-180°$ as the frequency becomes large. The phase angle asymptotes are shown on the plot of the previous page.

f. From the plot of the previous page we observe that the phase angle at the cutoff frequency is approximately $-63°$

g. The exact phase angle at the cutoff frequency $\omega_c = 16\ r/s$ is found from (1) with $s = j16$.

$$G(j16) = \frac{4(j16+25)}{(j16)^2 + 4(j16) + 100}$$

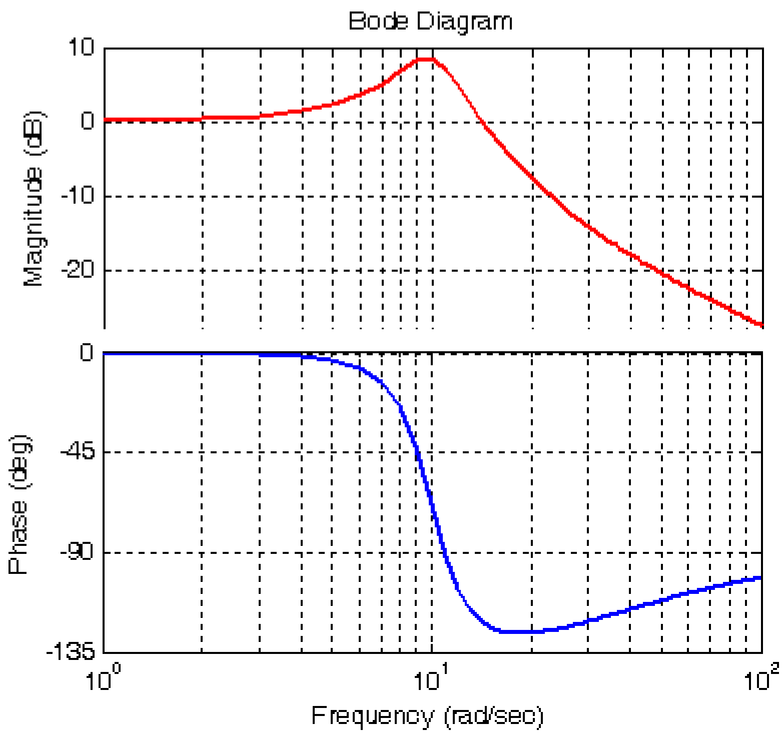
We need not simplify this expression; we will use the MATLAB script below.

```
g16=(64j+100)/((16j)^2+64j+100); angle(g16)*180/pi
```

```
ans =
  -125.0746
```

This value is about twice as that we observed from the asymptotic plot of the previous page. Errors such as this occur because of the high non-linearity between frequency intervals. Therefore, we should use the straight line asymptotes only to observe the shape of the phase angle. It is best to use MATLAB as shown below.

```
num=[0  4  100]; den=[1  4 100]; w=logspace(0,2,1000);bode(num,den,w)
```


Bode Diagram
10
0
-10
-20
Magnitude (dB)
0
-45
-90
-135
Phase (deg)
10^0
10^1
10^2
Frequency (rad/sec)

Chapter 9

Self and Mutual Inductances – Transformers

This chapter begins with the interactions between electric circuits and changing magnetic fields. It defines self and mutual inductances, flux linkages, induced voltages, the dot convention, Lenz's law, and magnetic coupling. It concludes with a detailed discussion on transformers.

9.1 Self–Inductance

About 1830, Joseph Henry, while working at the university which is now known as Princeton, found that electric current flowing in a circuit has a property analogous to mechanical momentum which is a measure of the motion of a body and it is equal to the product of its mass and velocity, i.e., Mv. In electric circuits this property is sometimes referred to as the *electrokinetic momentum* and it is equal to the product of Li where i is the current analogous to velocity and the *self–inductance* L is analogous to the mass M. About the same time, Michael Faraday visualized this property in a magnetic field in space around a current carrying conductor. This electrokinetic momentum is denoted by the symbol λ, that is,

$$\boxed{\lambda = Li} \tag{9.1}$$

Newton's second law states that the force necessary to change the velocity of a body with mass M is equal to the rate of change of the momentum, i.e.,

$$\mathbf{F} = \frac{d}{dt}(Mv) = M\frac{dv}{dt} = M\mathbf{a} \tag{9.2}$$

where $\mathbf{a}$ is the acceleration. The analogous electrical relation says that the voltage v necessary to produce a change of current in an inductive circuit is equal to the rate of change of electrokinetic momentum, i.e,

$$\boxed{v = \frac{d}{dt}(Li) = L\frac{di}{dt}} \tag{9.3}$$

9.2 The Nature of Inductance

Inductance is associated with the magnetic field which is always present when there is an electric current. Thus when current flows in an electric circuit, the conductors (wires) connecting the devices in the circuit are surrounded by a magnetic field. Figure 9.1 shows a simple loop of wire and its magnetic field which is represented by the small loops. The direction of the magnetic field (not shown) can be determined by the left–hand rule if conventional current flow is assumed, or by the right–hand rule if electron current flow is assumed. The magnetic field loops are circular in form and are called lines of *magnetic flux*. The unit of magnetic flux is the weber (Wb).

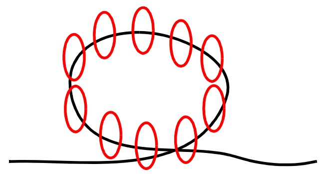


Figure 9.1. Magnetic field around a current carrying wire

In a loosely wound coil of wire such as the one shown in Figure 9.2, the current through the wound coil produces a denser magnetic field and many of the magnetic lines link the coil several times.

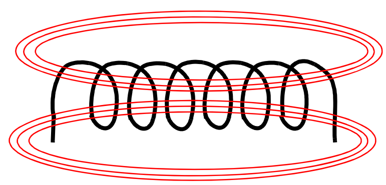


Figure 9.2. Magnetic field around a current carrying wound coil

The magnetic flux is denoted as φ and, if there are N turns and we assume that the flux φ passes through each turn, the total flux denoted as λ is called *flux linkage*. Then,

$$\lambda = N\varphi \tag{9.4}$$

By definition, a linear inductor one in which the flux linkage is proportional to the current through it, that is,

$$\lambda = Li \tag{9.5}$$

where the constant of proportionality L is called inductance in webers per ampere.
We now recall Faraday's law of electromagnetic induction which states that

$$v = \frac{d\lambda}{dt} \tag{9.6}$$

and from (9.3) and (9.5),

$$v = L\frac{di}{dt} \tag{9.7}$$

9.3 Lenz's Law

Heinrich F. E. Lenz was a German scientist who, without knowledge of the work of Faraday and Henry, duplicated many of their discoveries nearly simultaneously. The law which goes by his name, is a useful rule for predicting the direction of an induced current. *Lenz's law* states that:

Whenever there is a change in the amount of magnetic flux linking an electric circuit, an induced voltage of value directly proportional to the time rate of change of flux linkages is set up tending to produce a current in such a direction as to oppose the change in flux.

To understand Lenz's law, let us consider the transformer shown in Figure 9.3.

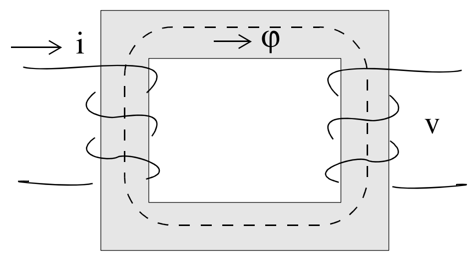


Figure 9.3. Basic transformer construction

Here, we assume that the current in the primary winding has the direction shown and it produces the flux φ in the direction shown in Figure 9.3 by the arrow below the dotted line. Suppose that this flux is decreasing. Then in the secondary winding there will be a voltage induced whose current will be in a direction to increase the flux. In other words, the current produced by the induced voltage will tend to prevent any decrease in flux. Conversely, if the flux produced by the primary winding in increasing, the induced voltage in the secondary will produce a current in a direction which will oppose an increase in flux.

9.4 Mutually Coupled Coils

Consider the inductor (coil) shown in Figure 9.4.

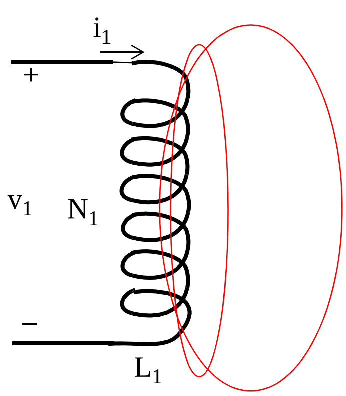


Figure 9.4. Magnetic lines linking a coil

There are many magnetic lines of flux linking the coil L_1 with N_1 turns but for simplicity, only two are shown in Figure 9.4. The current i_1 produces a magnetic flux φ_{11}. Then by (9.4) and (9.5), we obtain

$$\lambda_1 = N_1\varphi_{11} = L_1 i_1 \tag{9.8}$$

and by Faraday's law of (9.6), in terms of the *self–inductance* L_1,

$$v_1 = \frac{d\lambda_1}{dt} = N_1\frac{d\varphi_{11}}{dt} = L_1\frac{di_1}{dt} \tag{9.9}$$

Next, suppose another coil L_2 with N_2 turns is brought near the vicinity of coil L_1 and some lines of flux are also linking coil L_2 as shown in Figure 9.5.

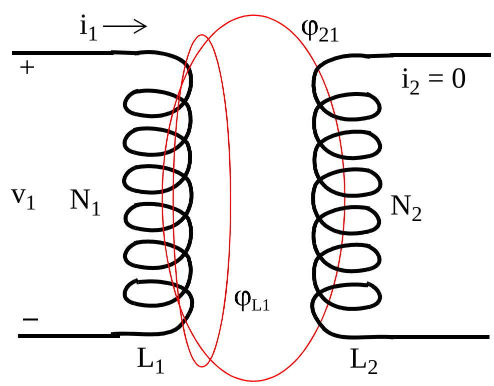


Figure 9.5. Lines of flux linking two coils

It is convenient to express the flux φ_{11} as the sum of two fluxes φ_{L1} and φ_{21}, that is,

$$\boxed{\varphi_{11} = \varphi_{L1} + \varphi_{21}} \tag{9.10}$$

where the *linkage flux* φ_{L1} is the flux which links coil L_1 only and not coil L_2, and the *mutual flux* φ_{21} is the flux which links both coils L_1 and L_2. We have assumed that the linkage and mutual fluxes φ_{L1} and φ_{21} link all turns of coil L_1 and the mutual flux φ_{21} links all turns of coil L_2.

The arrangement above forms an elementary transformer where coil L_1 is called the *primary winding* and coil L_2 the *secondary winding*.

In a *linear transformer* the mutual flux φ_{21} is proportional to the primary winding current i_1 and since there is no current in the secondary winding, the flux linkage in the secondary winding is by (9.8),

$$\lambda_2 = N_2 \varphi_{21} = M_{21} i_1 \tag{9.11}$$

where M_{21} is the *mutual inductance* (in Henries) and thus the open–circuit secondary winding voltage v_2 is

$$v_2 = \frac{d\lambda_2}{dt} = N_2 \frac{d\varphi_{21}}{dt} = M_{21} \frac{di_1}{dt} \tag{9.12}$$

In summary, when there is no current in the secondary winding the voltages are

$$\boxed{\begin{array}{c} v_1 = L_1 \dfrac{di_1}{dt} \text{ and } v_2 = M_{21} \dfrac{di_1}{dt} \\ \text{if } i_1 \neq 0 \text{ and } i_2 = 0 \end{array}} \tag{9.13}$$

Next, we will consider the case where there is a voltage in the secondary winding producing current i_2 which in turn produces flux φ_{22} as shown in Figure 9.6.

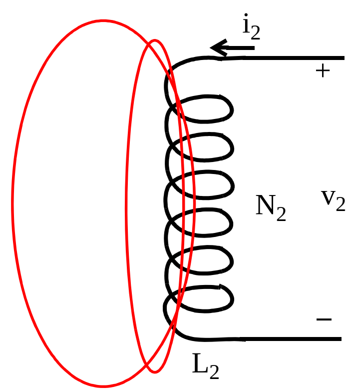


Figure 9.6. Flux in secondary winding

Then in analogy with (9.8) and (9.9)

$$\lambda_2 = N_2 \varphi_{22} = L_2 i_2 \tag{9.14}$$

and by Faraday's law in terms of the *self–inductance* L_2

$$v_2 = \frac{d\lambda_2}{dt} = N_2 \frac{d\varphi_{22}}{dt} = L_2 \frac{di_2}{dt} \tag{9.15}$$

If another coil L_1 with N_1 turns is brought near the vicinity of coil L_2, some lines of flux are also linking coil L_1 as shown in Figure 9.7.

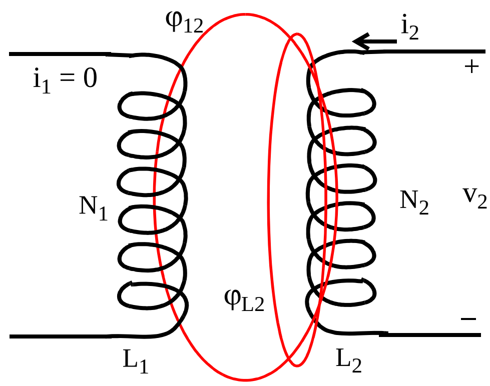


Figure 9.7. Lines of flux linking open primary coil

Following the same procedure as above, we express the flux φ_{22} as the sum of two fluxes φ_{L2} and φ_{12}, that is,

$$\varphi_{22} = \varphi_{L2} + \varphi_{12} \tag{9.16}$$

where the *linkage flux* φ_{L2} is the flux which links coil L_2 only and not coil L_1, and the *mutual flux* φ_{12} is the flux which links both coils L_2 and L_1. As before, we have assumed that the linkage and mutual fluxes link all turns of coil L_2 and the mutual flux links all turns of coil L_1.

Since there is no current in the primary winding, the flux linkage in the primary winding is

$$\lambda_1 = N_1\varphi_{12} = M_{12}i_2 \tag{9.17}$$

where M_{12} is the *mutual inductance* (in Henries) and thus the open–circuit primary winding voltage v_1 is

$$v_1 = \frac{d\lambda_1}{dt} = N_1\frac{d\varphi_{12}}{dt} = M_{12}\frac{di_2}{dt} \tag{9.18}$$

In summary, when there is no current in the primary winding, the voltages are

$$v_2 = L_2\frac{di_2}{dt} \text{ and } v_1 = M_{12}\frac{di_2}{dt} \quad \text{if } i_1 = 0 \text{ and } i_2 \neq 0 \tag{9.19}$$

We will see later that

$$M_{12} = M_{21} = M \tag{9.20}$$

The last possible arrangement is shown in Figure 9.8 where $i_1 \neq 0$ and also $i_2 \neq 0$.

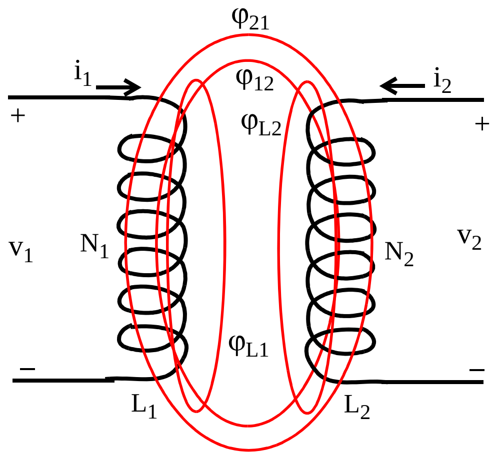


Figure 9.8. Flux linkages when both primary and secondary currents are present

The total flux φ_1 linking coil L_1 is

$$\varphi_1 = \varphi_{L1} + \varphi_{21} + \varphi_{12} = \varphi_{11} + \varphi_{12} \tag{9.21}$$

and the total flux φ_2 linking coil L_2 is

$$\varphi_2 = \varphi_{L2} + \varphi_{12} + \varphi_{21} = \varphi_{21} + \varphi_{22} \tag{9.22}$$

and since $\lambda = N\varphi$, we express (9.21) and (9.22) as

$$\lambda_1 = N_1\varphi_{11} + N_1\varphi_{12} \tag{9.23}$$

and

$$\lambda_2 = N_2\varphi_{21} + N_2\varphi_{22} \tag{9.24}$$

Differentiating (9.23) and (9.24) and using (9.13), (9.14), (9.19) and (9.20) we obtain:

$$\boxed{\begin{aligned} v_1 &= L_1\frac{di_1}{dt} + M\frac{di_2}{dt} \\ v_2 &= M\frac{di_1}{dt} + L_2\frac{di_2}{dt} \end{aligned}} \tag{9.25}$$

In (9.25) the voltage terms

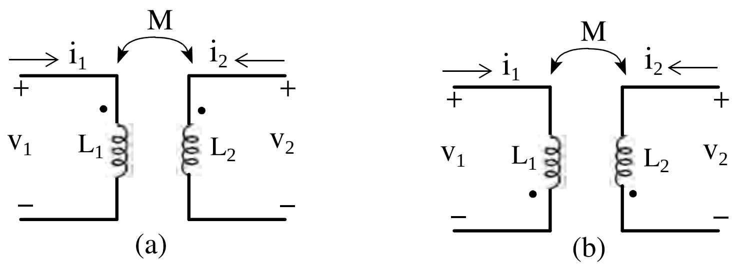
$$L_1\frac{di_1}{dt} \quad \text{and} \quad L_2\frac{di_2}{dt}$$

are referred to as *self–induced voltages* and the terms

$$M\frac{di_1}{dt} \quad \text{and} \quad M\frac{di_2}{dt}$$

are referred to as *mutual voltages*.

In our previous studies we used the passive sign convention as a basis to denote the polarity (+) and (–) of voltages and powers. While this convention can be used with the self–induced voltages, it cannot be used with mutual voltages because there are four terminals involved. Instead, the polarity of the mutual voltages is denoted by the *dot convention.* To understand this convention, we first consider the transformer circuit designations shown in Figures 9.9(a) and 9.9(b) where the dots are placed on the upper terminals and the lower terminals respectively.

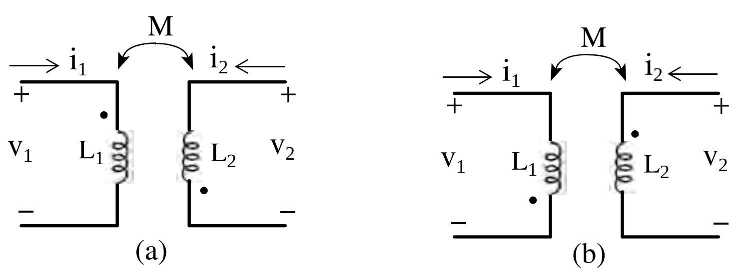


$$v_2 = M\frac{di_1}{dt}$$

for both networks

Figure 9.9. Arrangements where the mutual voltage has a positive sign

These designations indicate the condition that a current i entering the dotted (undotted) terminal of one coil induce a voltage across the other coil with positive polarity at the dotted (undotted) terminal of the other coil. Thus, the mutual voltage term has a positive sign. Following the same rule we see that in the circuits of Figure 9.10 (a) and 9.10(b) the mutual voltage has a negative sign.



$$v_2 = -M\frac{di_1}{dt}$$

for both networks

Figure 9.10. Arrangements where the mutual voltage has a negative sign

Example 9.1

For the circuit of Figure 9.11 find v_1 and v_2 if

a. $i_1 = 50\ mA$ and $i_2 = 25\ mA$

b. $i_1 = 0$ and $i_2 = 20\sin 377t\ mA$

c. $i_1 = 15\cos 377t\ mA$ and $i_2 = 40\sin(377t + 60°)\ mA$

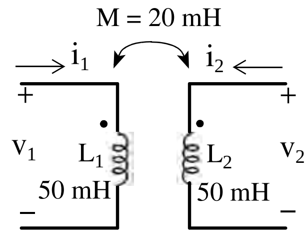


Figure 9.11. Circuit for Example 9.1

Solution:

a. Since both currents i_1 and i_2 are constants, their derivatives are zero, i.e.,

$$\frac{di_1}{dt} = \frac{di_2}{dt} = 0$$

and thus

$$v_1 = v_2 = 0$$

b. The dot convention in the circuit of Figure 9.11 shows that the mutual voltage terms are positive and thus

$$v_1 = L_1\frac{di_1}{dt} + M\frac{di_2}{dt} = 0.05 \times 0 + 20 \times 10^{-3} \times 20 \times 377 \times \cos 377t$$

$$= 150.8\cos 377t \text{ mV}$$

$$v_2 = M\frac{di_1}{dt} + L_2\frac{di_2}{dt} = 20 \times 10^{-3} \times 0 + 0.05 \times 20 \times 377 \times \cos 377t$$

$$= 377\cos 377t \text{ mV}$$

c.

$$v_1 = L_1\frac{di_1}{dt} + M\frac{di_2}{dt} = 0.05(-15 \times 377\sin 377t) + 0.02 \times 40 \times 377\cos(377t + 60°)$$

$$= -282.75\sin 377t + 301.6\cos(377t + 60°) \text{ mV}$$

$$v_2 = M\frac{di_1}{dt} + L_2\frac{di_2}{dt} = 0.02(-15 \times 377\sin 377t) + 0.05 \times 40 \times 377\cos(377t + 60°)$$

$$= -113.1\sin 377t + 754\cos(377t + 60°) \text{ mV}$$

Example 9.2

For the circuit of Figure 9.12 find the open–circuit voltage v_2 for $t > 0$ given that $i_1(0^-) = 0$.

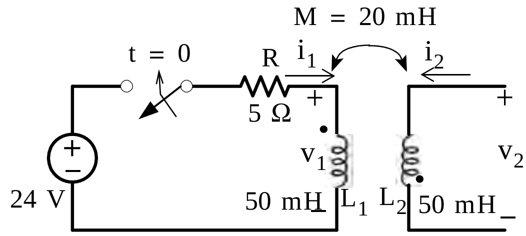


Figure 9.12. Circuit for Example 9.2

Solution:

For $t > 0$

$$L\frac{di_1}{dt} + Ri_1 = 24$$

$$0.05\frac{di_1}{dt} + 5i_1 = 24$$

$$\frac{di_1}{dt} + 100i_1 = 480$$

Also,

$$i_1 = i_f + i_n$$

where i_f is the forced response component of i_1 and it is obtained from

$$i_f = \frac{24}{5} = 4.8\ A$$

and i_n is the natural response component of i_1 and it is obtained from

$$i_n = Ae^{-Rt/L} = Ae^{-100t}$$

Then,

$$i_1 = i_f + i_n = 4.8 + Ae^{-100t}$$

and with the initial condition

$$i_1(0^+) = i_1(0^-) = 0 = 4.8 + Ae^0$$

we obtain $A = -4.8$

Therefore,

$$i_1 = i_f + i_n = 4.8 - 4.8e^{-100t}$$

and in accordance with the dot convention,

$$v_2 = -M\frac{di_1}{dt} = -0.02(480e^{-100t}) = -9.6e^{-100t}$$

9.5 Establishing Polarity Markings

In our previous discussion and in Examples 9.1 and 9.2, the polarity markings (dots) were given. There are cases, however, when these are not known. The following method is generally used to establish the polarity marking in accordance with the dot convention.

Consider the transformer and its circuit symbol shown in Figure 9.13.

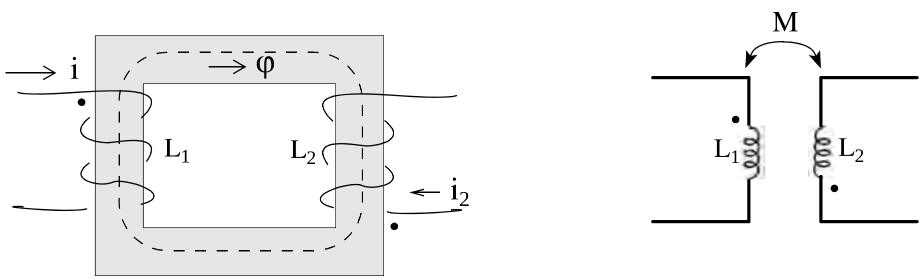


Figure 9.13. Establishing polarity markings

We recall that the direction of the flux φ can be found by the right–hand rule which states that if the fingers of the right hand encircle a winding in the direction of the current, the thumb indicates the direction of the flux. Let us place a dot at the upper end of L_1 and assume that the current i_1 enters the top end thereby producing a flux in the clockwise direction shown. Next, we want the current in L_2 to enter the end which will produce a flux in the same direction, in this case, clockwise. This will be accomplished if the current i_2 in L_2 enters the lower end as shown and thus we place a dot at that end.

Example 9.3

For the transformer shown in Figure 9.14, find v_1 and v_2.

Solution:

Let us first establish the dot positions as discussed above. Since the current i_2 has a negative sign, it leaves the upper terminal, or equivalently, enters the lower terminal and thus we enter a dot at the lower terminal. The dotted circuit now is as shown in Figure 9.15.

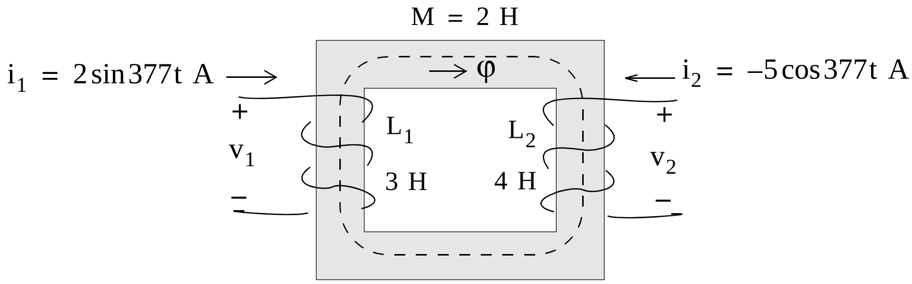


Figure 9.14. Network for Example 9.3

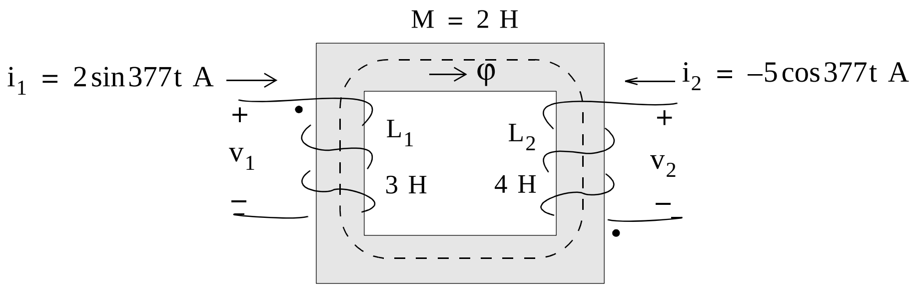


Figure 9.15. Figure for Example 9.3 with dotted markings

The current i_1 enters the upper terminal on the left side and i_2 leaves the upper terminal on the right side, the fluxes oppose each other. Therefore,

$$v_1 = L_1\frac{di_1}{dt} - M\frac{di_2}{dt} = 2262\cos 377t - 3770\sin 377t \text{ V}$$

$$v_2 = -M\frac{di_1}{dt} + L_2\frac{di_2}{dt} = -1508\cos 377t + 7540\sin 377t \text{ V}$$

Example 9.4

For the network in Figure 9.16 find the voltage ratio $|\mathbf{V}_2/\mathbf{V}_1|$.*

Solution:

The dots are given to us as shown. Now, we arbitrarily assign currents $\mathbf{I}_1$ and $\mathbf{I}_2$ as shown in Figure 9.17 and we write mesh equations for each mesh.

* *Henceforth we will be using bolded capital letters to denote phasor quantities.*

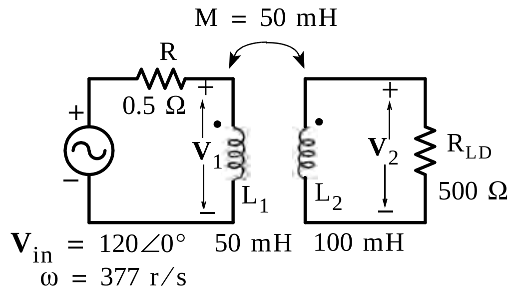


Figure 9.16. Circuit for Example 9.4

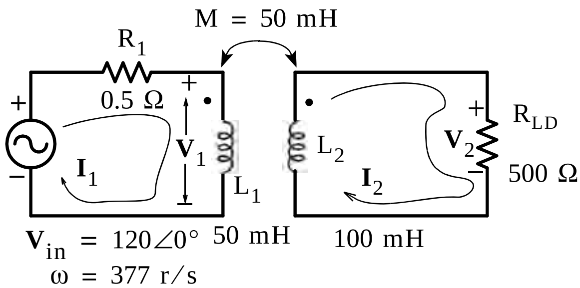


Figure 9.17. Mesh currents for the circuit of Example 9.4

With this current assignments $\mathbf{I_2}$ leaves the dotted terminal of the right mesh and therefore the mutual voltage has a negative sign. Then,

Mesh 1:

$$R_1\mathbf{I_1} + j\omega L_1\mathbf{I_1} - j\omega M\mathbf{I_2} = \mathbf{V}_{in}$$

or

$$(0.5 + j18.85)\mathbf{I_1} - j18.85\mathbf{I_2} = 120\angle 0° \tag{9.26}$$

Mesh 2:

$$-j\omega M\mathbf{I}_1 + j\omega L_2\mathbf{I_2} + R_{LOAD}\,\mathbf{I_2} = 0$$

or

$$-j18.85\mathbf{I_1} + (1000 + j37.7)\mathbf{I_2} = 0 \tag{9.27}$$

We will find the ratio $|\mathbf{V}_2/\mathbf{V}_1|$ using the MATLAB script below where

$$\mathbf{V}_1 = j\omega L_1\mathbf{I_1} = j18.85\mathbf{I_1}$$

```
Z=[0.5+18.85j  –18.85j; –18.85j  500+37.7j]; V=[120 0]'; I=Z\V;...
fprintf(' \n'); fprintf('V1 = %7.3f V \t', abs(18.85j*I(1))); fprintf('V2 = %7.3f V \t', abs(500*I(2)));...
fprintf('Ratio V2/V1 = %7.3f \t',abs((500*I(2))/(18.85j*I(1))))
```

```
V1 = 120.093 V   V2 = 119.753 V   Ratio V2/V1 =   0.997
```

That is,

$$\left|\frac{\mathbf{V_2}}{\mathbf{V_1}}\right| = \frac{119.75}{120.09} = 0.997 \tag{9.28}$$

and thus the magnitude of $\mathbf{V}_{LD} = \mathbf{V}_2$ is practically the same as the magnitude of $\mathbf{V}_{in}$. However, we suspect that $\mathbf{V}_{LD}$ will be out of phase with $\mathbf{V}_{in}$. We can find the phase of $\mathbf{V}_{LD}$ by adding the following statement to the MATLAB script above.

```
fprintf('Phase V2= %6.2f deg', angle(500*I(2))*180/pi)
```

```
Phase V2=  -0.64 deg
```

This is a very small phase difference from the phase of $\mathbf{V}_{in}$ and thus we see that both the magnitude and phase of $\mathbf{V}_{LOAD}$ are essentially the same as that of $\mathbf{V}_{in}$.

If we increase the load resistance R_{LD} to $1\ K\Omega$ we will find that again the magnitude and phase of $\mathbf{V}_{LOAD}$ are essentially the same as that of $\mathbf{V}_{in}$. Therefore, the transformer of this example is an *isolation transformer,* that is, it isolates the load from the source and the value of $\mathbf{V}_{in}$ appears across the load even though the load changes. An isolation transformer is also referred to as a 1:1 transformer.

If in a transformer the secondary winding voltage is considerably higher than the input voltage, the transformer is referred to as a *step−up transformer*. Conversely, if the secondary winding voltage is considerably lower than the input voltage, the transformer is referred to as a *step−down transformer*.

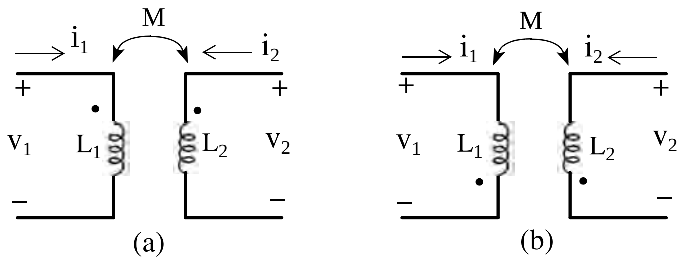
9.6 Energy Stored in a Pair of Mutually Coupled Inductors

We know that the energy stored in an inductor is

$$W(t) = \frac{1}{2}Li^2(t) \tag{9.29}$$

In the transformer circuits shown in Figure 9.18, the stored energy is the sum of the energies supplied to the primary and secondary terminals. From (9.25),

$$\begin{aligned} v_1 &= L_1\frac{di_1}{dt} + M\frac{di_2}{dt} \\ v_2 &= M\frac{di_1}{dt} + L_2\frac{di_2}{dt} \end{aligned} \tag{9.30}$$



$$v_2 = M\frac{di_1}{dt}$$

for both circuits

Figure 9.18. Transformer circuits for computation of the energy

and after replacing M with M_{12} and M_{21} in the appropriate terms, the instantaneous power delivered to these terminals are:

$$p_1 = v_1 i_1 = \left(L_1\frac{di_1}{dt} + M_{12}\frac{di_2}{dt}\right)i_1$$
$$p_2 = v_2 i_2 = \left(M_{21}\frac{di_1}{dt} + L_2\frac{di_2}{dt}\right)i_2 \qquad (9.31)$$

Next, let us suppose that at some reference time t_0, both currents i_1 and i_2 are zero, that is,

$$i_1(t_0) = i_2(t_0) = 0 \qquad (9.32)$$

In this case, there is no energy stored, and thus

$$W(t_0) = 0 \qquad (9.33)$$

Now, let us assume that at time t_1, the current i_1 is increased to some finite value, while i_2 is still zero. In other words, we let

$$i_1(t_1) = I_1 \qquad (9.34)$$

and

$$i_2(t_1) = 0 \qquad (9.35)$$

Then, the energy accumulated at this time is

$$W_1 = \int_{t_0}^{t_1}(p_1 + p_2)dt \qquad (9.36)$$

and since $i_2(t_1) = 0$, then $p_2(t_1) = 0$ and also $di_2/dt = 0$. Therefore, from (9.31) and (9.36) we obtain

$$W_1 = \int_{t_0}^{t_1} L_1 i_1 \frac{di_1}{dt}dt = L_1\int_{t_0}^{t_1} i_1 di_1 = \frac{1}{2}L_1 I_1^2 \qquad (9.37)$$

Finally, let us at some later time t_2, maintain i_1 at its previous value, and increase i_2 to a finite value, that is, we let

$$i_1(t_2) = I_1 \tag{9.38}$$

and

$$i_2(t_2) = I_2 \tag{9.39}$$

During this time interval, $di_1/dt = 0$ and using (9.31) the energy accumulated is

$$\begin{aligned} W_2 &= \int_{t_1}^{t_2}(p_1+p_2)dt = \int_{t_1}^{t_2}\left(M_{12}I_1\frac{di_2}{dt}+L_2i_2\frac{di_2}{dt}\right)dt \\ &= \int_{t_1}^{t_2}(M_{12}I_1+L_2i_2)di_2 = M_{12}I_1I_2+\frac{1}{2}L_2I_2^2 \end{aligned} \tag{9.40}$$

Therefore, the energy stored in the transformer from t_0 to t_2 is from (9.37) and (9.40),

$$W\Big|_{t_0}^{t_2} = \frac{1}{2}L_1I_1^2+M_{12}I_1I_2+\frac{1}{2}L_2I_2^2 \tag{9.41}$$

Now, let us reverse the order in which we increase i_1 and i_2. That is, in the time interval $t_0 \le t \le t_1$, we increase i_2 so that $i_2(t_1) = I_2$ while keeping $i_1 = 0$. Then, at $t = t_2$, we keep $i_2 = I_2$ while we increase i_1 so that $i_1(t_2) = I_1$. Using the same steps in equations (9.33) through (9.40), we obtain

$$W\Big|_{t_0}^{t_2} = \frac{1}{2}L_1I_1^2+M_{21}I_1I_2+\frac{1}{2}L_2I_2^2 \tag{9.42}$$

Since relations (9.41) and (9.42) represent the same energy, we must have

$$M_{12} = M_{21} = M \tag{9.43}$$

and thus we can express (9.41) and (9.42) as

$$W\Big|_{t_0}^{t_2} = \frac{1}{2}L_1I_1^2+MI_1I_2+\frac{1}{2}L_2I_2^2 \tag{9.44}$$

Relation (9.44) was derived with the dot markings of Figure 9.18 which is repeated below as Figure 9.19 for convenience.

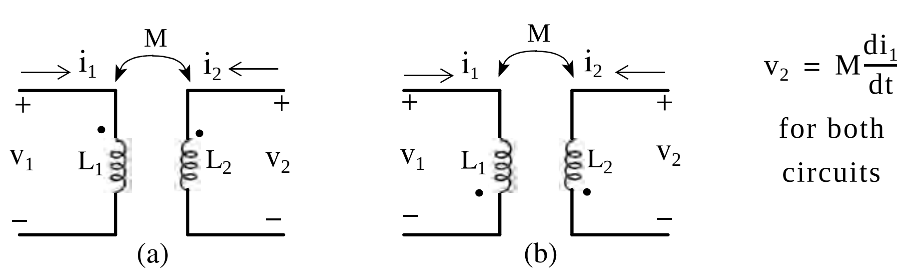


Figure 9.19. Transformer circuits of Figure 9.18

However, if we repeat the above procedure for the dot markings of the circuit of network 9.20 we will find that

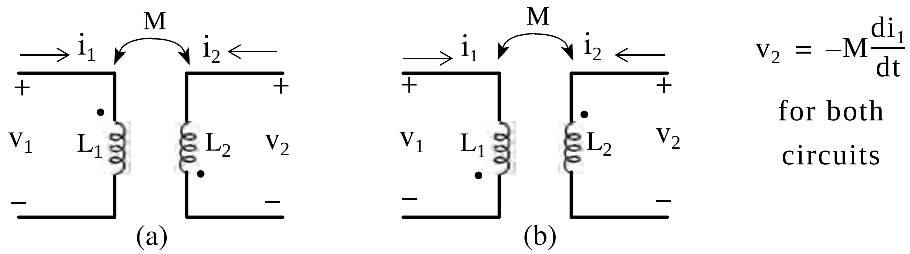


Figure 9.20. Transformer circuits with different dot arrangement from Figure 9.19

$$W\Big|_{t_0}^{t_2} = \frac{1}{2}L_1I_1^2 - MI_1I_2 + \frac{1}{2}L_2I_2^2 \tag{9.45}$$

and relations (9.44) and (9.45) can be combined to a single relation as

$$W\Big|_{t_0}^{t_2} = \frac{1}{2}L_1I_1^2 \pm MI_1I_2 + \frac{1}{2}L_2I_2^2 \tag{9.46}$$

where the sign of M is positive if both currents enter the dotted (or undotted) terminals, and it is negative if one current enters the dotted (or undotted) terminal while the other enters the undotted (or dotted) terminal.

The currents I_1 and I_2 are assumed constants and represent the final values of the instantaneous values of the currents i_1 and i_2 respectively. We may express (9.46) in terms of the instantaneous currents as

$$W\Big|_{t_0}^{t_2} = \frac{1}{2}L_1i_1^2 \pm Mi_1i_2 + \frac{1}{2}L_2i_2^2 \tag{9.47}$$

Obviously, the energy on the left side of (9.47) cannot be negative for any values of i_1, i_2, L_1, L_2, or M. Let us assume first that i_1 and i_2 are either both positive or both negative in which case their product is positive. Then, from (9.47) we see that the energy would be negative if

$$W\Big|_{t_0}^{t_2} = \frac{1}{2}L_1i_1^2 + \frac{1}{2}L_2i_2^2 - Mi_1i_2 \tag{9.48}$$

and the magnitude of the Mi_1i_2 is greater than the sum of the other two terms on the right side of that expression. To derive an expression relating the mutual inductance M to the self–inductances L_1 and L_2, we add and subtract the term $\sqrt{L_1L_2}i_1i_2$ on the right side of (9.47), and we complete the square. This expression then becomes

$$W\Big|_{t_0}^{t_2} = \frac{1}{2}(\sqrt{L_1}i_1 - \sqrt{L_2}i_2)^2 + \sqrt{L_1L_2}i_1i_2 - Mi_1i_2 \tag{9.49}$$

We now observe that the first term on the right side of (9.49) could be very small and could approach zero, but it can never be negative. Therefore, for the energy to be positive, the second and third terms on the right side of (9.48) must be such that $\sqrt{L_1L_2} \ge M$ or

$$M \le \sqrt{L_1L_2} \tag{9.50}$$

Expression (9.50) indicates that the mutual inductance can never be larger than the geometric mean of the inductances of the two coils between which the mutual inductance exists.

Note: The inequality in (9.49) was derived with the assumption that i_1 and i_2 have the same algebraic sign. If their signs are opposite, we select the positive sign of (9.47) and we find that (9.50) holds also for this case.

The ratio $M/\sqrt{L_1L_2}$ is known as the *coefficient of coupling* and it is denoted with the letter k, that is,

$$k = \frac{M}{\sqrt{L_1L_2}} \tag{9.51}$$

Obviously k must have a value between zero and unity, that is, $0 \le k \le 1$. Physically, k provides a measure of the proximity of the primary and secondary coils. If the coils are far apart, we say that they are *loose−coupled* and k has a small value, typically between 0.01 and 0.1. For *close−coupled* circuits, k has a value of about 0.5. Power transformers have a k between 0.90 and 0.95. The value of k is exactly unity only when the two coils are coalesced into a single coil.

Example 9.5

For the transformer of Figure 9.21 compute the energy stored at $t = 0$ if:

a. $i_1 = 50\ mA$ and $i_2 = 25\ mA$

b. $i_1 = 0$ and $i_2 = 20\sin 377t\ mA$

c. $i_1 = 15\cos 377t\ mA$ and $i_2 = 40\sin(377t + 60°)\ mA$

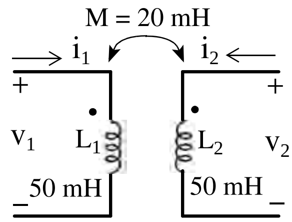


Figure 9.21. Transformer for Example 9.5

Solution:

Since the currents enter the dotted terminals, we use (9.47) with the plus (+) sign for the mutual inductance term, that is,

$$W(t) = \frac{1}{2}L_1 i_1^2 + M i_1 i_2 + \frac{1}{2}L_2 i_2^2 \tag{9.52}$$

Then,

a.

$$W|_{t=0} = 0.5 \times 50 \times 10^{-3} \times (50 \times 10^{-3})^2 + 20 \times 10^{-3} \times 50 \times 10^{-3} \times 25 \times 10^{-3}$$
$$+ 0.5 \times 50 \times 10^{-3} \times (25 \times 10^{-3})^2 = 103 \times 10^{-6}\ J = 103\ \mu J$$

b.

Since $i_1 = 0$ and $i_2 = 20\sin 377t|_{t=0} = 0$, it follows that

$$W|_{t=0} = 0$$

c.

$$W|_{t=0} = 0.5 \times 50 \times 10^{-3} \times (15 \times 10^{-3})^2 + 20 \times 10^{-3} \times 15 \times 10^{-3} \times 40 \times 10^{-3} \times \sin(60^\circ)$$
$$+ 0.5 \times 50 \times 10^{-3} \times (40 \times 10^{-3} \times \sin(60^\circ))^2 = 46 \times 10^{-6}\ J = 46\ \mu J$$

9.7 Circuits with Linear Transformers

A *linear transformer* is a four–terminal device in which the voltages and currents in the primary coils are linearly related.

The transformer shown in figure 9.22 a linear transformer. This transformer contains a voltage source in the primary, a load resistor in the secondary, and the resistors R_1 and R_2 represent the resistances of the primary and secondary coils respectively. Moreover, the primary is referenced to directly to ground, but the secondary is referenced to a DC voltage source V_0 and thus it is said that the secondary of the transformer has a DC *isolation*.

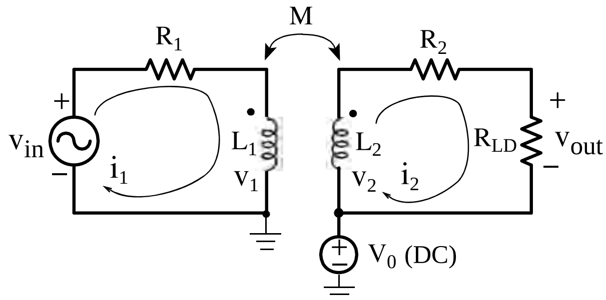


Figure 9.22. Transformer with DC isolation

Application of KVL around the primary and secondary circuits yields the loop equations

$$
\begin{aligned}
v_{in} &= R_1 i_1 + L_1 \frac{di_1}{dt} - M\frac{di_2}{dt} \quad * \\
0 &= -M\frac{di_1}{dt} + L_2\frac{di_2}{dt} + (R_2 + R_{LD})
\end{aligned}
\tag{9.53}
$$

and we see that the instantaneous values of the voltages and the currents are not affected by the presence of the DC voltage source V_0 since we would have obtained the same equations had we let $V_0 = 0$.

Example 9.6

For the transformer shown in Figure 9.23, find the total response of i_2 for $t > 0$ given that $i_1(0^-) = i_2(0^-) = 0$. Use MATLAB to sketch i_2 for $0 \le t \le 5\ s$.

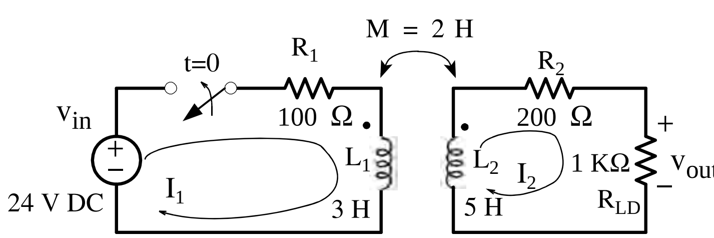


Figure 9.23. Transformer for Example 9.6

Solution:

The total response consists of the summation of the forced and natural responses, that is,

* *The mutual inductance terms* $M\frac{di_2}{dt}$ *and* $M\frac{di_1}{dt}$ *have a negative sign since the current* i_2 *is leaving the dotted terminal of the transformer secondary.*

$$i_{2T} = i_{2f} + i_{2n} \tag{9.54}$$

and since the applied voltage is constant (DC), no steady–state (forced) voltage is produced in the secondary and thus $i_{2f} = 0$.

For $t > 0$ the s–domain circuit is shown in Figure 9.24.

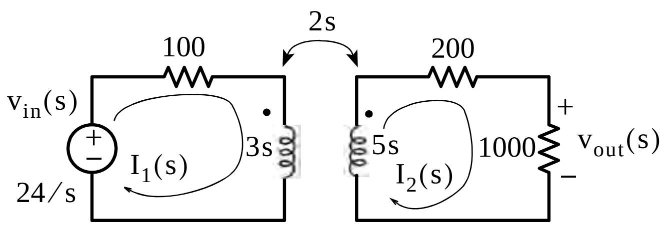


Figure 9.24. The s*–domain circuit for the transformer of Example 9.6 for* $t > 0$

The loop equations for this transformer are

$$\begin{aligned}(3s+100)I_1(s) - 2sI_2(s) &= 24/s \\ -2sI_1(s) + (5s+1200)I_2(s) &= 0\end{aligned} \tag{9.55}$$

Since we are interested only in $I_2(s)$, we will use Cramer's rule.

$$I_2(s) = \frac{\begin{bmatrix} 3s+100 & 24/s \\ -2s & 0 \end{bmatrix}}{\begin{bmatrix} 3s+100 & -2s \\ -2s & 5s+1200 \end{bmatrix}} = \frac{48}{11s^2 + 4100s + 120000} = \frac{4.36}{s^2 + 372.73s + 10909.01}$$

or

$$I_2(s) = \frac{4.36}{(s+340.71)(s+32.02)}$$

and by partial fraction expansion,

$$I_2(s) = \frac{4.36}{(s+340.71)(s+32.02)} = \frac{r_1}{s+340.71} + \frac{r_2}{s+32.02} \tag{9.56}$$

from which

$$r_1 = \left.\frac{4.36}{s+32.02}\right|_{s=-340.71} = -0.01 \tag{9.57}$$

$$r_2 = \left.\frac{4.36}{s+340.71}\right|_{s=-32.02} = 0.01 \tag{9.58}$$

By substitution into (9.56), we obtain

$$I_2(s) = \frac{0.01}{s + 32.02} + \frac{-0.01}{s + 340.71} \tag{9.59}$$

and taking the Inverse Laplace of (9.59) we obtain

$$i_{2n} = 0.01(e^{-32.02t} - e^{-340.71t}) \tag{9.60}$$

Using the following MATLAB script we obtain the plot shown on Figure 9.25.

```
t=0: 0.001: 0.2; i2n=0.01.*(exp(–32.02*t)–exp(–340.71.*t)); plot(t,i2n); grid
```

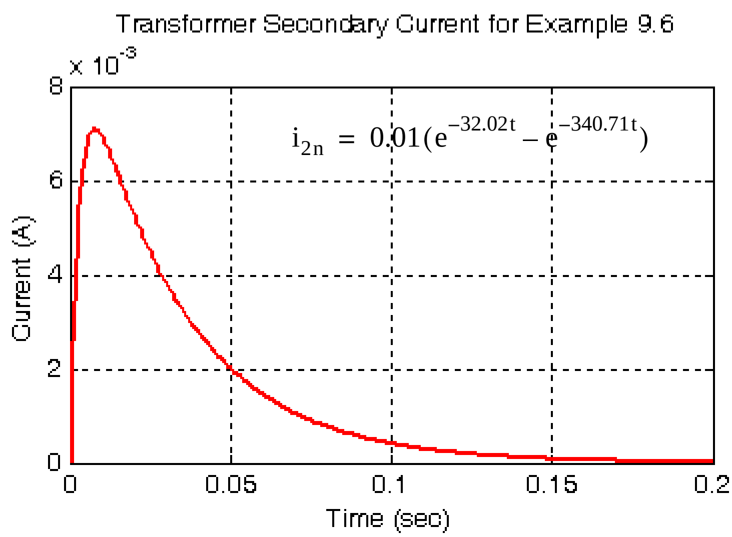


Figure 9.25. Plot for the secondary current of the transformer of Example 9.6

Example 9.7

For the transformer of Figure 9.26, find the steady–state (forced) response of v_{out}.

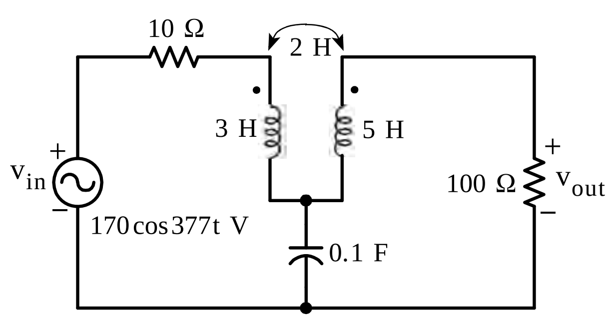


Figure 9.26. Circuit for Example 9.7

Solution:

The s–domain equivalent circuit is shown in Figure 9.27.

We could use the same procedure as in the previous example, but it is easier to work with the transfer function $G(s)$.

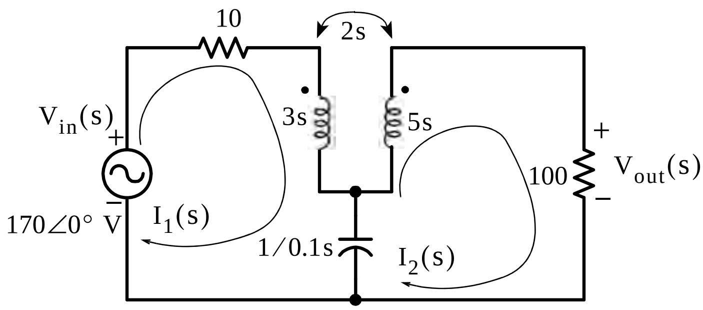


Figure 9.27. The s–domain equivalent circuit for Example 9.7

The loop equations for the transformer of Figure 9.27 are:

$$\begin{aligned}(3s + 10 + 1/0.1s)I_1(s) - (2s + 1/0.1s)I_2(s) &= V_{in}(s)\\ -(2s + 1/0.1s)I_1(s) + (5s + 100 + 1/0.1s)I_2(s) &= 0\end{aligned} \tag{9.61}$$

and by Cramer's rule,

$$I_2(s) = \frac{\begin{bmatrix}(3s+10+1/0.1s) & V_{in}(s)\\ -(2s+1/0.1s) & 0\end{bmatrix}}{\begin{bmatrix}(3s+10+1/0.1s) & -(2s+1/0.1s)\\ -(2s+1/0.1s) & (5s+100+1/0.1s)\end{bmatrix}}$$

or

$$I_2(s) = \frac{(2s+10/s)V_{in}(s)}{11s^2+350s+1040+1100/s} = \frac{(2s^2+10)V_{in}(s)}{11s^3+350s^2+1040s+1100}$$

$$= \frac{(0.18s^2+0.91)V_{in}(s)}{s^3+31.82s^2+94.55s+100}$$

From Figure 9.27 we observe that

$$V_{out}(s) = 100\cdot I_2(s) = 100\cdot\frac{(0.18s^2+0.91)V_{in}(s)}{s^3+31.82s^2+94.55s+100} = \frac{(18s^2+91)V_{in}(s)}{s^3+31.82s^2+94.55s+100} \tag{9.62}$$

and

$$G(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{18s^2+91}{s^3+31.82s^2+94.55s+100} \tag{9.63}$$

The input is a sinusoid, that is,

$$v_{in} = 170\cos 377t \text{ V}$$

and since we are interested in the steady−state response, we let

$$s = j\omega = j377$$

and thus

$$V_{in}(s) = V_{in}(j\omega) = 170\angle 0°$$

From (9.63) we obtain:

$$V_{out}(j\omega) = \frac{-2.56\times 10^6 + 91}{-j5.36\times 10^7 - 4.52\times 10^6 + j3.56\times 10^4 + 100}170\angle 0° = \frac{-4.35\times 10^8\angle 0°}{-4.52\times 10^6 - j5.36\times 10^7}$$

or

$$V_{out}(j\omega) = \frac{4.35\times 10^8\angle 180°}{5.38\times 10^7\angle -94.82°} = \frac{43.5\angle 180°}{5.38\angle -94.82°} = 8.09\angle 274.82° = 8.09\angle -85.18° \quad (9.64)$$

and in the t−domain,

$$v_{out}(t) = 8.09\cos(377t - 85.18°) \quad (9.65)$$

The expression of (9.65) indicates that the transformer of this example is a step−down transformer.

9.8 Reflected Impedance in Transformers

In this section, we will see how the load impedance of the secondary can be reflected into the primary.

Let us consider the transformer phasor circuit of Figure 9.28. We assume that the resistance of the primary and secondary coils is negligible.

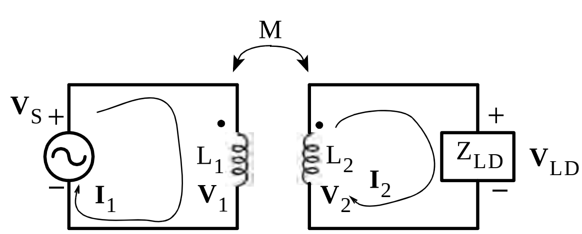


Figure 9.28. Circuit for the derivation of reflected impedance

By KVL the loops equations in phasor notation are:

$$j\omega L_1\mathbf{I}_1 - j\omega M\mathbf{I}_2 = \mathbf{V}_S \quad (9.66)$$

or

$$\mathbf{I}_2 = \frac{j\omega L_1\mathbf{I}_1 - \mathbf{V}_S}{j\omega M} \quad (9.67)$$

and

$$-j\omega M\mathbf{I}_1 + (j\omega L_2 + Z_{LD})\mathbf{I}_2 = 0 \tag{9.68}$$

or

$$\mathbf{I}_2 = \frac{j\omega M\mathbf{I}_1}{(j\omega L_2 + Z_{LOAD})} \tag{9.69}$$

Equating the right sides of (9.67) and (9.69) we obtain:

$$\frac{j\omega L_1\mathbf{I}_1 - \mathbf{V}_S}{j\omega M} = \frac{j\omega M\mathbf{I}_1}{(j\omega L_2 + Z_{LD})} \tag{9.70}$$

Solving for $\mathbf{V}_S$ we obtain:

$$\mathbf{V}_S = \left[j\omega L_1 - \frac{(j\omega M)^2}{(j\omega L_2 + Z_{LD})}\right]\mathbf{I}_1 \tag{9.71}$$

and dividing $\mathbf{V}_S$ by $\mathbf{I}_1$ we obtain the input impedance Z_{in} as

$$Z_{in} = \frac{\mathbf{V}_S}{\mathbf{I}_1} = j\omega L_1 + \frac{\omega^2 M^2}{j\omega L_2 + Z_{LD}} \tag{9.72}$$

The first term on the right side of (9.72) represents the reactance of the primary. The second term is a result of the mutual coupling and it is referred to as the *reflected impedance*. It is denoted as Z_R, i.e.,

$$Z_R = \frac{\omega^2 M^2}{j\omega L_2 + Z_{LD}} \tag{9.73}$$

From (9.73), we make two important observations:

1. The reflected impedance Z_R does not depend on the dot locations on the transformer. For instance, if either dot in the transformer of the previous page is placed on the opposite terminal, the sign of the mutual term changes from M to $-M$. But since Z_R varies as M^2, its sign remains unchanged.

2. Let $Z_{LD} = R_{LD} + jX_{LD}$. Then, we can express (9.73) as

$$Z_R = \frac{\omega^2 M^2}{j\omega L_2 + R_{LD} + jX_{LD}} = \frac{\omega^2 M^2}{R_{LD} + j(X_{LD} + \omega L_2)} \tag{9.74}$$

To express (9.74) as the sum of a real and an imaginary component, we multiply both numerator and denominator by the complex conjugate of the denominator. Then,

$$Z_R = \frac{\omega^2 M^2 R_{LD}}{R_{LD}^2 + (X_{LD} + \omega L_2)^2} - j\frac{\omega^2 M^2 (X_{LD} + \omega L_2)}{R_{LD}^2 + (X_{LD} + \omega L_2)^2} \tag{9.75}$$

The imaginary part of (9.75) represents the reflected reactance and we see that it is negative. That is, the reflected reactance is opposite to that of the net reactance $X_{LD} + \omega L_2$ of the secondary. Therefore, if X_{LD} is a capacitive reactance whose magnitude is less than ωL_2, or if it is an inductive reactance, then the reflected reactance is capacitive. However, if X_{LD} is a capacitive reactance whose magnitude is greater than ωL_2, the reflected reactance is inductive. In the case where the magnitude of X_{LD} is capacitive and equal to ωL_2, the reflected reactance is zero and the transformer operates at resonant frequency. In this case, the reflected impedance is purely real since (9.75) reduces to

$$Z_R = \frac{\omega^2 M^2}{R_{LD}} \tag{9.76}$$

Example 9.8

In the transformer circuit of Figure 9.29, Z_S represents the internal impedance of the voltage source V_S.

Find:

a. Z_{in}

b. $\mathbf{I}_1$

c. $\mathbf{I}_2$

d. $\mathbf{V}_1$

e. $\mathbf{V}_2$

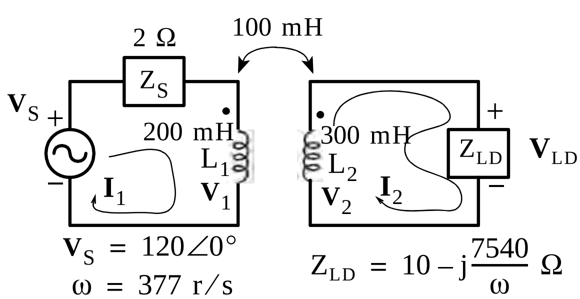


Figure 9.29. Transformer for Example 9.8

Solution:

a. From (9.72)

$$Z_{in} = \frac{\mathbf{V}_S}{\mathbf{I}_1} = j\omega L_1 + \frac{\omega^2 M^2}{j\omega L_2 + Z_{LD}}$$

and we must add $Z_s = 2\ \Omega$ to it. Therefore, for the transformer of this example,

$$Z_{in} = j\omega L_1 + \frac{\omega^2 M^2}{j\omega L_2 + Z_{LD}} + 2 = j75.4 + \frac{142129 \times 0.01}{j113.1 + 10 - j20} + 2$$

$$= 3.62 + j60.31 = 60.42\angle 86.56°\ \Omega$$

b.

$$\mathbf{I}_1 = \frac{\mathbf{V}_S}{Z_{in}} = \frac{120\angle 0°}{60.42\angle 86.56°\ \Omega} = 1.98\angle -86.56°\ \text{A}$$

c. By KVL

$$-j\omega M\mathbf{I}_1 + (j\omega L_2 + Z_{LD})\mathbf{I}_2 = 0$$

or

$$\mathbf{I}_2 = \frac{j\omega M}{j\omega L_2 + Z_{LD}}\mathbf{I}_1 = \frac{j37.7}{j113.1 + 10 - j20} 1.98\angle -86.56° = \frac{74.88\angle 3.04°}{93.64\angle 83.87°} = 0.8\angle -80.83°\ \text{A}$$

d.

$$\mathbf{V}_1 = j\omega L_1\mathbf{I}_1 - j\omega M\mathbf{I}_2 = 75.4\angle 90° \times 1.98\angle -86.56° - 37.7\angle 90° \times 0.8\angle -80.83°$$

$$= 149.29\angle 3.04° - 30.15\angle 9.17° = 149.08 + j7.92 - 30.15 - j4.8 = 118.9\angle 1.5°\ \text{V}$$

e.

$$V_2 = Z_{LD} \cdot \mathbf{I}_2 = (10 - j20)0.8\angle -80.83° = 22.36\angle -63.43° \times 0.8\angle -80.83° = 17.89\angle -144.26°\text{V}$$

9.9 The Ideal Transformer

An *ideal transformer* is one in which the coefficient of coupling is almost unity, and both the primary and secondary inductive reactances are very large in comparison with the load impedances. The primary and secondary coils have many turns wound around a laminated iron–core and are arranged so that the entire flux links all the turns of both coils.

An important parameter of an ideal transformer is the *turns ratio* a which is defined as the ratio of the number of turns on the secondary, N_2, to the number of turns of the primary N_1, that is,

$$a = \frac{N_2}{N_1} \tag{9.77}$$

The flux produced in a winding of a transformer due to a current in that winding is proportional to the product of the current and the number of turns on the winding. Therefore, letting α be a constant of proportionality which depends on the physical properties of the transformer, for the primary and secondary windings we have:

$$\phi_{11} = \alpha N_1 i_1$$
$$\phi_{22} = \alpha N_2 i_2 \qquad (9.78)$$

The constant α is the same for the primary and secondary windings because we have assumed that the same flux links both coils and thus both flux paths are identical. We recall from (9.8) and (9.14) that

$$\lambda_1 = N_1 \varphi_{11} = L_1 i_1$$
$$\lambda_2 = N_2 \varphi_{22} = L_2 i_2 \qquad (9.79)$$

Then, from (9.78) and (9.79) we obtain:

$$N_1 \varphi_{11} = L_1 i_1 = \alpha N_1^2 i_1$$
$$N_2 \varphi_{22} = L_2 i_2 = \alpha N_2^2 i_2 \qquad (9.80)$$

or

$$L_1 = \alpha N_1^2$$
$$L_2 = \alpha N_2^2 \qquad (9.81)$$

Therefore,

$$\frac{L_2}{L_1} = \left(\frac{N_2}{N_1}\right)^2 = a^2 \qquad (9.82)$$

From (9.69),

$$I_2 = \frac{j\omega M I_1}{(j\omega L_2 + Z_{LD})} \qquad (9.83)$$

or

$$\frac{I_2}{I_1} = \frac{j\omega M}{(j\omega L_2 + Z_{LD})} \qquad (9.84)$$

and since $j\omega L_2 \gg Z_{LD}$, (9.84) reduces to

$$\frac{I_2}{I_1} = \frac{j\omega M}{j\omega L_2} = \frac{M}{L_2} \qquad (9.85)$$

For the case of unity coupling,

$$k = \frac{M}{\sqrt{L_1 L_2}} = 1 \qquad (9.86)$$

or

$$M = \sqrt{L_1 L_2} \qquad (9.87)$$

and by substitution of (9.87) into (9.85) we obtain:

$$\frac{I_2}{I_1} = \frac{\sqrt{L_1 L_2}}{L_2} = \sqrt{\frac{L_1}{L_2}} \qquad (9.88)$$

From (9.82) and (9.88), we obtain the important relation

$$\boxed{\frac{I_2}{I_1} = \frac{1}{a}} \tag{9.89}$$

Also, from (9.77) and (9.89),

$$\boxed{N_1 I_1 = N_2 I_2} \tag{9.90}$$

and this relation indicates that if $N_2 < N_1$, the current I_2 is larger than I_1.

The primary and secondary voltages are also related to the turns ratio a. To find this relation, we define the secondary or load voltage V_2 as

$$V_2 = Z_{LD} I_2 \tag{9.91}$$

and the primary voltage V_1 across L_1 as

$$V_1 = Z_{in} I_1 \tag{9.92}$$

From (9.72),

$$Z_{in} = \frac{V_s}{I_1} = j\omega L_1 + \frac{\omega^2 M^2}{j\omega L_2 + Z_{LD}} \tag{9.93}$$

and for $k = 1$

$$M^2 = L_1 L_2$$

Then, (9.93) becomes

$$Z_{in} = j\omega L_1 + \frac{\omega^2 L_1 L_2}{j\omega L_2 + Z_{LD}} \tag{9.94}$$

Next, from (9.82)

$$L_2 = a^2 L_1 \tag{9.95}$$

Substitution of (9.95) into (9.94) yields

$$Z_{in} = j\omega L_1 + \frac{\omega^2 a^2 L_1^2}{j\omega a^2 L_1 + Z_{LD}} \tag{9.96}$$

and if we let $j\omega L_1 \to \infty$, both terms on the right side of (9.96) become infinite and we obtain an indeterminate result. To work around this problem, we combine these terms and we obtain:

$$Z_{in} = \frac{-\omega^2 a^2 L_1^2 + j\omega L_1 Z_{LD} + \omega^2 a^2 L_1^2}{j\omega a^2 L_1 + Z_{LD}} = \frac{j\omega L_1 Z_{LD}}{j\omega a^2 L_1 + Z_{LD}}$$

and as $j\omega L_1 \to \infty$,

$$Z_{in} = \frac{Z_{LD}}{a^2} \tag{9.97}$$

Finally, substitution of (9.97) into (9.92) yields

$$V_1 = \frac{Z_{LD}}{a^2} I_1 \tag{9.98}$$

and by division of (9.91) by (9.98) we obtain:

$$\frac{V_2}{V_1} = \frac{Z_{LD} I_2}{(Z_{LD}/a^2) I_1} = a^2 \cdot \frac{1}{a} = a \tag{9.99}$$

or

$$\boxed{\frac{V_2}{V_1} = a} \tag{9.100}$$

also, from the current and voltage relations of (9.88) and (9.99),

$$\boxed{V_2 I_2 = V_1 I_1} \tag{9.101}$$

that is, the volt–amperes of the secondary and the primary are equal.

An ideal transformer is represented by the network of Figure 9.30.

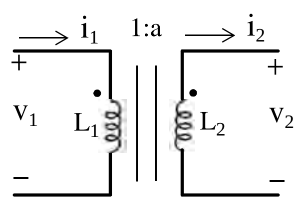


Figure 9.30. Ideal transformer representation

9.10 Impedance Matching

An ideal (iron–core) transformer can be used as an impedance level changing device. We recall from basic circuit theory that to achieve maximum power transfer, we must adjust the resistance of the load to make it equal to the resistance of the voltage source. But this is not always possible. A power amplifier for example, has an internal resistance of several thousand ohms. On the other hand, a speaker which is to be connected to the output of a power amplifier has a fixed resistance of just a few ohms. In this case, we can achieve maximum power transfer by inserting an iron–core transformer between the output of the power amplifier and the input of the speaker as shown in Figure 9.31 where $N_2 < N_1$.

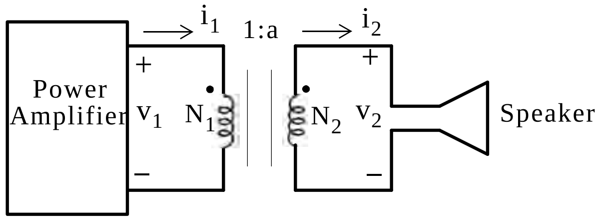


Figure 9.31. Transformer used as impedance matching device

Let us suppose that in Figure 9.31 the amplifier internal impedance is $80000\ \Omega$ and the impedance of the speaker is only $8\ \Omega$. We can find the appropriate turns ratio $N_2/N_1 = a$ using (9.97), that is,

$$Z_{in} = \frac{Z_{LD}}{a^2} \tag{9.102}$$

or

$$a = \frac{N_2}{N_1} = \sqrt{\frac{Z_{LD}}{Z_{in}}} = \sqrt{\frac{8}{80000}} = \sqrt{\frac{1}{10000}} = \frac{1}{100}$$

or

$$\frac{N_1}{N_2} = 100 \tag{9.103}$$

that is, the number of turns in the primary must be 100 times the number of the turns in the secondary.

9.11 Simplified Transformer Equivalent Circuit

In analyzing networks containing ideal transformers, it is very convenient to replace the transformer by an equivalent circuit before the analysis. Consider the transformer circuit of Figure 9.32.

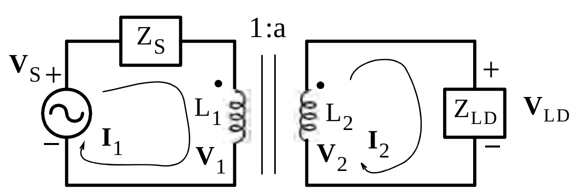


Figure 9.32. Circuit to be simplified

From (9.97)

$$Z_{in} = \frac{Z_{LD}}{a^2}$$

The input impedance seen by the voltage source $\mathbf{V_S}$ in the circuit of Figure 9.32 is

$$Z_{in} = Z_S + \frac{Z_{LD}}{a^2} \tag{9.104}$$

and thus the circuit of Figure 9.32 can be replaced with the simplified circuit shown in Figure 9.33.

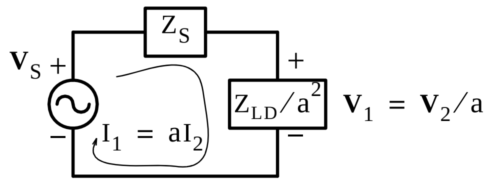


Figure 9.33. Simplified circuit for the transformer of Figure 9.32

The voltages and currents can now be found from the simple series circuit of Figure 9.33.

9.12 Thevenin Equivalent Circuit

Let us consider again the circuit of Figure 9.32. This time we want to find the Thevenin equivalent to the left of the secondary terminals and replace the primary by its Thevenin equivalent at points x and y as shown in Figure 9.34.

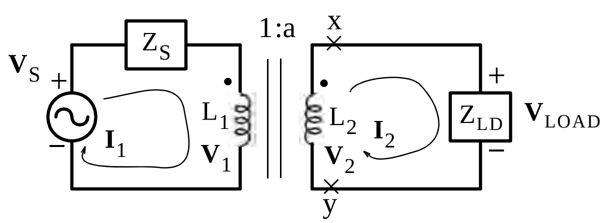


Figure 9.34. Circuit for the derivation of Thevenin's equivalent

If we open the circuit at points x and y as shown in Figure 9.34, we find the Thevenin voltage as $\mathbf{V}_{TH} = \mathbf{V}_{OC} = \mathbf{V}_{xy}$. Since the secondary is now an open circuit, we have $\mathbf{I}_2 = 0$, and also $\mathbf{I}_1 = 0$ because $\mathbf{I}_1 = a\mathbf{I}_2$. Since no voltage appears across Z_S, $\mathbf{V}_1 = \mathbf{V}_S$ and $\mathbf{V}_{2\ oc} = a\mathbf{V}_1 = a\mathbf{V}_S$. Then,

$$\boxed{\mathbf{V}_{TH} = \mathbf{V}_{OC} = \mathbf{V}_{xy} = a\mathbf{V}_S} \tag{9.105}$$

We will find the Thevenin impedance Z_{TH} from the relation

$$Z_{TH} = \frac{\mathbf{V}_{OC}}{\mathbf{I}_{SC}} \tag{9.106}$$

The short circuit current $\mathbf{I}_{SC}$ is found from

$$\mathbf{I}_{SC} = \mathbf{I}_2 = \frac{\mathbf{I}_1}{a} = \frac{\mathbf{V}_S/Z_S}{a} = \frac{\mathbf{V}_S}{aZ_S}\ * \tag{9.107}$$

and by substitution into (9.106),

$$Z_{TH} = \frac{a\mathbf{V}_S}{\mathbf{V}_S/aZ_S} = a^2 Z_S$$

The Thevenin equivalent circuit with the load connected to it is shown in Figure 9.35.

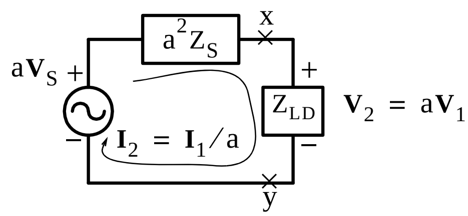


Figure 9.35. The Thevenin equivalent of the transformer circuit in Figure 9.34

The circuit of Figure 9.35 was derived with the assumption that the dots are placed as shown in Figure 9.34. If either dot is reversed, we simply replace a by $-a$.

Example 9.9

For the circuit of Figure 9.36, find $\mathbf{V}_2$.

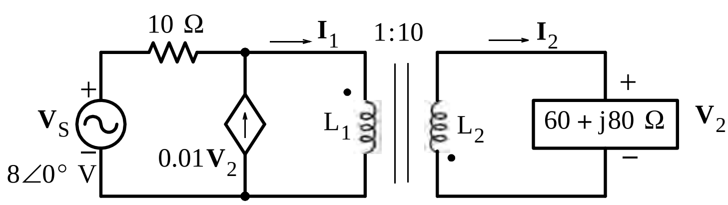


Figure 9.36. Circuit for Example 9.9

Solution:

We will replace the given circuit with its Thevenin equivalent. First, we observe that the dot in the secondary has been reversed, and therefore we will replace a by $-a$. The Thevenin equivalent is obtained by multiplying V_S by -10, dividing the dependent source by -10, and multiplying the $10\ \Omega$ resistor by $(-a)^2 = 100$. With these modifications we obtain the circuit of Figure 9.37.

* *Since* $V_2 = 0$ *and* $V_2/V_1 = a$ *or* $aV_1 = V_2$ *it follows that* $V_1 = 0$ *also.*

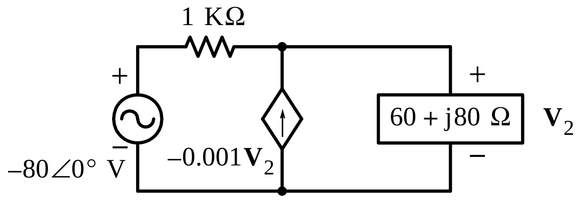


Figure 9.37. The Thevenin equivalent of the circuit of Example 9.9

Now, by application of KCL

$$\frac{\mathbf{V}_2-(-80\angle 0°)}{10^3}-(-10^{-3}\mathbf{V}_2)+\frac{\mathbf{V}_2}{60+j80} = 0$$

$$\frac{\mathbf{V}_2}{10^3}+\frac{\mathbf{V}_2}{10^3}+\frac{(60-j80)\mathbf{V}_2}{10000} = -\frac{80}{10^3}$$

$$2\mathbf{V}_2+(6-j8)\mathbf{V}_2 = -80$$

$$8(1-j1)\mathbf{V}_2 = 80\angle 180°$$

$$(\sqrt{2}\angle -45°)\mathbf{V}_2 = 10\angle 180°$$

or

$$\mathbf{V}_2 = \frac{10}{\sqrt{2}}\angle 225° = 5\sqrt{2}\angle -135°$$

Other equivalent circuits can be developed from the equations of the primary and secondary voltages and currents.

Consider for example, the linear transformer circuit of Figure 9.38.

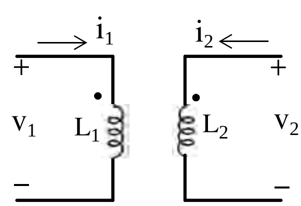


Figure 9.38. Linear transformer

From (9.30), the primary and secondary voltages and currents are:

$$v_1 = L_1\frac{di_1}{dt} + M\frac{di_2}{dt}$$
$$v_2 = M\frac{di_1}{dt} + L_2\frac{di_2}{dt} \quad (9.108)$$

and these equations are satisfied by the equivalent circuit shown in Figure 9.39.

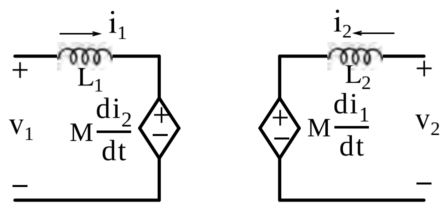


Figure 9.39. Network satisfying the expressions of (9.108)

If we rearrange the equations of (9.108) as

$$v_1 = (L_1 - M)\frac{di_1}{dt} + M\left(\frac{di_1}{dt} + \frac{di_2}{dt}\right)$$
$$v_2 = M\left(\frac{di_1}{dt} + \frac{di_2}{dt}\right) + (L_2 - M)\frac{di_2}{dt} \quad (9.109)$$

we find that these equations are satisfied by the circuit of Figure 9.40.

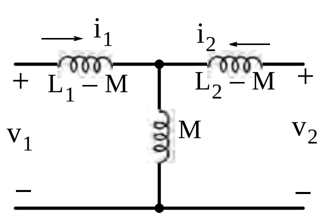


Figure 9.40. Network satisfying the expressions of (9.109)

Additional equivalent circuits are shown in Figure 9.41 and they are useful in the computations of transformer parameters computations from the open– and short–circuit tests, efficiency, and voltage regulation which will be discussed in subsequent sections in this chapter.

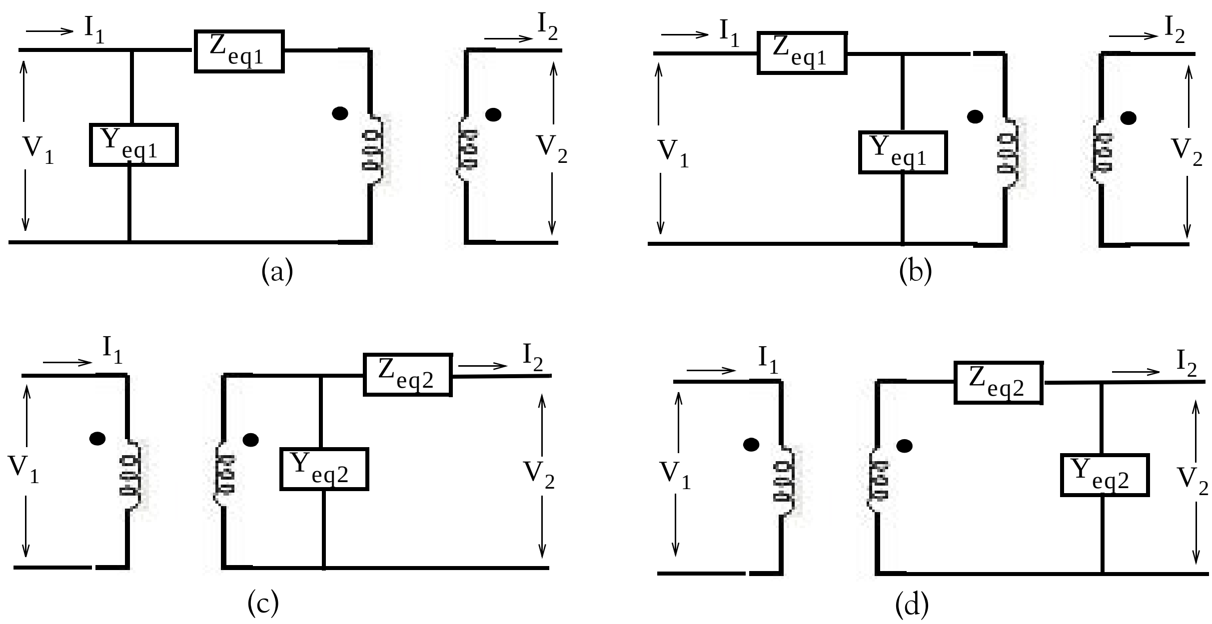
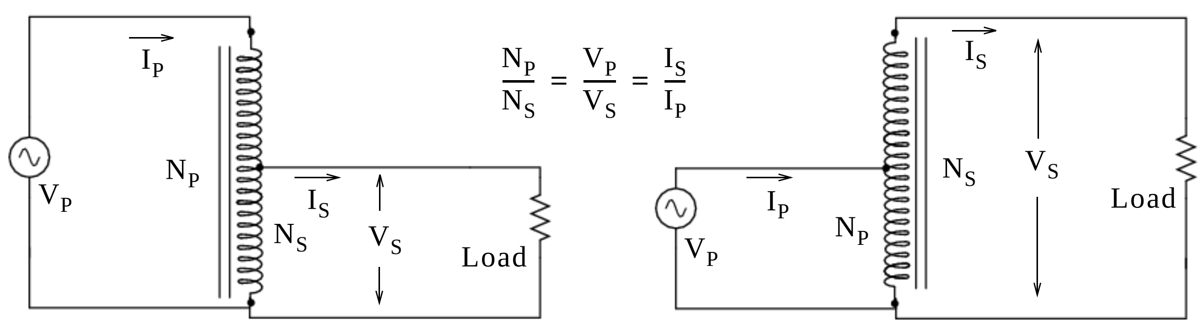


Figure 9.41. Other transformer equivalent circuits

9.13 Autotransformer

An autotransformer is a special transformer that shares a common winding, and can be configured either as a step−down or step−up transformer as shown in Figure 9.42.



(a) Step−down autotransformer

(b) Step−up autotransformer

Figure 9.42. Step−down and step−up center−tapped autotransformers

Autotransformers are not used in residential, commercial, or industrial applications because a break in the common winding may result in equipment damage and / or personnel injury.

A *variac* is an adjustable autotransformer, that is, its secondary voltage can be adjusted from zero to a maximum value y a wiper arm that slides over the common winding as shown in Figure 9.43.

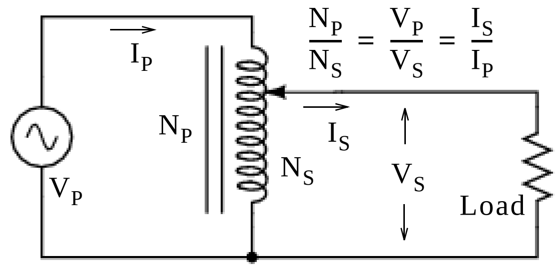


Figure 9.43. Variac

9.14 Transformers with Multiple Secondary Windings

Some transformers are constructed with a common primary winding and two or more secondary windings. These transformers are used in applications hen there is a need for two or more different secondary voltages with a common primary voltage. Figure 9.44 shows a transformer with one primary and two secondary windings.

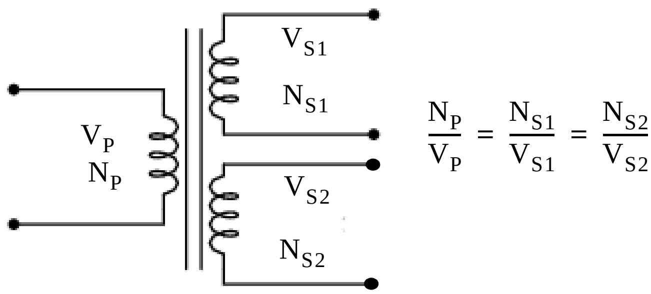


Figure 9.44. Transformer with common primary winding and two secondary windings

9.15 Transformer Tests

The analysis of the ideal transformer model provides approximate values. A practical transformer is shown in Figure 9.44 and makes provisions for core (hysteresis and eddy current l)* losses, winding losses, and magnetic flux leakages. The resistances $\mathbf{R_P}$ and $\mathbf{R_S}$ are the resistances of the primary and secondary windings respectively, the reactances $\mathbf{X_P}$ and $\mathbf{X_S}$ represent the leakage flux of the primary and secondary windings respectively, the resistance $\mathbf{R_C}$ is for the core loses, and the reactance $\mathbf{X_M}$, referred to as the magnetizing reactance, represents the transformer's main flux.

* *Exercise 11 at the end of this chapter provides a brief discussion and a method for the computation of hysteresis and eddy current losses,*

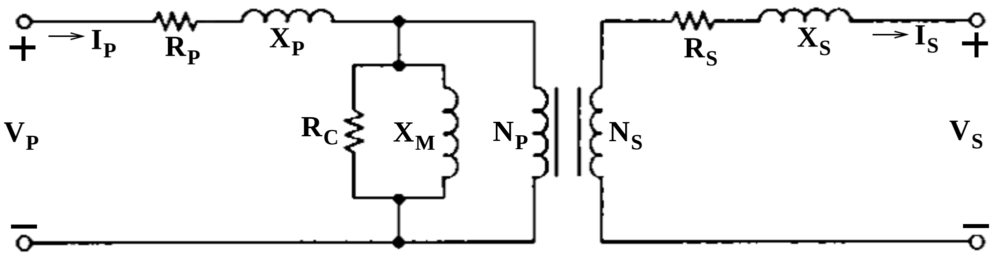


Figure 9.45. Equivalent circuit for practical transformer

Figure 9.46 shows the equivalent circuit in Figure 9.45 with the secondary quantities referred to the primary.

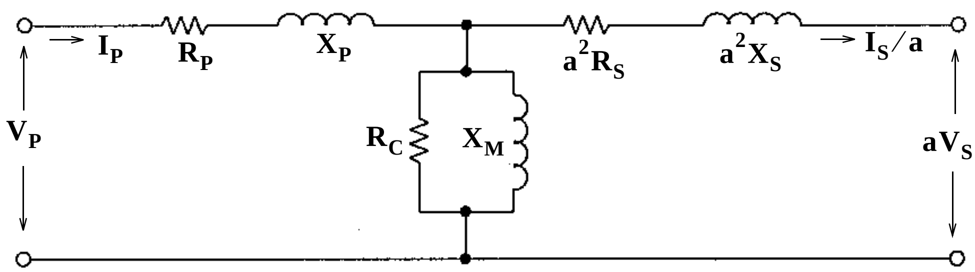


Figure 9.46. Equivalent circuit for practical transformer with secondary quantities referred to the primary

The resistance $\mathbf{R_P}$ in the primary winding and the resistance $\mathbf{R_S}$ in the secondary winding are read with an Ohmmeter. The other quantities are determined by the open–circuit and short–circuit tests described below.

I. Open–Circuit Test

The open–circuit test, also referred to as the no–load test, is used to determine the reactance $\mathbf{X_P}$ n the primary winding, the core resistance $\mathbf{R_C}$, and the magnetizing reactance $\mathbf{X_M}$. For this test, the secondary is left open, and an ammeter, a voltmeter, and a wattmeter are connected as shown in Figure 9.47.

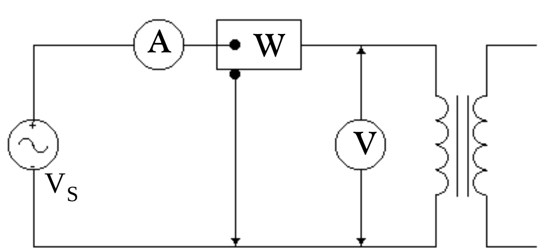


Figure 9.47. Configuration for transformer open–circuit test

In Figure 9.47, the value of the applied voltage V_S is set at its rated value*, and the voltmeter, ammeter, and wattmeter readings, denoted as V_{OC}, I_{OC}, and P_{OC} respectively, are measured and recorded. Then,

$$|Z_P| = \frac{V_{OC}}{I_{OC}} = \sqrt{R_P^2 + X_P^2} \tag{9.110}$$

from which

$$|X_P| = \sqrt{|Z_P|^2 - R_P^2} \tag{9.111}$$

The magnitude of the admittance Y_P in the excitation branch consisting of the parallel connection of R_C and X_M is found from

$$|Y_P| = \frac{V_{OC}}{I_{OC}} = \frac{I_{OC}}{V_{OC}} = \sqrt{G_C^2 + B_M^2} \tag{9.112}$$

where $G_C = 1/R_C$ and $B_M = 1/X_M$, and the phase angle θ_{OC} is found using the relation

$$\cos\theta_{OC} = \frac{P_{OC}}{V_{OC} \cdot I_{OC}} \tag{9.113}$$

from which

$$\theta_{OC} = \arccos\frac{P_{OC}}{V_{OC} \cdot I_{OC}} \tag{9.114}$$

Then,

$$\begin{aligned} G_C &= |Y_P|\cos\theta_{OC} \\ B_M &= |Y_P|\sin\theta_{OC} \end{aligned} \tag{9.115}$$

II. Short–Circuit Test

The short–circuit test is used to determine the magnitude of the series impedances referred to the primary side of the transformer denoted as Z_{SC} For this test, the secondary is shorted, and an ammeter, a voltmeter, and a wattmeter are connected as shown in Figure 9.48.

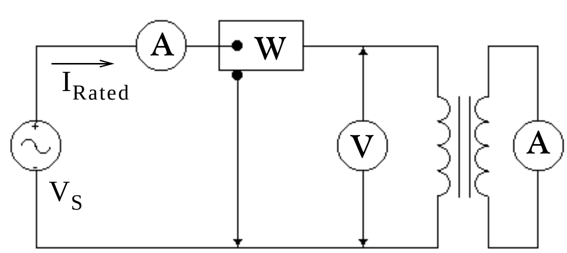


Figure 9.48. Configuration for transformer short–circuit test

* *It is important to use rated values so that the impedances and admittances will not have different values at different voltages.*

In Figure 9.48, the value of the applied voltage V_S is considerably less than the rated value of the transformer. It is set at a value such that the primary current denoted as I_{Rated} is the rated primary current value, and the voltmeter, ammeter, and wattmeter readings, denoted as V_{SC}, I_{SC}, and P_{SC} respectively, are measured and recorded. Then,

$$|Z_{SC}| = \frac{V_{SC}}{I_{SC}} \tag{9.116}$$

and the phase angle θ_{SC} is found using the relation

$$\cos\theta_{SC} = \frac{P_{SC}}{V_{SC} \cdot I_{SC}} \tag{9.117}$$

from which

$$\theta_{SC} = \arccos\frac{P_{SC}}{V_{SC} \cdot I_{SC}} \tag{9.118}$$

Then,

$$\begin{aligned} R_{SC} &= |Z_{SC}|\cos\theta_{SC} \\ X_{SC} &= |Z_{SC}|\cos\theta_{SC} \end{aligned} \tag{9.119}$$

Example 9.10

The open–circuit and short–circuit tests on a $100\ \text{KVA}$, $13.2/2.4\ \text{KV}$, $60\ \text{Hz}$ transformer produced the data shown in Table 9.1.

TABLE 9.1 Open– and Short–Circuit data for transformer in Example 9.10

Test	Voltage (V)	Current (A)	Power (W)
Open–circuit	2400	37	1100
Short–circuit	450	8.2	1600

The high–voltage side of this transformer is connected to a generator via a long transmission line, and the transmission line impedance is estimated to be $Z_{line} = 10 + j35\ \Omega$. A $75\ \text{KW}$ load at 0.8 lagging power factor is connected to the low–voltage side of the transformer, and it is desired that the voltage across the $75\ \text{KW}$ load be $2,300\ \text{V}$. Compute the terminal voltage V_{GEN} of the generator connected to the left end of the transmission line.

Solution:

The equivalent circuit of this system is shown in Figure 9.49, and all quantities are referred to the primary side.

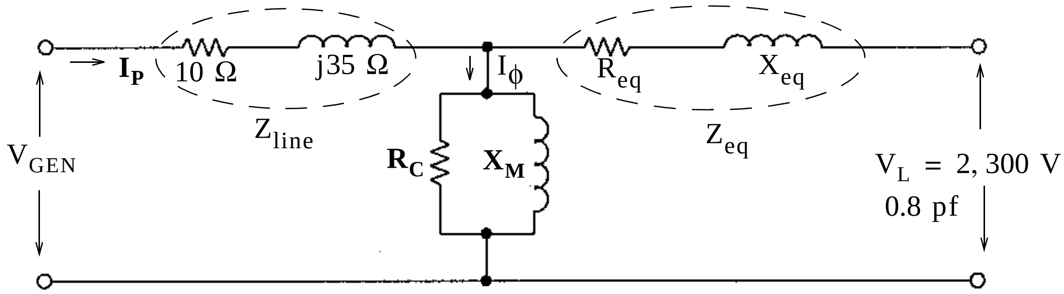


Figure 9.49. Circuit for Example 9.10

For this transformer, the ratio **a** is

$$a = \frac{N_P}{N_S} = \frac{V_P}{V_S} = \frac{13.2\ KV}{2.4\ KV} = 5.5 \tag{9.120}$$

From the short–circuit test,

$$R_{eq} = \frac{P_{SC}}{I_{SC}^2} = \frac{1600}{8.2^2} = 23.8\ \Omega \tag{9.121}$$

and

$$Z_{eq} = \frac{V_{SC}}{I_{SC}} = \frac{450}{8.2} = 54.9\ \Omega \tag{9.122}$$

Then,

$$X_{eq} = \sqrt{Z_{eq}^2 - R_{eq}^2} = \sqrt{54.9^2 - 23.8^2} = 49.5\ \Omega \tag{9.123}$$

The load current I_L referred to the primary is

$$I_L = \frac{\text{Load KW}}{\text{Load KV} \times a \times pf} = \frac{75}{2.3 \cdot 5.5 \cdot 0.8} = 7.4\ A \tag{9.124}$$

The excitation current I_ϕ referred to the primary is

$$I_\phi = \frac{I_{OC}}{a} = \frac{37}{5.5} = 6.73\ A \tag{9.125}$$

and its phase angle ϕ is

$$\phi = \arccos\frac{P_{OC}}{V_{OC} \cdot I_{OC}} = \arccos\frac{1100}{2400 \cdot 37} \approx 90\ deg \tag{9.126}$$

and since in a real transformer the angle of the current lags the angle of the voltage, we accept $\angle\phi = -90\ deg$, and thus

$$I_\phi = 6.73\angle{-90°} = -j6.73 \tag{9.127}$$

Therefore, the generator voltage V_{GEN} must be

$$V_{GEN} = I_\phi \cdot Z_{line} + I_L \cdot (Z_{line} + Z_{eq}) \cdot pf \cdot aV_L$$
$$= -j6.73 \times (10 + j35) + 7.4 \times (33.8 + j84.5) \times (0.8 - j0.6) + 5.5 \times 2300 = 13457 + j280.5$$

or

$$|V_{GEN}| = 13.46 \text{ KV} \tag{9.128}$$

9.16 Efficiency

Efficiency, denoted as η, is a dimensionless quantity defined as

$$\eta = \frac{P_{OUT}}{P_{IN}} = \frac{P_{IN} - P_{LOSS}}{P_{IN}} = 1 - \frac{P_{LOSS}}{P_{IN}} \tag{9.129}$$

or in terms of the output and losses

$$\eta = \frac{P_{OUT}}{P_{OUT} + P_{LOSS}} = 1 - \frac{P_{LOSS}}{P_{OUT} + P_{LOSS}} \tag{9.130}$$

The losses in a transformer are the summation of the core losses (hysteresis and eddy currents), and copper losses caused by the resistance of the conducting material of the coils, generally made of copper. The core losses can be obtained from the transformer equivalent circuit in Figure 9.50.

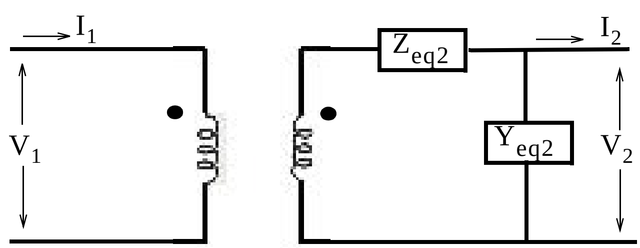


Figure 9.50. Transformer equivalent circuit for computation of the core losses

Thus, the core losses P_C are found from the relation

$$P_C = G_{C2} V_2^2 \tag{9.131}$$

The copper losses can be obtained from the transformer equivalent circuit in Figure 9.51.

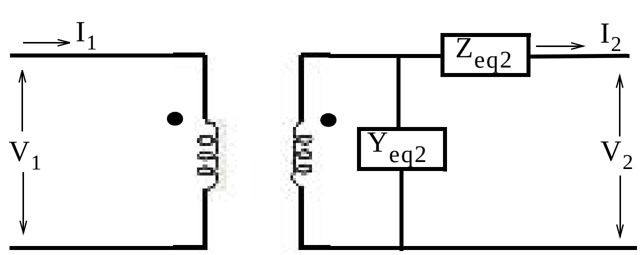


Figure 9.51. Transformer equivalent circuit for computation of the copper losses

Thus, the copper losses P_R are found from the relation

$$P_R = R_{eq2} I_2^2 \tag{9.132}$$

Therefore, using equation (9.130), we obtain

$$\eta = \frac{P_{OUT}}{P_{OUT} + P_{LOSS}} = \frac{V_2 I_2 \cos\theta_2}{V_2 I_2 \cos\theta_2 + G_{C2} V_2^2 + R_{eq2} I_2^2} \tag{9.133}$$

The efficiency varies with the load current I_2, and to find the maximum efficiency we differentiate (9.133) with respect to the load current I_2 * and we obtain

$$\frac{d\eta}{dI_2} = \frac{(V_2 I_2 \cos\theta_2 + G_{C2} V_2^2 + R_{eq2} I_2^2) \cdot V_2 \cos\theta_2 - (V_2 \cos\theta_2 + 2R_{eq2} I_2) \cdot V_2 I_2 \cos\theta_2}{(V_2 I_2 \cos\theta_2 + G_{C2} V_2^2 + R_{eq2} I_2^2)^2} = 0 \tag{9.134}$$

$$\frac{(V_2 I_2 \cos\theta_2 + G_{C2} V_2^2 + R_{eq2} I_2^2) \cdot V_2 \cos\theta_2 - (V_2 \cos\theta_2 + 2R_{eq2} I_2) \cdot V_2 I_2 \cos\theta_2}{(V_2 I_2 \cos\theta_2 + G_{C2} V_2^2 + R_{eq2} I_2^2)^2} = 0 \tag{9.135}$$

or

$$(V_2 I_2 \cos\theta_2 + G_{C2} V_2^2 + R_{eq2} I_2^2) - (V_2 \cos\theta_2 + 2R_{eq2} I_2) \cdot I_2 = 0 \tag{9.136}$$

and after simplification,

$$G_{C2} V_2^2 = R_{eq2} I_2^2 \tag{9.137}$$

That is, the efficiency attains its maximum value at that load at which the constant (core) losses are equal to the losses that vary with the load, i.e., the copper losses.

Example 9.11

A $1000\ KVA$, $13.2 / 4.16\ KV$ transformer has an equivalent series impedance $Z_{eq} = 1 + j4.2\ \Omega$ referred to the low–voltage side, and a core loss $2500\ w$ at rated terminal voltage. Find:

a. The value of the load current I_2 which will produce the maximum efficiency

b. The KVA output at maximum efficiency.

Solution:

a. From relation (9.137),

$$R_{eq} I_2^2 = G_C V_2^2 = 2500$$

* *The quantities* V_2 *and* $\cos\theta_2$ *are constant.*

and with $R_{eq} = 1\ \Omega$, we find that the maximum efficiency occurs when $I_2 = 50\ A$, and with (9.133) we find that the efficiency is

$$\eta = \frac{V_2 I_2 \cos\theta_2}{V_2 I_2 \cos\theta_2 + G_{C2} V_2^2 + R_{eq2} I_2^2} = \frac{4160 \times 50 \times 0.8}{4160 \times 50 \times 0.8 + 2500 + 2500} = 0.97 \quad \text{or} \quad 97\%$$

Figure 9.52 is a plot of the efficiency versus the load current, and we observe that the maximum efficiency occurs when the load current I_2 is $50\ A$.

The plot in Figure 9.51 was produced with the MATLAB script below.

```
i2=0:1:150; eff=4.16.*0.8.*i2./(4.16.*0.8.*i2+2.5+i2.^2./1000); plot(i2,eff); grid;...
xlabel('Load Current I2 (A)'); ylabel('Efficiency'); ...
title('Efficiency vs Load Current, Example 9.11')
```

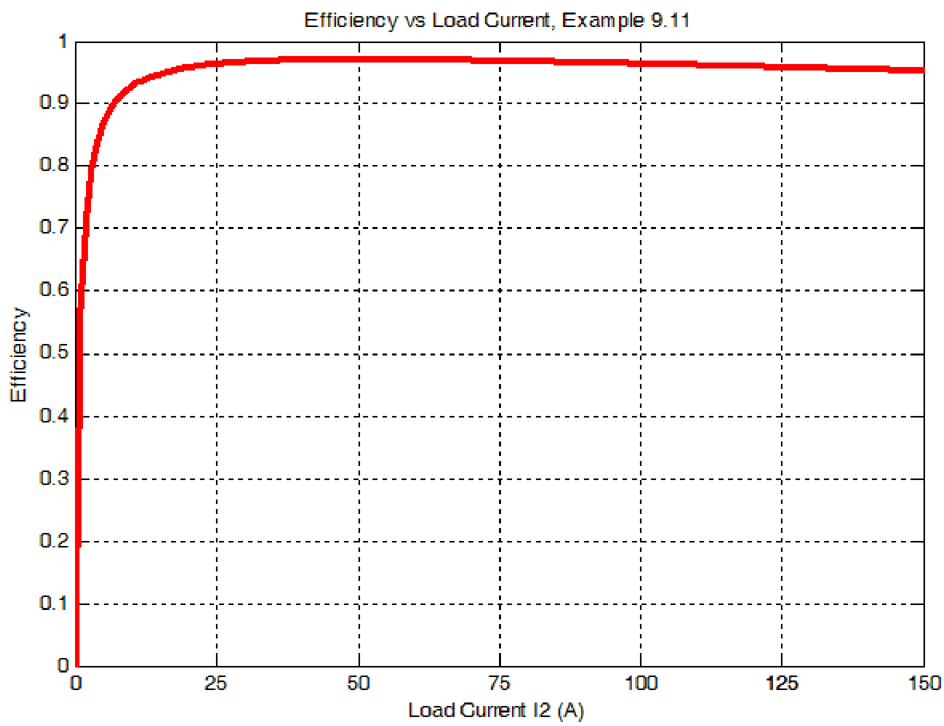


Figure 9.52. Efficiency vs. load current for the transformer in Example 9.11

b. At maximum efficiency the KVA output is

$$4.16\ KV \times 50\ A = 208\ KVA$$

It is reasonable to assume that whenever a transformer is intended to operate continuously, it should be designed to operate at its maximum efficiency at rated load. However, the loads supplied by the transformer vary from time to time, but in most cases follow the same pattern day after day. Thus, a more meaningful measure is a *energy efficiency*, denoted as η_W, for the entire day, and it is defined as

$$\eta_W = \frac{\int_{t_1}^{t_2} P_{OUT}\, dt}{\int_{t_1}^{t_2} P_{OUT}\, dt + \int_{t_1}^{t_2} P_C\, dt + \int_{t_1}^{t_2} P_R\, dt} \qquad (9.138)$$

where P_C = core losses and P_R = copper losses.

All–day efficiency is defined as the ratio of energy output to energy input for a 24–hour period.

Example 9.12

A $10KVA$, $2400/240$, $60\ Hz$ transformer is in operation 24 hours a day. The loads during the day are:

a. $10\ KVA$ at $pf = 1.0$ for 3 hours

b. $6\ KVA$ at $pf = 0.8$ for 5 hours

c. No load for 16 hours

Using the transformer equivalent circuit in Figure 9.53 where

$$Y_{eq1} = G_{C1} + jB_{m1} = 12.5 - j28.6\ \ \mu\Omega^{-1}$$

and

$$Z_{eq1} = R_{eq1} + jX_{eq1} = 8.4 + j13.7\ \Omega$$

compute the all–day efficiency.

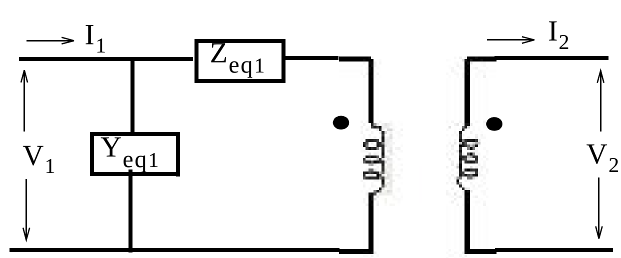


Figure 9.53. Transformer equivalent circuit for Example 9.12

Solution:

The all–day efficiency is readily found by evaluating the integrals in equation (9.138). Thus, denoting the energy as W, we obtain

$$W_{OUT} = (10000\times1.0)\times3+(6000\times0.8)\times5+0\times16 = 54,000 \text{ watt-hours} \tag{9.139}$$

The core losses P_C are the same for the entire 24–hour period and using (9.131) we obtain

$$P_C = G_{C2}V_2^2 = G_{C1}V_1^2 = (12.5\times10^{-6})\cdot(2400)^2 = 72 \text{ w}$$

and the energy W_C dissipated during the 24–hour period is

$$W_C = 72\times24 = 1728 \text{ watt-hours} \tag{9.140}$$

For the 3–hour period the energy dissipated due to copper losses is

$$W_{R\ 3-hr} = \left(\frac{10\ KVA}{2.4\ KV}\right)^2\cdot R_{eq1}\cdot3 = \left(\frac{10}{2.4}\right)^2\times8.4\times3 = 437.5 \text{ w}-\text{h} \tag{9.141}$$

For the 5–hour period the energy dissipated due to copper losses is

$$W_{R\ 5-hr} = \left(\frac{6\ KVA}{2.4\ KV}\right)^2\cdot R_{eq1}\cdot5 = \left(\frac{6}{2.4}\right)^2\times8.4\times5 = 262.5 \text{ w}-\text{h} \tag{9.142}$$

For the 16–hour period the energy dissipated due to copper losses is zero, that is,

$$W_{R\ 16-hr} = 0 \text{ w}-\text{h} \tag{9.143}$$

and from (9.141) through (9.143),

$$W_{R\ 24-hr} = 437.5+262.5+0 \text{ w} = 700 \text{ w-h} \tag{9.144}$$

Therefore, from (9.138) we find that all–day efficiency is

$$\eta_W = \frac{54000}{54000+1728+700} = 0.957$$

9.17 Voltage Regulation

The voltage regulation in a transformer is based on rated voltage and rated current at the secondary terminal. Accordingly, a transformer operates at rated conditions when the following conditions are satisfied.

$$V_2 = V_2(\text{rated}) \tag{9.145}$$

$$I_2 = I_2(\text{rated}) = \frac{KVA(\text{rated})}{V_2(\text{rated})} \tag{9.146}$$

$$\text{Turns ratio} = a = \frac{V_1(\text{rated})}{V_2(\text{rated})} \tag{9.147}$$

While the relation in (9.147) defines the turns ratio, the primary terminal voltage under rated conditions is not exactly $V_1(\text{rated})$ under normal operating conditions and thus it cannot be computed as $V_1 = aV_2$. Its actual value can be computed from a transformer equivalent circuit such as the one shown in Figure 9.52, Page 9–45, from which

$$\mathbf{V_1} = a\mathbf{V_2} + Z_{eq1} \cdot \frac{\mathbf{I_2}}{a} \tag{9.148}$$

and we must remember that V_2 and I_2 are the transformer rated values. Relation (9.148) can also be expressed as

$$\mathbf{V_1} = a(\mathbf{V_2} + Z_{eq2} \cdot \mathbf{I_2}) \tag{9.149}$$

if we use the equivalent circuit in Figure 9.50, Page 9–42.

The relations in (9.148) and (9.149) are phasor quantities. However, the transformer regulation, denoted as ε, is defined in terms of the magnitudes of V_1 as computed from relation (9.148) or (9.149), and the magnitude of rated secondary voltage V_2 as

$$\varepsilon = \frac{V_1 - aV_2}{aV_2} = \frac{V_1/a - V_2}{V_2} \tag{9.150}$$

The transformer voltage regulation can also be expressed in terms of the no–load and full–load voltages as

$$\varepsilon = \frac{V_2(\text{No Load}) - V_2(\text{Full Load})}{V_2(\text{Full Load})} = \frac{V_{2\,NL} - V_{2\,FL}}{V_{2\,FL}} \tag{9.151}$$

where $V_{2\,FL}$ represents the condition where the transformer operates under rated conditions, that is, V_2 and I_2 are the rated values defined in (9.145) and (9.146), and $V_{2\,NL}$ represents the condition where the load is disconnected in which case $I_2 = 0$, and the output voltage V_2 attains the value V_1/a.

Obviously, the transformer regulation depends on the power factor of the load. In Figure 9.53, a resistive load is represented by the phasor diagram (a), an inductive load is represented by the phasor diagram (b), and a capacitive load is represented by the phasor diagram (c).

When pu values are used the transformer ratio is unity. that is, $a = 1$. This is because the pu values are the same regardless of which side there are referred to, e.g., $Z_{eq1} = Z_{eq2}$. Accordingly, whenever pu values are use, the voltage regulation expression in (9.150) above, reduces to (9.152) below.

$$\varepsilon_{pu} = \frac{V_1 - V_2}{V_2} = \frac{V_1}{V_2} - 1 \tag{9.152}$$

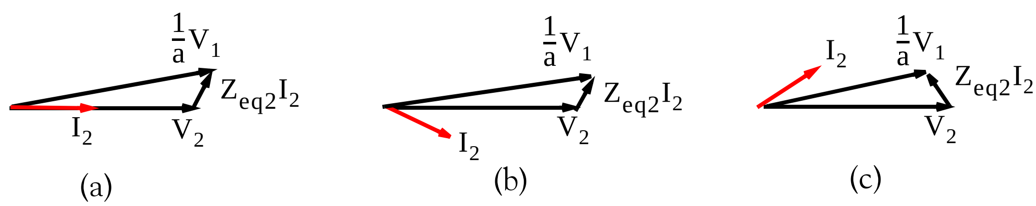


Figure 9.54. Transformer voltage regulation dependence on load power factor

Example 9.13

An equivalent circuit of a $10KVA$, $2400/240$, $60\ Hz$ transformer is shown in Figure 9.55 where

$$Y_{eq1} = G_{C1} + jB_{m1} = 12.5 - j28.6\ \ \mu\Omega^{-1}$$

and

$$Z_{eq1} = R_{eq1} + jX_{eq1} = 8.4 + j13.7\ \Omega$$

Compute the voltage regulation if the transformer operates at rated load and $pf = 08$ lagging.

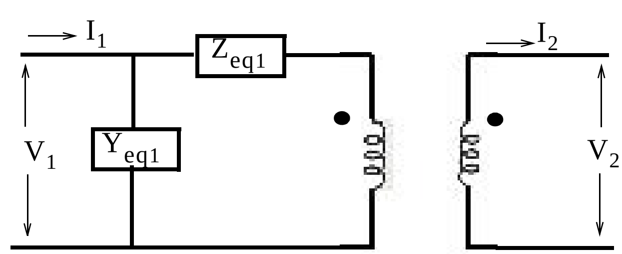


Figure 9.55. Transformer equivalent circuit for Example 9.13

Solution:

The voltage regulation is defined as in relation (9.150). Therefore we need to find the value of V_1 using relation (9.148). We choose the secondary rated voltage $V_2 = 240\angle 0°\ V$ as our reference. The magnitude of the rated current I_2 is found from (9.146), that is,

$$|I_2| = \frac{KVA(rated)}{V_2(rated)} = \frac{10\ KVA}{0.24\ KV} = 41.7\ A$$

and since $pf = \cos\theta = 0.8$, the power factor angle is $\theta = \cos^{-1}(0.8) = 36.9°$, and thus

$$\mathbf{I_2} = 41.7\angle{-36.9°} = 33.4 - j25.0$$

and since $a = 10/1 = 10$,

$$\frac{\mathbf{I_2}}{a} = \frac{33.4 - j25.0}{10} = 3.34 - j2.50$$

Also,

$$aV_2 = 2400\angle 0°\ V$$

and it is given that

$$Z_{eq1} = 8.4 + j13.7\ \Omega$$

Then, from (9.148)

$$\mathbf{V_1} = a\mathbf{V_2} + Z_{eq1} \cdot \frac{\mathbf{I_2}}{a} = 2400 + (8.4 + j13.7) \cdot (3.34 - j2.50) = 2462 + j25 = 2462\angle 0.58°$$

The voltage regulation is computed using only the magnitudes of the voltages $\mathbf{V_1}$ and $\mathbf{V_2}$. Thus from (9.150)

$$\varepsilon = \frac{V_1 - aV_2}{aV_2} = \frac{2462 - 2400}{2400} = 0.0258 \text{ or } 2.58\%$$

9.18 Transformer Modeling with Simulink® / SimPowerSystems®

The MathWorks™ Simulink / SimPowerSystems libraries include single–phase and three–phase transformer blocks. In this section we will model a single–phase transformer circuit, and in Chapter 11 we will model a three–phase transformer circuit. Introductions to Simulink and SimPowerSystems are presented in Appendices B and C respectively.

Example 9.14

We begin the creation of our model by performing the following steps:

1. At the MATLAB command prompt we enter powerlib and the SimPowerSystems library blocks window appears as shown in Figure 9.56.
2. At the upper left corner we click File>New>Model and the window shown in Figure 9.57 appears.

3. From the powerlib library in Figure 9.56, we drag the following blocks into the blank window in Figure 9.57

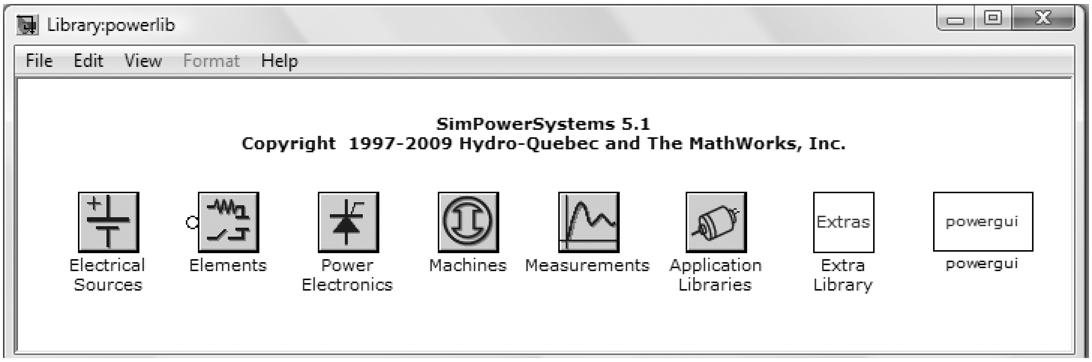


Figure 9.56. The powerlib library

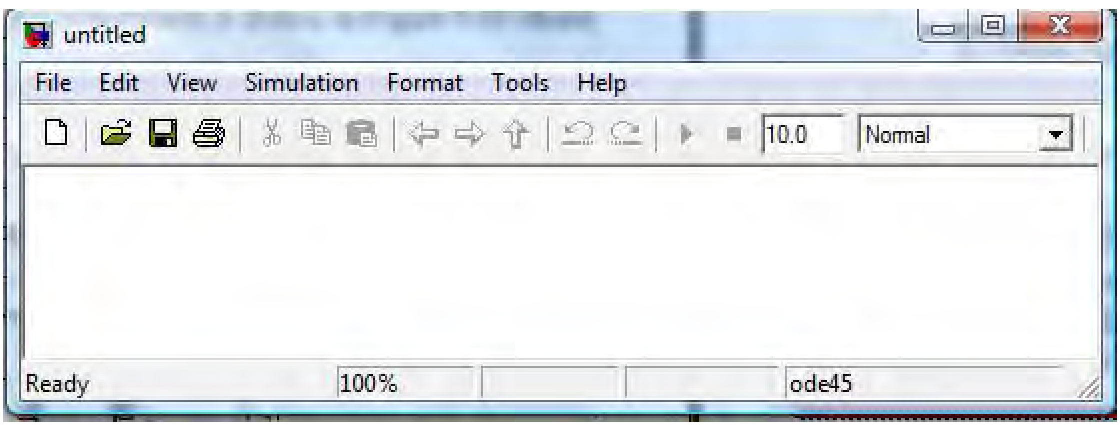


Figure 9.57. Window for new model

a. powergui

b. **Electrical Sources: Choose AC Voltage Source**

c. Elements: Choose Parallel RLC Load, Ground (copy 4 times), Linear Transformer

d. Measurements: Current Measurement, Voltage Measurement

e. From the Simulink Commonly Used Blocks: Scope (copy once)

When all the blocks are dragged, the new model window will appears as shown in Figure 9.58.

Next, we perform the following steps:

a. We double–click the Linear Transformer block and on the Block Parameters window we uncheck the Three windings transformer option. The transformer now appears as a two winding transformer.

b. We double–click the Parallel RLC Load and on the Block Parameters window we set the **Capacitive reactive power Qc** to zero. The block now is reduced to a parallel RL block. We rotate this block with Format>Rotate Block>Counterclockwise.

c. We interconnect the blocks and we rename them as shown in the model in Figure 9.59.

d. The parallel $40\ \mathrm{KW}/30\ \mathrm{KVAR}$ load is assumed to be a $pf = 0.8$ lagging load.

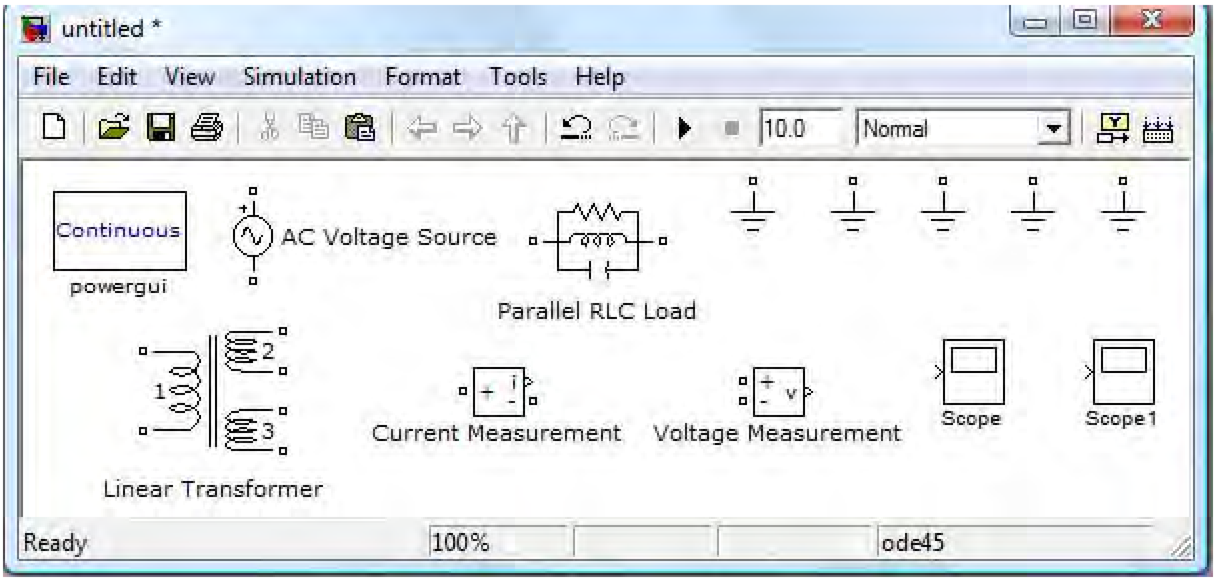


Figure 9.58. The blocks for the model for Example 9.14

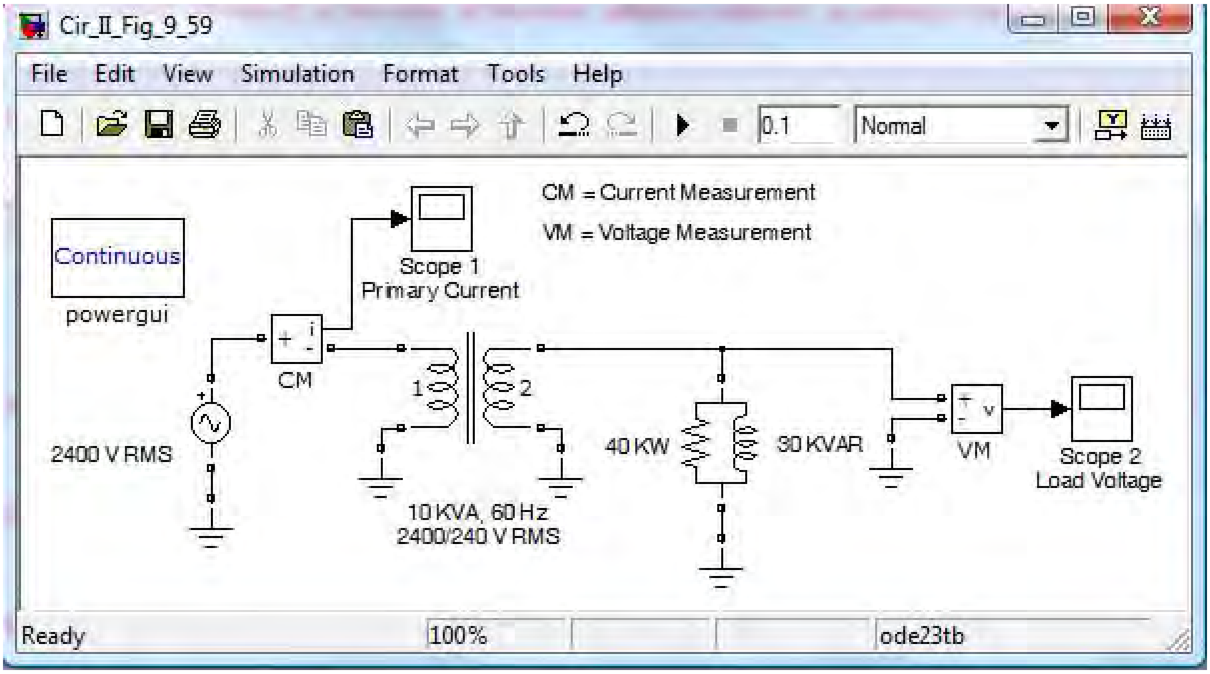


Figure 9.59. The model for Example 9.14

By default, the calculations are performed using the pu method but the parameters will automatically be converted if we change from pu to SI or vice versa. The Block Parameters for the transformer block are in pu values are shown in Figure 9.60. These values were obtained in the solution of Exercise 9.8 at the end of this chapter.

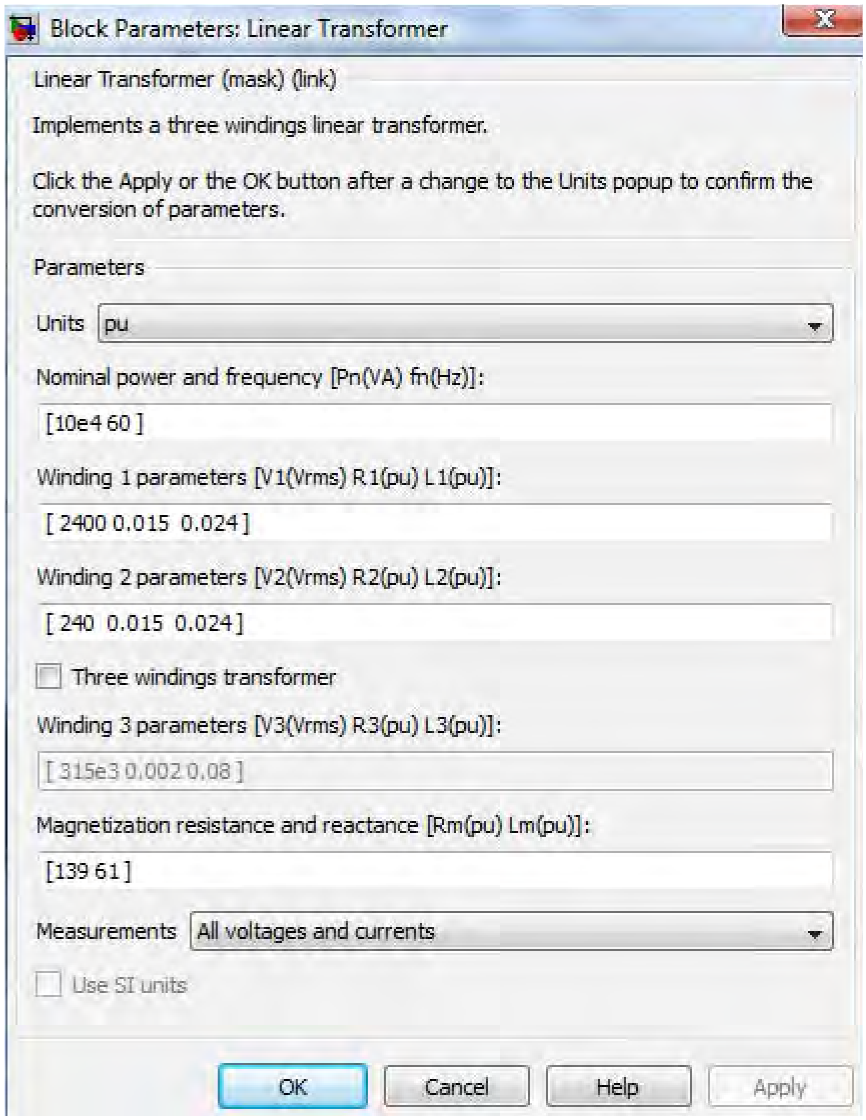


Figure 9.60. The Block Parameters dialog box for the transformer of the model in Figure 9.59

Before we issue the **Simulation Start** command for the model in Figure 9.59, we click **Simulation>Configuration Parameters>Solver**, and we select the **ode23b**(stiff/TR–BDF2) parameter. After the simulation command is executed the Scope 1 and Scope 2 blocks display the waveforms in Figures 9.61 and 9.62 respectively, noting that amplitudes are in peak values, i.e., $\text{Peak} = \text{RMS} \times \sqrt{2}$.

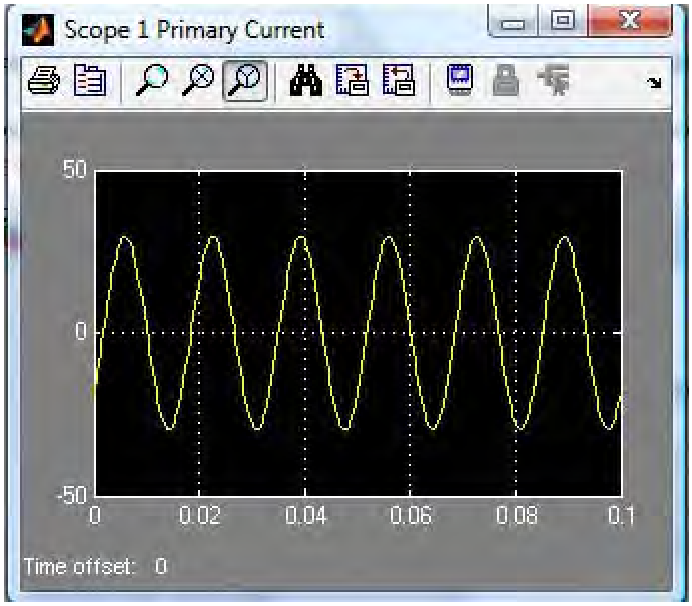


Figure 9.61. Waveform for the primary winding current

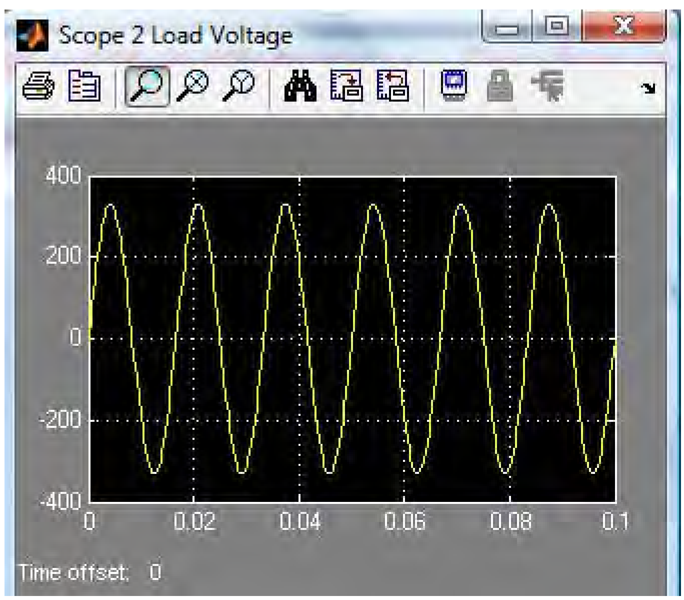


Figure 9.62. Waveform for the voltage across the load

The SimPowerSystems/Measurements library includes the **Multimeter** block which is now added to the model and the new model is shown in Figure 9.63. We double−click the Multimeter block and we observe that the left pane in the dialog box in Figure 9.64 displays 6 **Available Measurements** and as **Ub** (Parallel RLC Load), **Uw1** and **Uw2** (Primary and Secondary Winding Voltages), **Iw1** and **Iw2** (Primary and Secondary Winding Currents), and **Imag** (Magnetization Current). The last 5 measurement are displayed because in the Block Parameters dialog box for the

Linear Transformer block in Figure 9.60, in the **Measurements** parameter we selected the **All voltages and currents** option.

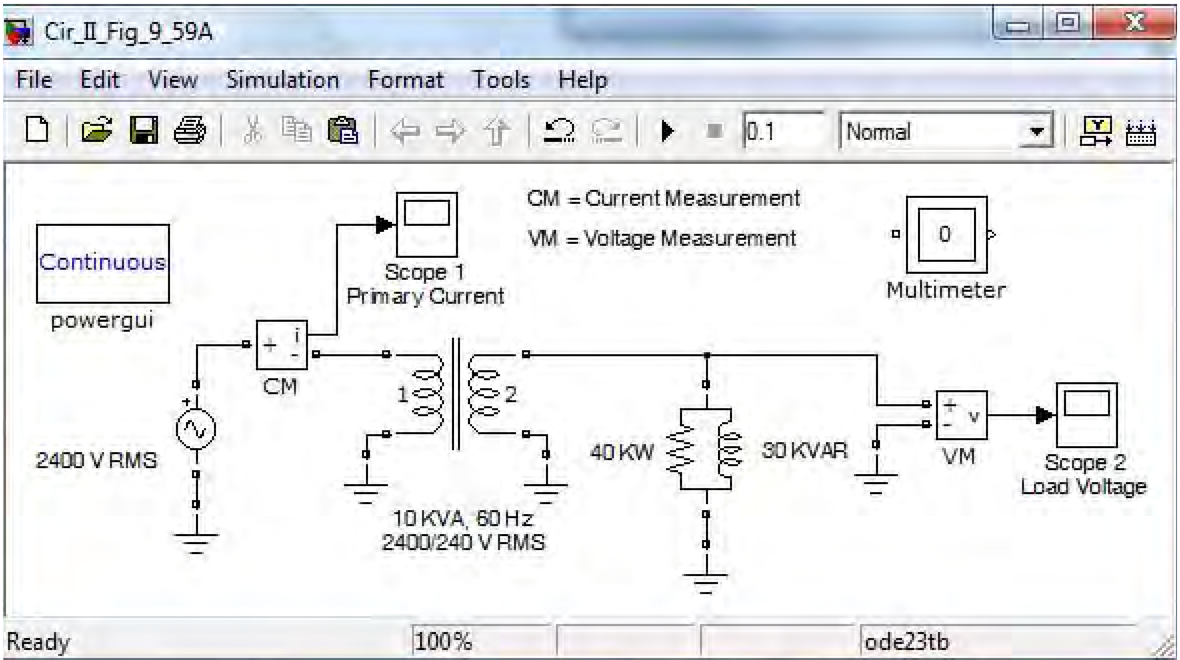


Figure 9.63. The model for Example 9.13 with the added Multimeter block

In the **Multimeter** dialog box in Figure 9.64, the **Available Measurements** in the left pane were highlighted to be selected, and were copied to the **Selected Measurements** pane on the right side by clicking the >> icon. The dialog box was then updated by clicking the **Update** button, and with the **Plot selected measurements** parameter selected, the Simulation Start command was issued producing the plots of the selected measurements shown in Figure 9.65, and we observed that the number 0 inside the **Multimeter** block was changed to 6.

As we have seen, with the use of the **Multimeter** block it was not necessary to use the Scope 1 and Scope 2 blocks since the primary current and the load voltage waveforms are also shown in Figure 9.65.

The output port of a Multimeter block can also be connected to a Scope block with multiple axes through a **Demux** block as shown in the SimPowerSystems documentation demo. It can be accessed by typing power_compensated at the MATLAB command prompt.

An example with a centered tapped transformer (3–winding) demo is also provided in the SimPowerSystems documentation. It can be accessed by typing power_transformer at the MATLAB command prompt

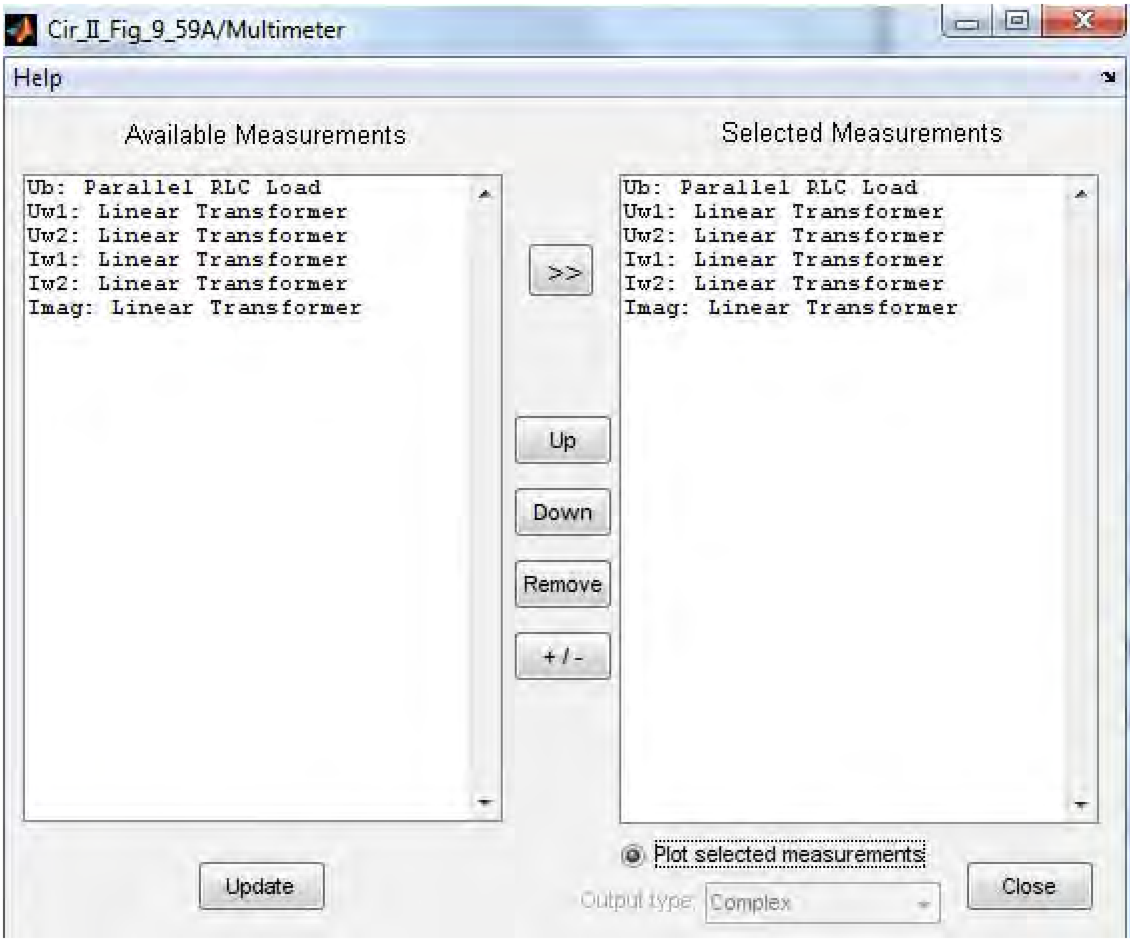


Figure 9.64. The Multimeter block dialog box

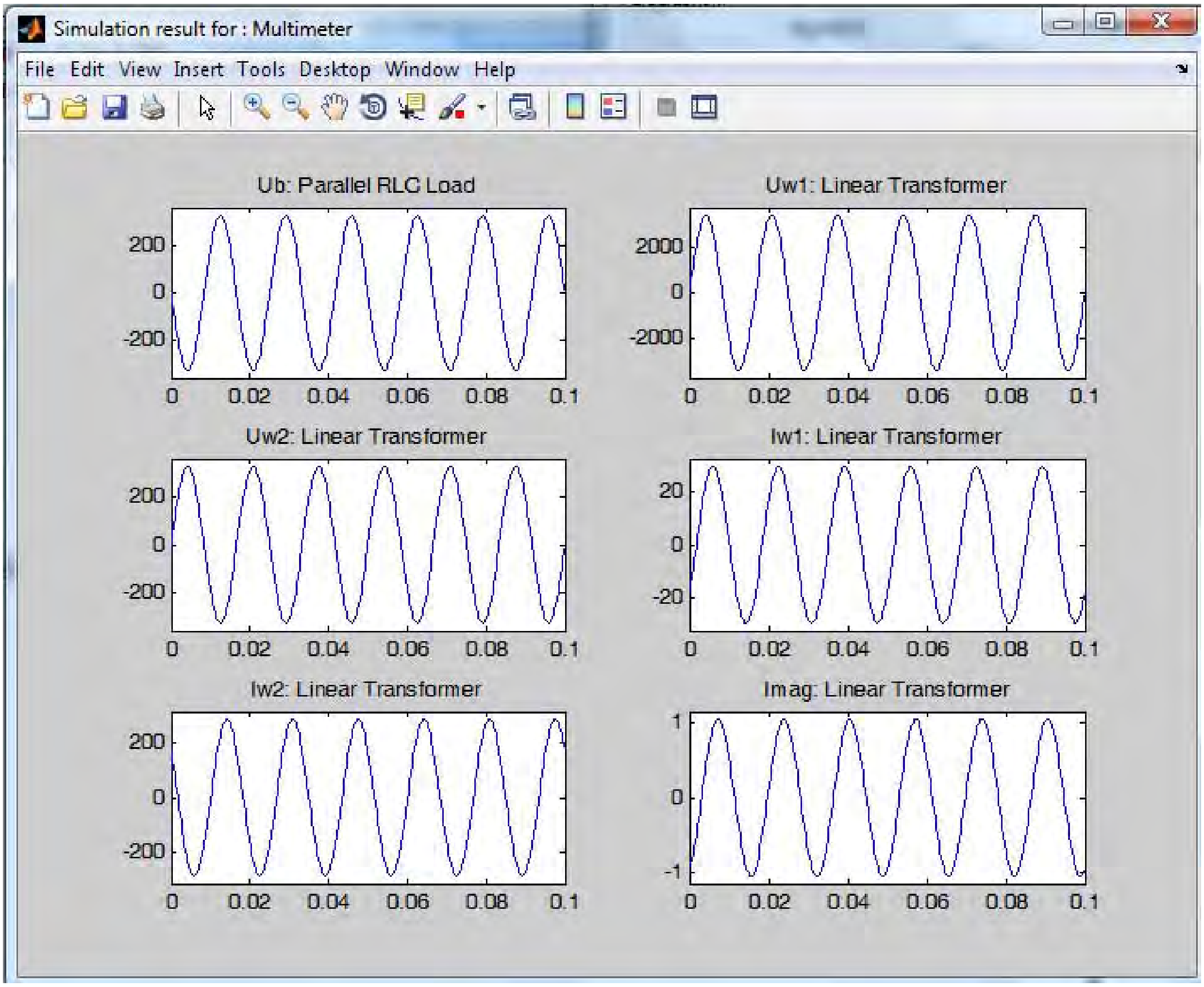


Figure 9.65. Waveforms for the six measurements provided by the Measurements block in Figure 9.63

9.19 Summary

- Inductance is associated with the magnetic field which is always present when there is an electric current.
- The magnetic field loops are circular in form and are called lines of magnetic flux.
- The magnetic flux is denoted as φ and the unit of magnetic flux is the weber (Wb).
- If there are N turns and we assume that the flux φ passes through each turn, the total flux denoted as λ is called flux linkage. Then,

$$\lambda = N\varphi$$

- A linear inductor one in which the flux linkage is proportional to the current through it, that is,

$$\lambda = Li$$

where the constant of proportionality L is called inductance in webers per ampere.

- Faraday's law of electromagnetic induction states that

$$v = \frac{d\lambda}{dt}$$

- Lenz's law states that whenever there is a change in the amount of magnetic flux linking an electric circuit, an induced voltage of value directly proportional to the time rate of change of flux linkages is set up tending to produce a current in such a direction as to oppose the change in flux.
- A linear transformer is a four−terminal device in which the voltages and currents in the primary coils are linearly related.
- In a linear transformer, when there is no current in the secondary winding the voltages are

$$v_1 = L_1\frac{di_1}{dt} \quad \text{and} \quad v_2 = M_{21}\frac{di_1}{dt}$$

$$\text{if } i_1 \neq 0 \text{ and } i_2 = 0$$

- In a linear transformer, when there is no current in the primary winding, the voltages are

$$v_2 = L_2\frac{di_2}{dt} \quad \text{and} \quad v_1 = M_{12}\frac{di_2}{dt}$$

$$\text{if } i_1 = 0 \text{ and } i_2 \neq 0$$

- In a linear transformer, when there is a current in both the primary and secondary windings, the voltages are

$$v_1 = L_1\frac{di_1}{dt} + M\frac{di_2}{dt}$$

$$v_2 = M\frac{di_1}{dt} + L_2\frac{di_2}{dt}$$

- The voltage terms

$$L_1\frac{di_1}{dt} \text{ and } L_2\frac{di_2}{dt}$$

are referred to as self–induced voltages.

- The voltage terms

$$M\frac{di_1}{dt} \text{ and } M\frac{di_2}{dt}$$

are referred to as mutual voltages.

- The polarity of the mutual voltages is denoted by the dot convention. If a current i entering the dotted (undotted) terminal of one coil induces a voltage across the other coil with positive polarity at the dotted (undotted) terminal of the other coil, the mutual voltage term has a positive sign. If a current i entering the undotted (dotted) terminal of one coil induces a voltage across the other coil with positive polarity at the dotted (undotted) terminal of the other coil, the mutual voltage term has a negative sign.

- If the polarity (dot) markings are not given, they can be established by using the right–hand rule which states that if the fingers of the right hand encircle a winding in the direction of the current, the thumb indicates the direction of the flux. Thus, in an ideal transformer with primary and secondary windings L_1 and L_2 and currents i_1 and i_2 respectively, we place a dot at the upper end of L_1 and assume that the current i_1 enters the top end thereby producing a flux in the clockwise direction. Next, we want the current in L_2 to enter the end which will produce a flux in the same direction, in this case, clockwise.

- The energy stored in a pair of mutually coupled inductors is given by

$$W\Big|_{t_0}^{t_2} = \frac{1}{2}L_1 i_1^2 \pm M i_1 i_2 + \frac{1}{2}L_2 i_2^2$$

where the sign of M is positive if both currents enter the dotted (or undotted) terminals, and it is negative if one current enters the dotted (or undotted) terminal while the other enters the undotted (or dotted) terminal.

- The ratio

$$k = \frac{M}{\sqrt{L_1 L_2}}$$

is known as the coefficient of coupling and k provides a measure of the proximity of the primary and secondary coils. If the coils are far apart, we say that they are *loose–coupled*, and k has a small value, typically between 0.01 and 0.1. For *close–coupled* circuits, k has a value of about 0.5. Power transformers have a k between 0.90 and 0.95. The value of k is exactly unity only when the two coils are coalesced into a single coil.

- If the secondary of a linear transformer is referenced to a DC voltage source V_0, it is said that the secondary has DC isolation.

- In a linear transformer, the load impedance of the secondary can be reflected into the primary can be reflected into the primary using the relation

$$Z_R = \frac{\omega^2 M^2}{j\omega L_2 + Z_{LD}}$$

where Z_R is referred to as the reflected impedance.

- An ideal transformer is one in which the coefficient of coupling is almost unity, and both the primary and secondary inductive reactances are very large in comparison with the load impedances. The primary and secondary coils have many turns wound around a laminated iron–core and are arranged so that the entire flux links all the turns of both coils.

- In an ideal transformer number of turns on the primary N_1 and the number of turns on the secondary N_2 are related to the primary and secondary currents I_1 and I_2 respectively as

$$N_1 I_1 = N_2 I_2$$

- An important parameter of an ideal transformer is the turns ratio a which is defined as the ratio of the number of turns on the secondary, N_2, to the number of turns of the primary N_1, that is,

$$a = \frac{N_2}{N_1}$$

- In an ideal transformer the turns ratio a relates the primary and secondary currents as

$$\frac{I_2}{I_1} = \frac{1}{a}$$

- In an ideal transformer the turns ratio a relates the primary and secondary voltages as

$$\frac{V_2}{V_1} = a$$

- In an ideal transformer the volt–amperes of the primary and the secondary are equal, that is,

$$V_2 I_2 = V_1 I_1$$

- An ideal transformer can be used as an impedance matching device by specifying the appropriate turns ratio $N_2/N_1 = a$. Then,

$$Z_{in} = \frac{Z_{LD}}{a^2}$$

- In analyzing networks containing ideal transformers, it is very convenient to replace the transformer by an equivalent circuit before the analysis. One method is presented in Section 9.11.
- An ideal transformer can be replaced by a Thevenin equivalent as discussed in Section 9.12.
- Four transformer equivalent circuits are shown in Figure 9.41 and they are useful in the computations of transformer parameters computations from the open– and short–circuit tests, efficiency, and voltage regulation.
- An autotransformer is a special transformer that shares a common winding, and can be configured either as a step–down or step–up transformer as shown in Figure 9.42.
- Autotransformers are not used in residential, commercial, or industrial applications because a break in the common winding may result in equipment damage and / or personnel injury.
- A *variac* is an adjustable autotransformer, that is, its secondary voltage can be adjusted from zero to a maximum value y a wiper arm that slides over the common winding as shown in Figure 9.43.
- Some transformers are constructed with a common primary winding and two or more secondary windings. These transformers are used in applications hen there is a need for two or more different secondary voltages with a common primary voltage.
- The transformer open–circuit test, also referred to as the no–load test, is used to determine the reactance $\mathbf{X_P}$ n the primary winding, the core resistance $\mathbf{R_C}$, and the magnetizing reactance $\mathbf{X_M}$. For this test, the secondary is left open, and an ammeter, a voltmeter, and a wattmeter are connected as shown in Figure 9.47.
- The transformer short–circuit test is used to determine the magnitude of the series impedances referred to the primary side of the transformer denoted as Z_{SC} For this test, the secondary is shorted, and an ammeter, a voltmeter, and a wattmeter are connected as shown in Figure 9.48.

- Efficiency, denoted as η, is a dimensionless quantity defined as

$$\eta = \frac{P_{OUT}}{P_{IN}} = \frac{P_{IN} - P_{LOSS}}{P_{IN}} = 1 - \frac{P_{LOSS}}{P_{IN}}$$

or in terms of the output and losses

$$\eta = \frac{P_{OUT}}{P_{OUT} + P_{LOSS}} = 1 - \frac{P_{LOSS}}{P_{OUT} + P_{LOSS}}$$

- The losses in a transformer are the summation of the core losses (hysteresis and eddy currents), and copper losses caused by the resistance of the conducting material of the coils, generally made of copper.

- Energy efficiency, denoted as η_W, for the entire day, and it is defined as

$$\eta_W = \frac{\int_{t_1}^{t_2} P_{OUT}\,dt}{\int_{t_1}^{t_2} P_{OUT}\,dt + \int_{t_1}^{t_2} P_C\,dt + \int_{t_1}^{t_2} P_R\,dt}$$

where $P_C = \text{core losses}$ and $P_R = \text{copper losses}$.

- All–day efficiency is defined as the ratio of energy output to energy input for a 24–hour period.

- The transformer voltage regulation, denoted as ε, is defined in terms of the magnitudes of V_1 as computed from relation (9.148) or (9.149), and the magnitude of rated secondary voltage V_2 as

$$\varepsilon = \frac{V_1 - aV_2}{aV_2} = \frac{V_1/a - V_2}{V_2}$$

The transformer voltage regulation can also be expressed in terms of the no–load and full–load voltages as

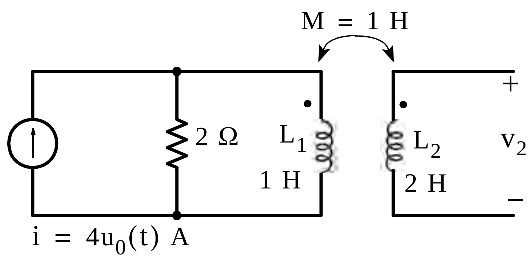
$$\varepsilon = \frac{V_2(\text{No Load}) - V_2(\text{Full Load})}{V_2(\text{Full Load})} = \frac{V_{2\,NL} - V_{2\,FL}}{V_{2\,FL}}$$

where $V_{2\,FL}$ represents the condition where the transformer operates under rated conditions, that is, V_2 and I_2 are the rated values defined in (9.145) and (9.146), and $V_{2\,NL}$ represents the condition where the load is disconnected in which case $I_2 = 0$, and the output voltage V_2 attains the value V_1/a.

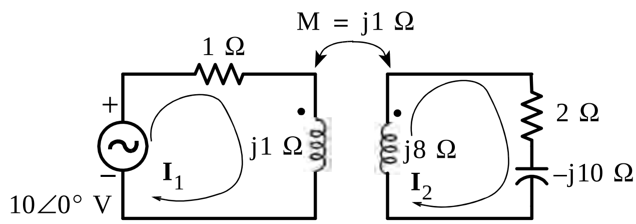
- The MathWorks Simulink / SimPowerSystems libraries include single–phase and three–phase transformer blocks. A model with a single–phase transformer is presented in this chapter.

9.20 Exercises

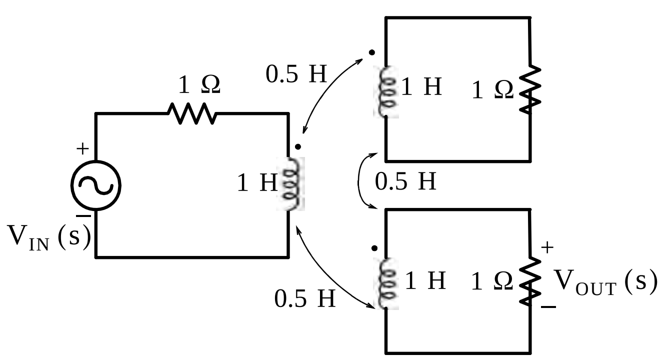
1. For the transformer below find v_2 for $t > 0$.



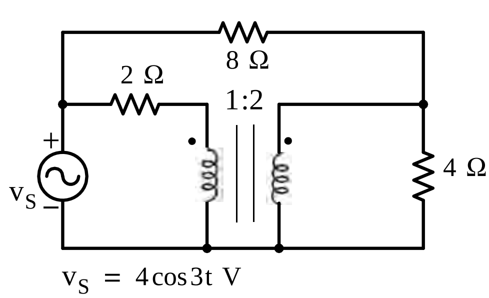
2. For the transformer below find the phasor currents $\mathbf{I}_1$ and $\mathbf{I}_2$.



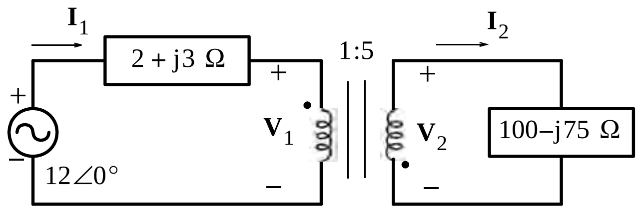
3. For the network below find the transfer function $G(s) = V_{OUT}(s)/V_{IN}(s)$.



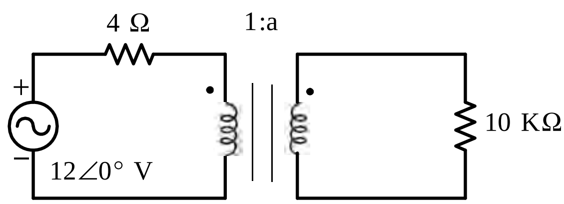
4. For the transformer below find the average power delivered to the $4\ \Omega$ resistor.



5. Replace the transformer below by a Thevenin equivalent and then compute $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{I}_1$ and $\mathbf{I}_2$



6. For the circuit below compute the turns ratio a so that maximum power will be delivered to the 10 KΩ resistor.

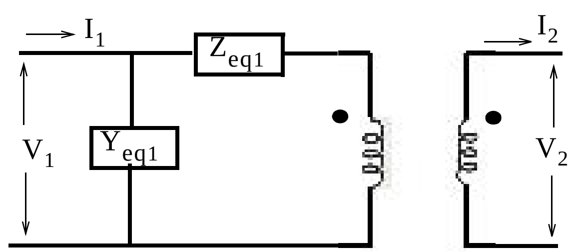


7. The recorded open– and short–circuit test data for a 10KVA, 2400 / 240, 60 Hz transformer are as follows:

Open–circuit test with input to the low side: 240 V, 0.75 A, 72 W

Short–circuit test with input to the high side: 80.5 V, 5 A, 210 W

Compute the parameters for the approximate equivalent circuit shown below.



8. Repeat Exercise 7 above using per–unit values.

9. Using the data in Exercise 7 above, compute the voltage regulation for power factor 0.8 leading using per–unit values.

10. Using the data in Exercise 7 above, compute the efficiency for power factor 0.8 lagging at half load using per–unit values.

11. As mentioned earlier, the core losses in a transformer consist of hysteresis losses and eddy current losses. The hysteresis loss is computed a

$$P_h = k_h \upsilon f B_{max}^n$$

where the factor k_h and the exponent n vary with the core material used, υ is the volume of the core, f is the frequency in Hz, and B is the magnetic flux density.

The eddy current loss is approximated by the relation

$$P_e = k_e \upsilon \tau^2 f^2 B_{max}^2$$

where τ is the thickness of the laminated cores, and the other variables are as in the hysteresis loss expression above.

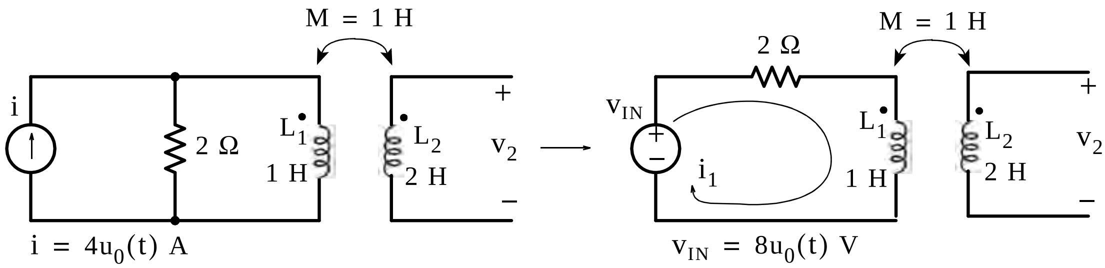
Since for a given core the volume υ and the thickness τ of the laminated cores are constant, it is convenient to lump together the hysteresis losses and eddy current losses as core losses P_C, that is,

$$P_C = P_h + P_e = k'_h f B_{max}^n + k'_e f^2 B_{max}^2$$

Now, suppose that the total core losses (hysteresis and eddy current) for a transformer core are $500\ W$ at $f_1 = 25\ Hz$. If the maximum flux density B_{max} remains unchanged while the frequency increases to $f_2 = 50\ Hz$, the total core losses increase to $1400\ W$. Compute the hysteresis and eddy current losses for both frequencies.

9.21 Solutions to End–of–Chapter Exercises

1.



Application of KVL in the primary yields

$$2i_1 + L_1\frac{di_1}{dt} = 8u_0(t)$$

$$1 \cdot \frac{di_1}{dt} + 2i_1 = 8 \qquad t > 0 \quad (1)$$

The total solution of i_1 is the sum of the forced component i_{1f} and the natural response i_{1n}, i.e.,

$$i_1 = i_{1f} + i_{1n}$$

From (1) we find that $i_{1f} = 8/2 = 4$, and i_{1n} is found from the characteristic equation $s + 2 = 0$ from which $s = -2$ and thus $i_{1n} = Ae^{-2t}$. Then,

$$i_1 = 4 + Ae^{-2t} \quad (2)$$

Since we are not told otherwise, we will assume that $i_1(0^-) = 0$ and from (2) $0 = 4 + Ae^0$ or $A = -4$ and by substitution into (2)

$$i_1 = 4(1 - 4e^{-2t})$$

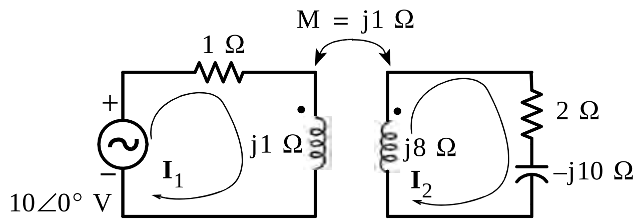
The voltage v_2 is found from

$$v_2 = M\frac{di_1}{dt} + L_2\frac{di_2}{dt}$$

and since $i_2 = 0$,

$$v_2 = 1 \cdot \frac{di_1}{dt} = \frac{d}{dt}[4(1 - 4e^{-2t})] = 8e^{-2t}\ V$$

2.



The mesh equations for primary and secondary are:

$$(1+j1)\mathbf{I}_1 - j1\mathbf{I}_2 = 10\angle 0°$$
$$-j1\mathbf{I}_1 + (2-j2)\mathbf{I}_2 = 0$$

By Cramer's rule,

$$\mathbf{I}_1 = D_1/\Delta \qquad \mathbf{I}_2 = D_2/\Delta$$

where

$$\Delta = \begin{bmatrix} (1+j1) & -j1 \\ -j1 & (2-j2) \end{bmatrix} = 5$$

$$D_1 = \begin{bmatrix} 10\angle 0° & -j1 \\ 0 & (2-j2) \end{bmatrix} = 20(1-j)$$

$$D_2 = \begin{bmatrix} (1+j1) & 10\angle 0° \\ -j1 & 0 \end{bmatrix} = j10$$

Thus,

$$\mathbf{I}_1 = \frac{20(1-j)}{5} = 4(1-j) = 4\sqrt{2}\angle -45°\ \mathrm{A}$$

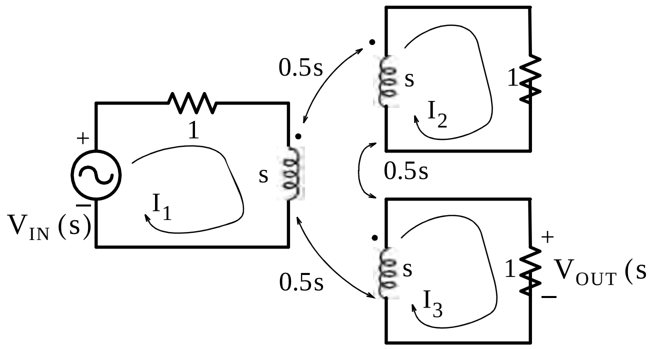
$$\mathbf{I}_2 = \frac{j10}{5} = j2 = 2\angle 90°\ \mathrm{A}$$

Check with MATLAB:

```
Z=[1+j –j; –j 2–2j]; V=[10 0]'; I=Z\V;
fprintf('magI1 = %5.2f A \t', abs(I(1))); fprintf('phaseI1 = %5.2f deg ',angle(I(1))*180/pi);...
fprintf(' \n');...
fprintf('magI2 = %5.2f A \t', abs(I(2))); fprintf('phaseI2 = %5.2f deg ',angle(I(2))*180/pi);...
fprintf(' \n')
```

```
magI1 =  5.66 A   phaseI1 = -45.00 deg
magI2 =  2.00 A   phaseI2 = 90.00 deg
```

3.



We will find $V_{OUT}(s)$ from $V_{OUT}(s) = (1\ \Omega)I_3$. The three mesh equations in matrix form are:

$$\begin{bmatrix} (s+1) & -0.5s & -0.5s \\ -0.5s & (s+1) & -0.5s \\ -0.5s & -0.5s & (s+1) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot V_{IN}(s)$$

We will use MATLAB to find the determinant Δ of the 3×3 matrix.

```
syms s
delta=[s+1  –0.5*s  –0.5*s; –0.5*s  s+1  –0.5*s;  –0.5*s  –0.5*s  s+1]; det_delta=det(delta)
```

```
det_delta =
9/4*s^2+3*s+1
```

```
d3=[s+1  –0.5*s  –0.5*s;  –0.5*s  s+1  –0.5*s;  1  0  0]; det_d3=det(d3)
```

```
det_d3 =
3/4*s^2+1/2*s
```

```
I3=det_d3/det_delta
```

```
I3 =
(3/4*s^2+1/2*s)/(9/4*s^2+3*s+1)
```

```
simplify(I3)
```

```
ans =
s/(3*s+2)
```

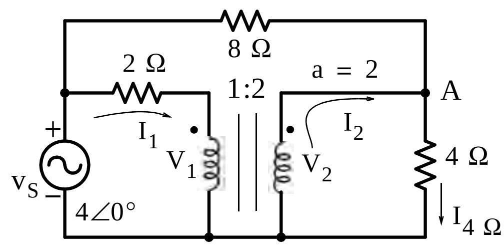
Therefore,

$$V_{OUT}(s) = 1 \cdot I_3 \cdot V_{IN}(s) = s/(3s+2) \cdot V_{IN}(s)$$

and

$$G(s) = V_{OUT}(s)/V_{IN}(s) = s/(3s+2)$$

4.



For this exercise, $P_{ave\ 4\ \Omega} = \frac{1}{2}(I_{4\Omega})^2 4$ and thus we need to find $I_{4\ \Omega}$.

At Node A,

$$\frac{V_2}{4} + \frac{V_2 - 4\angle 0°}{8} - I_2 = 0$$

$$\frac{3V_2}{8} - I_2 = \frac{1}{2} \quad (1)$$

From the primary circuit,

$$2I_1 + V_1 = 4 \quad (2)$$

Since $I_2/I_1 = 1/a$, $V_2/V_1 = a$, and $a = 2$, it follows that $I_1 = 2I_2$ and $V_1 = V_2/2$. By substitution into (2) we obtain

$$4I_2 + \frac{V_2}{2} = 4$$

$$I_2 + \frac{V_2}{8} = 1 \quad (3)$$

Addition of (1) and (3) yields

$$\frac{3V_2}{8} + \frac{V_2}{8} = \frac{1}{2} + 1$$

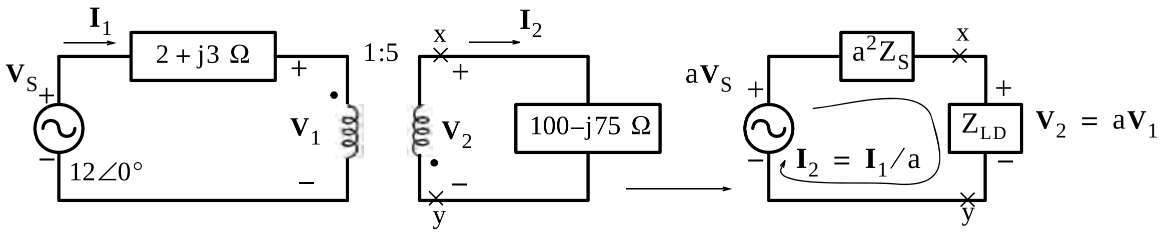
from which $V_2 = 3$. Then,

$$I_{4\ \Omega} = \frac{V_2}{4} = \frac{3}{4}$$

and

$$P_{ave\ 4\ \Omega} = \frac{1}{2}\left(\frac{3}{4}\right)^2 4 = \frac{9}{8}\ w$$

5.



Because the dot on the secondary is at the lower end, $a = -5$. Then,

$$a\mathbf{V}_S = -5 \times 12\angle 0° = -60\angle 0° = 60\angle 180°$$

$$a^2 Z_S = 25(2 + j3) = 50 + j75 = 90.14\angle 56.31°\ \Omega$$

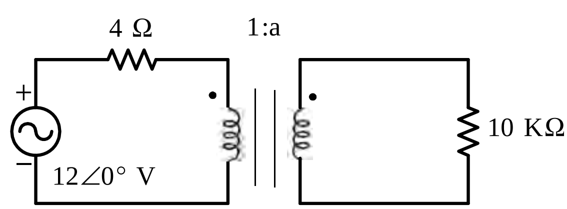
$$Z_{LD} = 100 - j75 = 125\angle -36.87°\ \Omega$$

$$I_2 = \frac{a\mathbf{V}_S}{a^2 Z_S + Z_{LD}} = \frac{60\angle 180°}{50 + j75 + 100 - j75} = \frac{60\angle 180°}{150} = \frac{2}{5}\angle 180°$$

and

$$\mathbf{V}_2 = Z_{LD} \cdot I_2 = 125\angle -36.87° \times \frac{2}{5}\angle 180° = 50\angle 143.13°\ \mathrm{V}$$

6.



From (9.102)

$$Z_{in} = \frac{Z_{LD}}{a^2}$$

Then,

$$a^2 = \frac{Z_{LD}}{Z_{in}} = \frac{10000}{4} = 2500$$

or

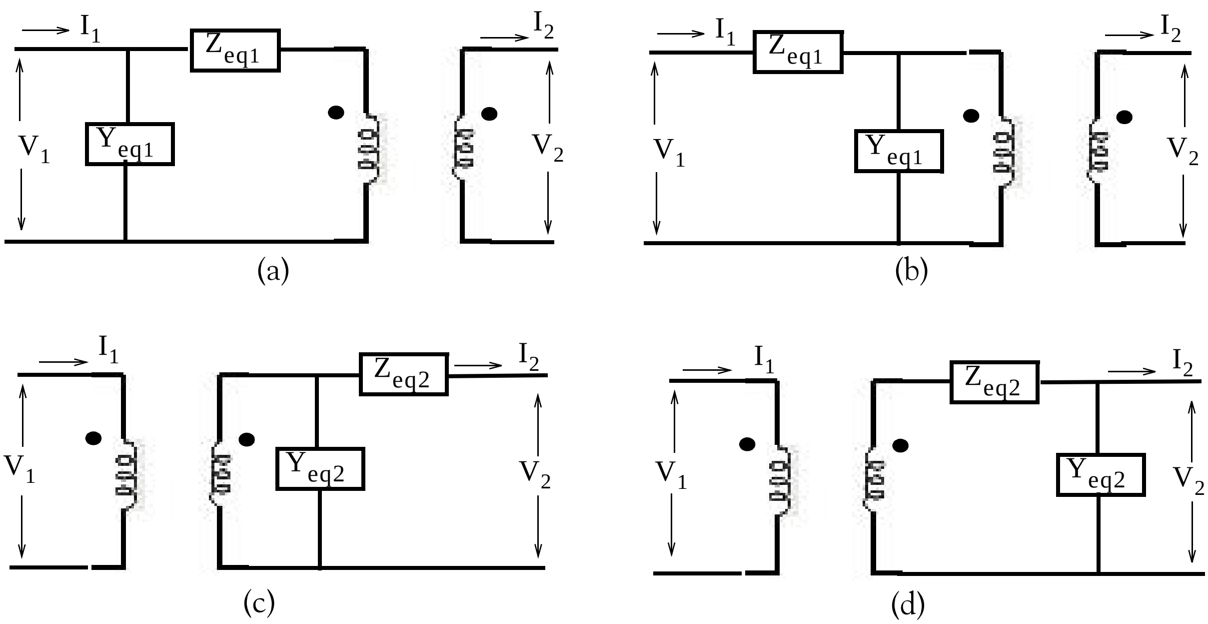
$$a = 50$$

7. We are told that open– and short–circuit test data for a 10KVA, 2400 / 240, 60 Hz transformer are as follows:

Open–circuit test with input to the low side: 240 V, 0.75 A, 72 W

Short–circuit test with input to the high side: 80.5 V, 5 A, 210 W

The given equivalent circuit is the circuit (a) in Figure 9.41 which is repeated below for convenience and we are asked to compute Y_{eq1} and Z_{eq1}. The equivalent circuits (b) and (d) will also be useful for the solution of this exercise.



Since the input for the open–circuit test is measured at the low side, we will compute the admittance Y_{eq2} in circuit (d) above, and we then refer it to the high side in Figure (a) using the relation $Y_{eq1} = Y_{eq2}/a^2$.

From the open–circuit test data, the admittance Y_{eq2} is

$$|Y_{eq2}| = \frac{I_{2\,OC}}{V_{2\,OC}} = \frac{0.75}{240} = 3.1 \times 10^{-3}\ \Omega^{-1}$$

and the phase angle θ_{OC} is found from

$$\cos\theta_{OC} = \frac{P_{OC}}{V_{2\,OC} \cdot I_{2\,OC}} = \frac{72}{240 \times 0.75} = 0.4$$

$$\theta_{OC} = \cos^{-1}(0.4) = -66.4° \ \text{(lagging)}$$

Then, with $a = 2400/240 = 10$

$$\mathbf{Y}_{eq1} = \frac{Y_{eq2}}{a^2} = \frac{3.1\times10^{-3}\ \Omega^{-1}\angle -66.4^\circ}{100} = (12.4 - j28.4)\times10^{-6}\ \Omega^{-1}$$

from which

$$G_{C1} = 12.4\times10^{-6}\ \Omega^{-1}$$

and

$$B_{M1} = -28.4\times10^{-6}\ \Omega^{-1}$$

The measurements for the short–circuit test were made at the high side, and thus we will use the equivalent circuit (b) above. The impedance $|Z_{eq1}|$ is found from

$$|Z_{eq1}| = \frac{V_{1\ SC}}{I_{1\ SC}} = \frac{80.5}{5} = 16.1\ \Omega$$

and the phase angle θ_{SC} is found from

$$\cos\theta_{SC} = \frac{P_{SC}}{V_{1\ SC}\cdot I_{1\ SC}} = \frac{210}{80.5\times5} = 0.52$$

$$\theta_{SC} = \cos^{-1}(0.52) = 58.7^\circ\ \ \text{(lagging)}$$

Then,

$$\mathbf{Z}_{eq1} = 16.1\angle 58.7^\circ = 8.36 + j13.76\ \Omega$$

from which

$$R_{eq1} = 8.36\ \Omega$$

and

$$X_{eq1} = 13.76\ \Omega$$

8. We begin with establishing the bases below.

$$P_{base} = P_{a\ base} = 10000\ VA \qquad V_{1\ base} = 2400\ V \qquad V_{2\ base} = 240\ V$$

$$I_{1\ base} = \frac{10000\ VA}{2400\ V} = 4.17\ A \qquad I_{2\ base} = \frac{10000\ VA}{240\ V} = 41.7\ A$$

Next, we convert all test data into per–unit values.

$$V_{OC\ pu} = \frac{V_{OC}}{V_{2\ base}} = \frac{240\ V}{240\ V} = 1\ pu \qquad I_{OC\ pu} = \frac{I_{OC}}{I_{2\ base}} = \frac{0.75\ A}{41.7\ A} = 0.018\ pu$$

$$P_{OC\ pu} = \frac{P_{OC}}{P_{a\ base}} = \frac{72\ W}{10000\ VA} = 0.0072\ pu \qquad V_{SC\ pu} = \frac{V_{SC}}{V_{1\ base}} = \frac{80.5\ V}{2400\ V} = 0.0335\ pu$$

$$I_{SC\ pu} = \frac{I_{SC}}{I_{1\ base}} = \frac{5\ A}{4.17\ A} = 1.2\ pu \qquad P_{SC\ pu} = \frac{P_{SC}}{P_{a\ base}} = \frac{210\ W}{10000\ VA} = 0.021\ pu$$

Following the same procedure as in Exercise 7, we obtain:

From the open–circuit test data, the magnitude of the admittance $Y_{eq2\ pu}$ is

$$|Y_{eq2\ pu}| = \frac{I_{OC\ pu}}{V_{OC\ pu}} = \frac{0.018}{1} = 0.018\ pu$$

$$\cos\theta_{OC\ pu} = \frac{P_{OC\ pu}}{V_{OC\ pu}\cdot I_{OC\ pu}} = \frac{0.0072}{1\times 0.018} = 0.4^{*}$$

$$\theta_{OC\ pu} = \cos^{-1}(0.4) = -66.4°\ \ (\text{lagging})$$

$$\sin\theta_{OC\ pu} = \sin(-66.4°) = -0.916$$

$$G_{C1\ pu} = |Y_{eq2\ pu}|\cos\theta_{OC\ pu} = 0.018\times 0.4 = 0.0072\ pu$$

$$B_{M1\ pu} = |Y_{eq2\ pu}|\sin\theta_{OC\ pu} = 0.018\times(-0.916) = -0.0165\ pu$$

$$Y_{eq2\ pu} = 0.0072 - j0.0165$$

From he short–circuit test data, the magnitude of the impedance $Z_{eq1\ pu}$ is

$$|Z_{eq1\ pu}| = \frac{V_{SC\ pu}}{I_{SC\ pu}} = \frac{0.0335}{1.2} = 0.028\ pu$$

$$\cos\theta_{SC\ pu} = \frac{P_{SC\ pu}}{V_{SC\ pu}\cdot I_{SC\ pu}} = \frac{0.021}{0.0335\times 1.2} = 0.522$$

$$\theta_{SC\ pu} = \cos^{-1}(0.522) = 58.5°$$

$$\sin\theta_{SC\ pu} = \sin(58.5°) = 0.853$$

$$R_{eq1\ pu} = |Z_{eq1\ pu}|\cos\theta_{SC\ pu} = 0.028\times 0.522 = 0.01456\ pu$$

$$X_{eq1\ pu} = |Z_{eq1\ pu}|\sin\theta_{SC\ pu} = 0.028\times 0.853 = 0.0238\ pu$$

$$Z_{eq1\ pu} = 0.0146 + j0.0238\ pu$$

Check:

$$Z_{eq1\ base} = \frac{V_{1\ base}}{I_{1\ base}} = \frac{2400\ V}{4.17\ A} = 575.54\ \Omega$$

* *Conversion to pu values applies only to magnitudes, angles remain the same as working with actual values.*

$$Z_{eq1\ SC}(\text{actual}) = \frac{V_{SC}}{I_{SC}} = \frac{80.5}{5} = 16.1\ \Omega$$

$$Z_{eq1\ pu} = \frac{Z_{eq1\ SC}(\text{actual})}{Z_{eq1\ base}} = \frac{16.1}{575.54} = 0.028$$

and the other quantities can be verified similarly.

9. The voltage regulation is computed using only the magnitudes of the voltages $\mathbf{V_1}$ and $\mathbf{V_2}$, and since we are using pu values, we will use (9.152), i.e.,

$$\varepsilon_{pu} = \frac{V_1}{V_2} - 1$$

We choose $\mathbf{V_2}$ as the reference phasor, and we let $\mathbf{V}_2 = V_{OC} = 240\ V$, and in pu,

$$\mathbf{V}_{2\ pu} = \frac{V_{OC}}{\mathbf{V}_2} = 1\angle 0°\ \text{pu}$$

and since the current leads the voltage by a leading power factor. we have $I_2 = I_{SC} = 5\ A$, and in pu,

$$I_{2\ pu} = \frac{I_{SC}}{I_2} = 1\angle 36.9°\ \text{pu} = 0.8 + j0.6$$

With $a = 1$, relation (9.148) reduces to:

$$\mathbf{V_1} = \mathbf{V_2} + Z_{eq} \cdot \mathbf{I_2}$$

where from the solution of Exercise 8,

$$Z_{eq1\ pu} = 0.01456 + j0.0238\ \text{pu}$$

Thus,

$$\mathbf{V_1} = 1 + (0.01456 + j0.0238) \cdot (0.8 + j0.6) = 0.9974 + 0.0278 = 0.9978$$

and

$$\varepsilon_{pu} = \frac{V_1}{V_2} - 1 = 0.9978 - 1 = -0.0022$$

As expected, the voltage regulation is negative because of the leading load.

10. Choosing $\mathbf{I_2}$ as our reference vector, that is, $\mathbf{I_2} = \mathbf{I_2}\angle 0°\ \text{pu}$, at half–load,

$$\mathbf{I_2}_{\ HL} = 0.5\ \text{pu}$$

and

$$P_{HL} = V_{Load} \cdot \mathbf{I}_{2\ HL} \cdot pf = 1 \times 0.5 \times 0.8 = 0.4\ pu$$

From the solution of Exercise 8, $G_C = 0.018\ pu$, and thus the core losses are

$$P_C = G_C V_{OUT}^2 = 0.018 \times 1^2 = 0.018\ pu$$

Also from the solution of Exercise 8, $R_{eq} = 0.01456\ pu$, and thus the copper losses are

$$P_R = R_{eq} I^2 = 0.01456 \times 0.5^2 = 0.0036\ pu$$

Thus, the efficiency is

$$\eta = \frac{P_{HL}}{P_{HL} + P_C + P_R} = \frac{0.4}{0.4 + 0.018 + 0.0036} = 0.949$$

11.

$$P_C = P_h + P_e = k'_h f B_{max}^n + k'_e f^2 B_{max}^2$$

Since B_{max} is constant, we let $x_1 = k'_h B_{max}^n$ and $x_2 = k'_e B_{max}^2$. Then,

$$P_{C\ 25\ Hz} = 25x_1 + (25)^2 x_2 = 25x_1 + 625x_2 = 500\ W$$

and

$$P_{C\ 50\ Hz} = 50x_1 + (50)^2 x_2 = 50x_1 + 2500x_2 = 1400\ W$$

or

$$x_1 + 25x_2 = 20$$

and

$$x_1 + 50x_2 = 28\ W$$

Simultaneous solution of the last two equations yields

$$x_1 = 12 \qquad x_2 = 0.32$$

and thus the individual losses are:

$$P_{h\ 25\ Hz} = 25 \times 12 = 300\ W \qquad P_{e\ 25\ Hz} = (25)^2 \times 0.32 = 200\ W$$

$$P_{h\ 50\ Hz} = 50 \times 12 = 600\ W \qquad P_{e\ 50\ Hz} = (50)^2 \times 0.32 = 800\ W$$

Chapter 10

One– and Two–Port Networks

This chapter begins with the general principles of one and two–port networks. The z, y, h, and g parameters are defined. Several examples are presented to illustrate their use. It concludes with a discussion on reciprocal and symmetrical networks.

10.1 Introduction and Definitions

Generally, a network has two pairs of terminals; one pair is denoted as the *input terminals*, and the other as the *output terminals*. Such networks are very useful in the design of electronic systems, transmission and distribution systems, automatic control systems, communications systems, and others where electric energy or a signal enters the input terminals, it is modified by the network, and it exits through the output terminals.

A *port* is a pair of terminals in a network at which electric energy or a signal may enter or leave the network. A network that has only one pair a terminals is called a *one–port network*. In an one–port network, the current that enters one terminal must exit the network through the other terminal. Thus, in Figure 10.1, $i_{in} = i_{out}$.

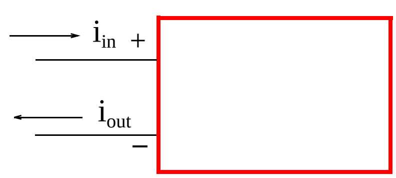


Figure 10.1. One–port network

Figures 10.2 and 10.3 show two examples of practical one–port networks.

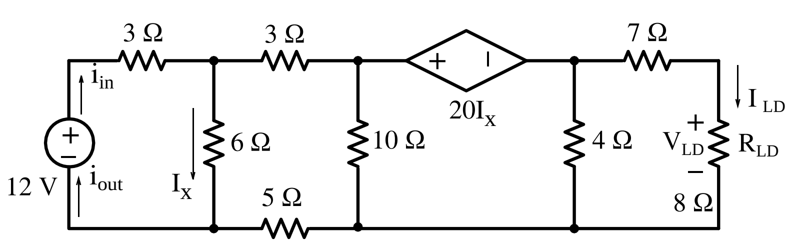


Figure 10.2. An example of an one–port network

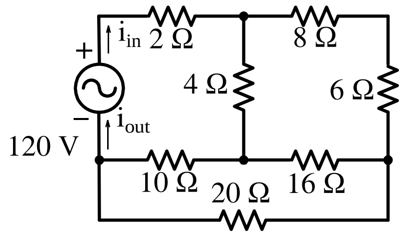


Figure 10.3. Another example of an one–port network

A two–port network has two pairs of terminals, that is, four terminals as shown in Figure 10.4 where $i_1 = i_3$ and $i_2 = i_4$

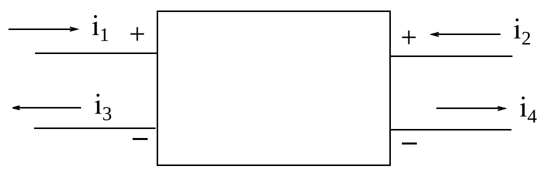


Figure 10.4. Two–port network

10.2 One–Port Driving–Point and Transfer Admittances

Let us consider an n – port network and write the mesh equations for this network in terms of the impedances Z. We assume that the subscript of each current corresponds to the loop number and KVL is applied so that the sign of each Z_{ii} is positive. The sign of any Z_{ij} for $i \neq j$ can be positive or negative depending on the reference directions of i_i and i_j.

$$\begin{aligned} Z_{11}i_1 + Z_{12}i_2 + Z_{13}i_3 + \ldots + Z_{1n}i_n &= v_1 \\ Z_{21}i_1 + Z_{22}i_2 + Z_{23}i_3 + \ldots + Z_{2n}i_n &= v_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots&\ldots\ldots \\ Z_{n1}i_1 + Z_{n2}i_2 + Z_{n3}i_3 + \ldots + Z_{nn}i_n &= v_n \end{aligned} \qquad (10.1)$$

In (10.1) each current can be found by Cramer's rule. For instance, the current i_1 is found by

$$i_1 = \frac{D_1}{\Delta} \qquad (10.2)$$

where

$$\Delta = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} & \ldots & Z_{1n} \\ Z_{21} & Z_{22} & Z_{23} & \ldots & Z_{2n} \\ Z_{31} & Z_{32} & Z_{33} & \ldots & Z_{3n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ Z_{n1} & Z_{n2} & Z_{n3} & \ldots & Z_{nn} \end{bmatrix} \qquad (10.3)$$

$$D_1 = \begin{bmatrix} V_1 & Z_{12} & Z_{13} & \dots & Z_{1n} \\ V_2 & Z_{22} & Z_{23} & \dots & Z_{2n} \\ V_3 & Z_{32} & Z_{33} & \dots & Z_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ V_n & Z_{n2} & Z_{n3} & \dots & Z_{nn} \end{bmatrix} \tag{10.4}$$

Next, we recall that the value of the determinant of a matrix A is the sum of the products obtained by multiplying each element of *any* row or column by its *cofactor**. The cofactor, with the proper sign, is the matrix that remains when both the row and the column containing the element are eliminated. The sign is plus (+) when the sum of the subscripts is even, and it is minus (–) when it is odd. Mathematically, if the cofactor of the element a_{qr} is denoted as A_{qr}, then

$$A_{qr} = (-1)^{q+r} M_{qr} \tag{10.5}$$

where M_{qr} is the *minor* of the element a_{qr}. We recall also that the minor is the cofactor without a sign.

Example 10.1

Compute the determinant of A from the elements of the first row and their cofactors given that

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & -4 & 2 \\ -1 & 2 & -6 \end{bmatrix}$$

Solution:

$$\det A = 1\begin{bmatrix} -4 & 2 \\ 2 & -6 \end{bmatrix} - 2\begin{bmatrix} 2 & 2 \\ -1 & -6 \end{bmatrix} - 3\begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix} = 1 \times 20 - 2 \times (-10) - 3 \times 0 = 40$$

Using the cofactor concept, and denoting the cofactor of the element a_{ij} as C_{ij}, we find that the cofactors of Z_{11}, Z_{12}, and Z_{21} of (10.1) are respectively,

$$C_{11} = \begin{bmatrix} Z_{22} & Z_{23} & \dots & Z_{2n} \\ Z_{32} & Z_{33} & \dots & Z_{3n} \\ \dots & \dots & \dots & \dots \\ Z_{n2} & Z_{n3} & \dots & Z_{nn} \end{bmatrix} \tag{10.6}$$

* *A detailed discussion on cofactors is included in Appendix E.*

$$C_{12} = -\begin{bmatrix} Z_{21} & Z_{23} & \dots & Z_{2n} \\ Z_{31} & Z_{33} & \dots & Z_{3n} \\ \dots & \dots & \dots & \dots \\ Z_{n1} & Z_{n3} & \dots & Z_{nn} \end{bmatrix} \qquad (10.7)$$

$$C_{21} = -\begin{bmatrix} Z_{12} & Z_{13} & \dots & Z_{1n} \\ Z_{32} & Z_{33} & \dots & Z_{3n} \\ \dots & \dots & \dots & \dots \\ Z_{n2} & Z_{n3} & \dots & Z_{nn} \end{bmatrix} \qquad (10.8)$$

Therefore, we can express (10.2) as

$$i_1 = \frac{D_1}{\Delta} = \frac{C_{11}v_1}{\Delta} + \frac{C_{21}v_2}{\Delta} + \frac{C_{31}v_3}{\Delta} + \dots + \frac{C_{n1}v_n}{\Delta} \qquad (10.9)$$

Also,

$$i_2 = \frac{D_2}{\Delta} = \frac{C_{12}v_1}{\Delta} + \frac{C_{22}v_2}{\Delta} + \frac{C_{32}v_3}{\Delta} + \dots + \frac{C_{n2}v_n}{\Delta} \qquad (10.10)$$

and the other currents i_3, i_4, and so on can be written in similar forms.

In network theory the y_{ij} *parameters* are defined as

$$y_{11} = \frac{C_{11}}{\Delta} \qquad y_{12} = \frac{C_{21}}{\Delta} \qquad y_{13} = \frac{C_{31}}{\Delta} \qquad \dots \qquad (10.11)$$

Likewise,

$$y_{21} = \frac{C_{12}}{\Delta} \qquad y_{22} = \frac{C_{22}}{\Delta} \qquad y_{23} = \frac{C_{32}}{\Delta} \qquad \dots \qquad (10.12)$$

and so on. By substitution of the y parameters into (10.9) and (10.10) we obtain:

$$i_1 = y_{11}v_1 + y_{12}v_2 + y_{13}v_3 + \dots + y_{1n}v_n \qquad (10.13)$$

$$i_2 = y_{21}v_1 + y_{22}v_2 + y_{23}v_3 + \dots + y_{2n}v_n \qquad (10.14)$$

If the subscripts of the y–parameters are alike, such as y_{11}, y_{22} and so on, they are referred to as *driving–point admittances*. If they are unlike, such as y_{12}, y_{21} and so on, they are referred to as *transfer admittances*.

If a network consists of only two loops such as in Figure 10.5 below,

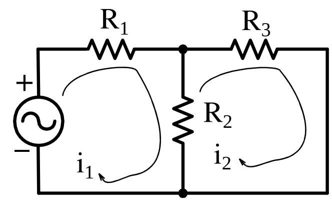


Figure 10.5. Two loop network

the equations of (10.13) and (10.14) will have only two terms each, that is,

$$i_1 = y_{11}v_1 + y_{12}v_2 \tag{10.15}$$

$$i_2 = y_{21}v_1 + y_{22}v_2 \tag{10.16}$$

From Figure 10.5 we observe that there is only one voltage source, v_1; there is no voltage source in Loop 2 and thus $v_2 = 0$. Then, (10.15) and (10.16) reduce to

$$i_1 = y_{11}v_1 \tag{10.17}$$

$$i_2 = y_{21}v_1 \tag{10.18}$$

Relation (10.17) reveals that the driving–point admittance y_{11} is the ratio i_1/v_1. That is, the driving–point admittance, as defined by (10.17), is the admittance seen by a voltage source that is present in the respective loop, in this case, Loop 1. Stated in other words, *the driving–point admittance is the ratio of the current in a given loop to the voltage source in that loop when there are no voltage sources in any other loops of the network.*

Transfer admittance is the ratio of the current in some other loop to the driving voltage source, in this case v_1. As indicated in (10.18), the transfer admittance y_{21} is the ratio of the current in Loop 2 to the voltage source in Loop 1.

Example 10.2

For the circuit of Figure 10.6, find the driving–point and transfer admittances and the current through each resistor.

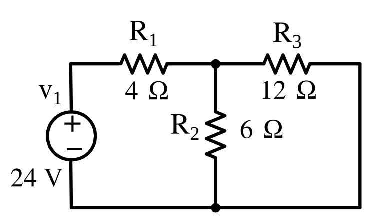


Figure 10.6. Circuit for Example 10.2

Solution:

We assign currents as shown in Figure 10.7.

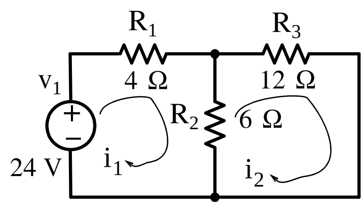


Figure 10.7. Loop equations for the circuit of Example 10.2

The loop equations are

$$\begin{aligned} 10i_1 - 6i_2 &= 24 \\ -6i_1 + 18i_2 &= 0 \end{aligned} \qquad (10.19)$$

The driving–point admittance is found from (10.11), that is,

$$y_{11} = \frac{C_{11}}{\Delta} \qquad (10.20)$$

and the transfer admittance from (10.12), that is,

$$y_{21} = \frac{C_{12}}{\Delta} \qquad (10.21)$$

For this example,

$$\Delta = \begin{bmatrix} 10 & -6 \\ -6 & 18 \end{bmatrix} = 180 - 36 = 144 \qquad (10.22)$$

The cofactor C_{11} is obtained by inspection from the matrix of (10.22), that is, eliminating the first row and first column we are left with 18 and thus $C_{11} = 18$. Similarly, the cofactor C_{12} is found by eliminating the first row and second column and changing the sign of -6. Then, $C_{12} = 6$. By substitution into (10.20) and (10.21), we obtain

$$y_{11} = \frac{C_{11}}{\Delta} = \frac{18}{144} = \frac{1}{8} \qquad (10.23)$$

and

$$y_{21} = \frac{C_{12}}{\Delta} = \frac{6}{144} = \frac{1}{24} \qquad (10.24)$$

Then, by substitution into (10.17) and (10.18) we obtain

$$i_1 = y_{11}v_1 = \frac{1}{8} \times 24 = 3 \text{ A} \qquad (10.25)$$

$$i_2 = y_{21}v_1 = \frac{1}{24} \times 24 = 1 \text{ A} \qquad (10.26)$$

Finally, the we observe that the current through the $4\ \Omega$ resistor is $3\ A$, through the $12\ \Omega$ is $1\ A$ and through the $6\ \Omega$ is $i_1 - i_2 = 3 - 1 = 2A$.

Of course, there are other simpler methods of computing these currents. However, the intent here was to illustrate how the driving–point and transfer admittances are applied. These allow easy computation for complicated network problems.

10.3 One–Port Driving–Point and Transfer Impedances

Now, let us consider an $n - \text{port}$ network and write the nodal equations for this network in terms of the admittances Y. We assume that the subscript of each current corresponds to the loop number and KVL is applied so that the sign of each Y_{ii} is positive. The sign of any Y_{ij} for $i \neq j$ can be positive or negative depending on the reference polarities of v_i and v_j.

$$\begin{aligned}Y_{11}v_1 + Y_{12}v_2 + Y_{13}v_3 + \ldots + Y_{1n}v_n &= i_1\\ Y_{21}v_1 + Y_{22}v_2 + Y_{23}v_3 + \ldots + Y_{2n}v_n &= i_2\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ Y_{n1}v_1 + Y_{n2}v_2 + Y_{n3}v_3 + \ldots + Y_{nn}v_n &= i_n\end{aligned} \qquad (10.27)$$

In (10.27), each voltage can be found by Cramer's rule. For instance, the voltage v_1 is found by

$$v_1 = \frac{D_1}{\Delta} \qquad (10.28)$$

where

$$\Delta = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & \ldots & Y_{1n}\\ Y_{21} & Y_{22} & Y_{23} & \ldots & Y_{2n}\\ Y_{31} & Y_{32} & Y_{33} & \ldots & Y_{3n}\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ Y_{n1} & Y_{n2} & Y_{n3} & \ldots & Y_{nn}\end{bmatrix} \qquad (10.29)$$

$$D_1 = \begin{bmatrix} V_1 & Y_{12} & Y_{13} & \ldots & Y_{1n}\\ V_2 & Y_{22} & Y_{23} & \ldots & Y_{2n}\\ V_3 & Y_{32} & Y_{33} & \ldots & Y_{3n}\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ V_n & Y_{n2} & Y_{n3} & \ldots & Y_{nn}\end{bmatrix} \qquad (10.30)$$

As in the previous section, we find that the nodal equations of (10.27) can be expressed as

$$v_1 = z_{11}i_1 + z_{12}i_2 + z_{13}i_3 + \dots + z_{1n}i_n \tag{10.31}$$

$$v_2 = z_{21}i_1 + z_{22}i_2 + z_{23}i_3 + \dots + z_{2n}i_n \tag{10.32}$$

$$v_3 = z_{31}i_1 + z_{32}i_2 + z_{33}i_3 + \dots + z_{3n}i_n \tag{10.33}$$

and so on, where

$$z_{11} = \frac{C_{11}}{\Delta} \qquad z_{12} = \frac{C_{21}}{\Delta} \qquad z_{13} = \frac{C_{31}}{\Delta} \qquad \dots \tag{10.34}$$

$$z_{21} = \frac{C_{12}}{\Delta} \qquad z_{22} = \frac{C_{22}}{\Delta} \qquad z_{23} = \frac{C_{32}}{\Delta} \qquad \dots \tag{10.35}$$

$$z_{31} = \frac{C_{13}}{\Delta} \qquad z_{32} = \frac{C_{23}}{\Delta} \qquad z_{33} = \frac{C_{33}}{\Delta} \qquad \dots \tag{10.36}$$

and so on. The matrices C_{ij} represent the cofactors as in the previous section.

The coefficients of (10.31), (10.32), and (10.33) with like subscripts are referred to as *driving–point impedances*. Thus, z_{11}, z_{22} and so on, are driving–point impedances. The remaining coefficients with unlike subscripts, such as z_{12}, z_{21} and so on, are called *transfer impedances*.

To understand the meaning of the driving–point and transfer impedances, we examine the network of Figure 10.8 where 0 is the reference node and nodes 1 and 2 are independent nodes. The driving point impedance is the ratio of the voltage across the nodes 1 and 0 to the current that flows through the branch between these nodes. In other words,

$$z_{11} = \frac{v_1}{i_1} \tag{10.37}$$

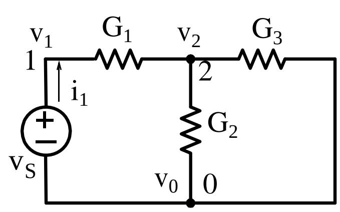


Figure 10.8. Circuit to illustrate the definitions of driving–point and transfer impedances.

The transfer impedance between nodes 2 and 1 is the ratio of the voltage v_2 to the current at node 1 *when there are no other current (or voltage) sources in the network.* That is,

$$z_{21} = \frac{v_2}{i_1} \tag{10.38}$$

Example 10.3

For the network of Figure 10.9, compute the driving–point and transfer impedances and the voltages across each conductance in terms of the current source.

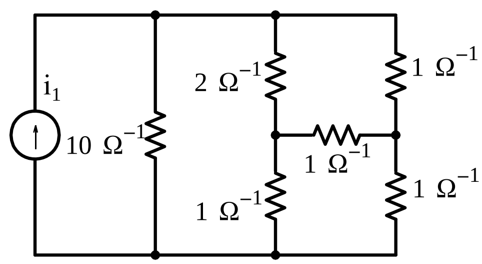


Figure 10.9. Network for Example 10.3.

Solution:

We assign nodes **0**, **1**, **2**, and **3** as shown in Figure 10.10.

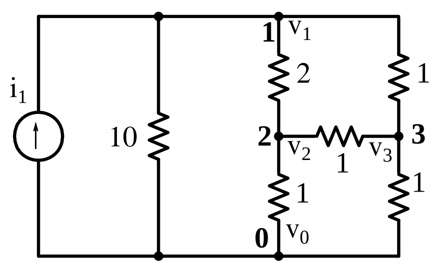


Figure 10.10. Node assignment for network of Example 10.3

The nodal equations are

$$
\begin{aligned}
10v_1 + 2(v_1 - v_2) + 1(v_1 - v_3) &= i_1 \\
2(v_2 - v_1) + 1(v_2 - v_3) + 1v_2 &= 0 \\
1(v_3 - v_1) + 1(v_3 - v_2) + 1v_3 &= 0
\end{aligned}
\tag{10.39}
$$

Simplifying and rearranging we obtain:

$$
\begin{aligned}
13v_1 - 2v_2 - v_3 &= i_1 \\
-2v_1 + 4v_2 - v_3 &= 0 \\
-v_1 - v_2 + 3v_3 &= 0
\end{aligned}
\tag{10.40}
$$

The driving–point impedance z_{11} is found from (10.34), that is,

$$z_{11} = \frac{C_{11}}{\Delta} \tag{10.41}$$

and the transfer impedances z_{21} and z_{31} from (10.35) and (10.36), that is,

$$z_{21} = \frac{C_{12}}{\Delta} \tag{10.42}$$

$$z_{31} = \frac{C_{13}}{\Delta} \tag{10.43}$$

For this example,

$$\Delta = \begin{bmatrix} 13 & -2 & -1 \\ -2 & 4 & -1 \\ -1 & -1 & 3 \end{bmatrix} = 156 - 2 - 2 - 4 - 13 - 12 = 123 \tag{10.44}$$

The cofactor C_{11} is

$$C_{11} = \begin{bmatrix} 4 & -1 \\ -1 & 3 \end{bmatrix} = 12 - 1 = 11 \tag{10.45}$$

Similarly, the cofactors C_{12} and C_{13} are

$$C_{12} = -\begin{bmatrix} -2 & -1 \\ -1 & 3 \end{bmatrix} = -(-6 - 1) = 7 \tag{10.46}$$

and

$$C_{13} = \begin{bmatrix} -2 & 4 \\ -1 & -1 \end{bmatrix} = 2 + 4 = 6 \tag{10.47}$$

By substitution into (10.41), (10.42), and (10.43), we obtain

$$z_{11} = \frac{C_{11}}{\Delta} = \frac{11}{123} \tag{10.48}$$

$$z_{21} = \frac{C_{12}}{\Delta} = \frac{7}{123} \tag{10.49}$$

$$z_{31} = \frac{C_{13}}{\Delta} = \frac{6}{123} \tag{10.50}$$

Then, by substitution into (10.31), (10.32), and (10.33) we obtain:

$$v_1 = z_{11}i_1 + z_{12}i_2 + z_{13}i_3 = \frac{11}{123}i_1 \tag{10.51}$$

$$v_2 = z_{21}i_1 + z_{22}i_2 + z_{23}i_3 = \frac{7}{123}i_1 \tag{10.52}$$

$$v_3 = z_{31}i_1 + z_{32}i_2 + z_{33}i_3 = \frac{6}{123}i_1 \tag{10.53}$$

As stated earlier, there are other simpler methods of computing these voltages. However, the intent here was to illustrate how the driving–point and transfer impedances are applied. These allow easy computation for complicated network problems.

10.4 Two–Port Networks

Figure 10.11 shows a two–port network with external voltages and currents specified.

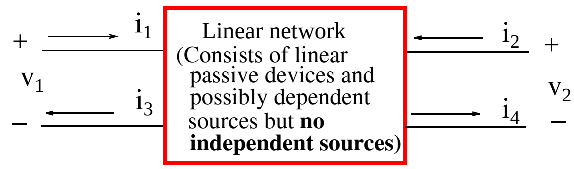


Figure 10.11. Two–port network

Here, we assume that $i_1 = i_3$ and $i_2 = i_4$. We also assume that i_1 and i_2 are obtained by the superposition of the currents produced by both v_1 and v_2.

Next, we will define the y, z, h, and g parameters.

10.4.1 The y Parameters

The two–port network of Figure 10.11 can be described by the following set of equations.

$$i_1 = y_{11}v_1 + y_{12}v_2 \tag{10.54}$$

$$i_2 = y_{21}v_1 + y_{22}v_2 \tag{10.55}$$

In two–port network theory, the y coefficients are referred to as the y *parameters*.
Let us assume that v_2 is shorted, that is, $v_2 = 0$. Then, (10.54) reduces to

$$i_1 = y_{11}v_1 \tag{10.56}$$

or

$$y_{11} = \frac{i_1}{v_1} \tag{10.57}$$

and y_{11} is referred to as the *short circuit input admittance* at the left port when the right port of Figure 10.11 is short–circuited.

Let us again consider (10.54), that is,

$$i_1 = y_{11}v_1 + y_{12}v_2 \tag{10.58}$$

This time we assume that v_1 is shorted, i.e., $v_1 = 0$. Then, (10.58) reduces to

$$i_1 = y_{12}v_2 \tag{10.59}$$

or

$$y_{12} = \frac{i_1}{v_2} \tag{10.60}$$

and y_{12} is referred to as the *short circuit transfer admittance* when the left port of Figure 10.11 is short–circuited. It represents the transmission from the right to the left port. For instance, in amplifiers where the left port is considered to be the input port and the right to be the output, the parameter y_{12} represents the internal feedback inside the network.

Similar expressions are obtained when we consider the equation for i_2, that is,

$$i_2 = y_{21}v_1 + y_{22}v_2 \tag{10.61}$$

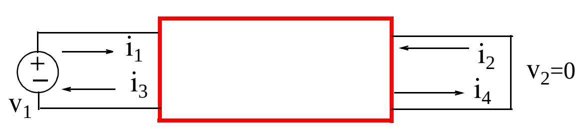
In an amplifier, the parameter y_{21} is also referred to as the short circuit transfer admittance and represents transmission from the left (input) port to the right (output) port. It is a measure of the so–called *forward gain*.

The parameter y_{22} is called the *short circuit output admittance.*

The y parameters and the conditions under which they are computed are shown in Figures 10.12 through 10.16.

$$i_1 = y_{11}v_1 + y_{12}v_2$$
$$i_2 = y_{21}v_1 + y_{22}v_2$$

Figure 10.12. The y parameters for $v_1 \neq 0$ *and* $v_2 \neq 0$



$$y_{11} = \left.\frac{i_1}{v_1}\right|_{v_2 = 0}$$

Figure 10.13. Network for the definition of the y_{11} *parameter*

$$y_{12} = \left.\frac{i_1}{v_2}\right|_{v_1 = 0}$$

Figure 10.14. Network for the definition of the y_{12} *parameter*

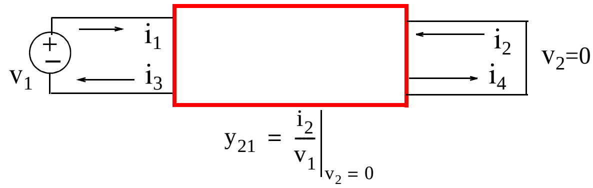


Figure 10.15. Network for the definition of the y_{21} *parameter*

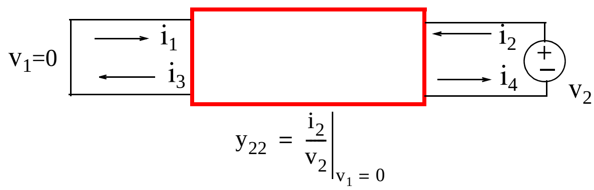


Figure 10.16. Network for the definition of the y_{22} *parameter*

Example 10.4

For the network of Figure 10.17, find the y parameters.

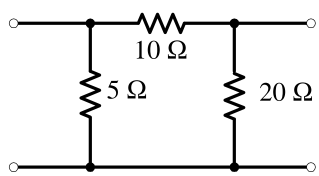


Figure 10.17. Network for Example 10.4

Solution:

a. The short circuit input admittance y_{11} is found from the network of Figure 10.18 where we have assumed that $v_1 = 1\ V$ and the resistances, for convenience, have been replaced with conductances in mhos.

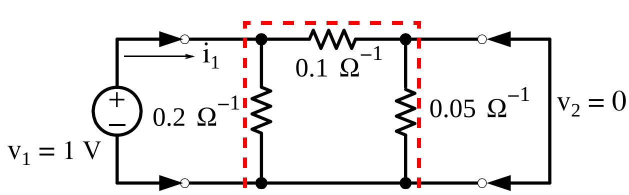


Figure 10.18. Network for computing y_{11}

We observe that the $0.05\ \Omega^{-1}$ conductance is shorted out and thus the current i_1 is the sum of the currents through the $0.2\ \Omega^{-1}$ and $0.1\ \Omega^{-1}$ conductances. Then,

$$i_1 = 0.2v_1 + 0.1v_1 = 0.2 \times 1 + 0.1 \times 1 = 0.3 \text{ A}$$

and thus the short circuit input admittance is

$$y_{11} = i_1 / v_1 = 0.3/1 = 0.3 \ \Omega^{-1} \tag{10.62}$$

b. The short circuit transfer admittance y_{12} when the left port is short–circuited, is found from the network of Figure 10.19.

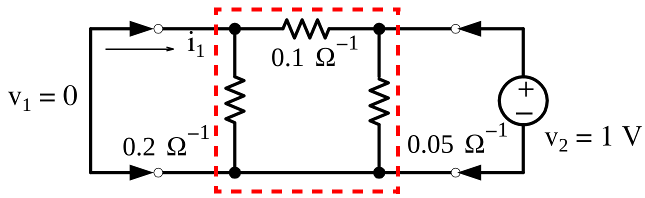


Figure 10.19. Network for computing y_{12}

We observe that the $0.2 \ \Omega^{-1}$ conductance is shorted out and thus the $0.1 \ \Omega^{-1}$ conductance is in parallel with the $0.05 \ \Omega^{-1}$ conductance. The current i_1, with a minus (–) sign, now flows through the $0.1 \ \Omega^{-1}$ conductance. Then,

$$i_1 = -0.1v_2 = -0.1 \times 1 = -0.1 \text{ A}$$

and

$$y_{12} = i_1 / v_2 = -0.1/1 = -0.1 \ \Omega^{-1} \backslash \tag{10.63}$$

c. The short circuit transfer admittance y_{21} when the right port is short–circuited, is found from the network of Figure 10.20.

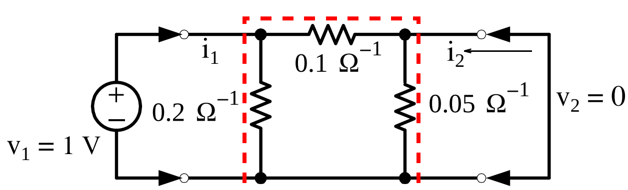


Figure 10.20. Network for computing y_{21}

We observe that the $0.05 \ \Omega^{-1}$ conductance is shorted out and thus the $0.1 \ \Omega^{-1}$ conductance is in parallel with the $0.2 \ \Omega^{-1}$ conductance. The current i_2, with a minus (–) sign, now flows through the $0.1 \ \Omega^{-1}$ conductance. Then,

$$i_2 = -0.1v_1 = -0.1 \times 1 = -0.1 \text{ A}$$

and

$$y_{21} = i_2/v_1 = -0.1/1 = -0.1\ \Omega^{-1} \tag{10.64}$$

d. The short circuit output admittance y_{22} at the right port when the left port is short–circuited, is found from the network of 10.21.

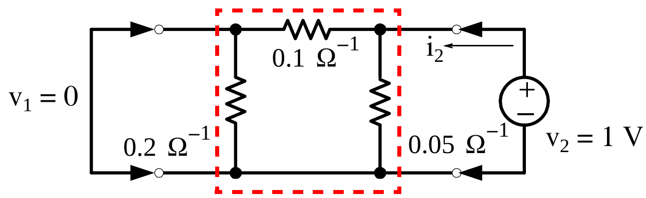


Figure 10.21. Network for computing y_{22}

We observe that the $0.2\ \Omega^{-1}$ conductance is shorted out and thus the current i_2 is the is the sum of the currents through the $0.05\ \Omega^{-1}$ and $0.1\ \Omega^{-1}$ conductances. Then,

$$i_2 = 0.05v_2 + 0.1v_2 = 0.05 \times 1 + 0.1 \times 1 = 0.15\ \text{A}$$

and

$$y_{22} = i_2/v_2 = 0.15/1 = 0.15\ \Omega^{-1} \tag{10.65}$$

Therefore, the two–port network of Figure 10.10 can be described by the following set of equations.

$$\begin{aligned} i_1 &= y_{11}v_1 + y_{12}v_2 = 0.3v_1 - 0.1v_2 \\ i_2 &= y_{21}v_1 + y_{22}v_2 = -0.1v_1 + 0.3v_2 \end{aligned} \tag{10.66}$$

Note:

In Example 10.4, we found that the short circuit transfer admittances are equal, that is,

$$y_{21} = y_{12} = -0.1 \tag{10.67}$$

This is not just a coincidence; this is true whenever a two–port network is *reciprocal* (or *bilateral*). A network is reciprocal if the *reciprocity theorem* is satisfied. This theorem states that:

If a voltage applied in one branch of a linear, two–port passive network produces a certain current in any other branch of this network, the same voltage applied in the second branch will produce the same current in the first branch.

The reverse is also true, that is, if current applied at one node produces a certain voltage at another, the same current at the second node will produce the same voltage at the first. An example is given at the end of this chapter. Obviously, if we know that the two–port network is reciprocal, only three computations are required to find the y parameters.

If in a reciprocal two–port network its ports can be interchanged without affecting the terminal voltages and currents, the network is said to be also *symmetric*. In a symmetric two–port network,

$$y_{22} = y_{11} \qquad y_{21} = y_{12} \tag{10.68}$$

The network of Figure 10.17 is not symmetric since $y_{22} \neq y_{11}$.

We will present examples of reciprocal and symmetric two–port networks at the last section of this chapter.

The following example illustrates the applicability of two–port network analysis in more complicated networks.

Example 10.5

For the network of Figure 10.22, compute v_1, v_2, i_1, and i_2.

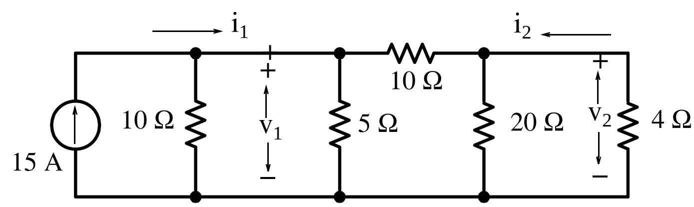


Figure 10.22. Network for Example 10.5

Solution:

We recognize the portion of the network enclosed in the dotted square, shown in Figure 10.23, as that of the previous example.

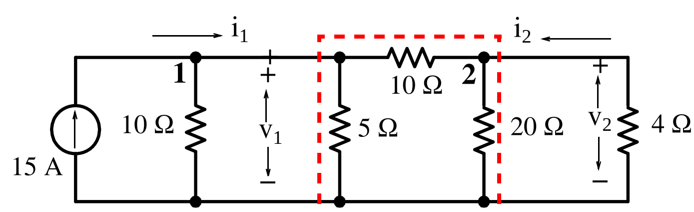


Figure 10.23. Portion of the network for which the y parameters are known.

For the network of Figure 10.23, at Node **1**,

$$i_1 = 15 - v_1/10 \tag{10.69}$$

and at Node **2**,

$$i_2 = -v_2/4 \tag{10.70}$$

By substitution of (10.69) and (10.70) into (10.66), we obtain:

$$i_1 = y_{11}v_1 + y_{12}v_2 = 0.3v_1 - 0.1v_2 = 15 - v_1/10$$
$$i_2 = y_{21}v_1 + y_{22}v_2 = -0.1v_1 + 0.3v_2 = -v_2/4 \qquad (10.71)$$

or

$$0.4v_1 - 0.1v_2 = 15$$
$$-0.1v_1 + 0.4v_2 = 0 \qquad (10.72)$$

We will use MATLAB to solve the equations of (10.72) to become more familiar with it.

```
syms v1 v2; [v1 v2]=solve(0.4*v1–0.1*v2–15, –0.1*v1+0.4*v2)
% Must have Symbolic Math Toolbox installed
```

```
v1 = 40
v2 = 10
```

and thus

$$v_1 = 40 \text{ V}$$
$$v_2 = 10 \text{ V} \qquad (10.73)$$

The currents i_1 and i_2 are found from (10.69) and (10.70).

$$i_1 = 15 - 40/10 = 11 \text{ A}$$
$$i_2 = -10/4 = -2.5 \text{ A} \qquad (10.74)$$

10.4.2 The z parameters

A two–port network such as that of Figure 10.24 can also be described by the following set of equations.

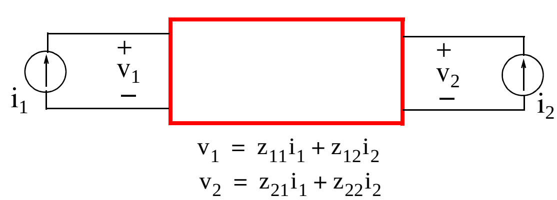


Figure 10.24. The z parameters for $i_1 \neq 0$ *and* $i_2 \neq 0$

$$v_1 = z_{11}i_1 + z_{12}i_2 \qquad (10.75)$$

$$v_2 = z_{21}i_1 + z_{22}i_2 \qquad (10.76)$$

In two–port network theory, the z_{ij} coefficients are referred to as the z *parameters* or as *open circuit impedance parameters*.

Let us assume that v_2 is open, that is, $i_2 = 0$ as shown in Figure 10.25.

$$z_{11} = \left.\frac{v_1}{i_1}\right|_{i_2 = 0}$$

Figure 10.25. Network for the definition of the z_{11} *parameter*

Then, (10.75) reduces to

$$v_1 = z_{11}i_1 \tag{10.77}$$

or

$$z_{11} = \frac{v_1}{i_1} \tag{10.78}$$

and this is the *open circuit input impedance* when the right port of Figure 10.25 is open.
Let us again consider (10.75), that is,

$$v_1 = z_{11}i_1 + z_{12}i_2 \tag{10.79}$$

This time we assume that the terminal at v_1 is open, i.e., $i_1 = 0$ as shown in Figure 10.26.

$$z_{12} = \left.\frac{v_1}{i_2}\right|_{i_1 = 0}$$

Figure 10.26. Network for the definition of the z_{12} *parameter*

Then, (10.75) reduces to

$$v_1 = z_{12}i_2 \tag{10.80}$$

or

$$z_{12} = \frac{v_1}{i_2} \tag{10.81}$$

and this is the *open circuit transfer impedance* when the left port is open as shown in Figure 10.26.

Similar expressions are obtained when we consider the equation for v_2, that is,

$$v_2 = z_{21}i_1 + z_{22}i_2 \tag{10.82}$$

Let us assume that v_2 is open, that is, $i_2 = 0$ as shown in Figure 10.27.
Then, (10.82) reduces to

$$v_2 = z_{21}i_1 \tag{10.83}$$

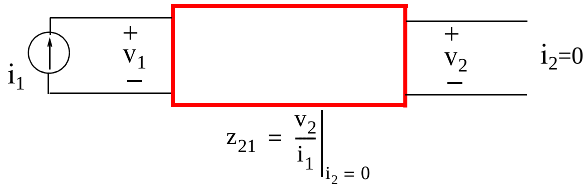


Figure 10.27. Network for the definition of the z_{21} *parameter*

or

$$z_{21} = \frac{v_2}{i_1} \tag{10.84}$$

The parameter z_{21} is referred to as *open circuit transfer impedance* when the right port is open as shown in Figure 10.27.

Finally, let us assume that the terminal at v_1 is open, i.e., $i_1 = 0$ as shown in Figure 10.28.

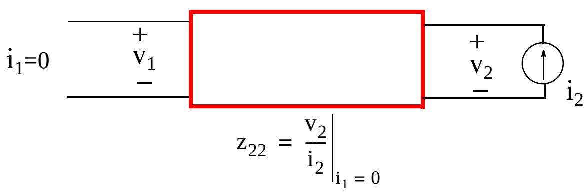


Figure 10.28. Network for the definition of the z_{22} *parameter*

Then, (10.82) reduces to

$$v_2 = z_{22}i_2 \tag{10.85}$$

or

$$z_{22} = \frac{v_2}{i_2} \tag{10.86}$$

The parameter z_{22} is called the *open circuit output impedance.*

We observe that the z parameters definitions are similar to those of the y parameters if we substitute voltages for currents and currents for voltages.

Example 10.6

For the network of Figure 10.29, find the z parameters.

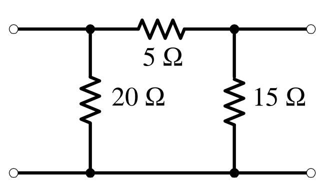


Figure 10.29. Network for Example 10.6

Solution:

a. The open circuit input impedance z_{11} is found from the network of Figure 10.30 where we have assumed that $i_1 = 1\ A$.

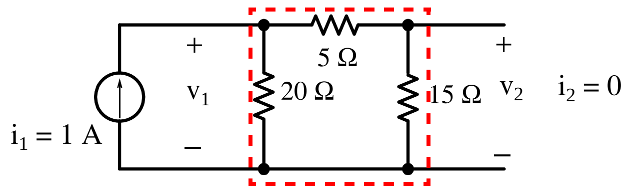


Figure 10.30. Network for computing z_{11} *for the network of Figure 10.29*

We observe that the 20 Ω resistor is in parallel with the series combination of the 5 Ω and 15 Ω resistors. Then, by the current division expression, the current through the 20 Ω resistor is 0.5 A and the voltage across that resistor is

$$v_1 = 20 \times 0.5 = 10\ V$$

Therefore, the open circuit input impedance z_{11} is

$$z_{11} = v_1/i_1 = 10/1 = 10\ \Omega \tag{10.87}$$

b. The open circuit transfer impedance z_{12} is found from the network of Figure 10.31.

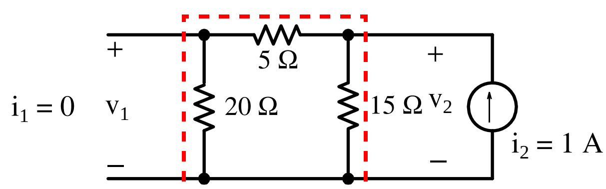


Figure 10.31. Network for computing z_{12} *for the network of Figure 10.29*

We observe that the 15 Ω resistance is in parallel with the series combination of the 5 Ω and 20 Ω resistances. Then, the current through the 20 Ω resistance is

$$i_{20\Omega} = \frac{15}{15+5+20} i_2 = \frac{15}{40} \times 1 = 3/8\ A$$

and the voltage across this resistor is

$$\frac{3}{8} \times 20 = \frac{60}{8} = 15/2\ V$$

Therefore, the open circuit transfer impedance z_{12} is

$$z_{12} = \frac{v_1}{i_2} = \frac{15/2}{1} = 7.5\ \Omega \tag{10.88}$$

c. The open circuit transfer impedance z_{21} is found from the network of Figure 10.32.

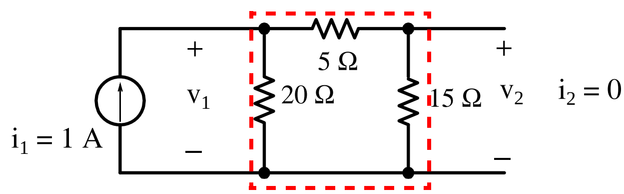


Figure 10.32. Network for computing z_{21} for the network of Figure 10.29

In Figure 10.32 the current that flows through the $15\ \Omega$ resistor is

$$i_{15\Omega} = \frac{20}{20+5+15} i_1 = \frac{20}{40} \times 1 = 1/2\ \text{A}$$

and the voltage across this resistor is

$$v_2 = \frac{1}{2} \times 15 = 15/2\ \text{V}$$

Therefore, the open circuit transfer impedance z_{21} is

$$z_{21} = \frac{v_2}{i_1} = \frac{15/2}{1} = 7.5\ \Omega \tag{10.89}$$

We observe that

$$z_{21} = z_{12} \tag{10.90}$$

d. The open circuit output impedance z_{22} is found from the network of Figure 10.33.

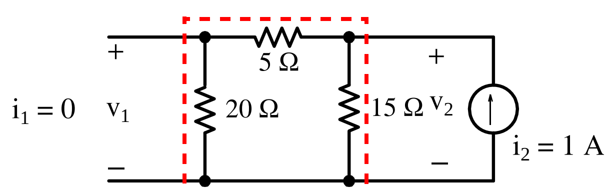


Figure 10.33. Network for computing z_{22} for the network of Figure 10.29

We observe that the $15\ \Omega$ resistance is in parallel with the series combination of the $5\ \Omega$ and $20\ \Omega$ resistances. Then, the current through the $15\ \Omega$ resistance is

$$i_{15\Omega} = \frac{20+5}{20+5+15} i_2 = \frac{25}{40} \times 1 = 5/8\ \text{A}$$

and the voltage across that resistor is

$$\frac{5}{8} \times 15 = 75/8 \text{ V}$$

Therefore, the open circuit output impedance z_{22} is

$$z_{22} = \frac{v_1}{i_2} = \frac{75/8}{1} = 75/8 \ \Omega \tag{10.91}$$

10.4.3 The h Parameters

A two–port network can also be described by the set of equations

$$v_1 = h_{11} i_1 + h_{12} v_2 \tag{10.92}$$

$$i_2 = h_{21} i_1 + h_{22} v_2 \tag{10.93}$$

as shown in Figure 10.34.

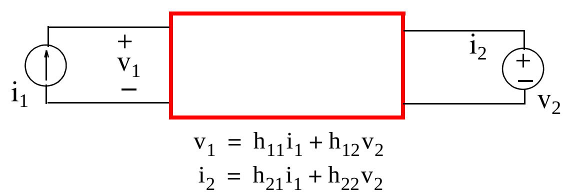


Figure 10.34. The h parameters for $i_1 \neq 0$ *and* $v_2 \neq 0$

The h parameters represent an impedance, a voltage gain, a current gain, and an admittance. For this reason they are called *hybrid* (different) parameters.

Let us assume that $v_2 = 0$ as shown in Figure 10.35.

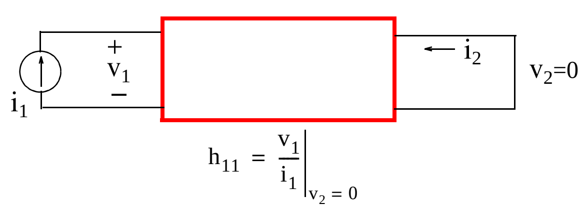


Figure 10.35. Network for the definition of the h_{11} *parameter*

Then, (10.92) reduces to

$$v_1 = h_{11} i_1 \tag{10.94}$$

or

$$h_{11} = \frac{v_1}{i_1} \tag{10.95}$$

Therefore, the parameter h_{11} represents the input impedance of a two–port network.

Let us assume that $i_1 = 0$ as shown in Figure 10.36.

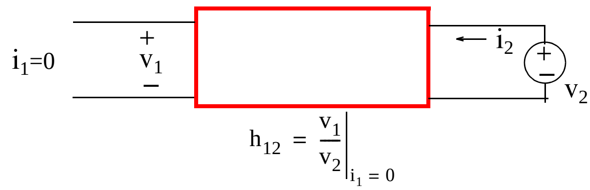


Figure 10.36. Network for computing h_{12} *for the network of Figure 10.34*

Then, (10.92) reduces to

$$v_1 = h_{12} v_2 \tag{10.96}$$

or

$$h_{12} = \frac{v_1}{v_2} \tag{10.97}$$

Therefore, in a two–port network the parameter h_{12} represents a voltage gain (or loss).

Let us assume that $v_2 = 0$ as shown in Figure 10.37.

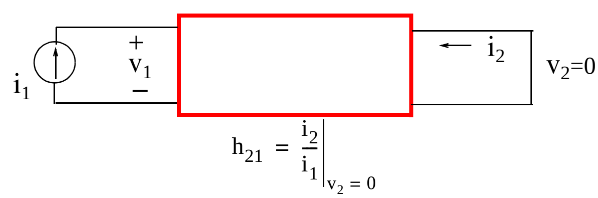


Figure 10.37. Network for computing h_{21} *for the network of Figure 10.34*

Then, (10.93) reduces to

$$i_2 = h_{21} i_1$$

or

$$h_{21} = \frac{i_2}{i_1}$$

Therefore, in a two–port network the parameter h_{21} represents a current gain (or loss).
Finally, let us assume that the terminal at v_1 is open, i.e., $i_1 = 0$ as shown in Figure 10.38.

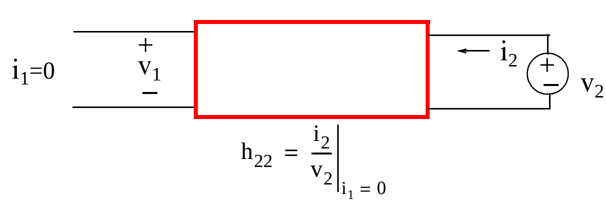


Figure 10.38. Network for computing h_{22} *for the network of Figure 10.34*

Then, (10.93) reduces to

$$i_2 = h_{22}v_2$$

or

$$h_{22} = \frac{i_2}{v_2}$$

Therefore, in a two–port network the parameter h_{22} represents an output admittance.

Example 10.7

For the network of Figure 10.39, find the h parameters.

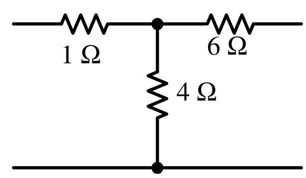


Figure 10.39. Network for Example 10.7

Solution:

a. The short circuit input impedance h_{11} is found from the network of Figure 10.40 where we have assumed that $i_1 = 1\ A$.

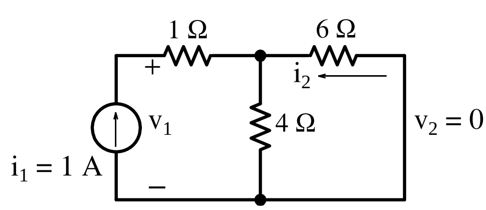


Figure 10.40. Network for computing h_{11} for the network of Figure 10.39

From the network of Figure 10.40 we observe that the $4\ \Omega$ and $6\ \Omega$ resistors are in parallel yielding an equivalent resistance of $2.4\ \Omega$ in series with the $1\ \Omega$ resistor. Then, the voltage across the current source is

$$v_1 = 1 \times (1 + 2.4) = 3.4\ V$$

Therefore, the short circuit input impedance h_{11} is

$$h_{11} = \frac{v_1}{i_1} = \frac{3.4}{1} = 3.4\ \Omega \tag{10.98}$$

b. The voltage gain h_{12} is found from the network of Figure 10.41.

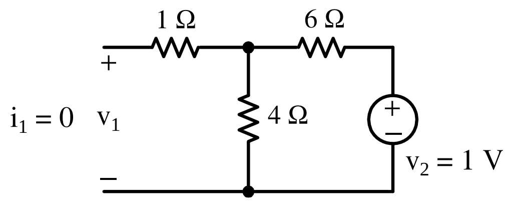


Figure 10.41. Network for computing h_{12} *for the network of Figure 10.39.*

Since no current flows through the $1\ \Omega$ resistor, the voltage v_1 is the voltage across the $4\ \Omega$ resistor. Then, by the voltage division expression,

$$v_1 = \frac{4}{6+4}v_2 = \frac{4}{10}\times 1 = 0.4 \text{ V}$$

Therefore, the voltage gain h_{12} is the dimensionless number

$$h_{12} = \frac{v_1}{v_2} = \frac{0.4}{1} = 0.4 \tag{10.99}$$

c. The current gain h_{21} is found from the network of Figure 10.42.

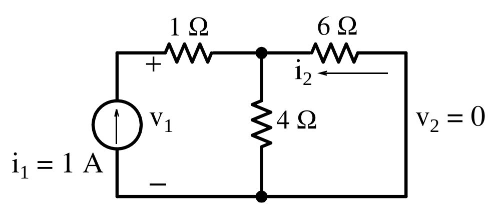


Figure 10.42. Network for computing h_{21} *for the network of Figure 10.39.*

We observe that the $4\ \Omega$ and $6\ \Omega$ resistors are in parallel yielding an equivalent resistance of $2.4\ \Omega$. Then, the voltage across the $2.4\ \Omega$ parallel combination is

$$v_{2.4\Omega} = 2.4\times 1 = 2.4 \text{ V}$$

The current i_2 is the current through the $6\ \Omega$ resistor. Thus,

$$i_2 = -\frac{2.4}{6} = -0.4 \text{ A}$$

Therefore, the current gain h_{21} is the dimensionless number

$$h_{21} = \frac{i_2}{i_1} = \frac{-0.4}{1} = -0.4$$

We observe that

$$h_{21} = -h_{12} \qquad (10.100)$$

and this is a consequence of the fact that the given network is reciprocal.

d. The open circuit admittance h_{22} is found from the network of Figure 10.43.

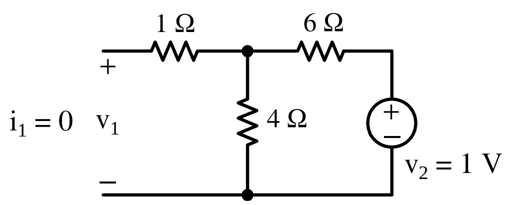


Figure 10.43. Network for computing h_{22} *for the network of Figure 10.39.*

Since no current flows through the $1\ \Omega$ resistor, the current i_2 is found by Ohm's law as

$$i_2 = \frac{v_2}{6+4} = \frac{1}{10} = 0.1\ \text{A}$$

Therefore, the open circuit admittance h_{22} is

$$h_{22} = \frac{i_2}{v_2} = \frac{0.1}{1} = 0.1\ \Omega^{-1} \qquad (10.101)$$

Note:

The h parameters and the g parameters (to be discussed next), are used extensively in networks consisting of transistors*, and feedback networks. The h parameters are best suited with series–parallel feedback networks, whereas the g parameters are preferred in parallel–series amplifiers.

10.4.4 The g Parameters

A two–port network can also be described by the set of equations

$$i_1 = g_{11}v_1 + g_{12}i_2 \qquad (10.102)$$

$$v_2 = g_{21}v_1 + g_{22}i_2 \qquad (10.103)$$

* *Transistors are three–terminal devices. However, they can be represented as large–signal equivalent two–port networks circuits and also as small–signal equivalent two–port networks where linearity can be applied. For a detailed discussion on transistors, please refer to Electronic Devices and Amplifier Circuits with MATLAB Applications, ISBN 978–1–934404–13–3.*

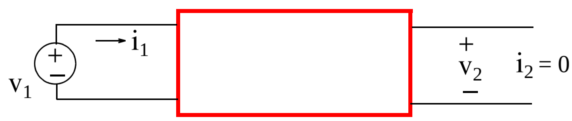
as shown in Figure 10.44.

$$i_1 = g_{11}v_1 + g_{12}i_2$$
$$v_2 = g_{21}v_1 + g_{22}i_2$$

Figure 10.44. The g parameters for $v_1 \neq 0$ *and* $i_2 \neq 0$

The g *parameters,* also known as *inverse hybrid parameters*, represent an admittance, a current gain, a voltage gain and an impedance.

Let us assume that $i_2 = 0$ as shown in Figure 10.45.



$$g_{11} = \left.\frac{i_1}{v_1}\right|_{i_2 = 0}$$

Figure 10.45. Network for computing g_{11} *for the network of Figure 10.44*

Then, (10.102) reduces to

$$i_1 = g_{11}\, v_1 \tag{10.104}$$

or

$$g_{11} = \frac{i_1}{v_1} \tag{10.105}$$

Therefore, the parameter g_{11} represents the input admittance of a two–port network.

Let us assume that $v_1 = 0$ as shown in Figure 10.46.

$$g_{12} = \left.\frac{i_1}{i_2}\right|_{v_1 = 0}$$

Figure 10.46. Network for computing g_{12} *for the network of Figure 10.44*

Then, (10.102) reduces to

$$i_1 = g_{12}\, i_2 \tag{10.106}$$

or

$$g_{12} = \frac{i_1}{i_2} \tag{10.107}$$

Therefore, in a two–port network the parameter g_{12} represents a current gain (or loss).

Let us assume that $i_2 = 0$ as shown in Figure 10.47.

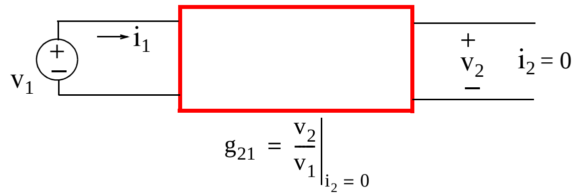


Figure 10.47. Network for computing g_{21} for the network of Figure 10.44

Then, (10.103) reduces to

$$v_2 = g_{21} v_1 \tag{10.108}$$

or

$$g_{21} = \frac{v_2}{i_1} \tag{10.109}$$

Therefore, in a two–port network the parameter g_{21} represents a voltage gain (or loss).

Finally, let us assume that v_1 is shorted, i.e., $v_1 = 0$ as shown in Figure 10.48.

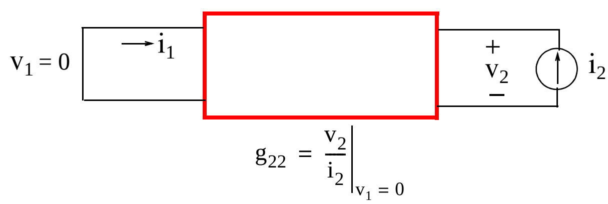


Figure 10.48. Network for computing g_{22} for the network of Figure 10.44

Then, (10.103) reduces to

$$v_2 = g_{22} i_2 \tag{10.110}$$

or

$$g_{22} = \frac{v_2}{i_2} \tag{10.111}$$

Thus, in a two–port network the parameter g_{22} represents the output impedance of that network.

Example 10.8

For the network of Figure 10.49, find the g parameters.

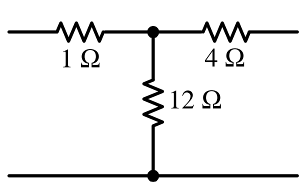


Figure 10.49. Network for Example 10.8

Solution:

a. The open circuit input admittance g_{11} is found from the network of Figure 10.50 where we have assumed that $v_1 = 1\ V$.

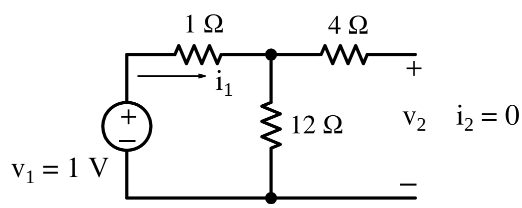


Figure 10.50. Network for computing g_{11} for the network of Figure 10.49.

There is no current through the $4\ \Omega$ resistor and thus by Ohm's law,

$$i_1 = \frac{v_1}{1+12} = \frac{1}{13}\ A$$

Therefore, the open circuit input admittance g_{11} is

$$g_{11} = \frac{i_1}{v_1} = \frac{1/13}{1} = \frac{1}{13}\ \Omega^{-1} \tag{10.112}$$

b. The current gain g_{12} is found from the network of Figure 10.51.

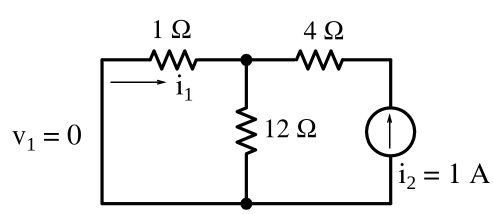


Figure 10.51. Network for computing g_{12} for the network of Figure 10.49.

By the current division expression, the current through the $1\ \Omega$ resistor is

$$i_1 = -\frac{12}{12+1}i_2 = -\frac{12}{13}\times 1 = -12/13\ A$$

Therefore, the current gain g_{12} is the dimensionless number

$$g_{12} = \frac{i_1}{i_2} = \frac{-12/13}{1} = -12/13 \tag{10.113}$$

c. The voltage gain g_{21} is found from the network of Figure 10.52.

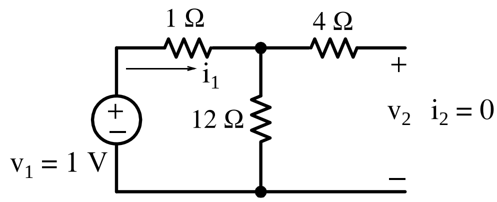


Figure 10.52. Network for computing g_{21} for the network of Figure 10.49.

Since there is no current through the $4\ \Omega$ resistor, the voltage v_2 is the voltage across the $12\ \Omega$ resistor. Then, by the voltage division expression,

$$v_2 = \frac{12}{1+12} \times 1 = 12/13 \text{ V}$$

Therefore, the voltage gain g_{21} is the dimensionless number

$$g_{21} = \frac{v_2}{v_1} = \frac{12/13}{1} = \frac{12}{13}$$

We observe that

$$g_{21} = -g_{12} \tag{10.114}$$

and this is a consequence of the fact that the given network is reciprocal.

d. The short circuit output impedance g_{22} is found from the network of Figure 10.53.

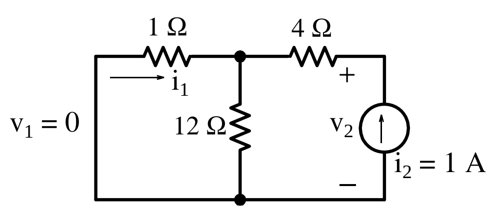


Figure 10.53. Network for computing g_{22} for the network of Figure 10.49.

The voltage v_2 is the sum of the voltages across the $4\ \Omega$ resistor and the voltage across the $12\ \Omega$ resistor. By the current division expression, the current through the $12\ \Omega$ resistor is

$$i_{12\Omega} = \frac{1}{1+12} i_2 = \frac{1}{13} \times 1 = 1/13 \text{ A} \tag{10.115}$$

Then,

$$v_{12\Omega} = \frac{1}{13} \times 12 = 12/13 \text{ V}$$

and

$$v_2 = \frac{12}{13} + 4 = 64/13 \text{ V}$$

Therefore, the short circuit output impedance g_{22} is

$$g_{22} = \frac{v_2}{i_2} = \frac{64/13}{1} = 64/13\ \Omega \tag{10.116}$$

10.5 Reciprocal Two−Port Networks

If any of the following relationships exist in a a two−port network,

$$\begin{aligned} z_{21} &= z_{12} \\ y_{21} &= y_{12} \\ h_{21} &= -h_{12} \\ g_{21} &= -g_{12} \end{aligned} \tag{10.117}$$

the network is said to be *reciprocal*.

If, in addition to (10.117), any of the following relationship exists

$$\begin{aligned} z_{22} &= z_{11} \\ y_{22} &= y_{11} \\ h_{11}h_{22} - h_{12}h_{21} &= 1 \\ g_{11}g_{22} - g_{12}g_{21} &= 1 \end{aligned} \tag{10.118}$$

the network is said to be *symmetric*.

Examples of reciprocal two−port networks are the tee, π, bridged (lattice), and bridged tee. These are shown in Figure 10.54. Examples of symmetric two−port networks are shown in Figure 10.55.

Let us review the reciprocity theorem and its consequences before we present an example. This theorem states that:

If a voltage applied in one branch of a linear, two−port passive network produces a certain current in any other branch of this network, the same voltage applied in the second branch will produce the same current in the first branch.

The reverse is also true, that is, if current applied at one node produces a certain voltage at another, the same current at the second node will produce the same voltage at the first.

It was also stated earlier that if we know that the two−port network is reciprocal, only three computations are required to find the y, z, h, and g parameters as shown in (10.117). Furthermore, if we know that the two−port network is symmetric, we only need to make only two computations as shown in (10.118).

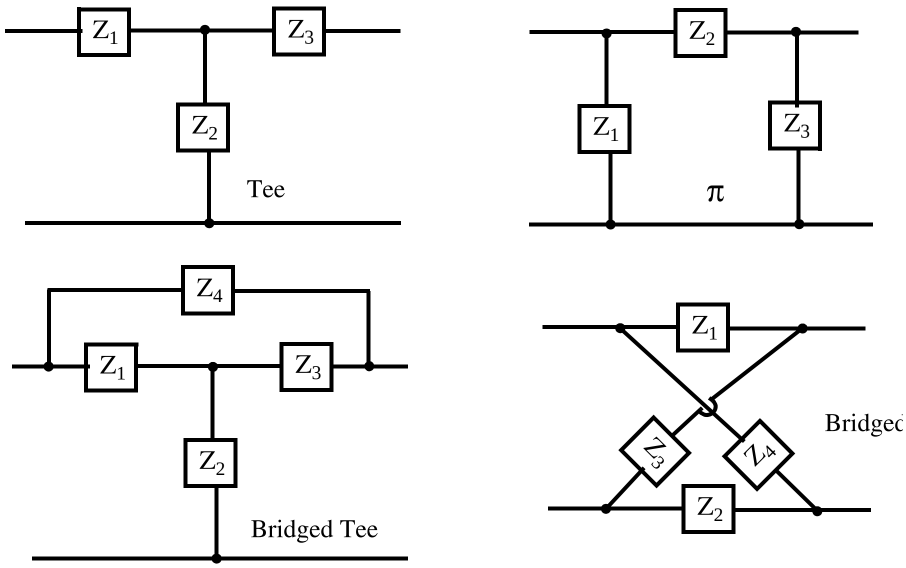


Figure 10.54. Examples of reciprocal two–port networks

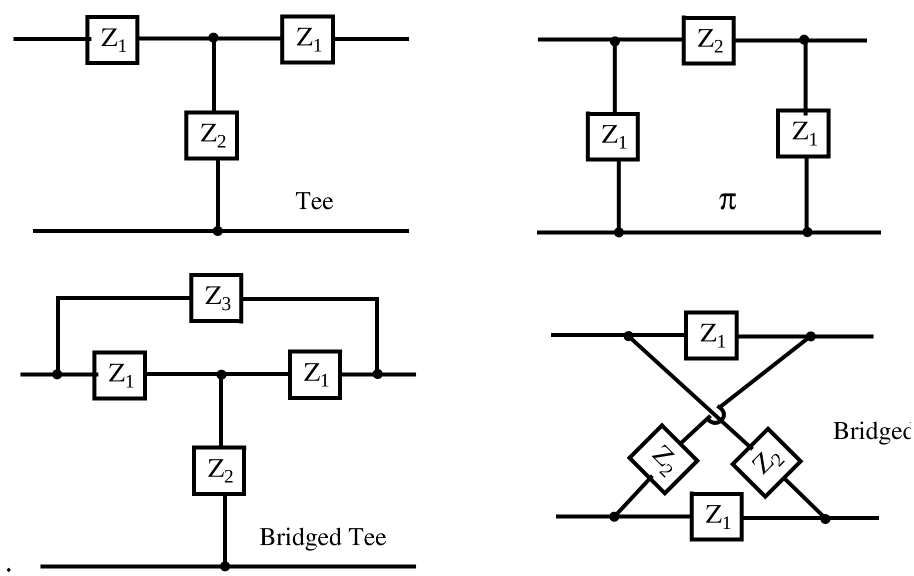


Figure 10.55. Examples of symmetric two–port networks.

Example 10.9

In the two–port network of Figure 10.56, the voltage source v_S connected at the left end of the network is set for $15\ V$, and all impedances are resistive with the values indicated. On the right side of the network is connected a DC ammeter denoted as A. Assume that the ammeter is ideal, that is, has no internal resistance.

a. Compute the ammeter reading.

b. Interchange the positions of the voltage source and recompute the ammeter reading.

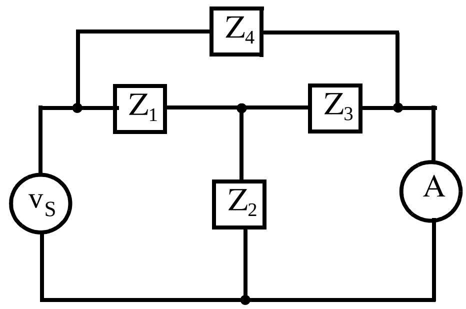


Figure 10.56. Network for Example 10.9.

Solution:

a. Perhaps the easiest method of solution is by nodal analysis since we only need to solve one equation.

The given network is redrawn as shown in Figure 10.57.

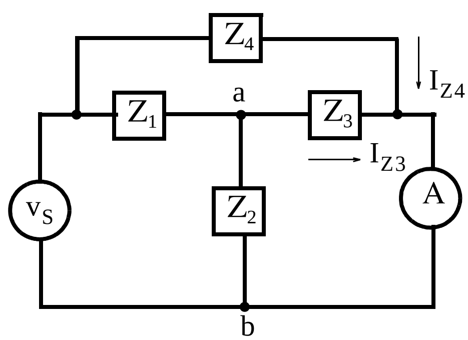


Figure 10.57. Network for solution of Example 10.9 by nodal analysis

By KCL at node a,

$$\frac{V_{ab}-15}{30}+\frac{V_{ab}}{60}+\frac{V_{ab}}{20} = 0$$

or

$$\frac{6}{60}V_{ab} = \frac{15}{30}$$

or

$$V_{ab} = 5\ \mathrm{V}$$

The current through the ammeter is the sum of the currents I_{Z3} and I_{Z4}. Thus, denoting the current through the ammeter as I_A we obtain:

$$I_A = I_{Z3}+I_{Z4} = \frac{V_{ab}}{Z_3}+\frac{V}{Z_4} = \frac{5}{20}+\frac{15}{10} = 0.25+1.50 = 1.75\ \mathrm{A} \tag{10.119}$$

b. With the voltage source and ammeter positions interchanged, the network is as shown in Figure 10.58.

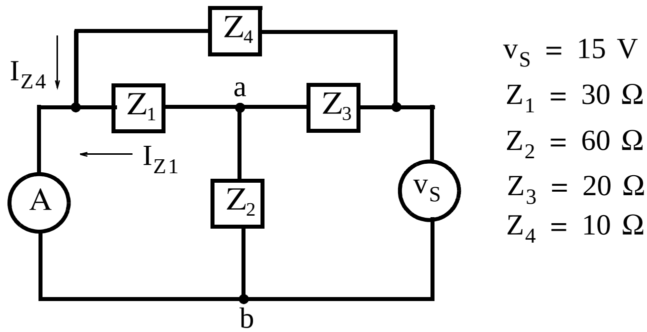


Figure 10.58. Network of Figure 10.57 with the voltage source and ammeter interchanged.

Applying KCL for the network of Figure 10.58, we obtain:

$$\frac{V_{ab}}{30}+\frac{V_{ab}}{60}+\frac{V_{ab}-15}{20} = 0$$

or

$$\frac{6}{60}V_{ab} = \frac{15}{20}$$

or

$$V_{ab} = 7.5\ V$$

The current through the ammeter this time is the sum of the currents I_{Z1} and I_{Z4}. Thus, denoting the current through the ammeter as I_A we obtain:

$$I_A = I_{Z1}+I_{Z4} = \frac{V_{ab}}{Z_1}+\frac{V}{Z_4} = \frac{7.5}{30}+\frac{15}{10} = 0.25+1.50 = 1.75\ A \qquad (10.120)$$

We observe that (10.119) and (10.120 yield the same value and thus we can say that the given network is reciprocal.

10.6 Summary

- A port is a pair of terminals in a network at which electric energy or a signal may enter or leave the network.
- A network that has only one pair a terminals is called a one–port network. In an one–port network, the current that enters one terminal must exit the network through the other terminal.
- A two–port network has two pairs of terminals, that is, four terminals.
- For an $n-\text{port}$ network the y parameters are defined as

$$i_1 = y_{11}v_1 + y_{12}v_2 + y_{13}v_3 + \dots + y_{1n}v_n$$

$$i_2 = y_{21}v_1 + y_{22}v_2 + y_{23}v_3 + \dots + y_{2n}v_n$$

$$i_3 = y_{31}v_1 + y_{32}v_2 + y_{33}v_3 + \dots + y_{2n}v_n$$

and so on.

- If the subscripts of the y–parameters are alike, such as y_{11}, y_{22} and so on, they are referred to as driving–point admittances. If they are unlike, such as y_{12}, y_{21} and so on, they are referred to as transfer admittances.
- For a $2-\text{port}$ network the y parameters are defined as

$$i_1 = y_{11}v_1 + y_{12}v_2$$
$$i_2 = y_{21}v_1 + y_{22}v_2$$

- In a $2-\text{port}$ network where the right port is short–circuited, that is, when $v_2 = 0$, the y_{11} parameter is referred to as the short circuit input admittance. In other words,

$$y_{11} = \left.\frac{i_1}{v_1}\right|_{v_2 = 0}$$

- In a $2-\text{port}$ network where the left port is short–circuited, that is, when $v_1 = 0$, the y_{12} parameter is referred to as the short circuit transfer admittance. In other words,

$$y_{12} = \left.\frac{i_1}{v_2}\right|_{v_1 = 0}$$

- In a $2-\text{port}$ network where the right port is short–circuited, that is, when $v_2 = 0$, the y_{21} parameter is referred to as the short circuit transfer admittance. In other words,

$$y_{21} = \left.\frac{i_2}{v_1}\right|_{v_2 = 0}$$

- In a $2-\text{port}$ network where the left port is short–circuited, that is, when $v_1 = 0$, the y_{22} parameter is referred to as the short circuit output admittance. In other words,

$$y_{22} = \left.\frac{i_2}{v_1}\right|_{v_1 = 0}$$

- For a $n-\text{port}$ network the z parameters are defined as

$$v_1 = z_{11}i_1 + z_{12}i_2 + z_{13}i_3 + \dots + z_{1n}i_n$$

$$v_2 = z_{21}i_1 + z_{22}i_2 + z_{23}i_3 + \dots + z_{2n}i_n$$

$$v_3 = z_{31}i_1 + z_{32}i_2 + z_{33}i_3 + \dots + z_{3n}i_n$$

and so on.

- If the subscripts of the z–parameters are alike, such as z_{11}, z_{22} and so on, they are referred to as driving–point impedances. If they are unlike, such as z_{12}, z_{21} and so on, they are referred to as transfer impedances.

- For a $2-\text{port}$ network the z parameters are defined as

$$v_1 = z_{11}i_1 + z_{12}i_2$$

$$v_2 = z_{21}i_1 + z_{22}i_2$$

- In a $2-\text{port}$ network where the right port is open, that is, when $i_2 = 0$, the z_{11} parameter is referred to as the open circuit input impedance. In other words,

$$z_{11} = \left.\frac{v_1}{i_1}\right|_{i_2 = 0}$$

- In a $2-\text{port}$ network where the left port is open, that is, when $i_1 = 0$, the z_{12} parameter is referred to as the open circuit transfer impedance. In other words,

$$z_{12} = \left.\frac{v_1}{i_2}\right|_{i_1 = 0}$$

- In a $2-\text{port}$ network where the right port is open, that is, when $i_2 = 0$, the z_{21} parameter is referred to as the open circuit transfer impedance. In other words,

$$z_{21} = \left.\frac{v_2}{i_1}\right|_{i_2 = 0}$$

- In a $2-\text{port}$ network where the left port is open, that is, when $i_1 = 0$, the z_{22} parameter is referred to as the open circuit output impedance. In other words,

$$z_{22} = \left.\frac{v_2}{i_2}\right|_{i_1 = 0}$$

- A two–port network can also be described in terms of the h parameters with the equations

$$v_1 = h_{11}i_1 + h_{12}v_2$$

$$i_2 = h_{21}i_1 + h_{22}v_2$$

- The h parameters represent an impedance, a voltage gain, a current gain, and an admittance. For this reason they are called hybrid (different) parameters.
- In a $2-\text{port}$ network where the right port is shorted, that is, when $v_2 = 0$, the h_{11} parameter represents the input impedance of the two–port network. In other words,

$$h_{11} = \left.\frac{v_1}{i_1}\right|_{v_2 = 0}$$

- In a $2-\text{port}$ network where the left port is open, that is, when $i_1 = 0$, the h_{12} parameter represents a voltage gain (or loss) in the two–port network. In other words,

$$h_{12} = \left.\frac{v_1}{v_2}\right|_{i_1 = 0}$$

- In a $2-\text{port}$ network where the right port is shorted, that is, when $v_2 = 0$, the h_{21} parameter represents a current gain (or loss). In other words,

$$h_{21} = \left.\frac{i_2}{i_1}\right|_{v_2 = 0}$$

- In a $2-\text{port}$ network where the left port is open, that is, when $i_1 = 0$, the h_{22} parameter represents an output admittance. In other words,

$$h_{22} = \left.\frac{i_2}{v_2}\right|_{i_1 = 0}$$

- A two–port network can also be described in terms of the g parameters with the equations

$$i_1 = g_{11}v_1 + g_{12}i_2$$

$$v_2 = g_{21}v_1 + g_{22}i_2$$

- The g parameters, also known as inverse hybrid parameters, represent an admittance, a current gain, a voltage gain and an impedance.
- In a $2-\text{port}$ network where the right port is open, that is, when $i_2 = 0$, the g_{11} parameter represents the input admittance of the two–port network. In other words,

$$g_{11} = \left.\frac{i_1}{v_1}\right|_{i_2 = 0}$$

- In a $2-\text{port}$ network where the left port is shorted, that is, when $v_1 = 0$, the g_{12} parameter represents a current gain (or loss) in the two–port network. In other words,

$$g_{12} = \left.\frac{i_1}{i_2}\right|_{v_1 = 0}$$

- In a $2-\text{port}$ network where the right port is open, that is, when $i_2 = 0$, the g_{21} parameter represents a voltage gain (or loss). In other words,

$$g_{21} = \left.\frac{v_2}{v_1}\right|_{i_2 = 0}$$

- In a $2-\text{port}$ network where the left port is shorted, that is, when $v_1 = 0$, the g_{22} parameter represents an output impedance. In other words,

$$g_{22} = \left.\frac{v_2}{i_2}\right|_{v_1 = 0}$$

- The reciprocity theorem states that if a voltage applied in one branch of a linear, two–port passive network produces a certain current in any other branch of this network, the same voltage applied in the second branch will produce the same current in the first branch. The reverse is also true, that is, if current applied at one node produces a certain voltage at another, the same current at the second node will produce the same voltage at the first.

- A two–port network is said to be reciprocal if any of the following relationships exists.

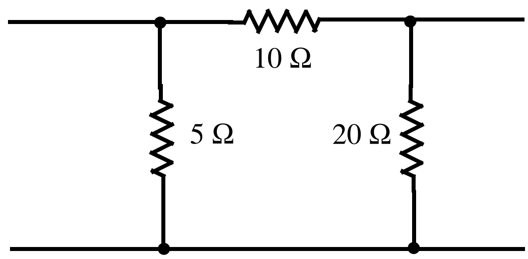
$$z_{21} = z_{12}$$
$$y_{21} = y_{12}$$
$$h_{21} = -h_{12}$$
$$g_{21} = -g_{12}$$

- A two–port network is said to be symmetrical if any of the following relationships exists.

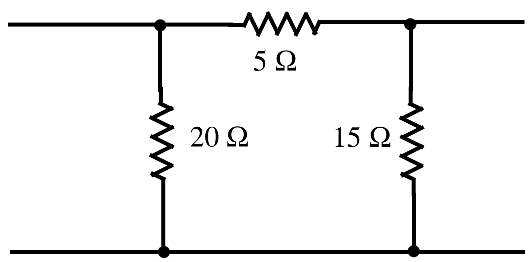
$$z_{21} = z_{12} \text{ and } z_{22} = z_{11}$$
$$y_{21} = y_{12} \text{ and } y_{22} = y_{11}$$
$$h_{21} = -h_{12} \text{ and } h_{11}h_{22} - h_{12}h_{21} = 1$$
$$g_{21} = -g_{12} \text{ and } g_{11}g_{22} - g_{12}g_{21} = 1$$

10.7 Exercises

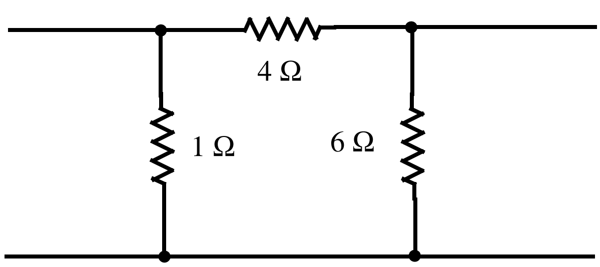
1. For the network below find the z parameters.



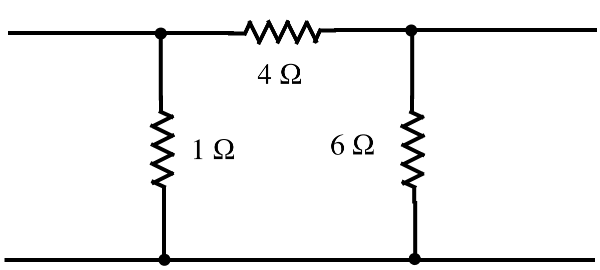
2. For the network below find the y parameters.



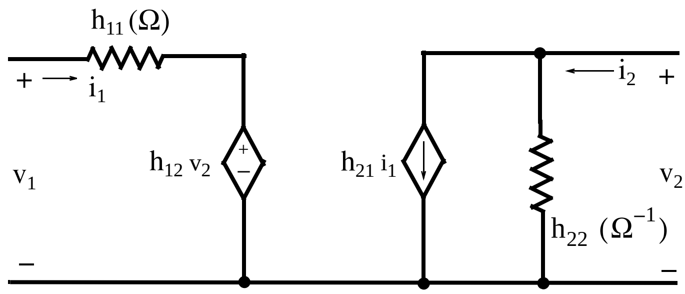
3. For the network below find the h parameters.



4. For the network below find the g parameters.



5. The equations describing the h parameters can be used to represent the network below. This network is a transistor equivalent circuit for the common–emitter configuration and the h parameters given are typical values for such a circuit. Compute the voltage gain and current gain for this network if a voltage source of $v_1 = \cos\omega t \text{ mV}$ in series with $800\ \Omega$ is connected at the input (left side), and a $5\ \text{K}\Omega$ load is connected at the output (right side).



$$h_{11} = 1.2\ \text{K}\Omega$$

$$h_{12} = 2 \times 10^{-4}$$

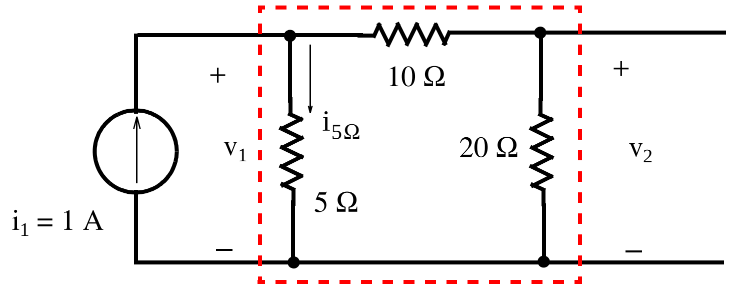
$$h_{21} = 50$$

$$h_{22} = 50 \times 10^{-6}\ \Omega^{-1}$$

10.8 Solutions to End–0f–Chapter Exercises

1.

$$z_{11} = \left.\frac{v_1}{i_1}\right|_{i_2 = 0}$$

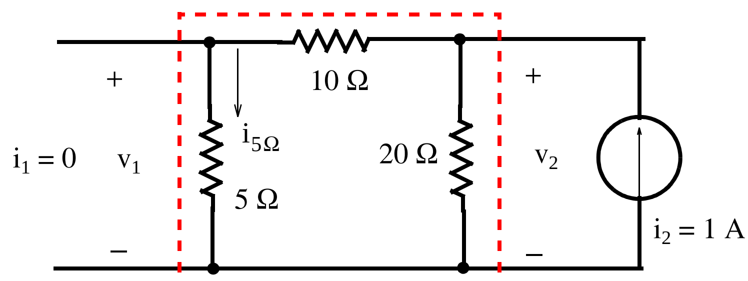


$$i_{5\Omega} = \frac{(10+20)}{(5+10+20)}i_1 = \frac{30}{35}\times 1 = 6/7 \text{ A}$$

$$v_1 = 5i_{5\Omega} = 5\times 6/7 = 30/7 \text{ V}$$

$$z_{11} = \frac{v_1}{i_1} = \frac{30/7}{1} = 30/7\ \Omega$$

$$z_{12} = \left.\frac{v_1}{i_2}\right|_{i_1 = 0}$$

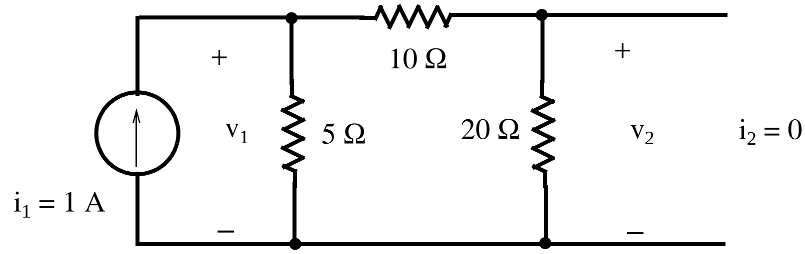


$$i_{5\Omega} = \frac{20}{(20+5+10)}i_2 = \frac{20}{35}\times 1 = 4/7 \text{ A}$$

$$v_1 = 5\times\frac{4}{7} = 20/7 \text{ V}$$

$$z_{12} = \frac{v_1}{i_2} = \frac{20/7}{1} = 20/7\ \Omega$$

$$z_{21} = \left.\frac{v_2}{i_1}\right|_{i_2 = 0}$$



$$i_{20\Omega} = \frac{5}{(5+10+20)}i_1 = \frac{5}{35}\times 1 = 1/7\ \text{A}$$

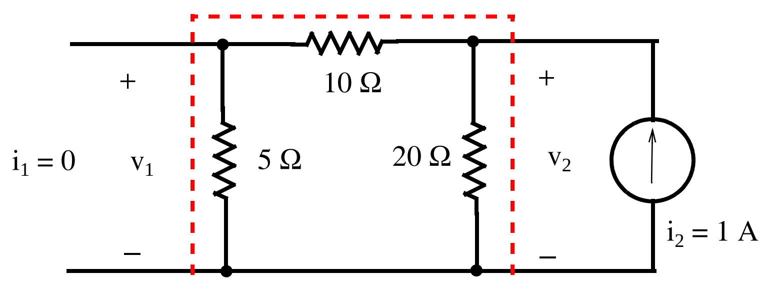
$$v_2 = 20\times\frac{1}{7} = 20/7\ \text{V}$$

$$z_{21} = \frac{v_2}{i_1} = \frac{20/7}{1} = 20/7\ \Omega$$

We observe that

$$z_{21} = z_{12}$$

$$z_{22} = \left.\frac{v_2}{i_2}\right|_{i_1 = 0}$$



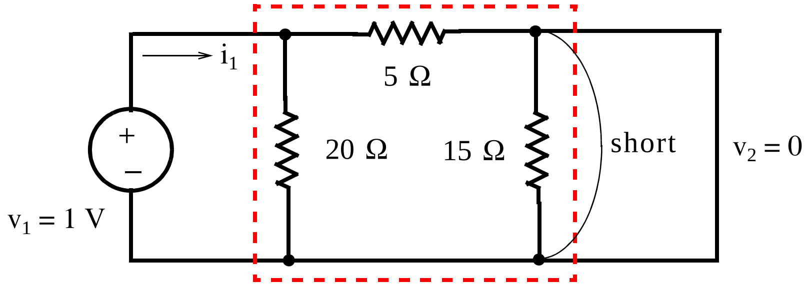
$$i_{20\Omega} = \frac{(10+5)}{(20+10+5)}i_2 = \frac{15}{35}\times 1 = 3/7\ \text{A}$$

$$v_2 = 20\times\frac{3}{7} = 60/7\ \text{V}$$

$$z_{22} = \frac{v_1}{i_2} = \frac{60/7}{1} = 60/7\ \Omega$$

2.

$$y_{11} = \left.\frac{i_1}{v_1}\right|_{v_2 = 0}$$

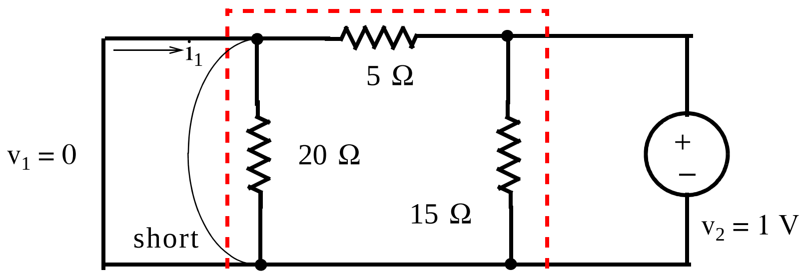


$$R_{eq} = 5 \| 20 = 4\ \Omega$$

$$i_1 = v_1 / R_{eq} = 1/4\ A$$

$$y_{11} = i_1/v_1 = \frac{1/4}{1} = 1/4\ \Omega^{-1}$$

$$y_{12} = \left.\frac{i_1}{v_2}\right|_{v_1 = 0}$$

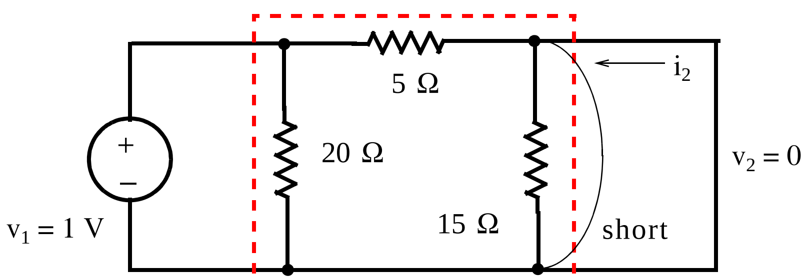


$$v_{5\Omega} = v_2 = 1\ V$$

$$i_1 = -v_{5\Omega}/5 = -1/5\ A$$

$$y_{12} = i_1/v_2 = -1/5/1 = -1/5\ \Omega^{-1}$$

$$y_{21} = \left.\frac{i_2}{v_1}\right|_{v_2 = 0}$$



$$v_{5\Omega} = v_1 = 1\ V$$

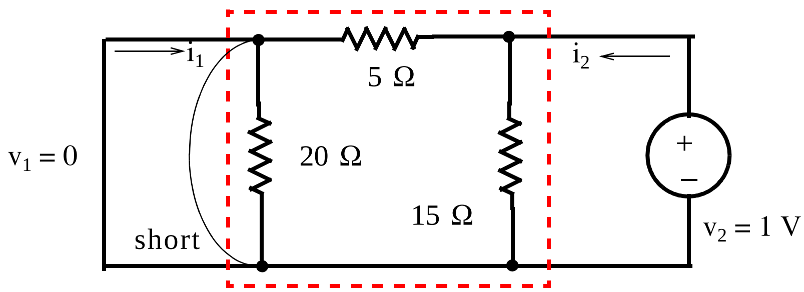
$$i_2 = -v_{5\Omega}/5 = -1/5\ A$$

$$y_{21} = i_2/v_1 = -1/5/1 = -1/5\ \Omega^{-1}$$

We observe that

$$y_{21} = y_{12}$$

$$y_{22} = \left.\frac{i_2}{v_2}\right|_{v_1 = 0}$$

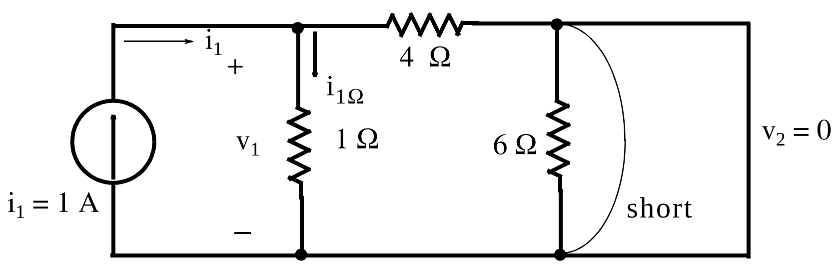


$$i_2 = v_2/R_{eq} = 1/(5 \| 15) = 1/(75/20) = 4/15\ A$$

$$y_{22} = i_2/v_2 = 4/15/1 = 4/15\ \Omega^{-1}$$

3.

$$h_{11} = \left.\frac{v_1}{i_1}\right|_{v_2 = 0}$$

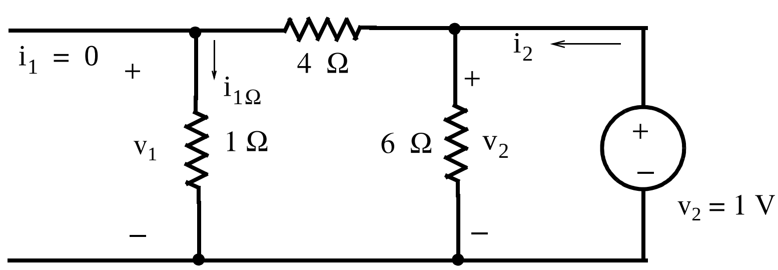


$$i_{1\Omega} = \frac{4}{(1+4)} i_1 = \frac{4}{5} \times 1 = 4/5\ A$$

$$v_1 = 1 \times i_{1\Omega} = 4/5\ V$$

$$h_{11} = \frac{v_1}{i_1} = \frac{4/5}{1} = 4/5\ \Omega$$

$$h_{12} = \left.\frac{v_1}{v_2}\right|_{i_1 = 0}$$

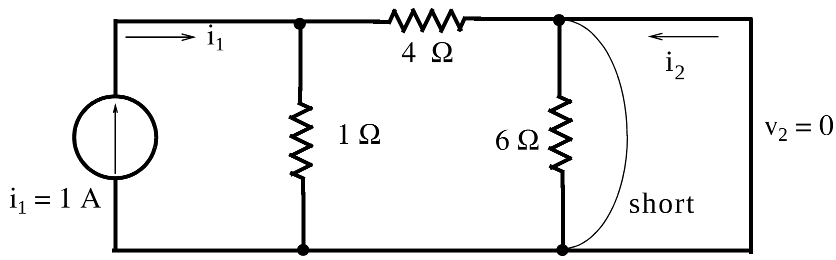


$$i_2 = \frac{v_2}{R_{eq}} = \frac{1}{6 \| (4+1)} = \frac{1}{30/11} = 11/30 \ \text{A}$$

$$v_1 = 1 \times i_{1\Omega} = 1 \times \frac{6}{(6+4+1)} \times i_2 = 1 \times \frac{6}{11} \times \frac{11}{30} = 1/5 \ \text{V}$$

$$h_{12} = \frac{v_1}{v_2} = \frac{1/5}{1} = 1/5 \ \text{(dimensionless)}$$

$$h_{21} = \left.\frac{i_2}{i_1}\right|_{v_2 = 0}$$



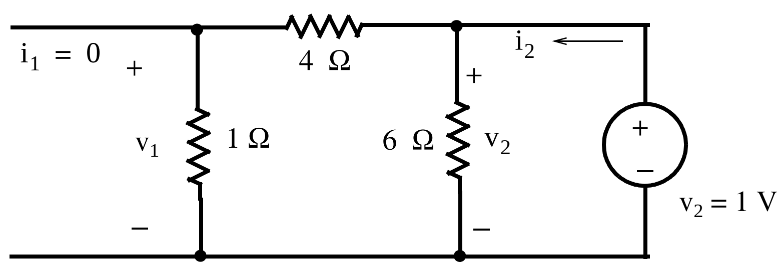
$$i_2 = \frac{1}{(1+4)} \times (-i_1) = \frac{1}{5} \times (-1) = -1/5 \ \text{A}$$

$$h_{21} = \frac{i_2}{i_1} = \frac{-1/5}{1} = -1/5$$

We observe that

$$h_{21} = -h_{12}$$

$$h_{22} = \left.\frac{i_2}{v_2}\right|_{i_1 = 0}$$

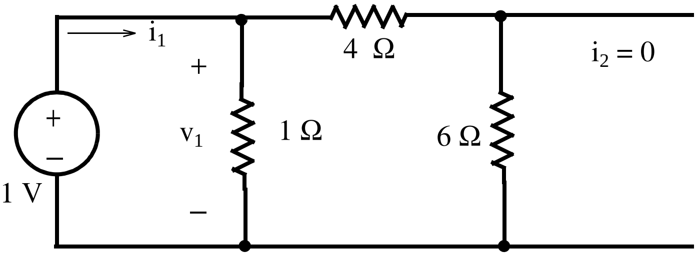


$$i_2 = \frac{v_2}{R_{eq}} = \frac{1}{6 \| (4+1)} = \frac{1}{30/11} = 11/30 \ \text{A}$$

$$h_{22} = \frac{i_2}{v_2} = \frac{11/30}{1} = 11/30 \ \Omega^{-1}$$

4.

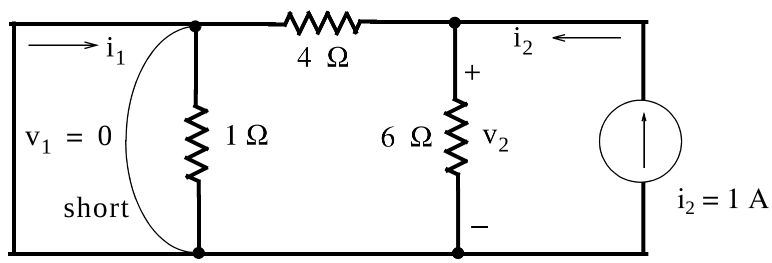
$$g_{11} = \left.\frac{i_1}{v_1}\right|_{i_2=0}$$



$$i_1 = \frac{v_1}{R_{eq}} = \frac{1}{1\,\|\,(4+6))} = \frac{1}{10/11} = 11/10\ \text{A}$$

$$g_{11} = \frac{i_1}{v_1} = \frac{11/10}{1} = 11/10\ \Omega^{-1}$$

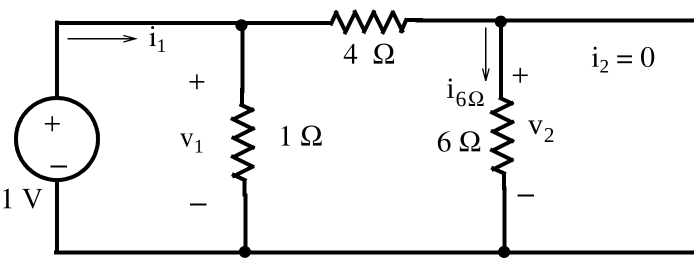
$$g_{12} = \left.\frac{i_1}{i_2}\right|_{v_1=0}$$



$$i_1 = \left(\frac{6}{6+4}\right)(-i_2) = -\frac{6}{10} = -3/5\ \text{A}$$

$$g_{12} = \frac{i_1}{i_2} = \frac{-3/5}{1} = -3/5\ (\text{dimensionless})$$

$$g_{21} = \left.\frac{v_2}{v_1}\right|_{i_2=0}$$



$$i_1 = \frac{v_1}{R_{eq}} = \frac{1}{1\,\|\,(4+6)} = \frac{1}{10/11} = 11/10\ \text{A}$$

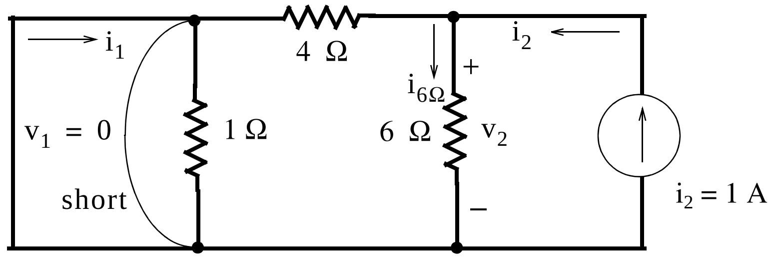
$$v_2 = 6 \times i_{6\Omega} = 6 \times \left(\frac{1}{1+4+6}\cdot\frac{11}{10}\right) = 3/5\ \text{V}$$

$$g_{21} = \frac{v_2}{v_1} = \frac{3/5}{1} = 3/5$$

We observe that

$$g_{21} = -g_{12}$$

$$g_{22} = \left.\frac{v_2}{i_2}\right|_{v_1 = 0}$$



$$v_2 = 6 \times i_{6\Omega} = 6 \times \left(\frac{4}{6+4} \times i_2\right) = \frac{24}{10} \times 1 = 12/5 \text{ V}$$

$$g_{22} = \frac{v_2}{i_2} = \frac{12/5}{1} = 12/5\ \Omega$$

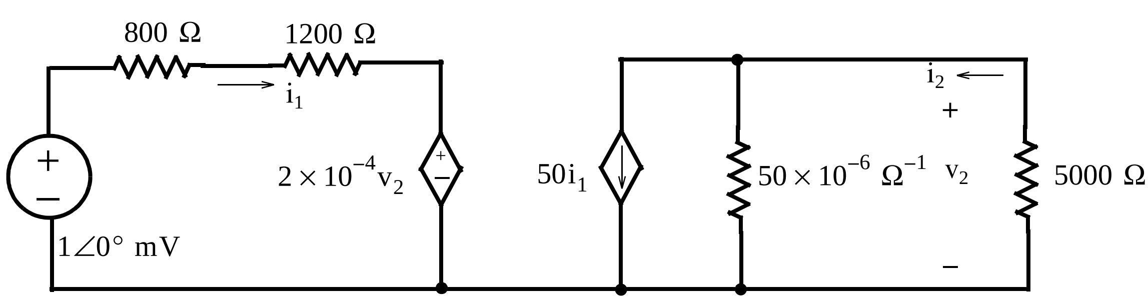
5.

We recall that

$$v_1 = h_{11}i_1 + h_{12}v_2 \quad (1)$$

$$i_2 = h_{21}i_1 + h_{22}v_2 \quad (2)$$

With the voltage source $v_1 = \cos\omega t$ mV in series with $800\ \Omega$ connected at the input and a $5\ K\Omega$ load connected at the output the network is as shown below.



The network above is described by the equations

$$(800 + 1200)i_1 + 2 \times 10^{-4}v_2 = 10^{-3}$$

$$50i_1 + 50 \times 10^{-6}v_2 = i_2 = \frac{-v_2}{5000}$$

or

$$2 \times 10^3 i_1 + 2 \times 10^{-4} v_2 = 10^{-3}$$

$$50 i_1 + 2 \times 10^{-4} v_2 = 0$$

We write the two equations above in matrix form and use MATLAB for the solution.

```
A=[2*10^3  2*10^(–4); 50  2*10^(–4)]; B=[10^(–3) 0]'; X=A\B;...
fprintf(' \n'); fprintf('i1 = %5.2e A \t',X(1)); fprintf('v2 = %5.2e V',X(2))
```

```
i1 = 5.13e-007 A v2 = -1.28e-001 V
```

Therefore,

$$i_1 = 0.513\ \mu A \quad (3)$$

$$v_2 = -128\ mV \quad (4)$$

Next, we use (1) and (2) to find the new values of v_1 and i_2

$$v_1 = 1.2 \times 10^3 \times 0.513 \times 10^{-6} + 2 \times 10^{-4} \times (-128 \times 10^{-3}) = 0.59\ mV$$

$$i_2 = 50 \times 0.513 \times 10^{-6} + 50 \times 10^{-6} \times (-128 \times 10^{-3}) = 19.25\ \mu A$$

The voltage gain is

$$G_V = \frac{v_2}{v_1} = \frac{-128\ mV}{0.59\ mV} = -217$$

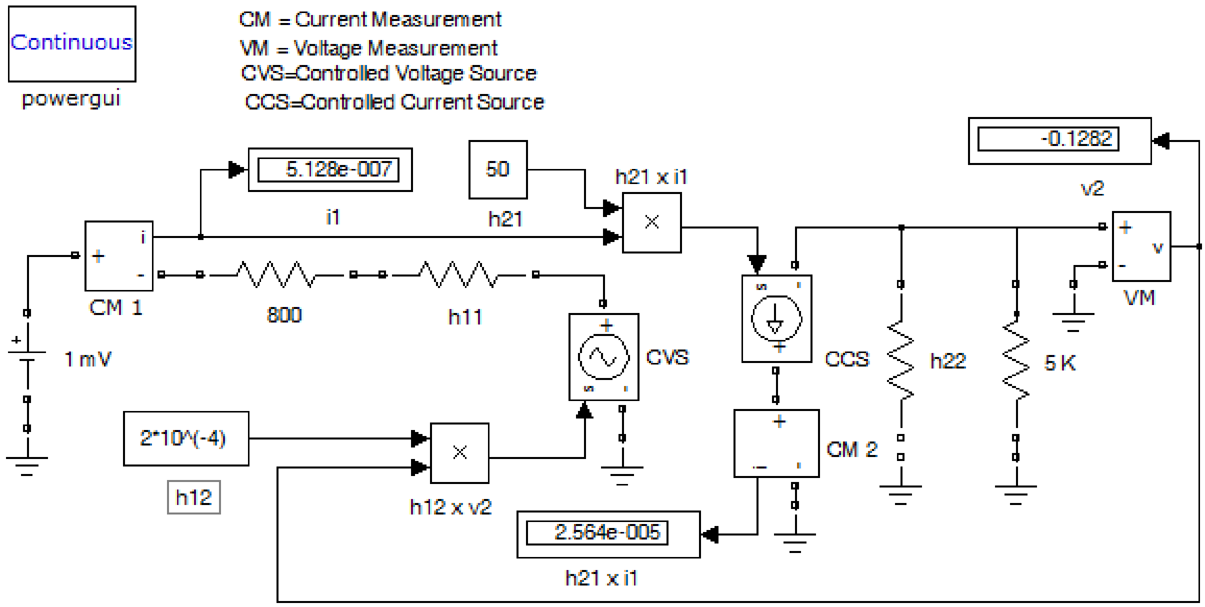
and the minus (–) sign indicates that the output voltage in $180°$ out–of–phase with the input.

The current gain is

$$G_I = \frac{i_2}{i_1} = \frac{19.25\ \mu A}{0.513\ \mu A} = 37.5$$

and the output current is in phase with the input.

The Simulink / SimPowerSystems model for this exercise is shown below.


Continuous
powergui
CM = Current Measurement
VM = Voltage Measurement
CVS=Controlled Voltage Source
CCS=Controlled Current Source
5.128e-007
i1
50
h21
h21 x i1
-0.1282
v2
CM 1
800
h11
CVS
CCS
h22
5 K
VM
1 mV
2*10^(-4)
h12
h12 x v2
CM 2
2.564e-005
h21 x i1

Chapter 11

Balanced Three–Phase Systems

This chapter is an introduction to three–phase power systems. The advantages of three–phase system operation are listed and computations of three phase systems are illustrated by several examples.

11.1 Advantages of Three–Phase Systems

The circuits and networks we have discussed thus far are known as *single–phase* systems and can be either DC or AC. We recall that AC is preferable to DC because voltage levels can be changed by transformers. This allows more economical transmission and distribution. The flow of power in a three–phase system is constant rather than pulsating. Three–phase motors and generators start and run more smoothly since they have constant torque. They are also more economical.

11.2 Three–Phase Connections

Figure 11.1 shows three single AC series circuits where, for simplicity, we have assumed that the internal impedance of the voltage sources and the wiring have been combined with the load impedance. We also have assumed that the voltage sources are $120°$ out–of–phase, the load impedances are the same, and thus the currents I_a, I_b, and I_c have the same magnitude but are $120°$ out–of–phase with each other as shown in Figure 11.2.

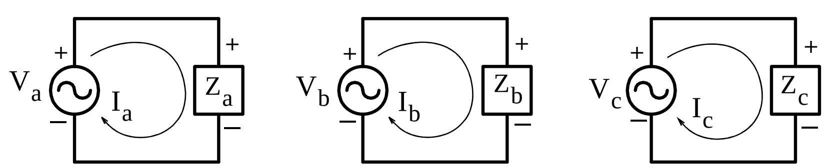


Figure 11.1. Three circuits with $120°$ out–of–phase voltage sources

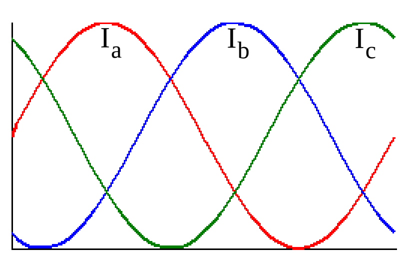


Figure 11.2. Waveforms for three $120°$ out–phase currents

Let us use a single wire for the return current of all three circuits as shown below. This arrangement is known as *four−wire, three−phase system.*

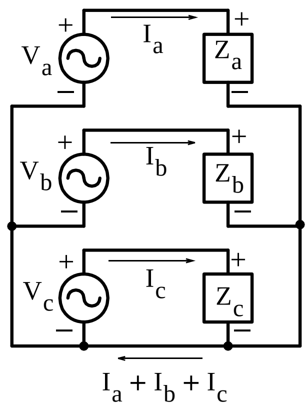


Figure 11.3. Four−wire, three−phase system

This arrangement shown in Figure 11.3 uses only 4 wires instead of the 6 wires shown in Figure 11.1. But now we must find the relative size of the common return wire that it would be sufficient to carry all three currents $I_a + I_b + I_c$

We have assumed that the voltage sources are equal in magnitude and $120°$ apart, and the loads are equal. Therefore, the currents will be *balanced* (equal in magnitude and $120°$ out−of phase). These currents are shown in the phasor diagram of Figure 11.4.

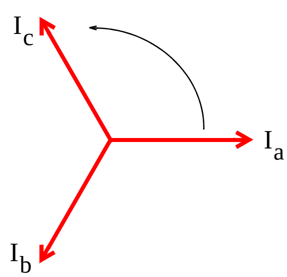


Figure 11.4. Phasor diagram for three−phase balanced system

From figure 11.4 we observe that the sum of these currents, added vectorially, is zero.* Therefore, under ideal (perfect balance) conditions, the common return wire carries no current at all. In a practical situation, however, is not balanced exactly and the sum is not zero. But still it is quite small and in a four−wire three−phase system the return wire is much smaller than the other three.

* *This can also be proved using trigonometric identities, and also the MATLAB statement* `x=sin(t); y=sin(t−2.*pi./3); z=sin(t−4.*pi./3); s=x+y+z`

Figure 11.5 shows a four−wire, three−phase $Y-\text{system}$ where $|V_a| = |V_b| = |V_c|$, the three loads are identical, and I_n is the current in the neutral (fourth) wire.

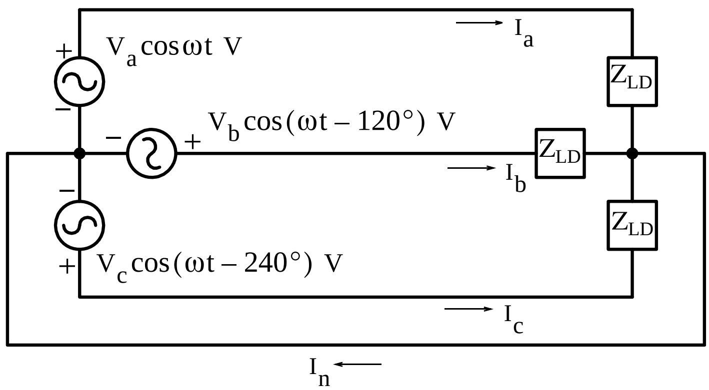


Figure 11.5. Four−wire, three−phase $Y-\text{system}$

A three−wire three−phase $Y-\text{system}$ is shown in Figure11.6 where $|V_a| = |V_b| = |V_c|$, and the three loads are identical.

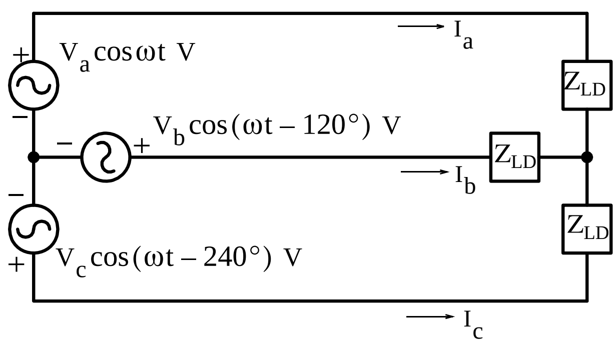


Figure 11.6. Three−wire, three−phase $Y-\text{system}$

This arrangement shown in Figure 11.6 could be used only if all the three voltage sources are perfectly balanced, and if the three loads are perfectly balanced also. This, of course, is a physical impossibility and therefore it is not used.

A three−wire three−phase $\Delta-\text{load}$ system is shown in Figure 11.7 where $|V_a| = |V_b| = |V_c|$, and the three loads are identical. We observe that while the voltage sources are connected as a $Y-\text{system}$, the loads are connected as a $\Delta-\text{system}$ and hence the name $\Delta-\text{load}$

The arrangement in Figure 11.7 offers the advantage that the Δ−connected loads need not be accurately balanced. However, a Δ−connection with only three voltages is not used for safety reasons, that is, it is a safety requirement to have a connection from the common point to the ground.

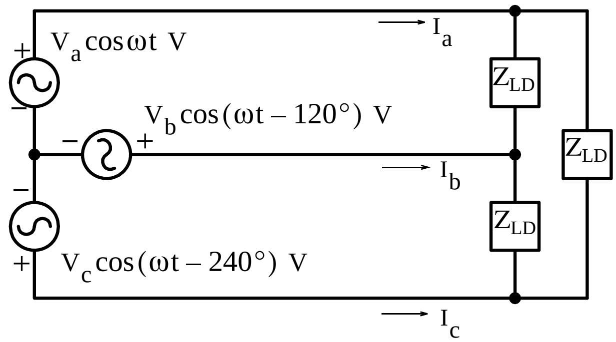


Figure 11.7. Three−wire, three−phase $\Delta - \text{load}$ *system*

11.3 Transformer Connections in Three−Phase Systems

Three−phase power systems use transformers to raise or to lower voltage levels. A typical generator voltage, typically 13.2 KV, is stepped up to hundreds of kilovolts for transmission over long distances. This voltage is then stepped down; for major distribution may be stepped down at a voltage level anywhere between 15 KV to 50 KV, and for local distribution anywhere between 2.4 KV to 12 KV Finally, the electric utility companies furnish power to industrial and commercial facilities at 480 V volts and 120 V and 240 V at residential areas. All voltage levels are in RMS values.

Figure 11.8 shows a bank of three single phase transformers where the primary is Δ−connected, while the secondary is Y−connected. This $\Delta - Y$ connection is typical of transformer installations at generating stations.

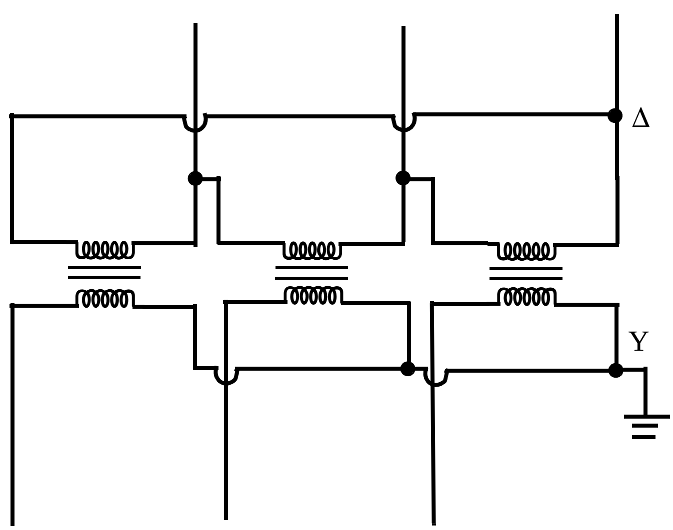


Figure 11.8. Three single−phase transformers use in three−phase systems

Figure 11.9 shows a *single–phase three–wire system* where the middle of the three wires is center–tapped at the transformer secondary winding. As indicated, voltage between the outer wires is 240 V while voltage from either of the two wires to the centered (neutral) wire is 120 V. This arrangement is used in residential areas.

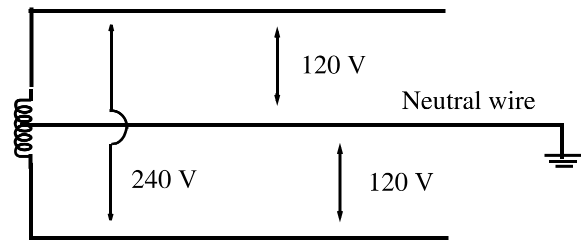


Figure 11.9. 240/120 volt single phase three–wire system

Industrial facilities need three–phase power for three–phase motors. Three–phase motors run smoother and have higher efficiency than single–phase motors. A $Y-\Delta$ connection is shown in Figure 11.10 where the secondary provides 240 V three–phase power to the motor, and one of the transformers of the secondary is center–tapped to provide 120 V to the lighting load.

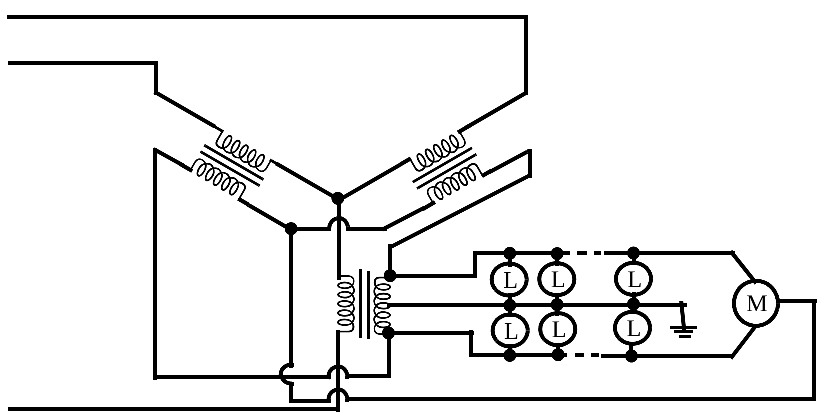


Figure 11.10. Typical 3–phase distribution system

11.4 Line–to–Line and Line–to–Neutral Voltages and Currents

We assume that the perfectly balanced Y–connected load of Figure 11.11 is perfectly balanced, that is, the three loads are identical. We also assume that the applied voltages are $120°$ out–of–phase but they have the same magnitude; therefore there is no current flowing from point n to the ground. The currents I_a, I_b and I_c are referred to as the *line currents* and the currents I_{an}, I_{bn}, and I_{cn} as the *phase currents*. Obviously, in a Y–connected load, the line and phase currents are the same.

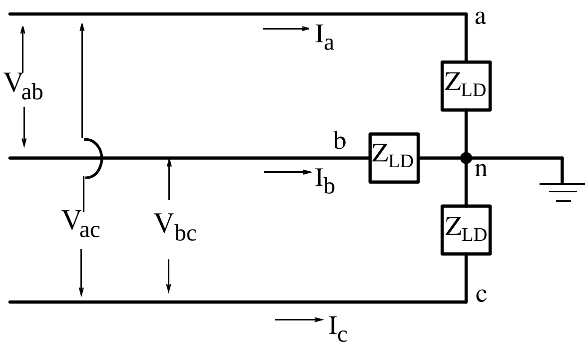


Figure 11.11. Perfectly balanced Y−connected load

Now, we consider the phasor diagram of Figure 11.12.

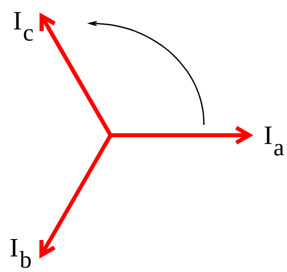


Figure 11.12. Phasor diagram for Y−connected perfectly balanced load

If we choose I_a as our reference, we have

$$I_a = I_a\angle 0° \quad (11.1)$$

$$I_b = I_a\angle -120° \quad (11.2)$$

$$I_c = I_a\angle +120° \quad (11.3)$$

These equations define the balance set of currents of *positive phase sequence* $a - b - c$.

Next, we consider the voltages. Voltages V_{ab}, V_{ac}, and V_{bc} are referred to as *line−to−line voltages* and voltages V_{an}, V_{bn}, and V_{cn} as *phase voltages*. We observe that in a Y−connected load, the line and phase voltages are not the same.

We will now derive the relationships between line and phase voltages in a Y−connected load.

Arbitrarily, we choose V_{an} as our reference phase voltage. Then,

$$V_{an} = V_{an}\angle 0° \quad (11.4)$$

$$V_{bn} = V_{an}\angle{-120°} \quad (11.5)$$

$$V_{cn} = V_{an}\angle{+120°} \quad (11.6)$$

These equations define a *positive phase sequence* $a-b-c$. These relationships are shown in Figure 11.13.

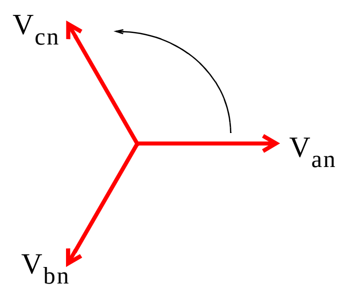


Figure 11.13. Phase voltages in a Y *–connected perfectly balanced load*

The Y –connected load in Figure 1.11 is repeated in Figure 11.14 below for convenience.

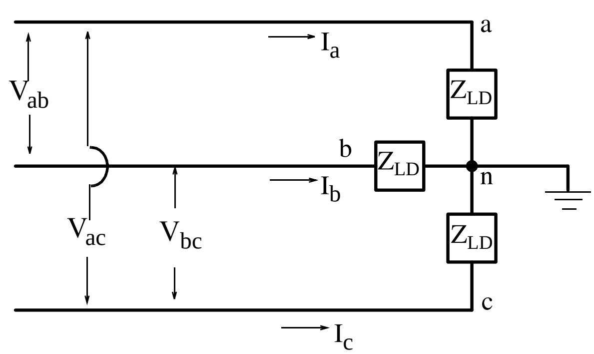


Figure 11.14. Y–connected load

From Figure 11.14

$$V_{ab} = V_{an} + V_{nb} = V_{an} - V_{bn} \quad (11.7)$$

$$V_{ca} = V_{cn} + V_{na} = V_{cn} - V_{an} \quad (11.8)$$

$$V_{bc} = V_{bn} + V_{nc} = V_{bn} - V_{cn} \quad (11.9)$$

These can also be derived from the phasor diagram of Figure 11.15.

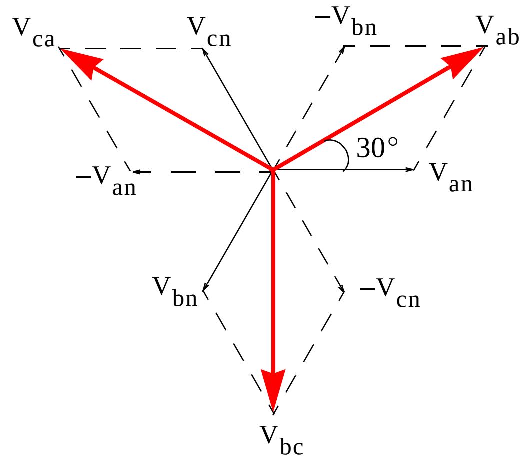


Figure 11.15. Phasor diagram for line−to−line and line−to−neutral voltages in Y load

From geometry and the law of sines we find that in a balanced three−phase, positive phase sequence Y−connected load, the line and phase voltages are related as

$$\boxed{\begin{array}{c} V_{ab} = \sqrt{3}V_{an}\angle 30° \\ Y-\text{connected load} \end{array}} \qquad (11.10)$$

The other two line−to−line voltages can be easily obtained from the phasor diagram in Figure 11.15.

Now, let us consider a Δ−connected load shown in Figure 11.16.

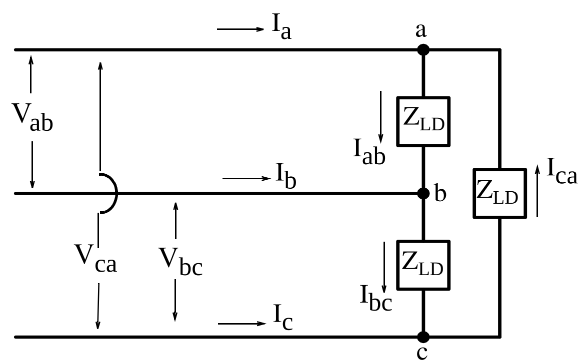


Figure 11.16. Line and phase currents in Δ−connected load

We observe that the line and phase voltages are the same, but the line and phase currents are not the same. To find the relationship between the line and phase currents, we apply KCL at point a and we obtain:

$$I_{ab} = I_a + I_{ca}$$

or

$$I_a = I_{ab} - I_{ca} \tag{11.11}$$

The line currents I_b and I_c are derived similarly, and the phase–to–line current relationship in a Δ–connected load is shown in the phasor diagram of Figure 11.17.

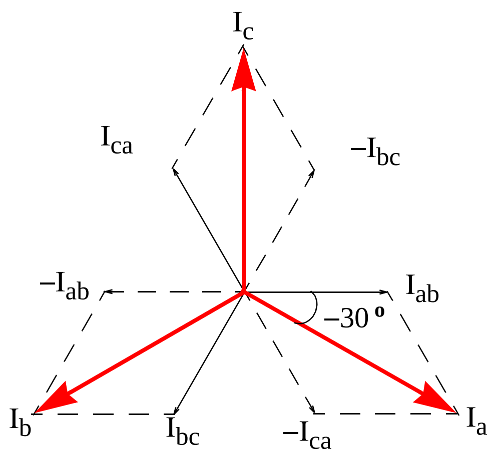


Figure 11.17. Phasor diagram for line and phase currents in Δ–connected load

From geometry and the law of sines we find that a balanced three–phase, positive phase sequence Δ–connected load, the line and phase currents are related as

$$I_a = \sqrt{3} I_{ab} \angle -30° \quad \Delta - \text{connected load} \tag{11.12}$$

The other two line currents can be easily obtained from the phasor diagram of Figure 11.17.

11.5 Equivalent Y and Δ Loads

In this section, we will establish the equivalence between the Y and Δ combinations shown in Figure 11.18.

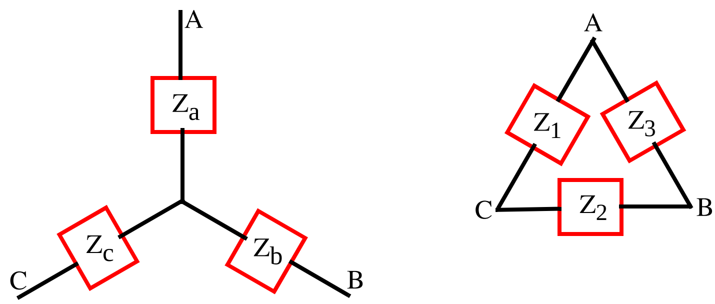


Figure 11.18. Equivalence for Δ and Y−connected loads

In the Y−connection, the impedance between terminals B and C is

$$Z_{BC\ Y} = Z_b + Z_c \tag{11.13}$$

and in the Δ−connection, the impedance between terminals B and C is Z_2 in parallel with the sum $Z_1 + Z_3$, that is,

$$Z_{BC\ \Delta} = \frac{Z_2(Z_1 + Z_3)}{Z_1 + Z_2 + Z_3} \tag{11.14}$$

Equating (11.13) and (11.14) we obtain

$$Z_b + Z_c = \frac{Z_2(Z_1 + Z_3)}{Z_1 + Z_2 + Z_3} \tag{11.15}$$

Similar equations for terminals AB and CA are derived by rotating the subscripts of (11.15) in a cyclical manner. Then,

$$Z_a + Z_b = \frac{Z_3(Z_1 + Z_2)}{Z_1 + Z_2 + Z_3} \tag{11.16}$$

and

$$Z_c + Z_a = \frac{Z_1(Z_2 + Z_3)}{Z_1 + Z_2 + Z_3} \tag{11.17}$$

Equations (11.15) and (11.17) can be solved for Z_a by adding (11.16) with (11.17), subtracting (11.15) from this sum, and dividing by two. That is,

$$2Z_a + Z_b + Z_c = \frac{Z_1Z_3 + Z_2Z_3 + Z_1Z_2 + Z_1Z_3}{Z_1 + Z_2 + Z_3} = \frac{2Z_1Z_3 + Z_2Z_3 + Z_1Z_2}{Z_1 + Z_2 + Z_3} \tag{11.18}$$

$$2Z_a + Z_b + Z_c - Z_b - Z_c = \frac{2Z_1Z_3 + Z_2Z_3 + Z_1Z_2 - Z_1Z_2 - Z_2Z_3}{Z_1 + Z_2 + Z_3} \tag{11.19}$$

$$2Z_a = \frac{2Z_1Z_3}{Z_1 + Z_2 + Z_3} \tag{11.20}$$

$$Z_a = \frac{Z_1Z_3}{Z_1 + Z_2 + Z_3} \tag{11.21}$$

Similar equations for Z_b and Z_c are derived by rotating the subscripts of (11.21) in a cyclical manner. Thus, the three equations that allow us to change any Δ–connection of impedances into a Y–connection are given by (11.22).

$$\boxed{\begin{aligned} Z_a &= \frac{Z_1Z_3}{Z_1 + Z_2 + Z_3} \\ Z_b &= \frac{Z_2Z_3}{Z_1 + Z_2 + Z_3} \\ Z_c &= \frac{Z_1Z_2}{Z_1 + Z_2 + Z_3} \end{aligned}} \tag{11.22}$$

$\Delta \rightarrow Y$ Conversion

Often, we wish to make the conversion in the opposite direction, that is, from Y to Δ.This conversion is performed as follows:

Consider the Y and Δ combinations of Figure 11.8 repeated for convenience as Figure 11.19.

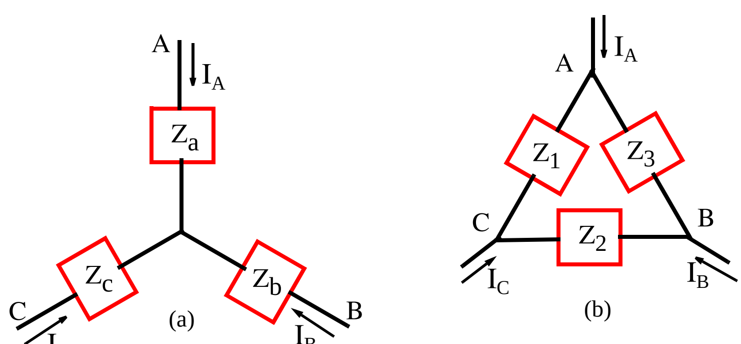


Figure 11.19. Y and Δ loads

From Figure (a),

$$V_{AB} = Z_aI_A - Z_bI_B \tag{11.23}$$

$$V_{BC} = Z_bI_B - Z_cI_C \tag{11.24}$$

$$V_{CA} = Z_cI_C - Z_aI_A \tag{11.25}$$

If we attempt to solve equations (11.23), (11.24) and (11.25) simultaneously, we will find that the determinant Δ of these sets of equations is singular, that is, $\Delta = 0$. This can be verified with Cramer's rule as follows:

$$\begin{aligned} Z_a I_A - Z_b I_B + 0 &= V_{AB} \\ 0 + Z_b I_B - Z_c I_C &= V_{BC} \\ -Z_a I_A + 0 + Z_c I_C &= V_{CA} \end{aligned} \tag{11.26}$$

$$\Delta = \begin{bmatrix} Z_a & -Z_b & 0 \\ 0 & Z_b & -Z_c \\ -Z_a & 0 & Z_c \end{bmatrix} = Z_a Z_b Z_c - Z_a Z_b Z_c + 0 + 0 + 0 + 0 = 0 \tag{11.27}$$

This result suggests that the equations of (11.26) are not independent and therefore, no solution exists. However, a solution can be found if, in addition to (11.23) through (11.25), we use the equation

$$I_A + I_B + I_C = 0 \tag{11.28}$$

Solving (11.28) for I_C we obtain:

$$I_C = -I_A - I_B \tag{11.29}$$

and by substitution into (11.25),

$$V_{CA} = -Z_c I_A - Z_c I_B - Z_a I_A = -(Z_a + Z_c) I_A - Z_c I_B \tag{11.30}$$

From (11.23) and (11.30),

$$\begin{aligned} Z_a I_A - Z_b I_B &= V_{AB} \\ -(Z_a + Z_c) I_A - Z_c I_B &= V_{CA} \end{aligned} \tag{11.31}$$

and by Cramer's rule,

$$I_A = \frac{D_1}{\Delta} \qquad I_B = \frac{D_2}{\Delta} \tag{11.32}$$

where

$$\Delta = \begin{bmatrix} Z_a & -Z_b \\ -(Z_a + Z_c) & -Z_c \end{bmatrix} = -Z_c Z_a - Z_a Z_b - Z_b Z_c \tag{11.33}$$

and

$$D_1 = \begin{bmatrix} V_{AB} & -Z_b \\ V_{CA} & -Z_c \end{bmatrix} = -Z_c V_{AB} + Z_b V_{CA} \tag{11.34}$$

Then,

$$I_A = \frac{D_1}{\Delta} = \frac{-Z_c V_{AB} + Z_b V_{CA}}{-Z_a Z_b - Z_b Z_c - Z_c Z_a} = \frac{Z_c V_{AB} - Z_b V_{CA}}{Z_a Z_b + Z_b Z_c + Z_c Z_a} \tag{11.35}$$

Similarly,

$$I_B = \frac{D_2}{\Delta} = \frac{Z_a V_{BC} - Z_c V_{AB}}{Z_a Z_b + Z_b Z_c + Z_c Z_a} \tag{11.36}$$

and by substitution of I_A and I_B into (11.28),

$$I_C = \frac{Z_b V_{CA} - Z_a V_{BC}}{Z_a Z_b + Z_b Z_c + Z_c Z_a} \tag{11.37}$$

Therefore, for the Y–connection which is repeated in Figure 11.20 for convenience, we have:

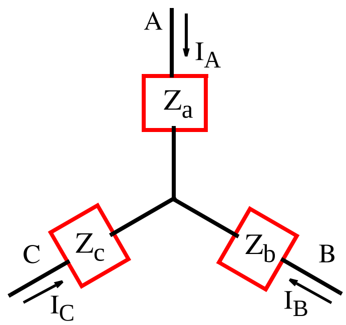


Figure 11.20. Currents in Y–connection

$$\begin{aligned} I_A &= \frac{Z_c V_{AB} - Z_b V_{CA}}{Z_a Z_b + Z_b Z_c + Z_c Z_a} \\ I_B &= \frac{Z_a V_{BC} - Z_c V_{AB}}{Z_a Z_b + Z_b Z_c + Z_c Z_a} \\ I_C &= \frac{Z_b V_{CA} - Z_a V_{BC}}{Z_a Z_b + Z_b Z_c + Z_c Z_a} \end{aligned} \tag{11.38}$$

For the Δ–connection, which is also repeated in Figure 11.21 for convenience, the line currents are:

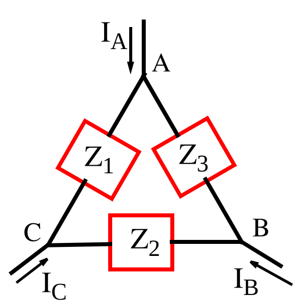


Figure 11.21. Currents in Δ–connection

$$I_A = \frac{V_{AB}}{Z_3} - \frac{V_{CA}}{Z_1}$$
$$I_B = \frac{V_{BC}}{Z_2} - \frac{V_{AB}}{Z_3} \qquad (11.39)$$
$$I_C = \frac{V_{CA}}{Z_1} - \frac{V_{BC}}{Z_2}$$

Now, the sets of equations of (11.38) and (11.39) are equal if

$$\frac{Z_c V_{AB} - Z_b V_{CA}}{Z_a Z_b + Z_b Z_c + Z_c Z_a} = \frac{V_{AB}}{Z_3} - \frac{V_{CA}}{Z_1} \qquad (11.40)$$

$$\frac{Z_a V_{BC} - Z_c V_{AB}}{Z_a Z_b + Z_b Z_c + Z_c Z_a} = \frac{V_{BC}}{Z_2} - \frac{V_{AB}}{Z_3} \qquad (11.41)$$

$$\frac{Z_b V_{CA} - Z_a V_{BC}}{Z_a Z_b + Z_b Z_c + Z_c Z_a} = \frac{V_{CA}}{Z_1} - \frac{V_{BC}}{Z_2} \qquad (11.42)$$

From (11.40)

$$\frac{Z_c}{Z_a Z_b + Z_b Z_c + Z_c Z_a} = \frac{1}{Z_3} \quad \text{and} \quad \frac{Z_b}{Z_a Z_b + Z_b Z_c + Z_c Z_a} = \frac{1}{Z_1} \qquad (11.43)$$

and from (11.41)

$$\frac{Z_a}{Z_a Z_b + Z_b Z_c + Z_c Z_a} = \frac{1}{Z_2} \qquad (11.44)$$

Rearranging, we obtain:

$$Z_1 = \frac{Z_a Z_b + Z_b Z_c + Z_c Z_a}{Z_b}$$
$$Z_2 = \frac{Z_a Z_b + Z_b Z_c + Z_c Z_a}{Z_a} \qquad (11.45)$$
$$Z_3 = \frac{Z_a Z_b + Z_b Z_c + Z_c Z_a}{Z_c}$$

$Y \rightarrow \Delta$ Conversion

Example 11.1

For the circuit of Figure 11.22, use the $Y \rightarrow \Delta$ conversion to find the currents in the various branches as indicated.*

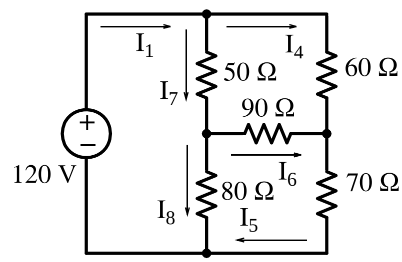


Figure 11.22. Circuit (a) for Example 11.1

Solution:

Let us indicate the nodes as a, b, c, and d, and denote the 90 Ω, 80 Ω, and 50 Ω resistances as R_a, R_b, and R_c respectively as shown in Figure 11.23.

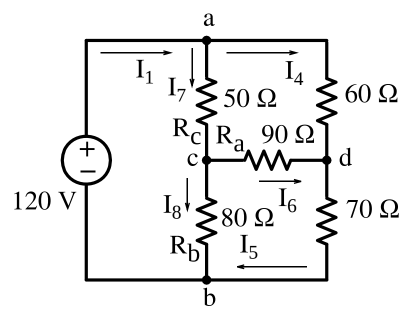


Figure 11.23. Circuit (b) for Example 11.1

Next, we replace the Y connection formed by a, b, c, and d with the equivalent Δ connection shown in Figure 11.24.

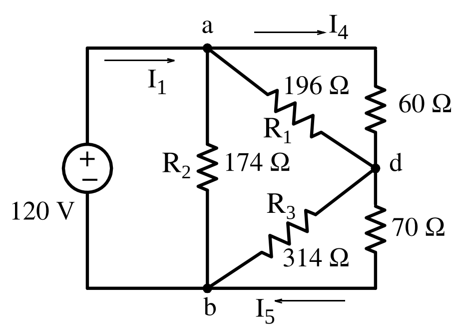


Figure 11.24. Circuit (c) for Example 11.1

Now, with reference to the circuits of Figures 11.23 and 11.24, and the relations of (11.45), we obtain:

* *The subscripts are assigned to be consistent with those in the solution steps.*

$$R_1 = \frac{R_aR_b + R_bR_c + R_cR_a}{R_b} = \frac{90 \times 80 + 80 \times 50 + 50 \times 90}{80} = \frac{15700}{80} \approx 196\ \Omega$$

$$R_2 = \frac{R_aR_b + R_bR_c + R_cR_a}{R_a} = \frac{15700}{90} \approx 174\ \Omega$$

$$R_3 = \frac{R_aR_b + R_bR_c + R_cR_a}{R_c} = \frac{15700}{50} = 314\ \Omega$$

Combination of parallel resistances in the circuit of Figure 11.24 yields

$$R_{bd} = \frac{196 \times 60}{196 + 60} \approx 46\ \Omega$$

and

$$R_{ad} = \frac{314 \times 70}{314 + 70} \approx 57\ \Omega$$

The circuit of Figure 11.24 reduces to the circuit in Figure 11.25.

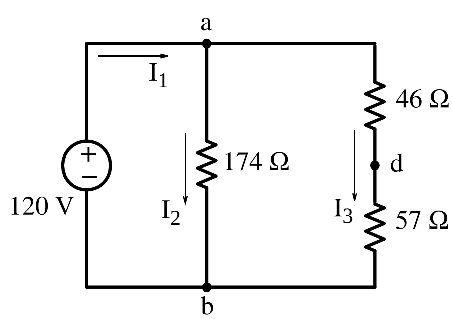


Figure 11.25. Circuit (d) for Example 11.1

The circuit of Figure 11.25 can be further simplified as shown in Figure 11.26.

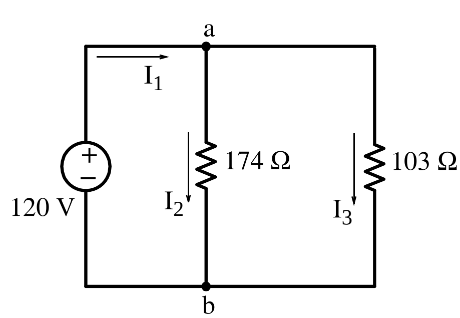


Figure 11.26. Circuit (e) for Example 11.1

From the circuit of Figure 11.26,

$$I_2 = \frac{120}{174} = 0.69\ A \tag{11.46}$$

$$I_3 = \frac{120}{103} = 1.17\ A \tag{11.47}$$

By addition of (11.46) and (11.47)

$$I_1 = I_2 + I_3 = 0.69 + 1.17 = 1.86 \tag{11.48}$$

To compute the other currents, we return to the circuit of Figure 11.25 which, for convenience, is repeated as Figure 11.27 and it is denoted as Circuit (f).

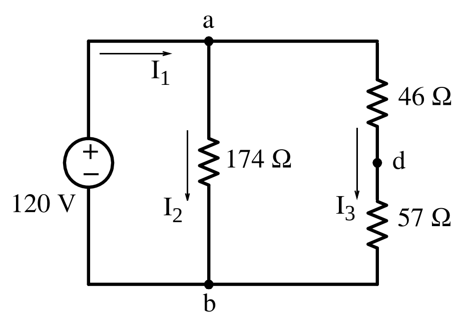


Figure 11.27. Circuit (f) for Example 11.1

For the circuit of Figure 11.27, by the voltage division expression

$$V_{ad} = \frac{46}{46+57} \times 120 = 53.6\ V \tag{11.49}$$

$$V_{db} = \frac{57}{46+57} \times 120 = 66.4\ V \tag{11.50}$$

Next, we return to the circuit of Figure 11.24 which, for convenience, is repeated as Figure 11.28 and denoted as Circuit (g).

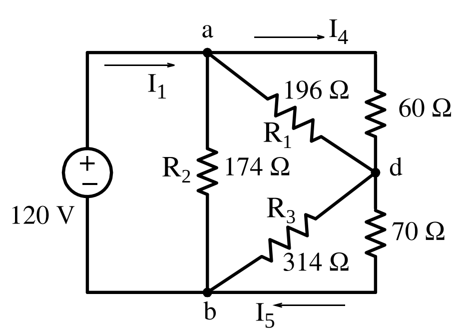


Figure 11.28. Circuit (g) for Example 11.1

From the circuit of figure 11.28,

$$I_4 = \frac{V_{ad}}{70} = \frac{53.6}{60} = 0.95\ \text{A} \tag{11.51}$$

and

$$I_5 = \frac{V_{db}}{60} = \frac{66.4}{70} = 0.89\ \text{A} \tag{11.52}$$

Finally, we return to the circuit of Figure 11.23 which, for convenience, is repeated as Figure 11.29 and denoted as Circuit (h).

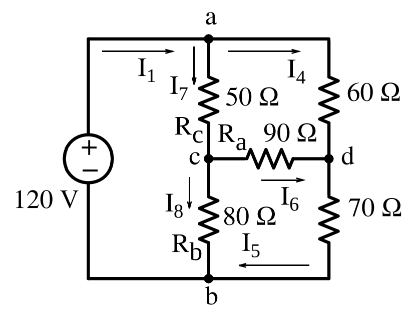


Figure 11.29. Circuit (h) for Example 11.1

For the circuit of Figure 11.29, by KCL,

$$I_7 = I_1 - I_4 = 1.86 - 0.95 = 0.91\ \text{A} \tag{11.53}$$

$$I_8 = I_1 - I_5 = 1.86 - 0.89 = 0.97\ \text{A} \tag{11.54}$$

and

$$I_6 = I_5 - I_4 = 0.89 - 0.95 = -0.06\ \text{A} \tag{11.55}$$

Of course, we could have found the branch currents with nodal or mesh analysis.

Quite often, the Y and Δ arrangements appear as shown in Figure 11.30 and they are referred to as the tee (T) and pi (π) circuits. Consequently, the formulas we developed for the Y and Δ arrangements can be used with the tee and π arrangements.

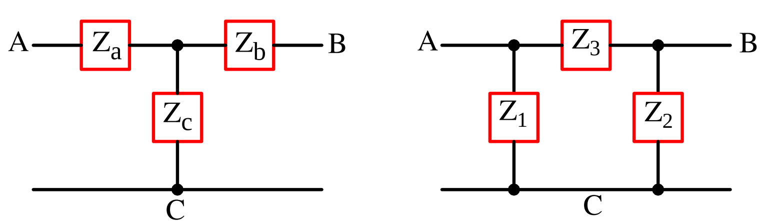


Figure 11.30. T and π circuits

In communications theory, the T and π circuits are symmetrical, i.e., $Z_a = Z_b$ and $Z_1 = Z_2$.

11.6 Computation by Reduction to Single Phase

When we want to compute the voltages, currents, and power in a balanced three–phase system, it is very convenient to use the Y–connection and work with one phase only. The other phases will have corresponding quantities (voltage, current, and power) exactly the same except for a time difference of 1/3 cycle. Thus, if current is found for phase a, the current in phase b will be 120° out–of–phase but it will have the same magnitude as phase a. Likewise, phase c will be 240° out–of–phase with phase a.

If the load happens to be Δ–connected, we use the $\Delta \rightarrow Y$ conversion shown in Figure 11.31 and the equations (11.56) below.

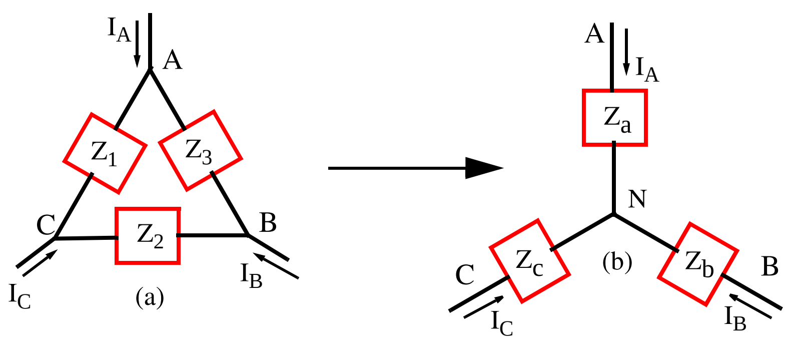


Figure 11.31. $\Delta \rightarrow Y$ conversion

$$Z_a = \frac{Z_1 Z_3}{Z_1 + Z_2 + Z_3} \qquad Z_b = \frac{Z_2 Z_3}{Z_1 + Z_2 + Z_3} \qquad Z_c = \frac{Z_1 Z_2}{Z_1 + Z_2 + Z_3} \tag{11.56}$$

$\Delta \rightarrow Y$ Conversion

Since the system is assumed to be balanced, the loads are equal, that is, $Z_1 = Z_2 = Z_3$ and $Z_a = Z_b = Z_c$. Therefore, the first equation in (11.56) reduces to:

$$Z_a = \frac{Z_1 Z_3}{Z_1 + Z_2 + Z_3} = \frac{Z_1^2}{3Z_1} = \frac{Z_1}{3} \tag{11.57}$$

and the same is true for the other phases.

11.7 Three−Phase Power

We can compute the power in a single phase and then multiply by three to find the total power in a three−phase system. Therefore, if a load is Y−connected, as in Figure 11.31 (b), the total three−phase power is given by

$$\boxed{P_{total} = 3|V_{AN}||I_A|\cos\theta} \qquad (11.58)$$
$$Y - \text{connected load}$$

where V_{AN} is the *line−to−neutral voltage*, I_A is the *line current*, $\cos\theta$ is the *power factor* of the load, and θ is the angle between V_{AN} and I_A.

If the load is Δ−connected as in Figure 11.31 (a), the total three−phase power is given by

$$\boxed{P_{total} = 3|V_{AB}||I_{AB}|\cos\theta} \qquad (11.59)$$
$$\Delta - \text{connected load}$$

We observe that relation (11.59) is given in terms of the line−to−neutral voltage and line current, and relation (11.58) in terms of the line−to−line voltage and phase current.
Quite often, the line−to−line voltage and line current of a three−phase systems are given. In this case, we substitute (11.12), i.e., $|I_A| = \sqrt{3}|I_{AB}|$ into (11.59) and we obtain

$$\boxed{P_{total} = \sqrt{3}|V_{AB}||I_A|\cos\theta_{LD}} \qquad (11.60)$$
$$Y \text{ or } \Delta - \text{connected load}$$

It is important to remember that the power factor $\cos\theta_{LD}$ in (11.60) refers to the load, that is, the angle θ is not the angle between V_{AB} and I_A.

Example 11.2

The three−phase generator of Figure 11.32 supplies $100\ kW$ at 0.9 lagging power factor to the three−phase load. The line−to−line voltage at the load is $2400\ V$. The resistance of the line is $4\ \Omega$ per conductor and the inductance and capacitance are negligible. What line−to−line voltage must the generator supply to the line?

Solution:

The load per phase at $0.9\ pf$ is

$$\frac{1}{3} \times 100 = 33.33\ kW$$

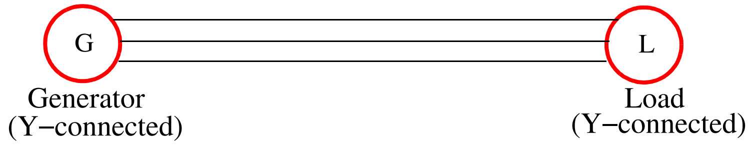


Figure 11.32. Circuit for Example 11.2

From (11.10),

$$V_{ab} = \sqrt{3}V_{an}\angle 30° \quad \text{Y − connected load} \tag{11.61}$$

Then, the magnitude of the line−to−neutral at the load end is

$$|V_{an\ load}| = \frac{|V_{ab\ load}|}{\sqrt{3}} = \frac{2400}{\sqrt{3}} = 1386\ \text{V} \tag{11.62}$$

and the KVA per phase at the load is

$$\frac{kW/phase}{pf} = \frac{33.33}{0.9} = 37.0\ \text{KVA} \tag{11.63}$$

The line current in each of the three conductors is

$$I_{line} = \frac{VA}{|V_{an\ load}|} = \frac{37000}{1386} = 26.7\ \text{A} \tag{11.64}$$

and the angle by which the line (or phase) current lags the phase voltage is

$$\theta = \cos^{-1}0.9 = 25.84° \tag{11.65}$$

Next, let us assume that the line current in phase a lies on the real axis. Then, the phasor of the line−to−neutral voltage at the load end is

$$V_{an\ load} = |V_{an}|\angle 25.84° = 1386(\cos 25.84° + j\sin 25.84°) = 1247 + j604\ \text{V} \tag{11.66}$$

The voltage drop across a conductor is in phase with the line current since it resistive in nature. Therefore,

$$V_{cond} = I_{line} \times R = 26.7 \times 4 = 106.8\ \text{V} \tag{11.67}$$

Now, the phasor line−to−neutral voltage at the generator end is

$$V_{an\ gen} = V_{an\ load} + V_{cond} = 1247 + j604 + 106.8 = 1354 + j604 \tag{11.68}$$

and its magnitude is

$$|V_{an\ gen}| = \sqrt{1354^2 + 604^2} = 1483\ \text{V} \tag{11.69}$$

Finally, the line–to–line voltage at the generator end is

$$|V_{line-line\ gen}| = \sqrt{3}\times|V_{an\ gen}| = \sqrt{3}\times 1483 = 2569\ V \tag{11.70}$$

11.8 Instantaneous Power in Three–Phase Systems

A significant advantage of a three–power system is that the total power in a balanced three–phase system is constant. This is proved as follows:

We assume that the load is purely resistive. Therefore, the voltage and current are always in–phase with each other. Now, let V_p and I_p be the peak (maximum) voltage and current respectively, and $|V|$ and $|I|$ the magnitude of their RMS values. Then, the instantaneous voltage and current in phase a are given by

$$v_a = V_p\cos\omega t = \sqrt{2}|V|\cos\omega t \tag{11.71}$$

$$i_a = I_p\cos\omega t = \sqrt{2}|I|\cos\omega t \tag{11.72}$$

Multiplication of (11.71) and (11.72) yields the instantaneous power, and using the trigonometric identity

$$\cos^2\omega t = (\cos 2\omega t + 1)/2 \tag{11.73}$$

we obtain

$$p_a = v_a i_a = 2|V||I|\cos^2\omega t = |V||I|(\cos 2\omega t + 1) \tag{11.74}$$

The voltage and current in phase b are equal in magnitude to those in phase a but they are $120°$ out–of–phase. Then,

$$v_b = \sqrt{2}|V|\cos(\omega t - 120°) \tag{11.75}$$

$$i_b = \sqrt{2}|I|\cos(\omega t - 120°) \tag{11.76}$$

$$p_b = v_b\cdot i_b = 2|V||I|\cos^2(\omega t - 120°) = |V||I|[\cos(2\omega t - 240°) + 1] \tag{11.77}$$

Similarly, the power in phase c is

$$p_c = v_c\cdot i_c = 2|V||I|\cos^2(\omega t - 240°) = |V||I|[\cos(2\omega t - 480°) + 1] \tag{11.78}$$

and the total instantaneous power is

$$p_{total} = p_a + p_b + p_c$$
$$= |V||I|[\cos 2\omega t + \cos(2\omega t - 240°) + \cos(2\omega t - 480°) + 3] \quad (11.79)$$

Recalling that

$$\cos(x - y) = \cos x \cos y + \sin x \sin y \quad (11.80)$$

we find that the sum of the three cosine terms in (11.79) is zero. Then,

$$\boxed{p_{total} = 3|V||I|} \quad (11.81)$$
$$\text{Three − phase Balanced System}$$

Therefore, the *instantaneous* total power is constant and it is equal three times the *average* power.

The proof can be extended to include any power factor; thus, (11.81) can be also expressed as

$$p_{total} = 3|V||I|\cos\theta \quad (11.82)$$

Example 11.3

Figure 11.33 shows a three−phase feeder with two loads; one consists of a bank of lamps connected line−to neutral and the rating is given in the diagram; the other load is Δ−connected and has the impedance shown. Find the current in the feeder lines and the total power absorbed by the two loads.

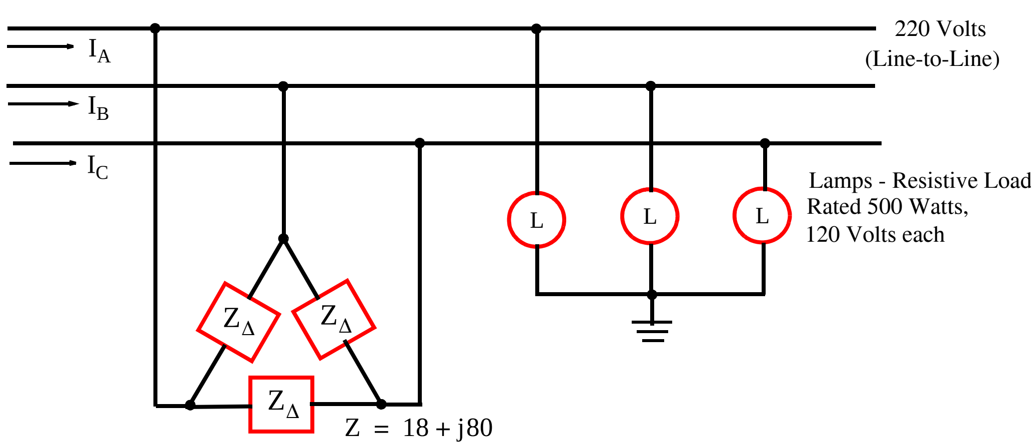


Figure 11.33. Diagram for Example 11.3

Solution:

To facilitate the computations, we will reduce the given circuit to one phase (phase a) taken as reference, i.e., at zero degrees, as shown in Figure 11.34.

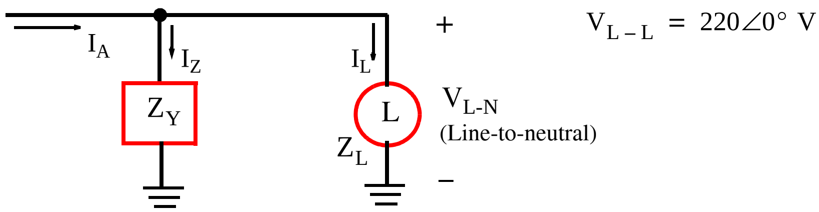


Figure 11.34. Single−phase representation of Figure 11.33

We first compute the impedance Z_Y. Using (11.56),

$$Z_Y = \frac{Z_\Delta}{3} = \frac{18 + j80}{3} = \frac{82\angle 77.32}{3} = 27.33\angle 77.32\ \Omega$$

Next, we compute the lamp impedance Z_L.

$$Z_L = R_{lamp} = \frac{V^2_{rated}}{P_{rated}} = \frac{120^2}{500} = 28.8\ \Omega$$

The line−to−line voltage is given as $V_{L-L} = 220\ V$; therefore, by (11.10), the line−to−neutral voltage V_{L-N} is

$$V_{L-N} = \frac{V_{L-L}}{\sqrt{3}} = \frac{220\angle 0°}{\sqrt{3}} = 127\angle 0°\ V$$

For convenience, we indicate these values in Figure 11.34 which now is as shown in Figure 11.35.

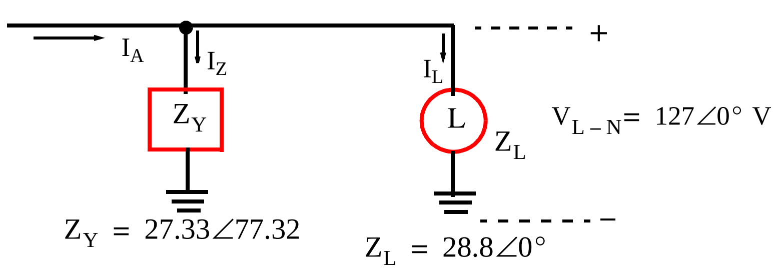


Figure 11.35. Diagram with computed values, Example 11.3

From Figure 11.35,

$$I_Z = \frac{V_{L-N}}{Z_Y} = \frac{127\angle 0°}{27.33\angle 77.32} = 4.65\angle -77.32 = 1.02 - j4.54$$

and

$$I_L = \frac{V_{L-N}}{Z_L} = \frac{127\angle 0°}{28.8\angle 0°} = 4.41\angle 0° = 4.41$$

Then,

$$I_Z + I_L = 1.02 - j4.54 + 4.41 = 5.43 - j4.54 = 7.08\angle{-39.9°}$$

and the power delivered by phase a is

$$P_A = V_{L-N} \cdot I_A = 127 \times 7.08 \times \cos(-39.9°) = 690 \text{ watts}$$

Finally, the total power delivered to the entire load is three times of P_A, that is,

$$P_{total} = 3 \times 690 = 2070 \text{ watts} = 2.07 \text{ Kw}$$

Check:

Each lamp is rated 120 V and 500 w but operates at 127 V. Thus, each lamp absorbs

$$\left(\frac{V_{oper}}{V_{rated}}\right)^2 = \frac{P_{oper}}{P_{rated}} \qquad P_{oper} = \left(\frac{127}{120}\right)^2 \times 500 = 560 \text{ w}$$

and the power absorbed by the three lamps is

$$P_{lamps} = 3 \times 560 = 1680 \text{ w}$$

The voltage across each impedance Z in the Δ−connected load is (see Figure 11.33) 220 V. Then, the current in each impedance Z is

$$I_Z = \frac{V_{L-L}}{18 + j80} = \frac{220}{82\angle 77.32} = 2.68\angle{-77.32°} \text{ A}$$

and the power absorbed by each impedance Z is

$$P = V_{L-L} I_Z \cos\theta = 220 \times 2.68 \times \cos(-77.32°) = 129.4 \text{ watts}$$

The total power absorbed by the Δ load is

$$P_\Delta = 3 \times 129.4 = 388 \text{ watts}$$

and the total power delivered to the two loads is

$$P_{TOTAL} = P_{lamps} + P_\Delta = 2068 \text{ watts} = 2.068 \text{ kw}$$

This value is in close agreement with the value on the previous page.

11.9 Measuring Three−Phase Power

A *wattmeter* is an instrument which measures power in watts or kilowatts. It is constructed with two sets of coils, a current coil and a voltage coil where the interacting magnetic fields of these coils produce a torque which is proportional to the $V \times I$ product. It would appear then that one would need three wattmeters to measure the total power in a three−phase system. This is true in a four−wire system where the current in the neutral (fourth wire) is not zero. However, if the neutral carries no current, it can be eliminated thereby reducing the system to a three−wire three−phase system. In this section, we will show that the total power in a balanced three−wire, three phase system can be measured with just two wattmeters.

Figure 11.36 shows three wattmeters connected to a Y load* where each wattmeter has its current coil connected in one line, and its potential coil from that line to neutral. With this arrangement, Wattmeters 1, 2, and 3 measure power in phase a, b, and c respectively.

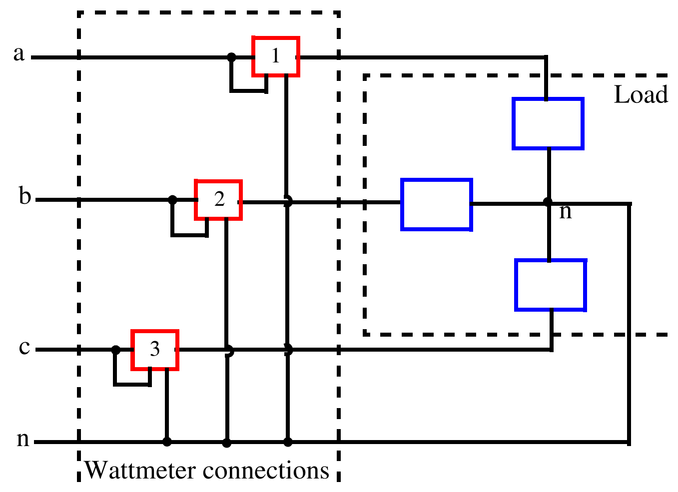


Figure 11.36. Wattmeter connections in four−wire, three−phase system

Figure 11.37 shows a three−wire, three−phase system without a neutral. This arrangement occurs in systems where the load, such as an induction motor, has only three terminals. The lower end of the voltage coils can be connected to any reference point, say p. We will now show that with this arrangement, the sum of the three wattmeters gives the correct total power even though the reference point was chosen as any reference point.

* *If the load were Δ−connected, each wattmeter would have its current coil in one side of the Δ and its potential coil from line to line.*

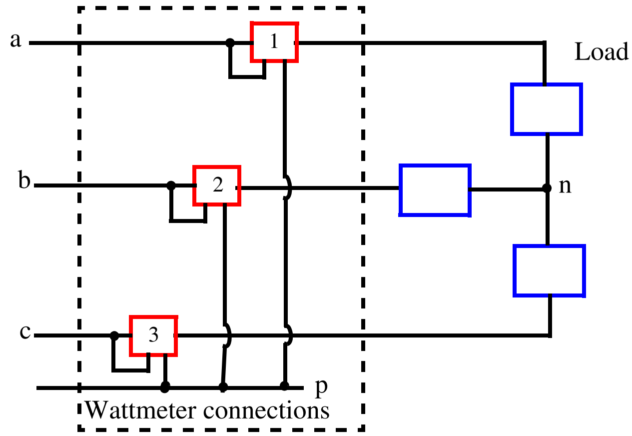


Figure 11.37. Wattmeter connections in three−wire, three−phase system

We recall that the average power P_{ave} is found from

$$P_{ave} = \frac{1}{T}\int_0^T p\,dt = \frac{1}{T}\int_0^T vi\,dt \tag{11.83}$$

Then, the total power absorbed by the load of Figure 11.36 is

$$P_{total} = \frac{1}{T}\int_0^T (v_{an}i_a + v_{bn}i_b + v_{cn}i_c)dt \tag{11.84}$$

This is the true power absorbed by the load, not power indicated by the wattmeters.

Now, we will compute the total power indicated by the wattmeters. Each wattmeter measures the average of the line current times the voltage to point p. Then,

$$P_{wattmeters} = \frac{1}{T}\int_0^T (v_{ap}i_a + v_{bp}i_b + v_{cp}i_c)dt \tag{11.85}$$

But

$$\begin{aligned} v_{ap} &= v_{an} + v_{np} \\ v_{bp} &= v_{bn} + v_{np} \\ v_{cp} &= v_{cn} + v_{np} \end{aligned} \tag{11.86}$$

and by substitution of these into (11.85), we obtain:

$$P_{wattmeters} = \frac{1}{T}\int_0^T [(v_{an}i_a + v_{bn}i_b + v_{cn}i_c) + v_{np}(i_a + i_b + i_c)]dt \tag{11.87}$$

and since

$$i_a + i_b + i_c = 0 \tag{11.88}$$

then (11.87) reduces to

$$P_{wattmeters} = \frac{1}{T}\int_0^T (v_{an}i_a + v_{bn}i_b + v_{cn}i_c)dt \tag{11.89}$$

This relation is the same as (11.84); therefore, the power indicated by the wattmeters and the true power absorbed by the load are the same.

Some thought about the location of the arbitrarily selected point p would reveal a very interesting result. No matter where this point is located, the power relation (11.87) reduces to (11.89). Suppose that we locate point p on line c. If we do this, the voltage coil of Wattmeter 3 is zero and thus the reading of this wattmeter is zero. Accordingly, we can remove this wattmeter and still obtain the true power with just Wattmeters 1 and 2 as shown in Figure 11.38.

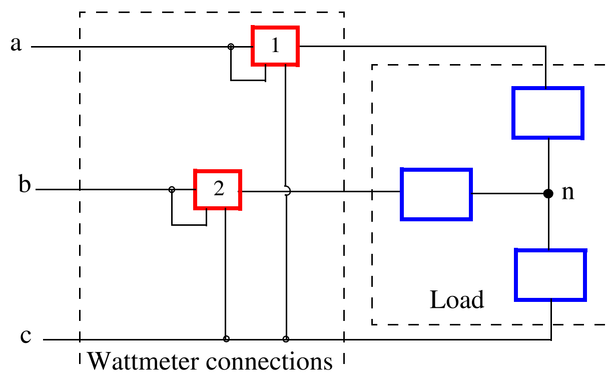


Figure 11.38. Two wattmeter method of reading three−phase power

11.10 Practical Three−Phase Transformer Connections

The four possible transformer connections and their applications are listed below.

The Δ−Δ connection is used in certain industrial applications.

The Δ−Y connection is the most common and it is used in both commercial and industrial applications.

The Y−Δ connection used for transmissions of high voltage power.

The Y−Y connection causes harmonics and balancing problems and thus is to be avoided.

If three phase transformation is needed and a three phase transformer of the proper size and turns ratio is not available, three single phase transformers can be connected to form a three phase bank. When three single phase transformers are used to make a three phase transformer bank, their primary and secondary windings are connected in a Y or Δ connection. The three transformer windings in *Figure 11.39* are labeled H1 and the other end is labeled H2. One end of each secondary lead is labeled X1 and the other end is labeled X2.

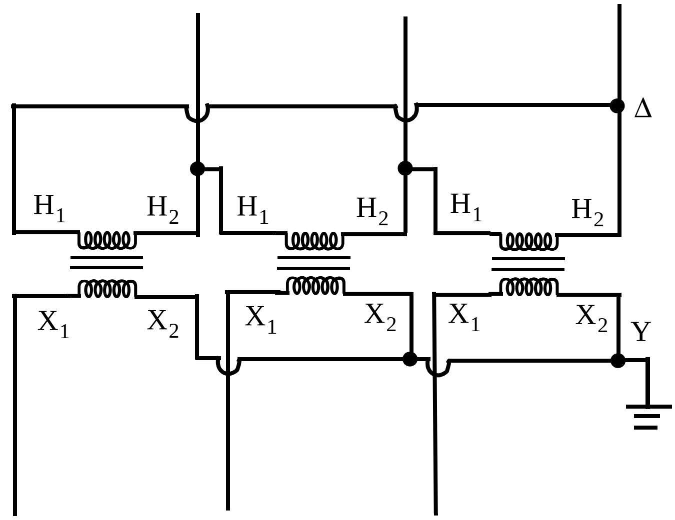


Figure 11.39. Primary and secondary leads labels in a transformer

11.11 Transformers Operated in Open Δ Configuration

In certain applications where large amounts of power are not required, the open Δ configuration is a viable alternative. The solution of Exercise 11.9 at the end of this chapter show that the input line currents form a symmetrical three–phase set and thus two transformers can also be used for a symmetrical three–phase system.

If in a closed Δ configuration one of the transformers is burnt out resulting in an open Δ configuration, the transformer bank KVA rating is reduced to about 58% of its original capacity. This is because in the open Δ configuration the line currents become phase currents and thus they are reduced to $I_{PHASE} = I_{LINE}/(\sqrt{3}) = 0.577 \cdot I_{LINE}$. For instance, if three $100\ KVA$ transformers were connected to form a closed Δ connection, the total output would be $300\ KVA$. If one of these transformers were removed and the transformer bank operated as an open delta connection, the output power would be reduced to 57.7% of its original capacity, that is, $300\ KVA \times 0.577 = 173.2\ KVA$.

If, in a bank o three transformers connected in Δ is burnt out and no replacement is readily available, capacitors with the proper rating can be used to prevent overloading as illustrated with Example 11.4 below.

Example 11.4

A bank of three $13200 / 4160\ V$ transformers each rated $833\ KVA$, $60\ Hz$ connected in Δ–Δ, feeds a short distribution line that is terminated in a bank of three $833\ KVA$, $4160 / 480\ V$ transformers with a $1600\ KVA$, and $0.8\ pf$ lagging load.

a. If one of the $13200 / 4160 \text{ V}$ transformers burns out, what would the voltage, current, and rating of capacitors on the secondary side of the $4160 / 480 \text{ V}$ transformers be to prevent overloading of any of the transformers?

b. What would the capacitor ratings be if installed at the 480 V side and what would the current be through this capacitor bank?

c. What would the capacitor ratings be if installed at the 4160 V side and what would the current be through this capacitor bank?

Assume that line and transformer impedances are negligible.

Solution:

a. With the assumption that the line and transformer impedances are negligible, the open Δ connection still forms a balanced symmetrical system.*

The rated current per transformer at 13200 V is

$$I_{rated} = \frac{833 \text{ KVA}}{13.2 \text{ KV}} = 63.1 \text{ A} \quad (1)$$

With the open Δ connection the 1600 KVA at 0.8 pf lagging load, the new KVA rating is $1600/\sqrt{3} = 923.8 \text{ KVA}$, and the actual current per transformer is

$$I_{actual} = \frac{923.8 \text{ KVA}}{13.2 \text{ KV}} = 70 \text{ A} \quad (2)$$

The reduction in KVA is found from the proportion of (1) and (2) above, i.e.,

$$\frac{63.1}{70} \times 1600 = 1443 \text{ KVA}$$

The real power P_{KW} (kilowatts) at 0.8 pf lagging load is

$$P_{Kw} = 1600 \times 0.8 = 1280 \text{ Kw}$$

and without capacitors the reactive power Q_{Kvar1} (kilovars) is

$$Q_{Kvar1} = \sqrt{KVA_{old}^2 - P_{Kw}^2} = \sqrt{1600^2 - 1280^2} = 960 \text{ Kvar}$$

With capacitors the reactive power Q_{Kvar2} (kilovars) will be

* *This is illustrated in Exercise 11.9 at the end of this chapter.*

$$Q_{Kvar2} = \sqrt{KVA_{new}^2 - P_{Kw}^2} = \sqrt{1443^2 - 1280^2} = 666 \text{ Kvar}$$

Therefore, the Kvar required to prevent overloading should be

$$Q_{Kvar1} - Q_{Kvar2} = 960 - 666 = 294 \approx 300 \text{ Kvar}$$

b. For installation at the 480 V side, three single–phase capacitors each rated 100 Kvar will be required, and the current through this capacitor bank must be $100/0.480 = 208$ A per phase.

c. For installation at the 13200 V side, three single–phase capacitors each rated 100 Kvar will be required, and the current through this capacitor bank must be $100/13.2 = 24$ A per phase.

11.12 Three–Phase Systems Modeling with Simulink / SimPowerSystems

The MathWorks Simulink / SimPowerSystems toolbox includes several three–phase transformer and they can be used with three–phase system models that include three–phase transformers. Two of these are shown in Figure 11.40 below.

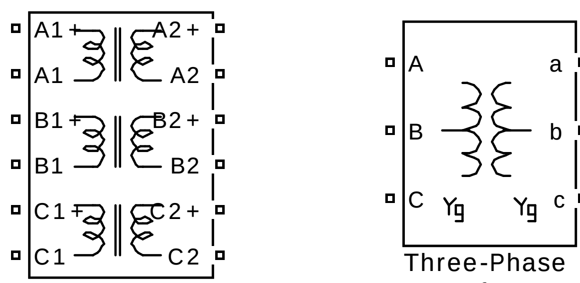


Figure 11.40. Two of the three–phase transformer blocks included in the Simulink / SimPowerSystems toolbox

Example 11.5

For the circuit in Figure 11.41, the three–phase transformer bank consists of three transformers each rated 5 KVA, 440 / 208 V, 60 Hz connected Δ–Y connection, and the lighting load is balanced. Each lamp is rated 500 w at 120 V. Assume that each lamp draws rated current. The three–phase motor draws 5.0 Kw at a power factor of 0.8 lagging. The secondary of the transformer is connected Y grounded and provides balanced 208 V line–to–line. The distance between the transformer and the loads is small and the wiring resistance and inductance can be neglected. The input voltages are:

$V_{AB} = 480\angle 0°$ $V_{BC} = 480\angle -120°$ $V_{CA} = 480\angle 120°$

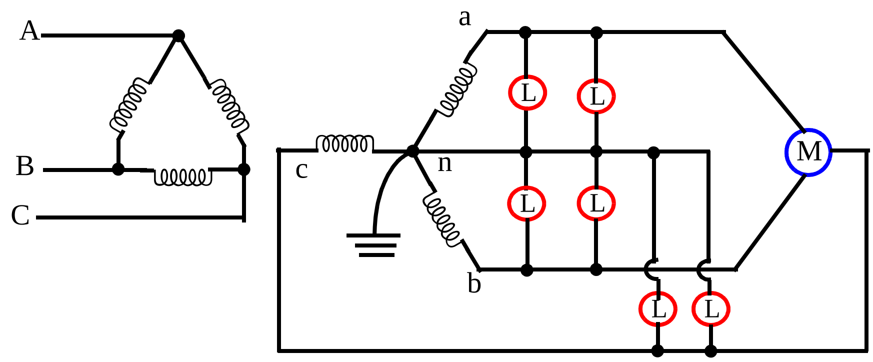


Figure 11.41. Three−phase circuit for Example 11.5

Create a Simulink / SimPowerSystems model to display all voltages and currents.

Solution:

The model is shown in Figure 11.42.

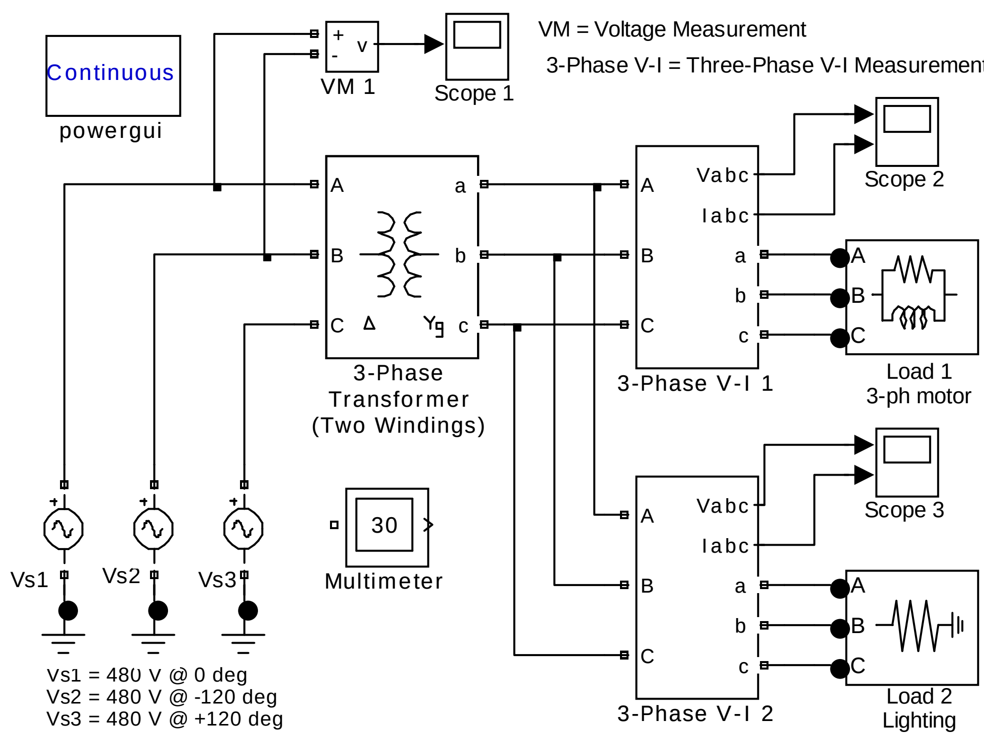


Figure 11.42. Simulink / SimPowerSystems model for the three−phase circuit in Figure 11.41

For the model in Figure 11.42, the default integration algorithm ode45 was changed to odetb23. This is done with **Simulation>Configuration Parameters>Solver>odetb23**.

The dialog box is configured as shown in Figure 11.43, and the dialog box for is shown in Figure 11.44.

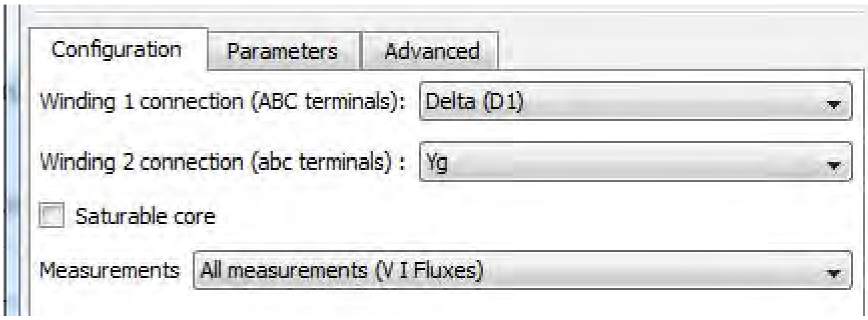


Figure 11.43. Block Parameters 3-Phase Transformer – Configuration tab

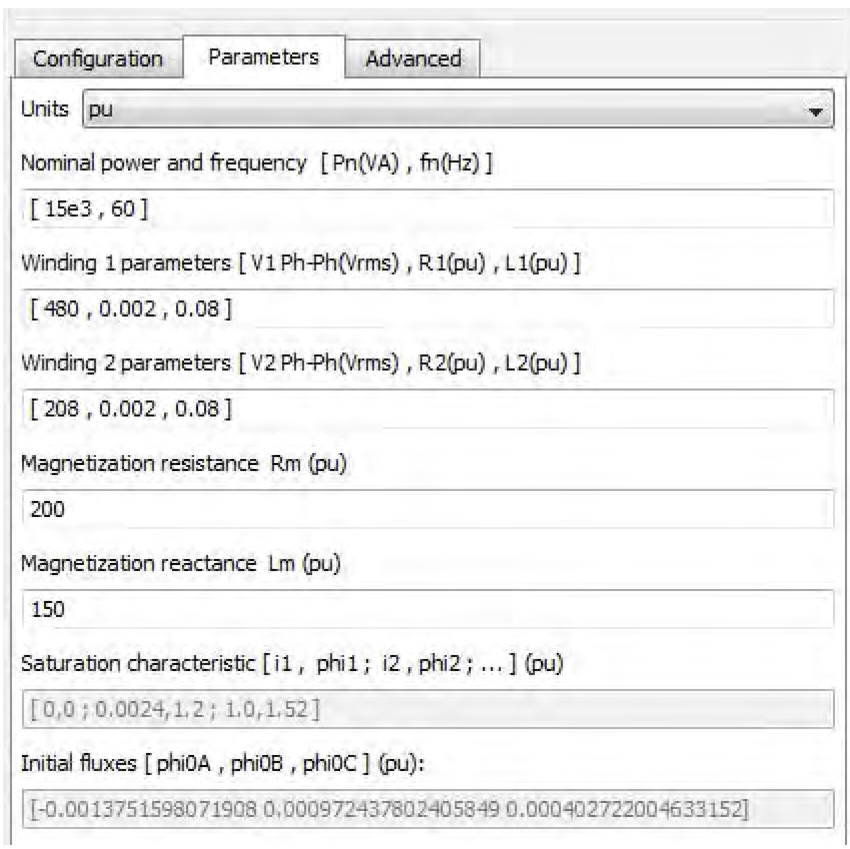


Figure 11.44. Block Parameters 3-Phase Transformer – Parameters tab

For the remaining blocks, the **Measurement** parameter has been set to **Voltage**, **Current**. **Voltage and Current**, or **All Measurements (V-I) fluxes** (indicated in Figure 11.43), and the **Multimeter** block in Figure 11.42 indicates that 30 measurements will be displayed when selected in the **Multimeter** block dialog box shown in Figure 11.45.

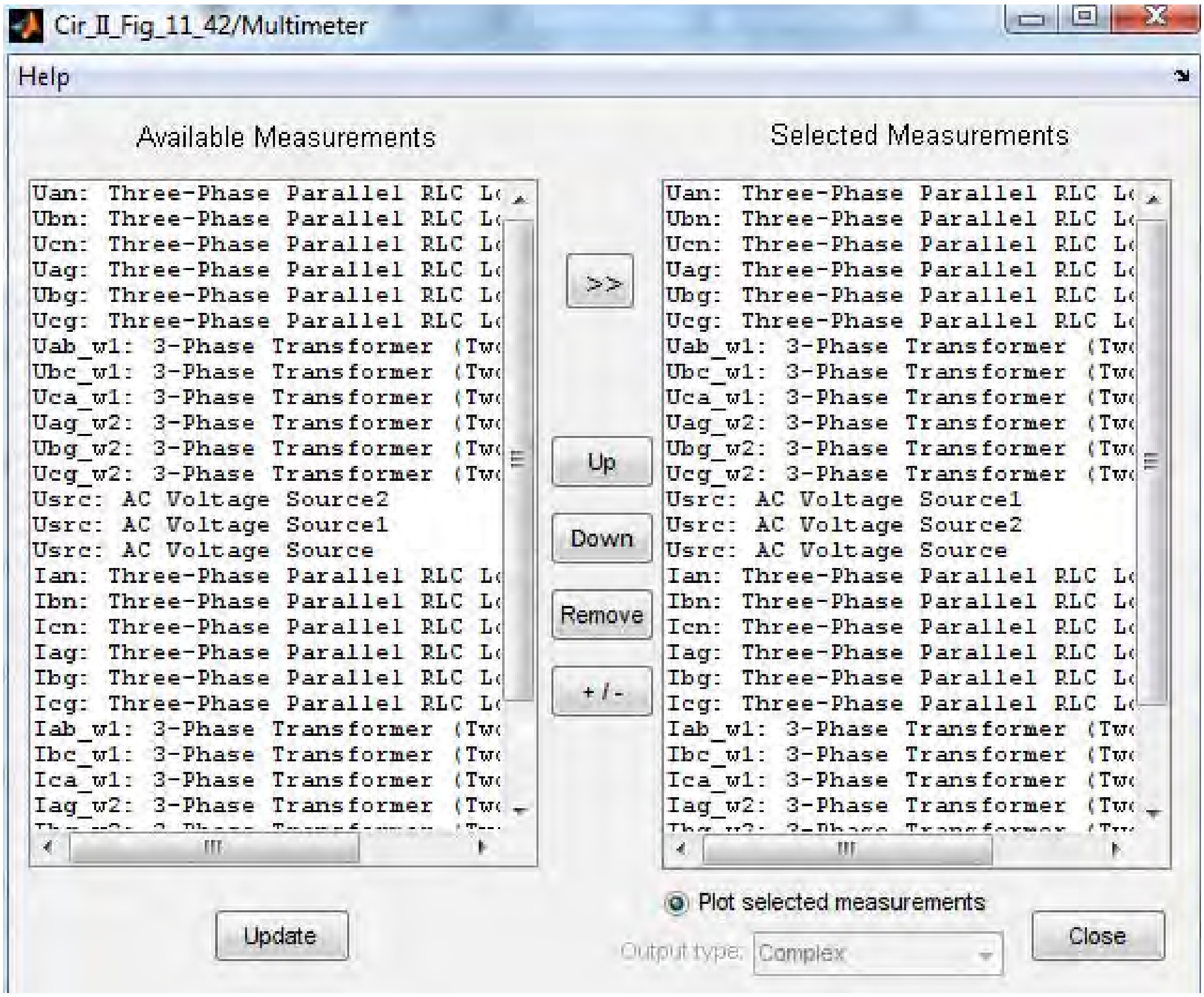


Figure 11.45. The Multimeter block dialog box

The SimPowerSystems **powerlib/Electrical Sources** library includes the Three−Phase Source block shown in Figure 11.46.

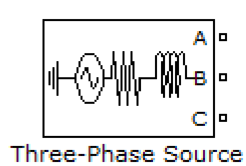


Figure 11.46. The SimPowerSystems Three−Phase Source block

This block is a balanced three−phase voltage source with an internal R−L impedance. It allows us to specify the source internal resistance and inductance either directly by entering R and L values or indirectly by specifying the source inductive short-circuit level* and X/R ratio. More details are provided in the Help menu for this block, and an example is provided by The MathWorks. It can be accessed by typing power_3phseriescomp at the MATLAB prompt.

Another three−phase voltage source block is the **Three−Phase Programmable Voltage Source** shown in Figure 11.47. This three−phase voltage source allows variation for the amplitude, phase, or frequency of the fundamental component of the source. Positive, negative, and zero sequences are discussed in Chapter 12.

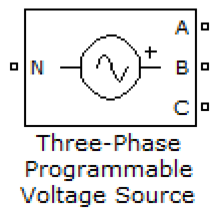


Figure 11.47. The SimPowerSystems Three−Phase Programmable Voltage Source block

More details are provided in the Help menu for this block, and an example is provided by The MathWorks. It can be accessed by typing power_3phsignalseq at the MATLAB prompt.

* *The short-circuit level is a function of the transformer rated VA, the rated secondary voltage, and the transformer impedance in percent. These parameters are provided by the transformer manufacturer. It is computed from the relation* $I_{SC} = \left(\frac{100\ \%}{Z\ \%} \times VA\right) / (\sqrt{3} \times V_{SEC})$. *Thus, for a* 100KVA, 2300 / 13800 V, Z = 7% *transformer, the short-circuit level will be* $\left(\frac{100}{7.5} \times 10^5\right) / (\sqrt{3} \times 13.8 \times 10^3) = 55.8\ A$

11.13 Summary

- AC is preferable to DC because voltage levels can be changed by transformers. This allows more economical transmission and distribution.
- The flow of power in a three−phase system is constant rather than pulsating. Three−phase motors and generators start and run more smoothly since they have constant torque. They are also more economical.
- If the voltage sources are equal in magnitude and 120° apart, and the loads are also equal, the currents will be balanced (equal in magnitude and 120° out−of−phase).
- Industrial facilities need three−phase power for three−phase motors. Three−phase motors run smoother and have higher efficiency than single−phase motors.
- The equations $I_a = I_a\angle 0°$, $I_b = I_a\angle -120°$, $I_c = I_a\angle +120°$ define a balanced set of currents of positive phase sequence $a - b - c$.
- The equations $V_{an} = V_{an}\angle 0°$, $V_{bn} = V_{an}\angle -120°$, and $V_{cn} = V_{an}\angle +120°$ also define a balanced set of voltages of positive phase sequence $a - b - c$.
- In a Y−connected system

$$V_{ab} = \sqrt{3}V_{an}\angle 30°$$

- In a Y−connected load, the line and phase currents are the same.
- In a Δ−connected system

$$I_a = \sqrt{3}I_{ab}\angle -30°$$

- In a Δ−connected load, the line and phase voltages are the same.
- For $\Delta \to Y$ Conversion we use the relations

$$Z_a = \frac{Z_1 Z_3}{Z_1 + Z_2 + Z_3} \qquad Z_b = \frac{Z_2 Z_3}{Z_1 + Z_2 + Z_3} \qquad Z_c = \frac{Z_1 Z_2}{Z_1 + Z_2 + Z_3}$$

- For $Y \to \Delta$ Conversion we use the relations

$$Z_1 = \frac{Z_a Z_b + Z_b Z_c + Z_c Z_a}{Z_b} \qquad Z_2 = \frac{Z_a Z_b + Z_b Z_c + Z_c Z_a}{Z_a} \qquad Z_3 = \frac{Z_a Z_b + Z_b Z_c + Z_c Z_a}{Z_c}$$

- When we want to compute the voltages, currents, and power in a balanced three−phase system, it is very convenient to use the Y−connection and work with one phase only.

- If a load is Y–connected, the total three–phase power is given by

$$\boxed{P_{TOTAL} = 3|V_{AN}||I_A|\cos\theta}$$
$$Y - \text{connected load}$$

- If the load is Δ–connected the total three–phase power is given by

$$\boxed{P_{TOTAL} = 3|V_{AB}||I_{AB}|\cos\theta}$$
$$\Delta - \text{connected load}$$

- For any load (Y or Δ – connected) the total three–phase power can be computed from

$$\boxed{P_{TOTAL} = \sqrt{3}|V_{AB}||I_A|\cos\theta_{LD}}$$
$$Y \text{ or } \Delta - \text{connected load}$$

  and it is important to remember that the power factor $\cos\theta_{LD}$ refers to the load, that is, the angle θ is not the angle between V_{AB} and I_A.

- Three–phase power can be measured with only two wattmeters.

- In a three–phase system, the Δ–Δ connection is preferred in certain industrial applications, the Δ–Y connection is the most common and it is used in both commercial and industrial applications, the Y–Δ connection used for transmissions of high voltage power, but the Y–Y connection causes harmonics and balancing problems and it is to be avoided.

- If a three–phase transformation is needed and a three phase transformer of the proper size and turns ratio is not available, three single phase transformers can be connected to form a three phase bank.

- A symmetrical three–phase system can also be formed with two transformers.

- If one of these transformers were removed and the transformer bank operated as an open delta connection, the output power would be reduced to 57.7% of its original capacity. To restore the system to its original capacity, capacitors can be added to the system.

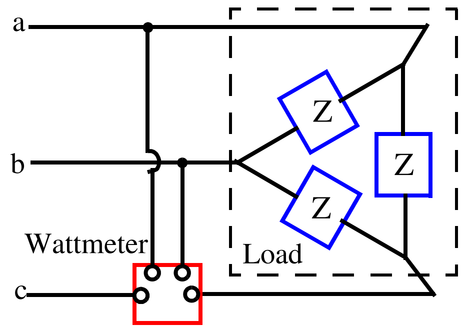
- The MathWorks Simulink / SimPowerSystems toolbox includes several three–phase transformer and they can be used with three–phase system models that include three–phase transformers.

11.14 Exercises

1. In the circuit below the line−to−line voltage is $100\ V$, the phase sequence is $a - b - c$, and each $Z = 10\angle 30°$. Compute:

   a. the total power absorbed by the three−phase load.

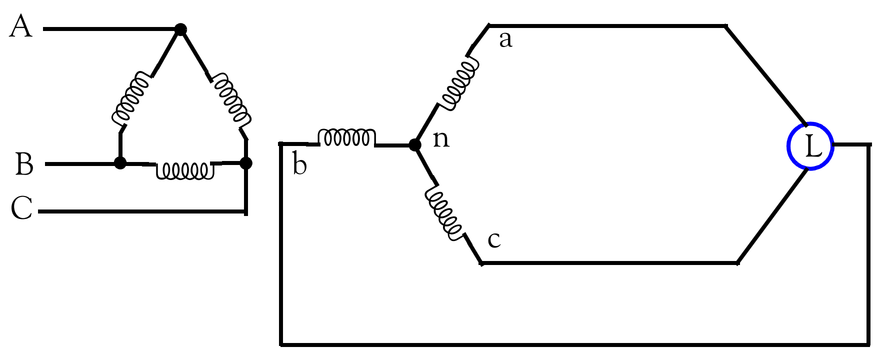
   b. the wattmeter reading.



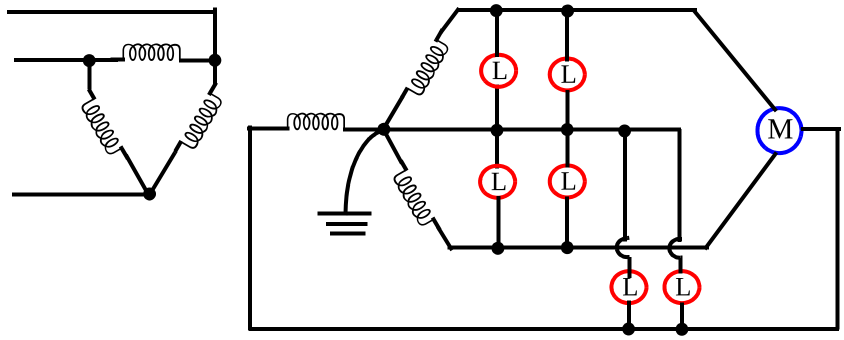
2. Three single−phase transformers are connected $\Delta - Y$ as shown below. Each transformer is rated $100\ KVA$, $2300/13800\ V\ RMS$, $60\ Hz$. The total three−phase load L is $270\ KVA$ with $pf = 0.866$ lagging. The input voltages are:

$$V_{AB} = 2300\angle 0° \qquad V_{BC} = 2300\angle -120° \qquad V_{CA} = 2300\angle 120°$$

   Find all voltages and currents assuming that the transformers are ideal, and the line−to−neutral voltages on the secondary are in phase with the input voltages.



3. In the circuit below the lighting load is balanced. Each lamp is rated $500\ w$ at $120\ V$. Assume constant resistance, that is, each lamp will draw rated current. The three−phase motor draws $5.0\ Kw$ at a power factor of 0.8 lagging. The secondary of the transformer provides balanced $208\ V$ line−to−line. The load is located 1500 feet from the three−phase transformer. The resistance and inductive reactance of the distribution line is $0.403\ \Omega$ and $0.143\ \Omega$ respectively per 1000 ft of the wire line. Compute line−to−line and line−to−neutral voltages at the load.



4. A three–phase motor and a single–phase motor are connected in a three–phase 208 volt, 60 Hz phase distribution system with neutral. The single–phase motor is connected between line c and the neutral, and there is no neutral connection for the three–phase motor. The phase sequence is $a - b - c$.

   The three–phase motor is rated 15 hp, 208 volts, 1740 rpm, 87% efficiency, and 0.866 pf.

   The single–phase motor is rated 3.5 hp, 115 volts, 1750 rpm, 85% efficiency, and 0.8 pf. How much current flows in each line and in the neutral when both motors are operating with full loads?

5. Three–phase power of 1 Mw is to be delivered over a distance of 100 miles to a Y–connected load whose power factor is 0.80 lagging. The operating frequency is 60 Hz and each line has a 30 Ω resistance and 30 mH inductance. The generator at the sending end is also Y–connected.

   What must the line–to–line voltages be at the sending end if the corresponding voltages at the load are to be 20, 000 V in magnitude?

6. A three–phase transmission line 20 miles long has a resistance of 0.6 Ω per mile of conductor and a reactance of 0.27 Ω per mile of conductor at 60 Hz. The transmission line delivers 1000 Kw to a Y–connected inductive load at a power factor of 0.80. The potential difference between line conductors at the load is 11000 V.

   a. Calculate the potential difference between line conductors at the input end of the line.

   b. Calculate the total rating in KVA of a bank of capacitors placed at the input of the line that will increase the power factor at that point to 0.90 lagging.

7. A potential difference of 66000 V is impressed between the conductors of a three–wire transmission line at its generator end. Each line conductor has an impedance of $80 + j60\ \Omega$. The load is Y–connected and the power absorbed by this load is 1000 Kw at a lagging power factor of 0.80. Calculate the potential difference between conductors at the load.

8. Each conductor of a three−phase, three−wire transmission line has an impedance of $15 + j20\ \Omega$ at $60\ Hz$. The potential difference between line conductors is $13200\ V$. The load connected to this system is balanced and absorbs $1000\ Kw$ at a lagging power factor that is to be determined. The current per conductor is $70\ A$. Find:

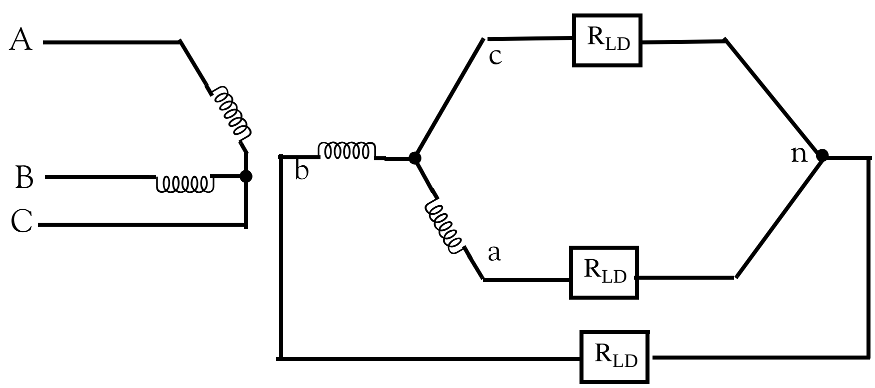
   a. the efficiency of transmission

   b. the potential difference between line conductors at the load

   c. the power factor at the load

9. Two transformers, each rated $20\ KVA$, $440 / 220\ V$, $60\ Hz$, are connected in open Δ configuration as shown below. Each load R_{LD} is a resistive load of $1.27\ \Omega$. The input voltages are:

$$V_{AB} = 440\angle 0°\ V \qquad V_{BC} = 440\angle -120°\ V \qquad V_{CA} = 440\angle 120°\ V$$



Assuming that the primary and secondary voltages are in phase, and the transformers are ideal, find:

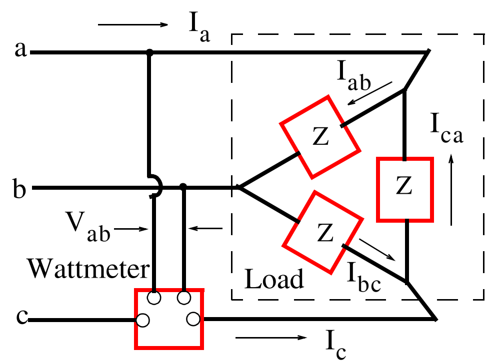
   a. the voltages on the secondary

   b. all currents

11.15 Solutions to End−of−Chapter Exercises

1.

a.



From the circuit above

$$I_{ab} = \frac{V_{ab}}{Z} = \frac{100\angle 0°}{10\angle 30°} = 10\angle{-30°} = 10 \times \frac{\sqrt{3}}{2} - j10 \times \frac{1}{2} = 5\sqrt{3} - j5$$

$$I_{ca} = \frac{V_{ca}}{Z} = \frac{100\angle{-240°}}{10\angle 30°} = 10\angle{-270°} = 10\angle 90° = j10$$

$$I_a = I_{ab} - I_{ca} = 5\sqrt{3} - j5 - j10 = 5\sqrt{3} - j15$$

and with MATLAB,

```
x=5*sqrt(3)–15j; fprintf(' \n');...
fprintf('mag = %5.2f A \t', abs(x)); fprintf('phase = %5.2f deg', angle(x)*180/pi)
```

```
mag = 17.32 A     phase = -60.00 deg
```

Thus, $|I_a| = 17.32\ A$

The phase sequence $a - b - c$ implies the phase diagram below.

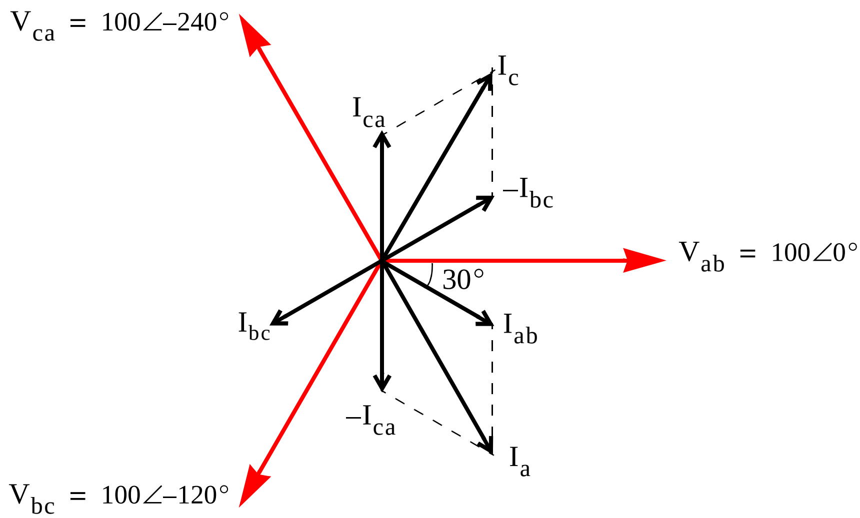
From (11.59)

$$P_{total} = \sqrt{3}|V_{ab}||I_a|(\text{load pf})$$
$$= \sqrt{3} \times 100 \times 17.32 \times \cos 30° = 2,598\ w$$

b.

The wattmeter reads the product $V_{ab} \times I_c$ where I_c is 240° behind I_a as shown on the phasor diagram below. Thus, the wattmeter reading is

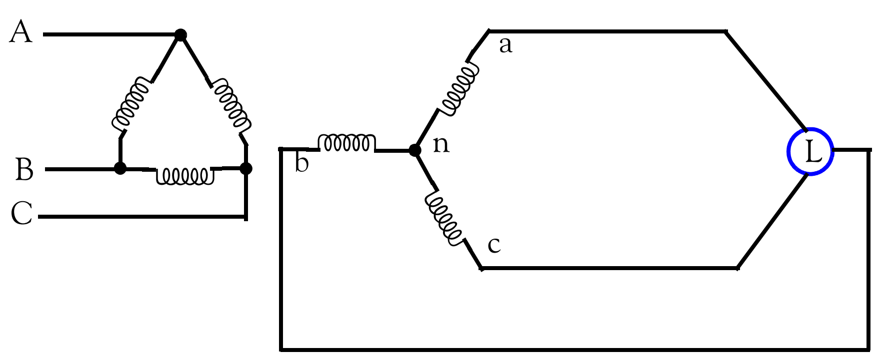
$$P_{wattmeter} = V_{ab} \times I_c = 100\angle 0° \times 10\sqrt{3} \times \cos(-60° - 240°)$$
$$= 100 \times 17.32 \times \cos(-300°) = 866\ w$$



and, as expected, this value is one−third of the total power.

2.

$$V_{AB} = 2300\angle 0° \qquad V_{BC} = 2300\angle -120° \qquad V_{CA} = 2300\angle 120°$$



Since the transformers are ideal, and the line−to−neutral voltages on the secondary are in phase with the input voltages, the line−to−neutral voltages on the secondary are:

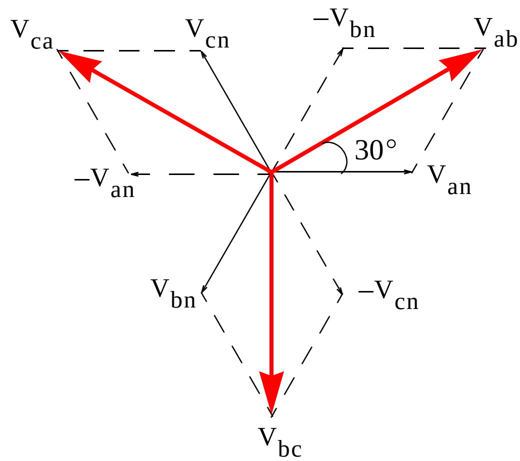
$$V_{an} = 13800\angle 0° \qquad V_{bn} = 13800\angle -120° \qquad V_{cn} = 13800\angle 120°$$

With reference to the phase diagram below, the line−to−line voltages on the secondary are:

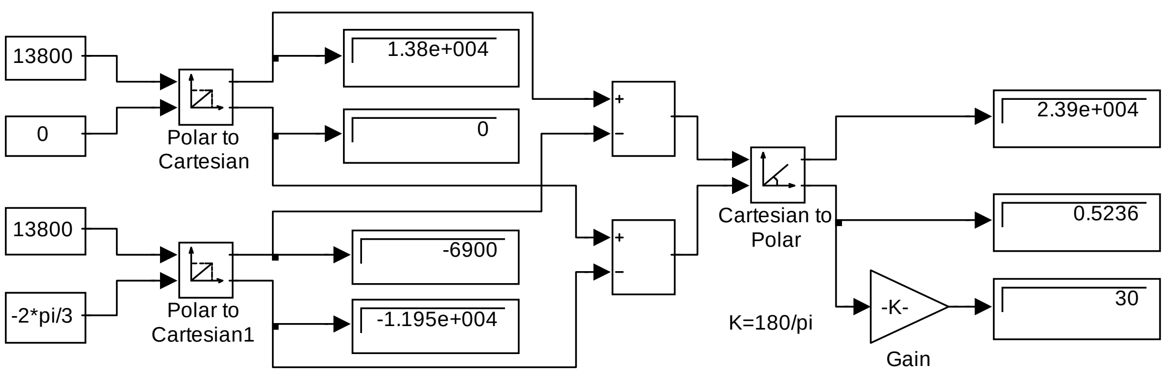
$$V_{ab} = V_{an} - V_{bn} = 13800\angle 0° - 13800\angle -120° = 23900\angle 30°$$

$$V_{bc} = V_{bn} - V_{cn} = 13800\angle -120° - 13800\angle -60° = 23900\angle -90°$$

$$V_{ca} = V_{cn} - V_{an} = 13800\angle 120° - 13800\angle 180° = 23900\angle 150°$$



For the above complex number operations and the others below, it is convenient to use the Simulink model below.*



The magnitude of the line currents on the secondary is determined by the current drawn by the load, that is, total three–phase load divided by 3, $270\ \mathrm{KVA}/3 = 90\ \mathrm{KVA}$, and thus

$$I_{LOAD}\ (\text{per phase}) = 90\ \mathrm{KVA}/13800\ \mathrm{V} = 6.52\ \mathrm{A}$$

The load power factor is 0.866 lagging and since $pf = \cos\theta = 0.866$, $\theta = \cos^{-1}0.866 = 30°$, and therefore the currents on the secondary lag the line–to–neutral voltages by $30°$. Then,

$$I_{na} = 6.52\angle 0° - 30° = 6.52\angle -30°$$

$$I_{nb} = 6.52\angle -120° - 30° = 6.52\angle -150°$$

$$I_{nc} = 6.52\angle 120° - 30° = 6.52\angle 90°$$

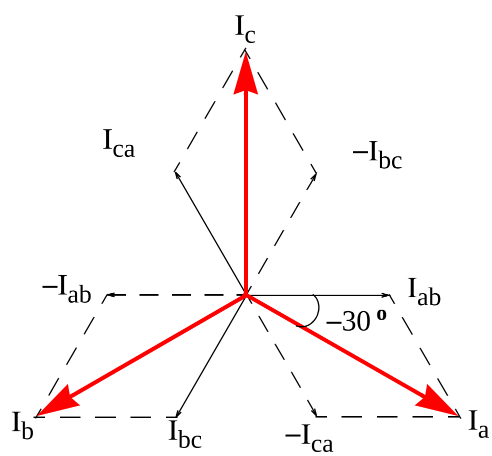
* *For the description of the Simulink blocks used in the model above, please consult The MathWorks, Inc. documentation, or refer to Introduction to Simulink with Engineering Applications, ISBN 978–1–934404–09–06.*

To find the values of the currents on the primary side, we make us of the transformers turns ratio, that is, $a = 2300/13800 = 1/6$. Then,

$$I_{AB} = (1/a) \cdot I_{na} = 39\angle -30°$$
$$I_{BC} = (1/a) \cdot I_{nb} = 39\angle -150°$$
$$I_{CA} = (1/a) \cdot I_{nc} = 39\angle 90°$$

With reference to the phase diagram below, the input line currents are:

$$I_A = I_{AB} - I_{CA} = 39\angle -30° - 39\angle 90° = 67.6\angle -60°$$
$$I_B = I_{BC} - I_{AB} = 39\angle -150° - 39\angle -30° = 67.6\angle 180°$$
$$I_C = I_{CA} - I_{BC} = 39\angle 90° - 39\angle -150° = 67.6\angle 60°$$



3.

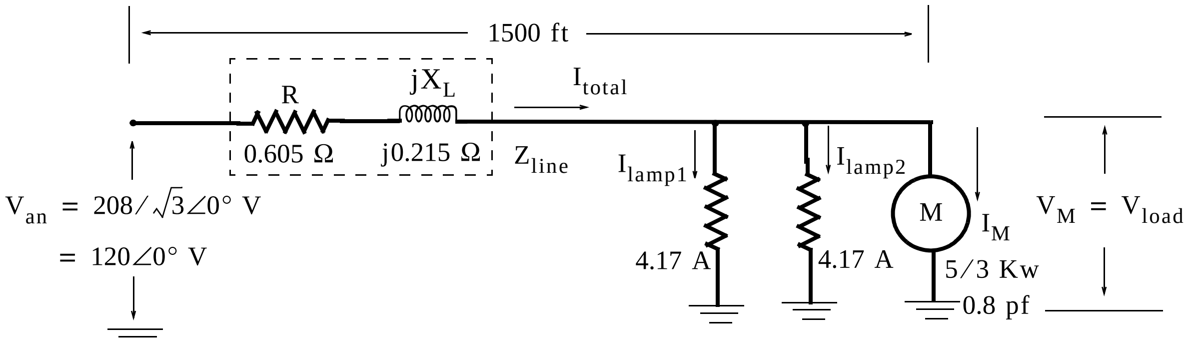
The single−phase equivalent circuit is shown below where

$$R = 0.403\ \Omega/1000\ \text{ft} \times 1500\ \text{ft} = 0.605\ \Omega$$

$$X_L = 0.143\ \Omega/1000\ \text{ft} \times 1500\ \text{ft} = 0.215\ \Omega$$

and thus

$$Z_{line} = 0.605 + j0.215$$



Also,

$$I_{lamp1} = I_{lamp2} = \frac{P_{rated}}{V_{rated}} = \frac{500}{120} = 4.17\ A$$

We recall that for a single phase system the real power is given by

$$P_{real} = |V_{RMS}||I_{RMS}|\cos\theta$$

where $\cos\theta = pf$

Then, we find the motor current I_M in terms of the motor voltage V_M as

$$|I_M| = \frac{5000/3}{0.8|V_M|} = \frac{2083}{|V_M|}$$

and since $\cos^{-1}0.8 = -36.9°$ (lagging pf), the motor current I_M is expressed as

$$I_M = \frac{2083}{V_M}\angle -36.9° = \frac{1}{V_M}(1666 - j1251)$$

The total current is

$$I_{total} = I_{lamp1} + I_{lamp2} + I_M = 2\times 4.17 + \frac{1}{V_M}(1666 - j1251) = \frac{1}{V_M}(8.34V_M + 1666 - j1251)$$

and the voltage drop across the 1500 ft line is

$$V_{line} = I_{total}\cdot Z_{line} = \frac{1}{V_M}(8.34V_M + 1666 - j1251)\cdot(0.605 + j0.215)$$

$$= \frac{1}{V_M}(5.05V_M + j1.79V_M + 1008 + j358.2 - j756.9 + 269.0)$$

$$= \frac{1}{V_M}[(5.05V_M + 1277) + j(1.79V_M - 398.7)]$$

Next,

$$V_{an} = 120\angle 0° = V_{line} + V_M = \frac{1}{V_M}[(5.05V_M + 1277) + j(1.79V_M - 398.7)] + V_M$$

or

$$120V_M = [(5.05V_M + 1277) + j(1.79V_M - 398.7)] + V_M^2$$

or

$$V_M^2 - (114.95 - j1.79)V_M + (1277 - j398.7) = 0$$

We solve this quadratic equation with the following MATLAB script:

```
p=[1 114.95–1.79j 1277–398.7j]; roots(p)
```

```
ans =
  1.0e+002 *
  1.0260 + 0.0238i
  0.1235 - 0.0417i
```

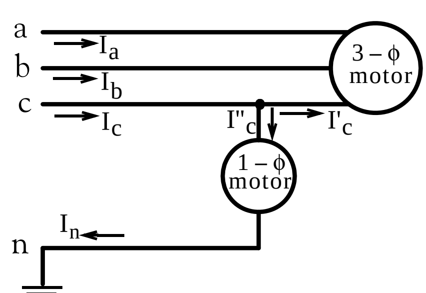
Then, $V_{M1} = 102.6 + j2.39 = 102.63\angle 1.33°$ and $V_{M2} = 12.35 - j4.17 = 13.4\angle -18.66°$. Of these, the value of V_{M2} is unrealistic and thus it is rejected. The positive phase angle in V_{M1} is a result of the fact that a motor is an inductive load. But since an inductive load has a lagging power factor, we denote this line−to neutral of line−to−ground voltage with a negative angle, that is,

$$V_M = V_{load} = 102.63\angle -1.33° \text{ V}$$

The magnitude of the line−to−line voltage is

$$|V_{l-l}| = \sqrt{3} \times V_M = \sqrt{3} \times 102.63 = 177.76 \text{ V}$$

4.



For the three−phase motor, the power is computed from the relation

$$P = \sqrt{3}|V_{ab}||I_a|\cos\theta_{LD}\eta$$

where $\cos\theta_{LD}$ is the load power factor, and η is the efficiency. Solving for the magnitude of the line current I_a we obtain

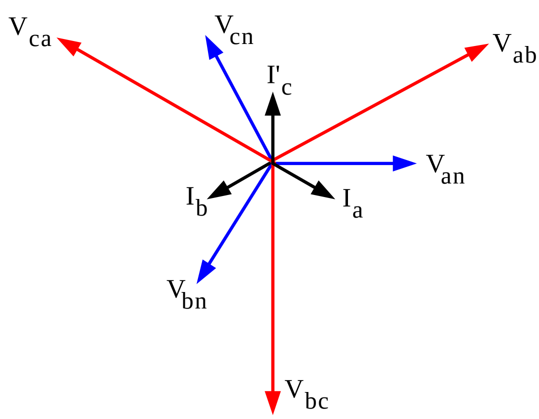
$$|I_a| = \frac{P}{\sqrt{3}|V_{ab}|\cos\theta_{LD}\eta} = \frac{15 \times 746}{\sqrt{3} \cdot 208 \cdot 0.866 \cdot 0.87} = 41.2 \text{ A}$$

Next, let us refer to the phasor diagram below where we have chosen V_{an} as the reference phase voltage. Then,

$$V_{an} = 120\angle 0°$$
$$V_{bn} = 120\angle -120°$$
$$V_{cn} = 120\angle 120°$$

as shown in the phasor diagram below. The position of the phase current I_a in the phasor diagram is determined by the load power factor $\cos\theta_{LD} = 0.866$ from which $\theta = -30°$ where the negative sign stems from the fact that the power factor is lagging. Therefore,

$$I_a = 41.2\angle -30°$$
$$I_b = 41.2\angle -150°$$
$$I'_c = 41.2\angle 90°$$



For the single–phase motor, the magnitude of the current I''_c is computed from the relation

$$|I''_c| = \frac{3.5 \times 746}{115 \cdot 0.8 \cdot 0.85} = 33.4 \text{ A}$$

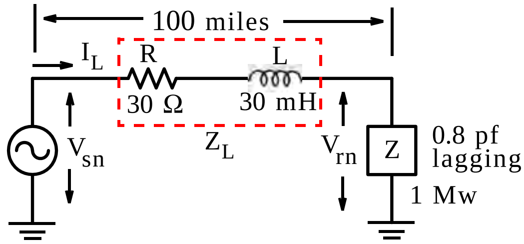
and since $\cos\theta'_{LD} = 0.8$ lagging, $\theta' = -36.9°$ and since I''_c is a component of the line current I_c which is $120°$ out–of–phase with the line current I_a, it follows that

$$I''_c = 33.4\angle -36.9° \times \angle 120° = 33.4\angle 83.1° \text{A}$$

and

$$I_c = I'_c + I''_c = 41.2\angle 90° + 33.4\angle 83.1° = j41.2 + 4 + j33.1 = 4 + j74.3 = 74.4\angle 87°$$

5. Since the system is balanced, we can find the solution treating it as a single–phase system as shown below.



We let:

V_{sn} = Voltage to neutral at sending end

$jX = j2\pi fL = j2\pi \times 60 \times 0.03 = 11.31\ \Omega$

$Z_L = R + jX = 30 + j11.31\ \Omega$

θ = power factor angle = $\cos^{-1} 0.80$

V_{rn} = Voltage to neutral at receiving end = $20,000/\sqrt{3} = 11,547\ V$

$$|I_L| = \frac{P}{\sqrt{3}V_{L-L}\cos\theta} = \frac{10^6}{\sqrt{3} \times 20,000 \times 0.8} = 36.1\ A$$

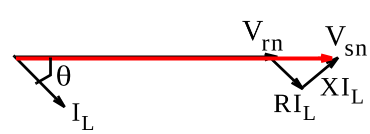
Then,

$$V_{sn} = Z_L I_L + V_{rn} = (30 + j11.31)(36.1)(0.8 - j0.6) + 11,547$$
$$= 12658 - j323 = 12662\angle -1.5°\ V$$

That is, the magnitude of voltage to neutral at the sending end is $12662\ V$, and the line–to–line voltages are

$$V_{L-L} = \angle 3 \times 12662 \approx 22,000\ V$$

The phasor diagram below shows the relevant voltages and currents. The angle of V_{sn} is very small and it is neglected.



6.

a. The line current I_{LN} is

$$I_{LN} = \frac{P}{\sqrt{3}V_L pf} = \frac{1000 \times 1000}{\sqrt{3} \times 11000 \times 0.8} = 65.6\ A$$

The line resistance R_L and the line reactance X_L for the entire length of 20 miles are

$$R_L = 20 \times 0.6 = 12\ \Omega \qquad X_L = 20 \times 0.27 = 5.4\ \Omega$$

and thus the line impedance Z_L is

$$Z_L = 12 + j5.4\ \Omega$$

The line–to–neutral voltage at the load end, denoted as V_{rn}, is

$$V_{rn} = \frac{11000}{\sqrt{3}} = 6350\ \text{V}$$

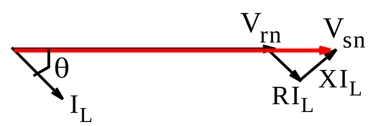
and the line–to–neutral voltage at the sending end, denoted as V_{sn}, is

$$V_{sn} = Z_L I_L + V_{rn} = (12 + j5.4)65.6(0.8 - j0.6) + 6350$$
$$= 7192 - j189 = 7195\angle{-1.5°}\ \text{V}$$

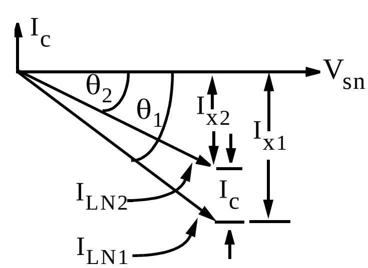
and the line–to–line voltage at the sending end, denoted as V_{L-L}, is

$$V_{L-L} = \sqrt{3}|V_{sn}| = \sqrt{3} \times 7195 \approx 12500\ \text{V}$$

The phasor diagram below shows the relevant voltages and currents. The angle of V_{sn} is very small and it is neglected.



b. The capacitor bank consumes no real power but it will cause the flow of a current that leads V_{sn} by $90°$ as shown in the phasor diagram below.



Original current:

$$I_{LN1} = 65.6(\cos\theta_1 - j\sin\theta_1) = 65.6(0.8 - j0.6) = 52.5 - j39.4$$

Original lagging reactive current:

$$I_{x1} = -j39.4$$

For improved power factor 0.9, $\text{acos}(0.9) = 25.8°$, $\sin(25.8°) = 0.436$. Then,

$$I_{LN2} = 65.6(\cos\theta_2 - j\sin\theta_2) = 65.6(0.9 - j0.436) = 59.0 - j28.6$$

Thus, final lagging reactive current is

$$I_{x2} = -j28.6$$

and leading reactive current by the capacitor bank is

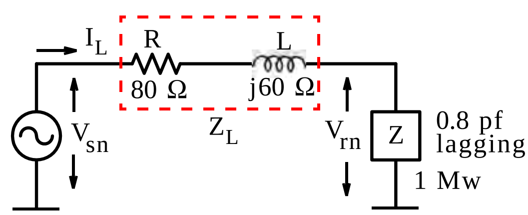
$$I_c = I_{x1} - I_{x2} = j39.4 - j28.6 = j10.8$$

Therefore, the KVA rating of the capacitor bank is

$$\text{Capacitor bank rating} = \frac{\sqrt{3} \cdot V_{L-L} \cdot I_c}{1000} = \frac{\sqrt{3} \times 12500 \times 10.8}{1000} = 234\ KVA$$

7.

The single−phase equivalent circuit is shown below where $V_{sn} = 66000/\sqrt{3} = 38100\ V$



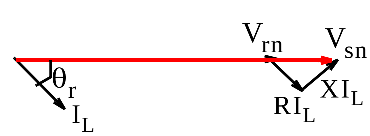
We recall that for a three−phase Y−connected load the three−phase power is given by

$$P_{total} = 3 \cdot |V_{rn}| \cdot |I_L| \cdot (\text{load pf})$$

and thus

$$|I_L| = \frac{P}{3 \cdot |V_{rn}| \cdot pf} = \frac{1000 \times 1000}{3 \cdot |V_{rn}| \cdot 0.8} = \frac{10^6}{2.4 \cdot |V_{rn}|}$$

We choose V_{rn} as a reference vector as shown in the phasor diagram below.



Then, I_L as a vector is

$$I_L = |I_L|(\cos\theta_r - \sin\theta_r) = \frac{10^6}{2.4 \cdot |V_{rn}|}(0.8 - j0.6)$$

Next,

$$V_{sn} = ZI_L + V_{rn} = (80 + j60) \cdot \frac{10^6}{2.4 \cdot |V_{rn}|}(0.8 - j0.6) + V_{rn}$$

and since $(80 + j60) \cdot (0.8 - j0.6) = 100$, V_{sn} and V_{rn} are in–phase and the expression above simplifies to

$$V_{sn} = \frac{10^6}{2.4 \cdot |V_{rn}|} \cdot 100 + V_{rn} = \frac{10^8}{2.4 \cdot |V_{rn}|} + V_{rn}$$

or

$$38100 = \frac{10^8}{2.4 \cdot |V_{rn}|} + V_{rn}$$

$$V_{rn}^2 - 38100V_{rn} + 41.7 \times 10^6 = 0$$

We will use MATLAB to solve this quadratic equation.

```
syms Vrn
solve(Vrn^2–38100*Vrn+41.7*10^6)
```

```
ans =
19050+50*128481^(1/2)
19050-50*128481^(1/2)
```

We can find the magnitude of V_{rn} from either of these two solutions. Thus,

```
a=19050+50*128481^(1/2); abs(a)
```

```
ans =
3.6972e+004
```

That is, $V_{rn} = 36972$, and denoting the potential difference between conductors at the load as V_r, we obtain

$$V_r = \sqrt{3} \times 36972 = 64037 \text{ V}$$

8.

a. The percent efficiency η of the systems is

$$\eta = \frac{\text{Power output}}{\text{Power output} + \text{Line copper losses}} = \frac{P_{out}}{P_{out} + 3I_L^2 R} = \frac{10^6}{10^6 + 3 \times 70^2 \times 15} = 82\%$$

b. The power factor at the sending end is

$$\text{Power factor} = \cos\theta_s = \frac{P_s/\eta}{\sqrt{3} \cdot V_s \cdot I_L} = \frac{10^6/0.82}{\sqrt{3} \times 13200 \times 70} = 0.762$$

Also, $\mathrm{acos}(0.762) = 40.36°$, $\sin\theta_s = \sin(40.36°) = 0.648$. Then,

$$V_{rn(phase)} = V_{sn(phase)} - I_{LN}Z_{LN} = 13200/(\sqrt{3}) - 70(\cos\theta_s - j\sin\theta_s)(15 + j20)$$
$$= 7621 - 70(0.762 - j0.648)(15 + j20) = 5914 - j386 = 5926\angle{-3.74°}\ \text{V}$$

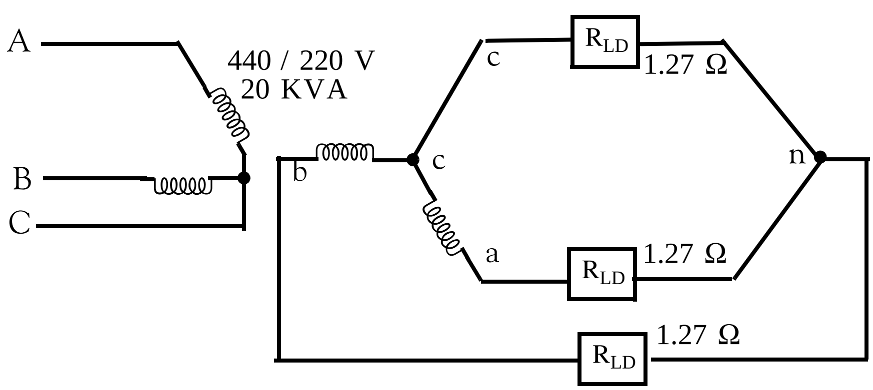
and

$$|V_{rn(line)}| = \sqrt{3} \cdot V_{rn(phase)} = \sqrt{3} \times 5926 = 10264\ \text{V}$$

c. The power factor at the load end is

$$pf_{LD} = \frac{P_{LD}}{\sqrt{3}|V_{rn(line)}||I_L|} = \frac{10^6}{\sqrt{3} \times 10264 \times 70} = 0.80$$

9.



$$V_{AB} = 440\angle 0°\ \text{V} \qquad V_{BC} = 440\angle{-120°}\ \text{V} \qquad V_{CA} = 440\angle 120°\ \text{V}$$

a. Since the voltages on the primary and secondary are in–phase, it follows that:

$$V_{bc} = 220\angle -120° \text{ V}$$

$$V_{ca} = 220\angle 120° \text{ V}$$

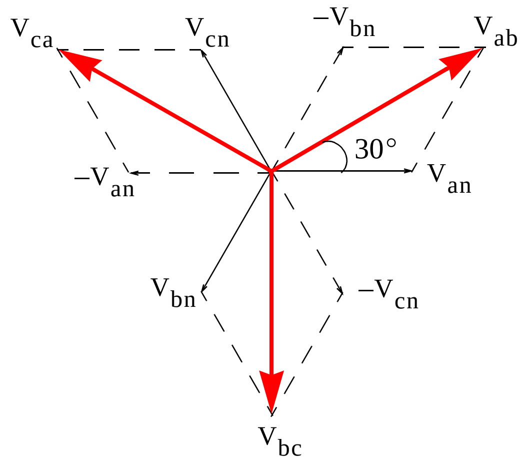
$$V_{ab} = V_{ac} + V_{cb} = -V_{ca} - V_{bc} = 220\angle 0° \text{ V}$$

and we observe that the secondary voltages form a symmetrical three–phase set.

b. The magnitude of the line–to–line voltages on the secondary side are 220 V and since the secondary is connected in Y, the phase voltages are $220/\sqrt{3}$. Accordingly, the magnitude of the current through each R_{LD} is

$$|I_{LD}| = \frac{220/\sqrt{3}}{1.27} = 100 \text{ A}$$

We found that $V_{ab} = 220\angle 0° \text{ V}$, then from the phasor diagram below,



$$V_{an} = \frac{V_{ab}}{\sqrt{3}}\angle -30°$$

$$I_{an} = \frac{V_{an}}{R_{LD}} = 100\angle -30° = I_{ca}$$

and since the secondary voltages form a symmetrical three–phase set, it follows that:

$$I_{bn} = \frac{V_{bn}}{R_{LD}} = 100\angle -150° = I_{ca}$$

$$I_{cn} = \frac{V_{cn}}{R_{LD}} = 100\angle -270° = 100\angle 90° = I_{bc} - I_{ca}$$

The ratio of transformation **a** is 2. Then, the primary currents are:

$$I_A = I_{AC} = (1/a)I_{ca} = 50\angle{-30°}$$

$$I_B = I_{BC} = (1/a)I_{cb} = 50\angle{-150°}$$

$$I_C = I_{CB} + I_{CA} = 50\angle{-270°} = 50\angle{90°}$$

These results show that the input line currents form a symmetrical three–phase set and thus two transformers can also be used for a symmetrical three–phase system.

Chapter 12

Unbalanced Three-Phase Systems

This chapter is an introduction to unbalanced three–phase power systems. It presents several practical examples of analysis applied to unbalanced three–phase systems and a number of observations are made based on the numerical examples. The method of symmetrical components is introduced and a phase sequence indicator serves as an illustration of a Y–connection with floating neutral.

12.1 Unbalanced Loads

Three–phase systems deliver power in enormous amounts to single–phase loads such as lamps, heaters, air–conditioners, and small motors. It is the responsibility of the power systems engineer to distribute these loads equally among the three–phases to maintain the demand for power fairly balanced at all times. While good balance can be achieved on large power systems, individual loads on smaller systems are generally unbalanced and must be analyzed as unbalanced three–phase systems.

Fortunately, many problems involving unbalanced loads can be handled as single–phase problems even though the computations can be three times as long as illustrated by the example below.

Example 12.1

In the three–phase system in Figure 12.1, the load consisting of electric heaters, draw currents as follows:

$$I_a = 150\ A \qquad I_b = 100\ A \qquad I_c = 50\ A$$

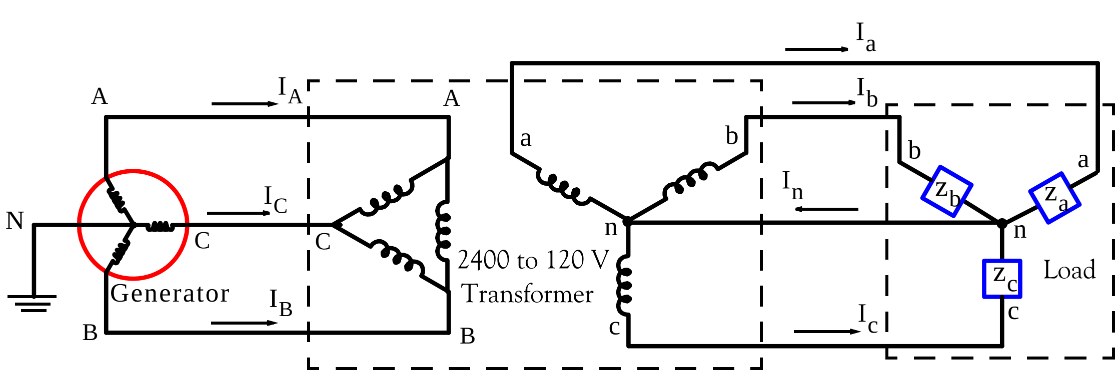


Figure 12.1. Three–phase system for Example 12.1

Find the current in each phase of the Y–connected generator.

Solution:

Let us assume that these currents are balanced in phase. Then,

$$I_a = 150\angle 0° = 150 + j0 \text{ A}$$
$$I_b = 100\angle -120° = -50 - j86.6 \text{ A} \qquad (12.1)$$
$$I_c = 50\angle +120° = -25 + j43.3 \text{ A}$$

and the current in the neutral connection is

$$I_n = I_a + I_b + I_c = 75 - j43.3 = 86.6\angle -30° \text{ A} \qquad (12.2)$$

The currents I_{AB}, I_{BC}, and I_{CA} on the primary side of each transformer is found from the known secondary currents I_a, I_b, and I_c and observing in Figure 12.1 that parallel coils belong to the same transformer, that is, the primary winding AB and the secondary winding nc are on the same transformer and so on, and observing that the transformer turn ratio is 2400 to 120, or 20 to 1, and thus the current ratio is 1 to 20.* Then, assuming that the polarity of the transformer windings is the same for the primary and the secondary, we have:

$$I_{AB} = \frac{I_{nc}}{20} = \frac{I_c}{20} = \frac{-25 + j43.3}{20} = -1.25 + j2.16 = 2.5\angle 120°$$
$$I_{BC} = \frac{I_{na}}{20} = \frac{I_a}{20} = \frac{150 + j0}{20} = 7.5 + j0 = 7.5\angle 0° \qquad (12.3)$$
$$I_{CA} = \frac{I_{nb}}{20} = \frac{I_b}{20} = \frac{-50 - j86.6}{20} = -2.5 - j4.33 = 5\angle -120°$$

Next, we compute the primary line currents I_A, I_B, and I_C which are also the generator phase currents. From Figure 12.1 we observe that

$$I_A = I_{AB} + I_{AC} = I_{AB} - I_{CA} = -1.25 + j2.16 - (-2.5 - j4.33) = 1.25 + j6.49 = 6.61\angle 79.1°$$
$$I_B = I_{BC} + I_{BA} = I_{BC} - I_{AB} = 7.5 + j0 - (-1.25 + j2.16) = 8.75 - j2.16 = 9.01\angle -13.87° \qquad (12.4)$$
$$I_C = I_{CA} + I_{CB} = I_{CA} - I_{BC} = -2.5 - j4.33 - (7.5 + j0) = -10 - j4.33 = 10.90\angle -156.59°$$

Therefore, the magnitude of the current in each phase of the Y–connected generator is 6.61 A, 9.01 A, and 10.90 A, and the rating of a generator to carry this load must have a rating of 11 A per phase or a total rating of $\sqrt{3} \times 2,400 \times 11 = 45.7$ KVA or more.

* *We recall from relation (9.89), Chapter 9, Page 9–29, that* $I_2/I_1 = 1/a$.

The current in the neutral connection of the generator is

$$I_N = I_A + I_B + I_C = 1.25 + j6.49 + 8.75 - j2.16 - 10 - j4.33 = 0 \tag{12.5}$$

as expected since there is no circuit in which it can flow.

The primary phase currents I_{AB}, I_{BC}, I_{CA} and the line currents I_A, I_B, I_C, are shown in the phasor diagram in Figure 12.2.

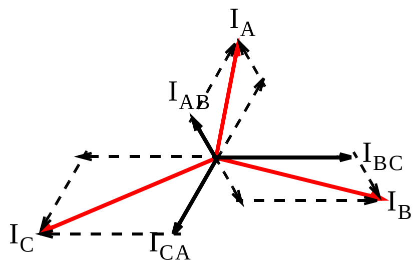


Figure 12.2. Phasor diagram for the primary phase and line currents in Example 12.1

12.2 Voltage Computations

In Example 12.1 above we did not consider the actual voltages at the load. If we assume that these voltages are 120 volts, line to neutral, and balanced, the voltage at the generator will be somewhat greater than the nominal value of 2400 volts because of the impedances in the system. This will be considered in Example 12.2 below.

Example 12.2

For the three−phase system in Figure 12.3, compute the generator voltages V_{AB}, V_{BC}, and V_{CA}. Assume that each transformer impedance on the high side is $j30\ \Omega$ and the transformer resistances are negligible. Assume also that the lines are very short and thus their impedances can are also negligible. The transformer secondary voltages are assumed to be as follows:

$$\begin{aligned} V_{an} &= 120\angle 0° = 120 + j0 \\ V_{bn} &= 120\angle -120° = -60 - j104 \\ V_{cn} &= 120\angle +120° = -60 + j104 \end{aligned} \tag{12.6}$$

Solution:

From Example 12.1, relation (12.3),

$$I_{AB} = -1.25 + j2.16 \qquad I_{BC} = 7.5 + j0 \qquad I_{CA} = -2.5 - j4.33$$

The voltage ratio is 20 to 1.* Therefore, the transformer primary voltages, line−to−line, are as follows:

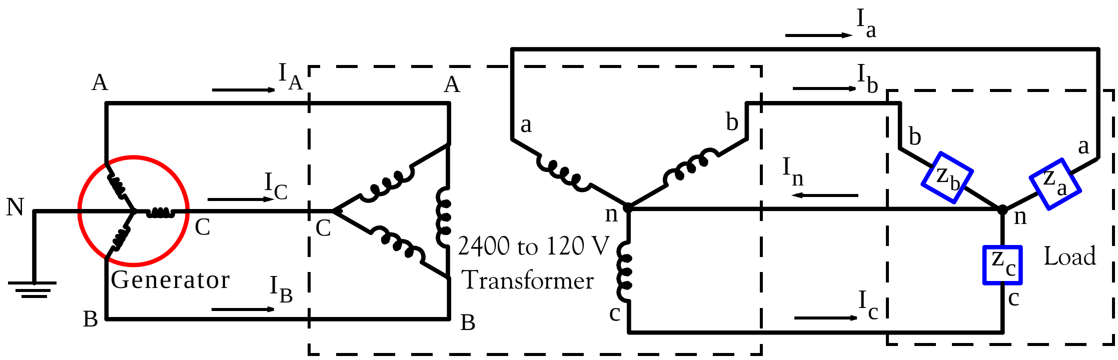


Figure 12.3. Three–phase system for Example 12.2

$$V_{AB} = 20V_{cn} + I_{AB}Z = (-1200 + j2080) + (-1.25 + j2.16)j30 = -1265 + j2043 = 2403\angle 121.8°$$

$$V_{BC} = 20V_{an} + I_{BC}Z = (2400) + (7.5 + j0)j30 = 2400 + j225 = 2411\angle 5.4°$$

$$V_{CA} = 20V_{bn} + I_{CA}Z = (-1200 - j2080) + (-2.5 - j4.33)j30 = -1270 - j2155 = 2406\angle -116.4°$$

Figure 12.4 below is the phasor diagram for these voltages.

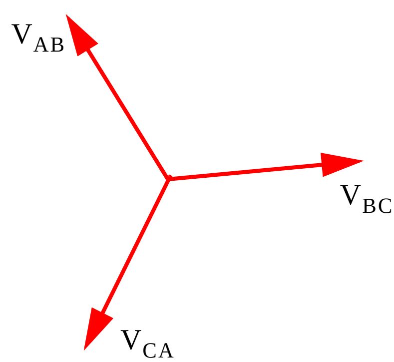


Figure 12.4. Phasor diagram for Example 12.2

The computations in Example 12.2 are accurate. However, the approach is not practical. A practical approach would be to begin with the assumption that the generator voltage is constant at 2400 volts and compute the load (heaters) voltages given their resistances. This can be done with loop or mesh equations and this approach will be used in the next example.

12.3 Phase–Sequence Indicator

The phase sequence is essential with rotating machines. The rotation of a generator in a clockwise direction may develop voltages of phase sequence $a - b - c$ while the rotation in a counterclock-

* *We recall from relation (9.99), Chapter 9, Page 9–30, that* $V_2/V_1 = a$.

wise direction will develop voltages of phase sequence $c-b-a$. The direction of rotation of an induction motor will be reversed if two line connections are interchanged. Using a device called *phase–sequence indicator*, we can prove that the currents in the three phases of an unbalanced Y–connected load are dependent on the phase sequence of the source. This will be illustrated with Example 12.3 below.

Example 12.3

Figure 12.5 shows a typical phase–indicator consisting of two resistors representing two light bulbs each rated 15 watts, 120 volts at 60 Hz frequency, and a 2 μF capacitor connected to a 120 volt three–phase system.

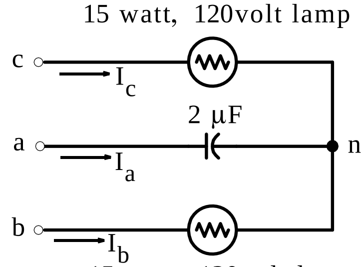


Figure 12.5. A phase–sequence indicator

The instructions provided by the manufacturer of this device states that after connecting the circuit as shown, we should attach line a to the middle (capacitor) terminal. Then, the lamp that lights is in line b. In the discussion that follows we will prove that only one of the lamps lights and which one.

Let us assign currents I_1 and I_2 as shown in Figure 12.6, and assume that

$$\begin{aligned} V_{ab} &= 120\angle 0° = 120 + j0 \\ V_{bc} &= 120\angle -120° = -60 - j104 \\ V_{ca} &= 120\angle +120° = -60 + j104 \end{aligned} \tag{12.7}$$

At the frequency $f = 60\ Hz$, the capacitive reactance is

$$X_C = -1/\omega C = -10^6/(2\pi \times 60 \times 2) = -1326\ \Omega$$

and the resistance of each lamp is

$$R = V^2/P = 120^2/15 = 960\ \Omega^*$$

* *For a balanced 3–phase load we must have* $|X_C| = R$

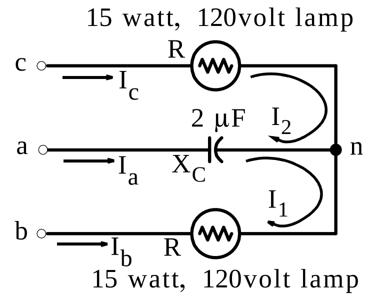


Figure 12.6. The phase−sequence indicator with assigned mesh currents

The mesh equations are

$$\begin{aligned}(R+jX_C)I_1 - jX_C I_2 &= V_{ab}\\ -jX_C I_1 + (R+jX_C)I_2 &= V_{ca}\end{aligned} \tag{12.8}$$

By Cramer's rule,

$$I_1 = \frac{\begin{bmatrix} V_{ab} & -jX_C \\ V_{ca} & R+jX_C \end{bmatrix}}{\begin{bmatrix} R+jX_C & -jX_C \\ -jX_C & R+jX_C \end{bmatrix}} = \frac{RV_{ab} + jX_C V_{ab} + jX_C V_{ca}}{(R+jX_C)^2 - (-X_C)^2} = \frac{RV_{ab} + jX_C(V_{ab}+V_{ca})}{(R+jX_C)^2 + X_C^2}$$

and since

$$V_{ab} + V_{ca} = V_{cb} = -V_{bc}\ ^*$$

we obtain

$$I_1 = \frac{RV_{ab} + jX_C(V_{cb})}{(R+jX_C)^2 + X_C^2} = \frac{RV_{ab} - jX_C(V_{bc})}{R^2 + j2RX_C}$$

and by substitution of numerical values we obtain

$$I_1 = \frac{960\times 120 - j(-1326)\times(-60 - j104)}{960^2 + 2j\times 960\times(-1326)} = 0.098\angle 52.6°\ A \tag{12.9}$$

By a similar procedure we obtain

$$I_2 = \frac{RV_{ca} - jX_C(V_{bc})}{R^2 + j2RX_C}$$

and by substitution of numerical values we obtain

$$I_2 = \frac{960\times(-60 + j104) - j(-1326)\times(-60 - j104)}{960^2 + 2j\times 960\times(-1326)} = 0.031\angle 84.3°\ A \tag{12.10}$$

* *See Figure 12.6*

The rated current for the $15-\text{watt}$ lamp is $15/120 = 0.125\text{ A}$ and from (12.8) we observe that the value of I_1 is approximately 80% of its rated current, and this is sufficient to light the lower lamp in Figure 12.6 though not to full brilliance. However, the value of I_2 is about one−fourth of the rated value of the lamp, and this is not sufficient to produce a noticeable brightness. Thus we have shown that one lamp lights brightly, and the other hardly at all, and that the lamp in line b is the bright one. More importantly, we have shown that the phase sequence does make a difference.

12.4 Δ−Y Transformation

We can substitute a Y−connected load such as that of the phase−sequence indicator in Figure 12.6, with a Δ−connected load and solve for phase and then for line currents.

Example 12.4

Figure 12.7(a) below is the same as the phase−sequence indicator as in Figure 12.6. We wish to find the equivalent Δ shown in 12.7(b).

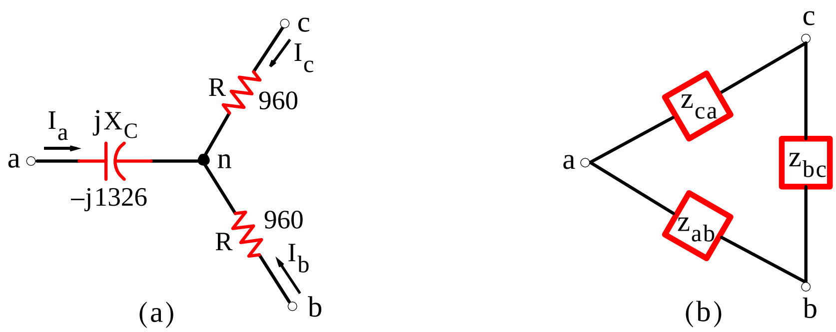


Figure 12.7. Y to Δ transformation for Example 12.4

Solution:

We begin with the application of the relations (11.45), Page 11−15, Chapter 11 which are repeated below for convenience, where we have substituted Z_1, Z_2, and Z_3 with Z_{ab}, Z_{bc}, and Z_{ca} respectively.

$$Z_{ab} = \frac{Z_aZ_b + Z_bZ_c + Z_cZ_a}{Z_b} \qquad Z_{bc} = \frac{Z_aZ_b + Z_bZ_c + Z_cZ_a}{Z_a} \qquad Z_{ca} = \frac{Z_aZ_b + Z_bZ_c + Z_cZ_a}{Z_c}$$

$$Y \to \Delta \text{ Conversion}$$

With reference to Figure 12.7, we obtain the following relations:

$$Z_{ab} = \frac{Z_aZ_b + Z_bZ_c + Z_cZ_a}{Z_b} = \frac{jXR + R^2 + jXR}{R} = R + j2X = 960 - j2652 = 2820\angle -70.1°$$

$$Z_{bc} = \frac{Z_aZ_b + Z_bZ_c + Z_cZ_a}{Z_a} = \frac{jXR + R^2 + jXR}{jX} = 2R - j\frac{R^2}{X} = 1920 + j695 = 2042\angle 19.9°$$

$$Z_{ca} = Z_{ab} = 2820\angle -70.1°$$

From (12.7),

$$V_{ab} = 120\angle 0° = 120 + j0$$

$$V_{bc} = 120\angle -120° = -60 - j104$$

$$V_{ca} = 120\angle +120° = -60 + j104$$

and the phase currents in the Δ connection are:

$$I_{ab} = \frac{V_{ab}}{Z_{ab}} = \frac{120\angle 0°}{2820\angle -70.1°} = 0.0426\angle 70.1° = 0.0145 + j0.0401$$

$$I_{bc} = \frac{V_{bc}}{Z_{bc}} = \frac{120\angle -120°}{2042\angle 19.9°} = 0.0588\angle -139.9° = 0.0450 - j0.0379$$

$$I_{ca} = \frac{V_{bc}}{Z_{bc}} = \frac{120\angle +120°}{2820\angle -70.1°} = 0.0426\angle 190.1° = -0.0419 - j0.0075$$

and the currents in Figure 12.7(a) or Figure 12.6 are:

$$I_a = I_{ab} - I_{ca} = 0.0564 + j0.0475 = 0.0736\angle 40.1°$$

$$I_b = I_{bc} - I_{ab} = -0.0595 - j0.079 = 0.0980\angle -127.4° \qquad (12.11)$$

$$I_c = I_{ca} - I_{bc} = 0.031 + j0.0304 = 0.0306\angle 84.2°$$

We observe that

$$I_a + I_b + I_c = 0$$

from (12.10) and (12.11)

$$I_2 = I_c$$

and from (12.9) and (12.11)

$$I_1 = -I_b$$

12.5 Practical and Impractical Connections

A Y−connected system with a floating neutral should be avoided because the load may become unbalanced. The reason becomes obvious by considering the phasor diagram in Figure 12.8.

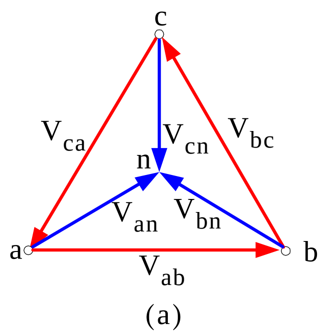


Figure 12.8. Phasor diagrams for balanced and unbalanced loads

In Figure 12.8(a) above the load is assumed to be balanced and thus the neutral point n is at the center of the triangle. However, if the load becomes unbalanced, the neutral point n moves away from the center as shown in Figure 12.8(b). An example where this may occur is the three−phase distribution system shown in Figure 12.9 below, and thus this arrangement is impractical and should be avoided.

Another example of an impractical distribution system is shown in Figure 12.10 where a $Y-Y$ transformer bank and a $Y-\Delta$ transformer bank are connected in parallel on both the primary and secondary sides. The problem here is that one transformer bank shifts the voltages $30°$ * and the other does not.

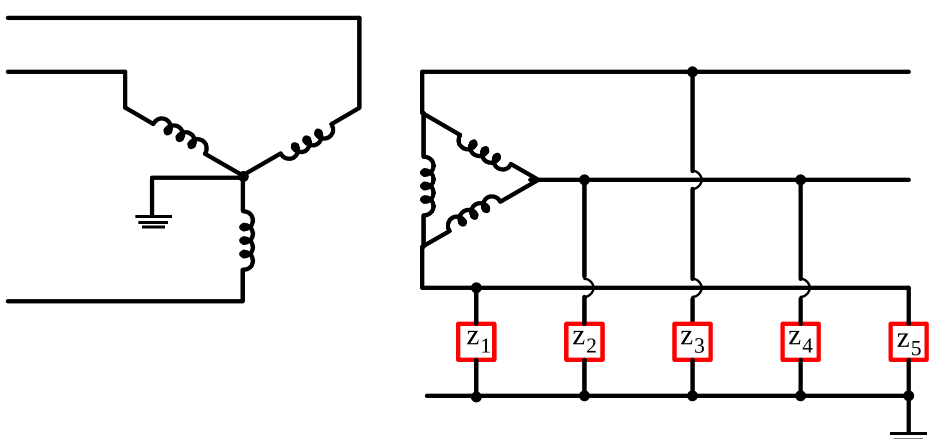


Figure 12.9. An impractical configuration for a three−phase distribution system

* *We recall that in a Y−connected system the line and phase voltages are different whereas in a Δ−connected system they are the same.*

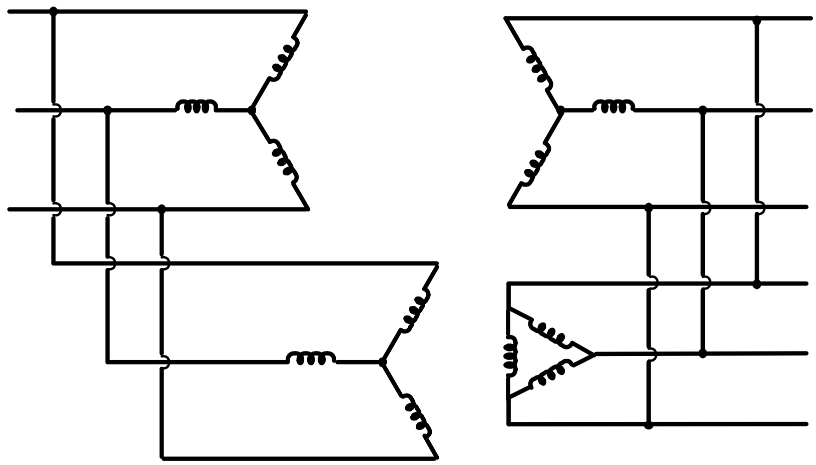


Figure 12.10. Another impractical configuration for a three−phase distribution system

Figure 12.11 shows an $open-\Delta$ connection on both the primary and secondary sides.

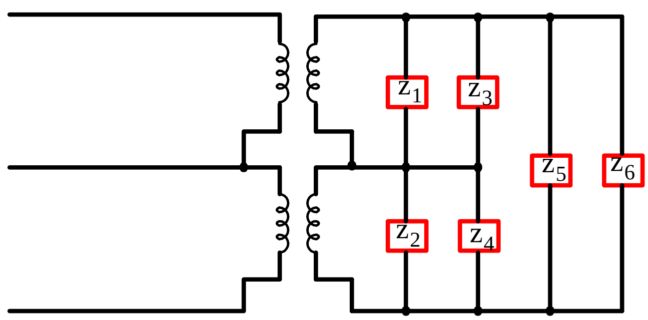


Figure 12.11. A practical open−Δ connection

This is the same as a standard $\Delta-\Delta$ connection but with one transformer omitted on both sides. This is a practical connection and it is convenient for temporary installations that are not heavily loaded. We observe that this arrangement provides three line−to−line voltages with the correct magnitude and phase.

12.6 Symmetrical Components

The analysis of unbalanced three−phase systems can be greatly simplified with the principle of *symmetrical components*. This principle states that any three vectors can be represented by three sets of balanced vectors. Thus, when applied to three−phase currents, any three current phasors can be replaced by three sets of balanced currents, and when applied to three−phase voltages, any three voltage phasors can be replaced by three sets of balanced voltages.

The voltages or currents at a point of unbalance in a three−phase system are determined and replaced by three sets of components known as *positive phase sequence*, *negative phase sequence*, and *zero phase sequence*. The positive phase sequence, negative phase sequence, and zero phase sequence voltages or currents are determined independently and the actual unbalanced voltages or currents are found by adding these three−phase sequences. Thus the solution of a difficult problem involving unbalanced voltages or currents is simplified to the solution of three easy problems involving only balanced voltages or currents.

Example 12.5

Show that the three unbalance current phasors in Figure 12.12(a) are the sum of the three balanced currents shown in Figure 12.12(b).

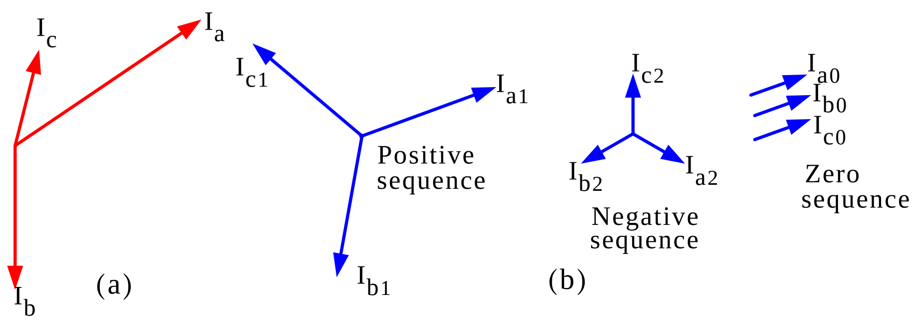


Figure 12.12. (a) Unbalanced currents and (b) their symmetrical components.

In symmetrical components, a symmetrical set of vectors as shown in Figure 12.12(b) above, are equal in length, and equally spaced in angle. The symmetrical sets of three vectors such as those shown in Figure 12.12(b) are related by equation (12.12) below.

$$I_{an} = I_{bn}\angle n \cdot 120° = I_{cn}\angle 2n \cdot 120° \tag{12.12}$$

For the positive−sequence we set $n = 1$, and thus

$$I_{a1} = I_{b1}\angle 120° = I_{c1}\angle 240° \tag{12.13}$$

In other words, for the positive−phase sequence set the order is $a - b - c - a - b - c - \ldots$ as shown in Figure 12.13 below.

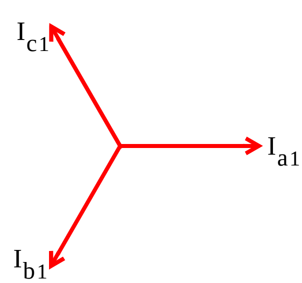


Figure 12.13. Positive sequence phasor diagram

For the negative−sequence we set $n = 2$, and thus

$$I_{a2} = I_{b2}\angle 240° = I_{c2}\angle 480° \quad (12.14)$$

or

$$I_{a2} = I_{b2}\angle -120° = I_{c2}\angle 120° \quad (12.15)$$

The same symmetrical set results by letting $n = -1$, and this accounts for the name of *negative* sequence. Thus,

$$I_{a2} = I_{b2}\angle -120° = I_{c2}\angle -240° \quad (12.16)$$

In other words, for the negative−phase sequence set the order is $c-b-a-c-b-a-\ldots$ as shown in Figure 12.14 below.

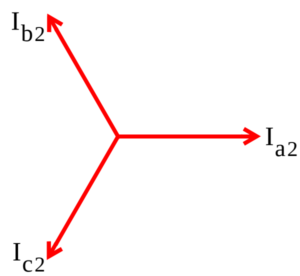


Figure 12.14. Negative sequence phasor diagram

For the zero−sequence we set $n = 3$ or $n = 0$, and the latter accounts for the *zero−sequence* name. The three components that comprise the zero−sequence set are equal in both magnitude and phase, and thus it is unnecessary to denote them as I_{a0}, I_{b0}, and I_{c0}. Instead, we use the single notation I_0 for any of the zero−sequence components, i.e.,

$$I_0 = I_{a0} = I_{b0} = I_{c0} \quad (12.17)$$

Now, let us return to Figure 12.12, Example 12.5, to prove that the addition of the positive−sequence, negative−sequence, a zero−sequence components in Figure 12.12(b) are added graphically to obtain the unbalanced set in Figure 12.12(a). The addition is shown in Figure 12.15 below.

The addition of the three symmetrical sets to obtain one unbalanced set is easy as shown in Figure 12.15. We will now derive three equations for finding the three symmetrical component sets of any three unbalanced phasors.

We begin the derivation with the definitions in the system of the three equations below.

$$\begin{aligned} I_{a1} + I_{a2} + I_0 &= I_a \\ I_{b1} + I_{b2} + I_0 &= I_b \\ I_{c1} + I_{c2} + I_0 &= I_c \end{aligned} \quad (12.18)$$

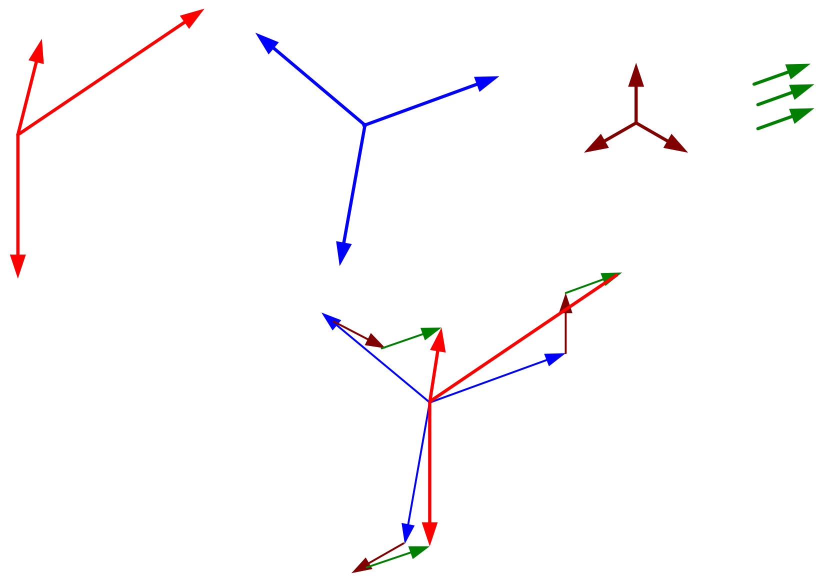


Figure 12.15. Addition of the symmetrical components to obtain an unbalanced three−phase set.

From (12.13),

$$I_{b1} = I_{a1}\angle{-120°} \tag{12.19}$$

and

$$I_{c1} = I_{a1}\angle{+120°} \tag{12.20}$$

From (12.15),

$$I_{b2} = I_{a2}\angle{+120°} \tag{12.21}$$

and

$$I_{c2} = I_{a2}\angle{-120°} \tag{12.22}$$

Substitution of (12.19) through (12.22) into (12.18) yields

$$\begin{aligned} I_{a1} \qquad\quad + I_{a2} \qquad\quad + I_0 &= I_a \\ I_{a1}\angle{-120°} + I_{a2}\angle{+120°} + I_0 &= I_b \\ I_{a1}\angle{+120°} + I_{a2}\angle{-120°} + I_0 &= I_c \end{aligned} \tag{12.23}$$

Adding the three equations in (12.23), we observe that the first two columns vanish, and thus

$$3I_0 = I_a + I_b + I_c$$

or

$$I_0 = \frac{1}{3}(I_a + I_b + I_c) \tag{12.24}$$

Next, we multiply the second equation in (12.23) by $1\angle{+120°}$ and the third equation by $1\angle{-120°}$ and we add again. This time the second and third columns in (12.23) vanish, leaving

$$3I_{a1} = I_a + I_b\angle{+120°} + I_c\angle{-120°}$$

or

$$I_{a1} = \frac{1}{3}(I_a + I_b\angle{+120°} + I_c\angle{-120°}) \tag{12.25}$$

Finally, we multiply the second equation in (12.23) by $1\angle{-120°}$ and the third equation by $1\angle{+120°}$ and we add again. We observe that the first and third columns in (12.23) vanish, leaving

$$I_{a2} = \frac{1}{3}(I_a + I_b\angle{-120°} + I_c\angle{+120°}) \tag{12.26}$$

Therefore, with (12.24) through (12.26) we can compute the symmetrical components of any unbalanced three−phase using the set of equations in (12.27) below.

$$\begin{aligned} I_0 &= \frac{1}{3}(I_a + I_b + I_c) \\ I_{a1} &= \frac{1}{3}(I_a + I_b\angle{+120°} + I_c\angle{-120°}) \\ I_{a2} &= \frac{1}{3}(I_a + I_b\angle{-120°} + I_c\angle{+120°}) \end{aligned} \tag{12.27}$$

It is customary to let $a = 1.0\angle 120°$ and $a^2 = 1.0\angle 240° = 1.0\angle{-120°}$ be unity vectors that apply the appropriate shift. Then, (12.27) can be expressed as

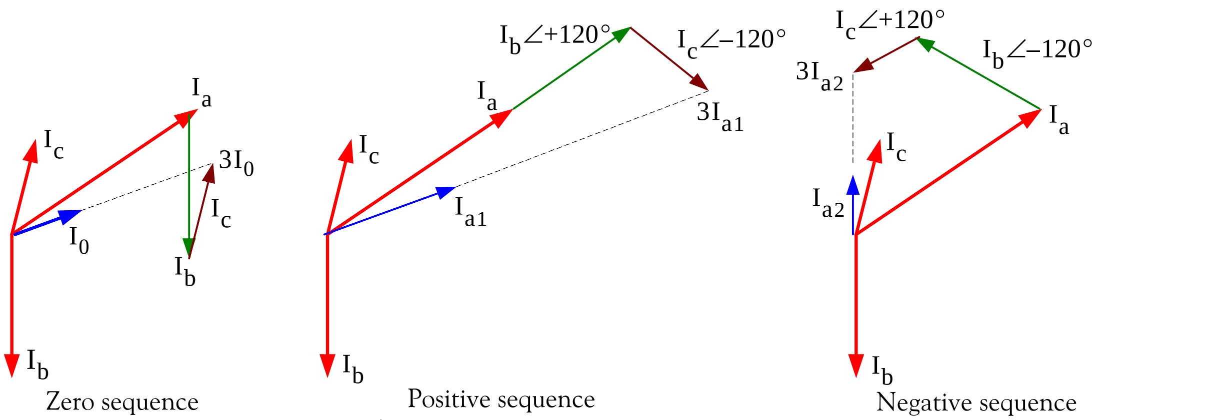
$$\begin{aligned} I_0 &= \frac{1}{3}(I_a + I_b + I_c) \\ I_{a1} &= \frac{1}{3}(I_a + aI_b + a^2 I_c) \\ I_{a2} &= \frac{1}{3}(I_a + a^2 I_b + aI_c) \end{aligned} \tag{12.28}$$

Example 12.6

In Example 12.5 the symmetrical components were presented without any explanation of where they came from. In this example, we will find the symmetrical components using (12.27).

Solution:

The method of analysis is illustrated in Figure 12.16 below where the phasors I_a, I_b, and I_c are the same as in Figure 12.15.



$I_0 = \frac{1}{3}(I_a + I_b + I_c)$ $\quad I_{a1} = \frac{1}{3}(I_a + I_b\angle{+120°} + I_c\angle{-120°})$ $\quad I_{a2} = \frac{1}{3}(I_a + I_b\angle{-120°} + I_c\angle{+120°})$

Figure 12.16. Analysis of an unbalanced three–phase set to find symmetrical components

The zero–sequence component I_0 is found by adding dashed lines equal to I_b and I_c at the tip of I_a, and one–third of the resultant is marked off as I_0 in accordance with the first equation in (12.27).

The positive–sequence component I_{a1} is found by adding a line equal to I_b rotated by $120°$ at the tip of I_a, and then a line equal to I_c rotated by $-120°$. In accordance with the second equation in (12.27), one–third of the resultant is I_{a1}.

The negative–sequence component I_{a2} is found by applying the third equation in (12.27) in a similar manner.

The complete symmetrical components system is by adding the phasors I_{b1} and I_{c1} after being rotated by the appropriate phase shift to the positive–sequence set, and by adding the phasors I_{b2} and I_{c2} after being rotated by the appropriate phase shift to the negative–sequence set as shown in Figure 12.17 below.

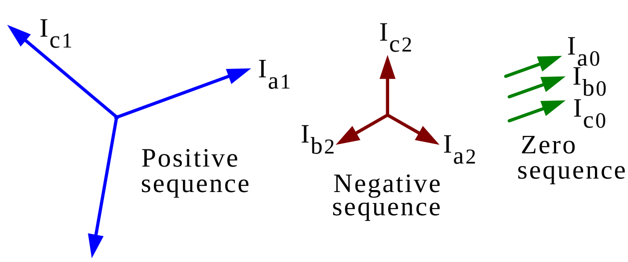


Figure 12.17. The complete symmetrical components set for Example 12.6

Three more problems on symmetrical components are given as exercises at the end of this chapter. Because symmetrical components are phasors, the computations can be facilitated with the use of MATLAB and /or Simulink as illustrated in Exercise 3 at the end of this chapter.

12.7 Cases where Zero−Sequence Components are Zero

Let us consider a Y−connected load with floating neutral shown in Figure 12.18.

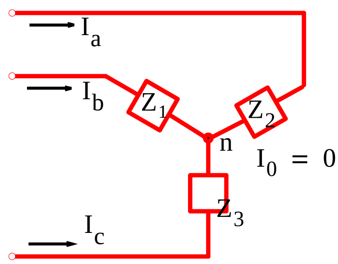


Figure 12.18. Y−connected load with floating neutral

The three−phase Y−connected load with floating neutral point n shown in Figure 12.18 can have no zero−sequence component. This can be shown from relation (12.24), i.e.,

$$I_0 = \frac{1}{3}(I_a + I_b + I_c)$$

and with a floating neutral, $I_a + I_b + I_c = 0$, and thus $I_0 = 0$ regardless whether the load impedances are unbalanced and what the applied voltages may be.

Next, let us consider a Y−connected load with the neutral point n connected to a ground as shown in Figure 12.19.

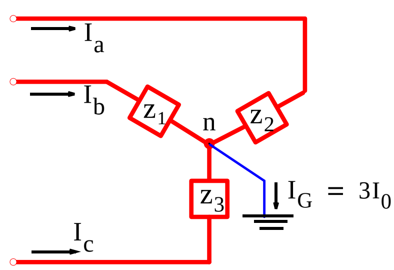


Figure 12.19. Y−connected load with grounded neutral

In Figure 12.19,

$$I_a + I_b + I_c = I_G$$

and since

$$I_0 = \frac{1}{3}(I_a + I_b + I_c)$$

it follows that

$$I_G = 3I_0$$

Now, let us consider the Δ–connected load shown in Figure 12.20.

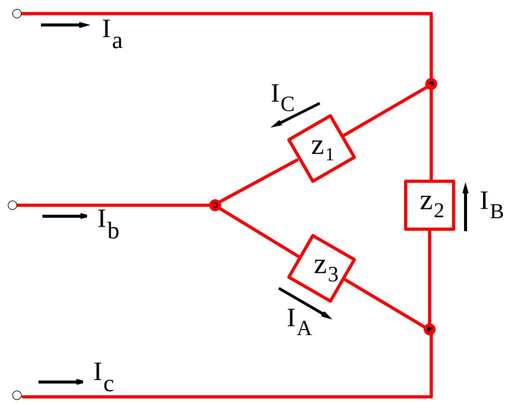


Figure 12.20. Δ–connected load showing line and phase currents

In Figure 12.20, the three line currents I_a, I_b, and I_c that supply the Δ–connected load have no zero–sequence component because $I_a + I_b + I_c = 0$. However, the sum of the phase currents I_A, I_B, and I_C do not necessarily add to zero; they may, they may not.

If there is a zero–sequence current in the Δ–connected load, it is a circulating current as indicated by the arrows for the phase currents I_A, I_B, and I_C. If there is only zero–sequence current flowing, these three currents are all in the arrow direction at the same instant. Then, they reverse all in the opposite direction together. In other words, the current flows first one way around the Δ–connected load, then the other way, but never gets our of the Δ.

A similarity applies to line–to–line voltages and line to neutral voltages. Zero–sequence voltage is one–third the sum of the three line–to–line voltages and these when circulated around a closed path always add to zero. But there may be a zero–sequence component of the line–to–neutral voltages.

Example 12.7

The three–phase generator in Figure 12.21 is connected to a transmission line through a transformer bank. There is no load at the other end of the transmission line system. One wire of the transmission line breaks and falls to the ground resulting in a line–to–ground short circuit. Derive the symmetrical component currents and total currents produced by the generator.

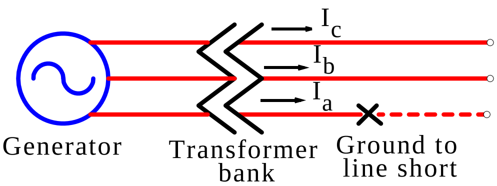


Figure 12.21. Three−phase system with a line−to−ground fault

Solution:

The system is balanced except at the point of fault indicated in Figure 12.21, and the fault current is I_a. Because no load is connected to the system, currents I_b and I_c are both zero.

The positive−, negative−, and zero−sequence currents at the point of fault are found from the system of equations of (12.27), i.e.,

$$I_0 = \frac{1}{3}(I_a + I_b + I_c)$$

$$I_{a1} = \frac{1}{3}(I_a + I_b\angle{+120°} + I_c\angle{-120°})$$

$$I_{a2} = \frac{1}{3}(I_a + I_b\angle{-120°} + I_c\angle{+120°})$$

and since $I_b = 0$ and $I_c = 0$, from the equations above we find that

$$I_0 = \frac{1}{3}I_a \qquad I_{a1} = \frac{1}{3}I_a \qquad I_{a2} = \frac{1}{3}I_a$$

Hence

$$I_{a1} = I_{a2} = I_0$$

as shown in Figure 12.22.

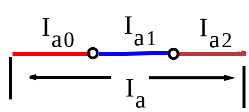


Figure 12.22. The symmetrical components for Example 12.7

Also, since the line currents I_b and I_c are both zero, we have

$$I_{b1} + I_{b2} + I_{a0} = 0$$

and

$$I_{c1} + I_{c2} + I_{a0} = 0$$

Symmetrical components are used in the calculation of fault currents since the total fault current is not symmetrical. It includes a DC component which depends on the point at which the fault is initiated.

The four types of faults that can occur in a three–phase system are shown in Figure 12.23

In the calculation of a three–phase fault only positive sequence components are considered, in the calculation of a line–to–line fault positive and negative sequence components are considered, and in the calculation of a line–to–neutral fault or in a line–to–ground fault, all three sequences, that is, positive, negative, and zero sequences are considered.

The calculation of fault currents is a laborious procedure since the degree of asymmetry is not the same in all three phases. Detailed discussion on this topic is beyond the scope of this book. This topic is discussed in power systems books, in General Electric™, Westinghouse™, and other reference books, and also in the Internet. Computer programs are available for the calculations and these can also be found in the Internet.

The MathWorks SimPowerSystems documentation contains several demos with three–phase faults. Four of them can be accessed by typing `power_machines`, `power_svc_pss`, `power_wind_dfig`, and `power_3phseriescomp` at the MATLAB command prompt. An example with a DC line fault can also be accessed by typing `power_hvdc12pulse` at the MATLAB command prompt.

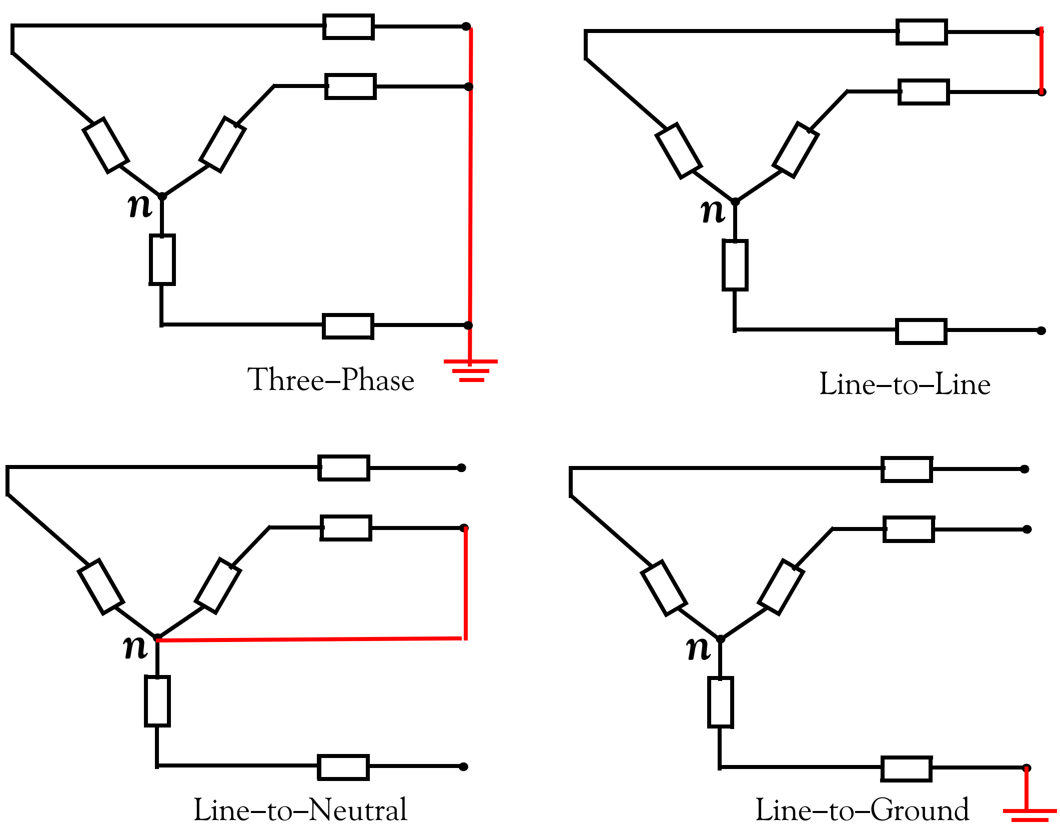


Figure 12.23. Types of faults in three–phase systems

12.8 Summary

- Loads connected to three−phase systems must be distributed equally among the three−phases to maintain the demand for power fairly balanced at all times.
- Loads are generally unbalanced and must be analyzed as unbalanced three−phase systems.
- Many problems involving unbalanced loads can be handled as single−phase problems even though the computations can be three times as long.
- A practical approach to compute load voltages, line currents, and load currents is to use loop or mesh equations.
- The phase sequence is essential with rotating machines. The rotation of a generator in a clockwise direction may develop voltages of phase sequence $a-b-c$ while the rotation in a counterclockwise direction will develop voltages of phase sequence $c-b-a$. The direction of rotation of an induction motor will be reversed if two line connections are interchanged.
- We can prove that the currents in the three phases of an unbalanced Y−connected load are dependent on the phase sequence of the source using a phase−sequence indicator.
- The analysis of unbalanced three−phase systems can be greatly simplified with the method of symmetrical components. This principle states that any three vectors can be represented by three sets of balanced vectors. Thus, when applied to three−phase currents, any three current phasors can be replaced by three sets of balanced currents, and when applied to three−phase voltages, any three voltage phasors can be replaced by three sets of balanced voltages.
- Using the method of symmetrical components the voltages or currents at a point of unbalance in a three−phase system are determined and replaced by three sets of components known as *positive phase sequence*, *negative phase sequence*, and *zero phase sequence*. The positive phase sequence, negative phase sequence, and zero phase sequence voltages or currents are determined independently and the actual unbalanced voltages or currents are found by adding these three−phase sequences. Thus the solution of a difficult problem involving unbalanced voltages or currents is simplified to the solution of three easy problems involving only balanced voltages or currents.
- In symmetrical components the vectors are equal in length, and equally spaced in angle. The symmetrical sets of three vectors are related by equation

$$I_{an} = I_{bn}\angle n\cdot 120° = I_{cn}\angle 2n\cdot 120°$$

For the positive−sequence we set $n = 1$, and thus

$$I_{a1} = I_{b1}\angle 120° = I_{c1}\angle 240°$$

In other words, for the positive−phase sequence set the order is $a-b-c-a-b-c-\ldots$.

- For the negative−sequence we set $n = 2$, and thus

$$I_{a2} = I_{b2}\angle 240° = I_{c2}\angle 480°$$

or

$$I_{a2} = I_{b2}\angle -120° = I_{c2}\angle 120°$$

The same symmetrical set results by letting $n = -1$, and this accounts for the name of negative sequence. Thus,

$$I_{a2} = I_{b2}\angle -120° = I_{c2}\angle -240°$$

In other words, for the negative−phase sequence set the order is $c-b-a-c-b-a-\ldots$.

- For the zero−sequence we set $n = 3$ or $n = 0$, and the latter accounts for the zero−sequence name. The three components that comprise the zero−sequence set are equal in both magnitude and phase, and thus it is unnecessary to denote them as I_{a0}, I_{b0}, and I_{c0}. Instead, we use the single notation I_0 for any of the zero−sequence components, i.e.,

$$I_0 = I_{a0} = I_{b0} = I_{c0}$$

- The three symmetrical sets are related as shown in the system of the three equations below.

$$I_{a1}+I_{a2}+I_0 = I_a \qquad I_{b1}+I_{b2}+I_0 = I_b \qquad I_{c1}+I_{c2}+I_0 = I_c$$

- We can compute the symmetrical components of any unbalanced three−phase using the set of equations below.

$$I_0 = \frac{1}{3}(I_a+I_b+I_c) \quad I_{a1} = \frac{1}{3}(I_a+I_b\angle +120°+I_c\angle -120°) \quad I_{a2} = \frac{1}{3}(I_a+I_b\angle -120°+I_c\angle +120°)$$

or in terms of the unity vectors $a = 1.0\angle 120°$ and $a^2 = 1.0\angle 240° = 1.0\angle -120°$

$$I_0 = \frac{1}{3}(I_a+I_b+I_c) \qquad I_{a1} = \frac{1}{3}(I_a+aI_b+a^2I_c) \qquad I_{a2} = \frac{1}{3}(I_a+a^2I_b+aI_c)$$

- A three−phase Y−connected load with floating neutral point can have no zero−sequence component regardless whether the load impedances are unbalanced and what the applied voltages may be.

- In a three−phase Y−connected load with neutral point connected to a ground, $I_G = 3I_0$ where I_G is the current flowing in the wire that connects the neutral point to the ground.

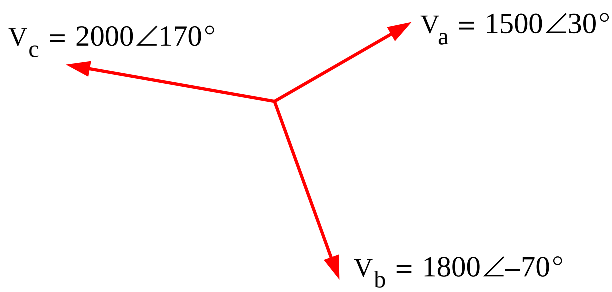
- In a three−phase system the three line currents I_a, I_b, and I_c that supply a Δ−connected load have no zero−sequence component because $I_a+I_b+I_c = 0$. However, the sum of the phase currents I_A, I_B, and I_C do not necessarily add to zero; they may, they may not.

12.9 Exercises

1. Balanced three–phase voltage 220 volts line–to–line, positive–phase sequence, is supplied to a load that is Y–connected, floating neutral, with $500\ \Omega$ resistors from neutral to lines a and b, and a capacitor whose capacitive reactance is $500\ \Omega$ to line c. Compute the current in each phase and draw a phasor diagram.

2. A good phase–sequence indicator operates with one lamp very bright and the other very dim. Using the same lamps as in Example 12.3, Page 12–5, but with a capacitor of different value, can you design a better indicator?

3. Resolve the unbalanced three–phase system shown below into its symmetrical components.



4. The voltages of an unbalanced three–phase supply are $V_a = 200 + j0$, $V_b = -j200$, and $V_c = -100 + j200$. Connected in Y across this supply are three equal impedances each $20 + j10\ \Omega$. There is no connection between the Y neutral and the supply neutral. Derive the symmetrical components of phase a and compute the three line currents.

5. The voltages of an unbalanced three–phase supply are $V_a = 150\angle 0°$, $V_b = 86.6\angle -90°$, and $V_c = 86.6\angle 90°$.

   a. Derive the symmetrical components of V_a.

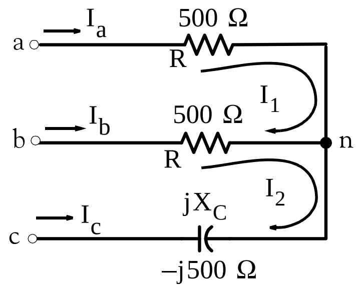
   b. Derive the symmetrical components of V_b and V_c from the symmetrical components of V_a found in part (a).

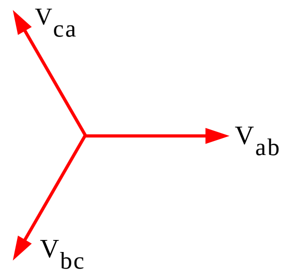
   c. Draw a phasor diagram showing all symmetrical components.

6. The currents in a three–phase system are $I_a = 5.00$, $I_b = -j8.66$, and $I_c = j10.00$. Compute I_{a1}, I_{a2}, and I_0. Sketch phasors of the three positive–sequence components, the three negative–sequence components, and the zero–sequence component.

12.10 Solutions to End–of–Chapter Exercises

1.





By KVL

$$\begin{aligned} 2RI_1 - RI_2 &= V_{ab} = 220\angle 0° = 220 + j0 \\ -RI_1 + (R + jXc)I_2 &= V_{bc} = 220\angle -120° = -110 - j190 \end{aligned} \tag{12.29}$$

and by Cramer's rule

$$I_1 = \frac{D_1}{\Delta} \qquad I_2 = \frac{D_2}{\Delta}$$

where the determinant Δ is

$$\Delta = \begin{vmatrix} 2R & -R \\ -R & R + jXc \end{vmatrix} = 2R^2 + j2RXc - R^2 = R^2 + j2RXc$$

and

$$D_1 = \begin{vmatrix} V_{ab} & -R \\ V_{bc} & R + jXc \end{vmatrix} = RV_{ab} + jXcV_{ab} + RV_{bc} = R(V_{ab} + V_{bc}) + jXcV_{ab}$$

Since

$$V_{ab} + V_{bc} = V_{ac} = -V_{ca}$$

$$D_1 = -RV_{ca} + jXcV_{ab}$$

Also,

$$D_2 = \begin{vmatrix} 2R & V_{ab} \\ -R & V_{bc} \end{vmatrix} = 2RV_{bc} + RV_{ab} + RV_{bc} = R(2V_{bc} + V_{ab})$$

Then,

$$I_1 = \frac{D_1}{\Delta} = \frac{-RV_{ca} + jXcV_{ab}}{R^2 + j2RXc} = 0.372 - j0.076$$

and

$$I_2 = \frac{D_2}{\Delta} = \frac{R(2V_{bc} + V_{ab})}{R^2 + j2RXc} = 0.304 - j0.152$$

From the three–phase network above, we observe that

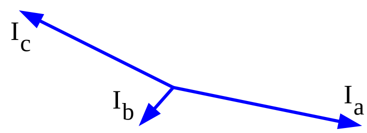
$$I_a = I_1 = 0.372 - j0.076 = 0.38\angle -11.5°$$

$$I_b = I_2 - I_1 = -0.068 - j0.076 = 0.102\angle -131.8°$$

and

$$I_c = -I_2 = -0.304 + j0.152 = 0.34\angle 153.4°$$

The phasor diagram for the three line currents is shown below.



2. The brightness or dimness of the lamps will depend on the magnitude, but not the phase of the current that flows through them. Accordingly let us choose a capacitor with capacitive reactance equal to the to the resistance R of each of the lamps as follows:

$$X_C = R$$

$$X_C = \frac{1}{2\pi fC}$$

$$C = \frac{1}{2\pi fX_C}$$

$$C = \frac{1}{2\pi fR}$$

and with $f = 60\ Hz$, C in μF, and R in $K\Omega$, the last expression above reduces to

$$C\ (\mu F) = \frac{2.65}{R\ (K\Omega)}$$

From Example 12.3

$$R = V^2/P = 120^2/15 = 960\ \Omega = 0.96\ K\Omega$$

and thus

$$C = \frac{2.65}{0.96} = 2.76\ \mu F$$

Replacing -1326 in Example 12.3 with -960, we obtain

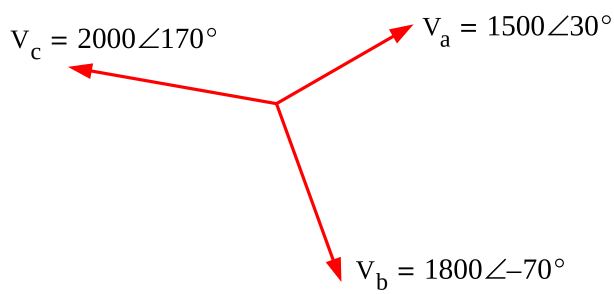
$$I_1 = \frac{960 \times 120 - j(-960) \times (-60 - j104)}{960^2 - 2j \times 960^2} = 0.108\angle 48.4°\ A$$

and

$$I_2 = \frac{960 \times (-60 + j104) - j(-960) \times (-60 - j104)}{960^2 - 2j \times 960^2} = 0.029\angle 108.4°\ A \qquad (12.30)$$

The rated current for the $15 - \text{watt}$ lamp is $15/120 = 0.125\ A$ and we observe that the value of I_1 is approximately 85% of its rated current, and this is an improvement in the lower lamp brilliance. The value of I_2 is only about 23% of the rated value of the lamp, and this is not sufficient to produce a noticeable brightness.

3.



$$V_{a1} = \frac{1}{3}(V_a + V_b\angle 120° + V_c\angle 240°)$$

$$V_{a2} = \frac{1}{3}(V_a + V_b\angle 240° + V_c\angle 120°) \quad (1)$$

$$V_{a0} = \frac{1}{3}(V_a + V_b + V_c)$$

where by definition

$$V_{a1} + V_{a2} + V_{a0} = V_a$$

$$V_{b1} + V_{b2} + V_{b0} = V_b$$

$$V_{c1} + V_{c2} + V_{c0} = V_c$$

Then,

$$V_{a1} = \frac{1}{3}(1500\angle 30° + 1800\angle(-70 + 120)° + 2000\angle(170 + 240)°) \qquad (2)$$

$$= \frac{1}{3}(1500\angle 30° + 1800\angle 50° + 2000\angle 410°)$$

$$= \frac{1}{3}(1299 + j750 + 1157 + j1379 + 1286 + j1532)$$

$$= \frac{1}{3}(3742 + j3661) = 1247 + j1220 = 1744\angle 44.37°$$

By definition, $V_{b1} = V_{a1}\angle -120°$, and for this exercise

$$V_{b1} = 1744\angle(44.4-120)° = 1744\angle -75.6° = 433 - j1689 \quad (3)$$

Also,

$$V_{c1} = 1744\angle(44.4+120)° = 1744\angle 164.4° = -1680 + j469 \quad (4)$$

Next,

$$V_{a2} = \frac{1}{3}(1500\angle 30° + 1800\angle(-70+240)° + 2000\angle(170+120)°) \quad (5)$$

$$= \frac{1}{3}(1500\angle 30° + 1800\angle 170° + 2000\angle 290)$$

$$= \frac{1}{3}(1299 + j750 - 1773 + j313 + 684 - j1879)$$

$$= \frac{1}{3}(210 - j816) = 70 - j272 = 281\angle -75.6°$$

$$V_{b2} = 281\angle(-75.6+120)° = 281\angle 44.4° = 201 + j197 \quad (6)$$

$$V_{c2} = 281\angle(-75.6+240)° = 281\angle 164.4° = -271 + j76 \quad (7)$$

Finally,

$$V_{a0} = \frac{1}{3}(1500\angle 30° + 1800\angle -70° + 2000\angle 170°) \quad (8)$$

$$= \frac{1}{3}(1299 + j750 + 616 - j1691 - 1970 + j347)$$

$$= \frac{1}{3}(-55 - j594) = -18.3 - j198 = 199\angle -95.3$$

and thus

$$V_{a0} = V_{b0} = V_{c0} = 199\angle -95.3 \quad (9)$$

Check:

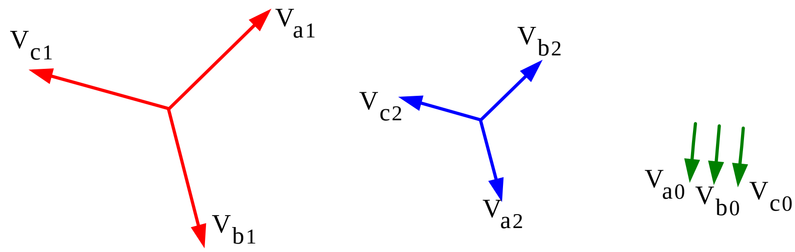
$$V_a = V_{a1} + V_{a2} + V_{a0} = 1247 + j1220 + 70 - j272 - 18 - j198 = 1299 + j750 \approx 1500\angle 30°$$

$$V_b = V_{b1} + V_{b2} + V_{b0} = 434 - j1689 + 201 + j197 - 18 - j198 = 617 - j1690 \approx 1800\angle -70°$$

$$V_c = V_{c1} + V_{c2} + V_{c0} = -1680 + j469 - 271 + j75.6 - 18 - j198 = -1969 + j347 \approx 2000\angle 170°$$

The symmetrical components in phasor diagrams are as shown below where we observe that for the positive−sequence the order of phases is $a-b-c-a-b-c-\ldots$, and that for the negative−sequence the order of phases is $c-b-a-c-b-a-\ldots$.

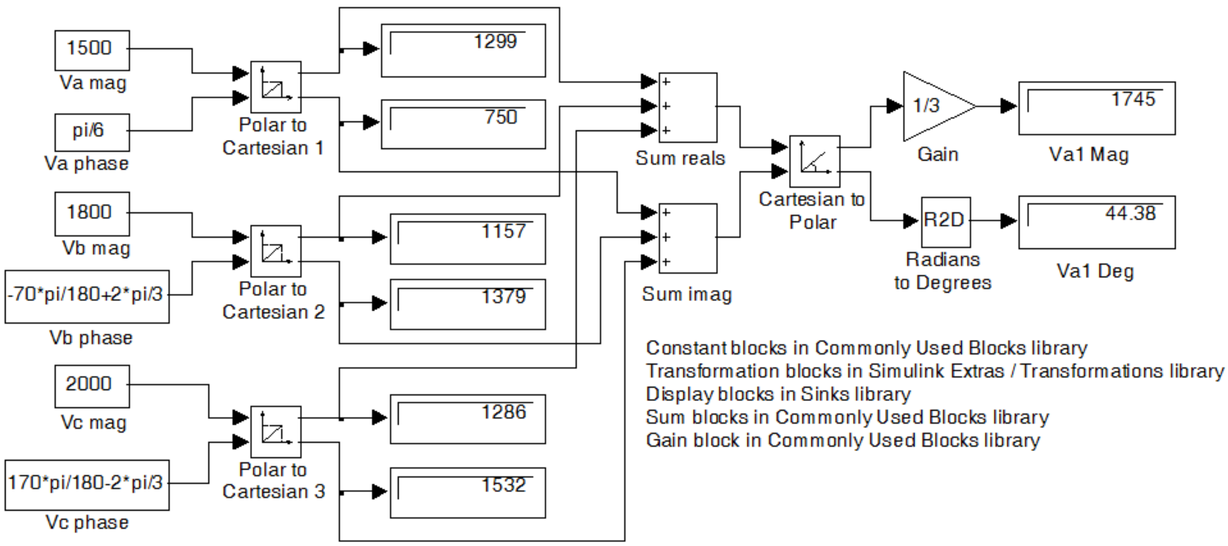
We can verify the computations for V_{a1} in (2) above with the following MATLAB script:




```
% Express Va, Vb rotated by 120 deg, and Vc rotated by 240 deg or by –120 deg, in accordance
% with (1) above

ReVa,ImVa]=pol2cart(30*pi/180, 1500),[ReVb,ImVb]=pol2cart((–70+120)*pi/180, 1800),
[ReVc,ImVc]=pol2cart((170–120)*pi/180, 2000)

ReVa = 1.2990e+003

ImVa = 750.0000

ReVb = 1.1570e+003

ImVb = 1.3789e+003

ReVc = 1.2856e+003

ImVc = 1.5321e+003
%
%  Add reals and imaginaries and divide by 3 to obtain Va1 in Cartesian form

Va1=(1/3)*(ReVa+ReVb+ReVc+j*(ImVa+ImVb+ImVc))

Va1 =  1.2472e+003 + 1.2203e+003i

%  To convert to polar form we define the real part ax x and the imaginary part as y

x=(1/3)*(ReVa+ReVb+ReVc), y=(1/3)*(ImVa+ImVb+ImVc)

x = 1.2472e+003

y = 1.2203e+003

[rad,mag]=cart2pol(x,y), deg=rad*180/pi

rad = 0.7745

mag = 1.7449e+003

deg = 44.3757
```

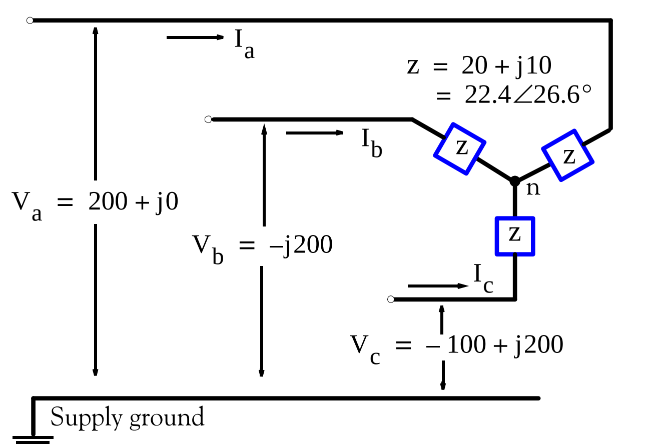
This script can be extended for the remaining calculations by repeated application of the [x,y]=pol2cart(theta,r) and [theta,r]=cart2pol(x,y) MATLAB functions.

The Simulink model below can also be used for the computations of V_{a1}.



This model can also be used for the computations of V_{a2} and V_{a0}

4.



$$a = 1\angle 120° \qquad a^2 = 1\angle 240° \qquad a^3 = 1\angle 360° = 1$$

For positive–phase sequence,

$$\begin{aligned}
V_{a1} &= \frac{1}{3}(V_a + aV_b + a^2V_c) = \frac{1}{3}(V_a + V_b\angle 120° + V_c\angle 240°) \\
&= \frac{1}{3}(200 - j200\angle 120° + (-100 + j200)\angle 240°) \\
&= \frac{1}{3}(200 + 200\angle 30° + 223.6\angle(116.6 + 240)°) \\
&= \frac{1}{3}(200 + 173.2 + j100 + 223.2 - j13.3) \\
&= \frac{1}{3}(596.4 + j86.7) = 198.8 + j28.9 = 200.9\angle 8.3°
\end{aligned}$$

For negative–phase sequence,

$$\begin{aligned}
V_{a2} &= \frac{1}{3}(V_a + a^2V_b + aV_c) = \frac{1}{3}(V_a + V_b\angle 240° + V_c\angle 120°) \\
&= \frac{1}{3}(200 - j200\angle 240° + (-100 + j200)\angle 120°) \\
&= \frac{1}{3}(200 + 200\angle 150° + 223.6\angle(116.6 + 120)°) \\
&= \frac{1}{3}(200 - 173.2 + j100 - 123.1 - j186.7) \\
&= \frac{1}{3}(-96.3 - j86.7) = -32.1 - j28.9 = 43.2\angle -138°
\end{aligned}$$

For zero–phase sequence,

$$V_{a2} = \frac{1}{3}(V_a + V_b + V_c) = \frac{1}{3}(200 - j200 - 100 + j200) = 33.3$$

Next,

$$I_{a1} = \frac{V_{a1}}{Z} = \frac{200.9\angle 8.3°}{22.4\angle 26.6°} = 8.97\angle -18.3° = 8.52 - j2.82$$

$$I_{a2} = \frac{V_{a2}}{Z} = \frac{43.2\angle -138°}{22.4\angle 26.6°} = 1.93\angle -164.6° = -1.86 - j0.51$$

There is no connection between the Y neutral point n and the supply ground, and thus

$$I_{a0} = 0$$

Now, for line current I_a,

$$I_a = I_{a1} + I_{a2} + I_{a0} = 8.52 - j2.82 - 1.86 - j0.51 + 0 = 6.66 - j3.33 = 7.45\angle -26.6°$$

For line current I_b,

$$I_b = a^2 I_{a1} + aI_{a2} + I_{a0} = (8.97\angle -18.3°)\angle -120° + (1.93\angle -164.6°)\angle 120°$$
$$= 8.97\angle -138.3° + 1.93\angle -44.6° = -6.70 - j5.97 + 1.37 - j1.36$$
$$= -5.33 - j7.33 = 9.06\angle -54°$$

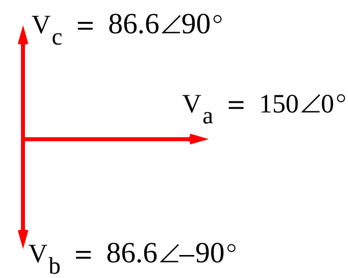
and for line current I_c,

$$I_c = aI_{a1} + a^2 I_{a2} + I_{a0} = (8.97\angle -18.3°)\angle 120° + (1.93\angle -164.6°)\angle -120°$$
$$= 8.97\angle 101.7° + 1.93\angle -284.6° = -1.82 + j8.78 + 0.49 + j1.87$$
$$= -1.33 + j10.65 = 10.73\angle 97.1°$$

Check:

$$I_a + I_b + I_c = 6.66 - j3.33 - 5.33 - j7.33 - 1.33 + j10.65 \approx 0$$

5.



a.

$$V_{a1} = \frac{1}{3}(V_a + aV_b + a^2 V_c) = \frac{1}{3}(V_a + V_b\angle 120° + V_c\angle -120°)$$
$$= \frac{1}{3}(150\angle 0° + 86.6\angle 30° + 86.6\angle -30°)$$
$$= \frac{1}{3}(150 + 86.6(\sqrt{3}/2) + j86.6(1/2) + 86.6(\sqrt{3}/2) - j86.6(1/2))$$
$$= \frac{1}{3}(150 + 150) = 100\angle 0°$$

b.

$$V_{b1} = a^2 V_{a1} = V_{a1}\angle -120° = 100\angle -120° = -50 - j86.6$$

$$V_{c1} = aV_{a1} = V_{a1}\angle 120° = 100\angle 120° = -50 + j86.6$$

Check:

$$V_{a1} + V_{b1} + V_{c1} = 100 - 50 - j86.6 - 50 + j86.6 = 0$$

Next,

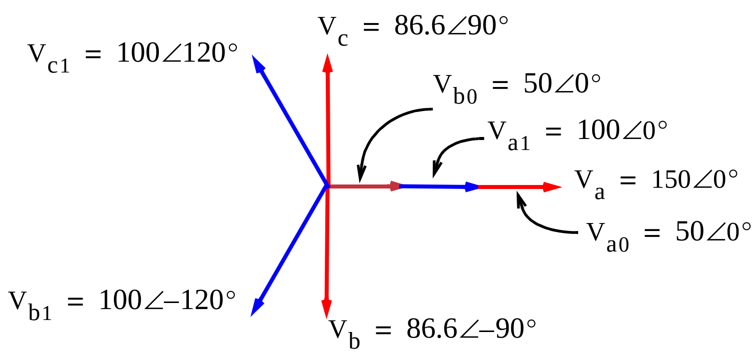
$$V_{a2} = \frac{1}{3}(V_a + a^2V_b + aV_c) = \frac{1}{3}(V_a + V_b\angle{-120°} + V_c\angle 120°)$$
$$= \frac{1}{3}(150\angle 0° + 86.6\angle 150° + 86.6\angle 210°)$$
$$= \frac{1}{3}(150 + 86.6(-\sqrt{3}/2) + j86.6(1/2) + 86.6(-\sqrt{3}/2) - j86.6(1/2))$$
$$= \frac{1}{3}(150 - 75 - 75) = 0$$

$$V_{a0} = \frac{1}{3}(V_a + V_b + V_c) = \frac{1}{3}(150\angle 0° + 86.6\angle{-90°} + 86.6\angle 90°)$$
$$= \frac{1}{3}(150\angle 0° + 0) = 50\angle 0°$$

Check:

$$V_a = V_{a1} + V_{a2} + V_{a0} = 100\angle 0° + 0 + 50\angle 0° = 150\angle 0°$$

and the phasor diagram is shown below.



6.

$I_a = 5.00 = 5\angle 0°$ A $\qquad I_b = -j8.66 = 8.66\angle{-90°}$ A $\qquad I_c = j10.00 = 10.00\angle 90°$ A

$$I_{a1} = \frac{1}{3}(I_a + aI_b + a^2I_c) = \frac{1}{3}(I_a + I_b\angle 120° + I_c\angle{-120°})$$
$$= \frac{1}{3}(5\angle 0° + 8.66\angle 30° + 10\angle{-30°})$$
$$= \frac{1}{3}(5 + 8.66(\sqrt{3}/2) + j8.66(1/2) + 10(\sqrt{3}/2) - j10(1/2))$$
$$= \frac{1}{3}(5 + 7.5 + j4.33 + 8.66 - j5) = \frac{1}{3}(21.16 - j0.67)$$
$$= 7.05 - j0.22 = 7.05\angle{-1.8°}$$

$$I_{b1} = a^2I_{a1} = I_{a1}\angle{-120°} = 7.05\angle{-121.8°} = -3.72 - j5.99$$

$$I_{c1} = aI_{a1} = I_{a1}\angle 120° = 7.05\angle 118.2° = -3.33 + j6.21$$

Next,

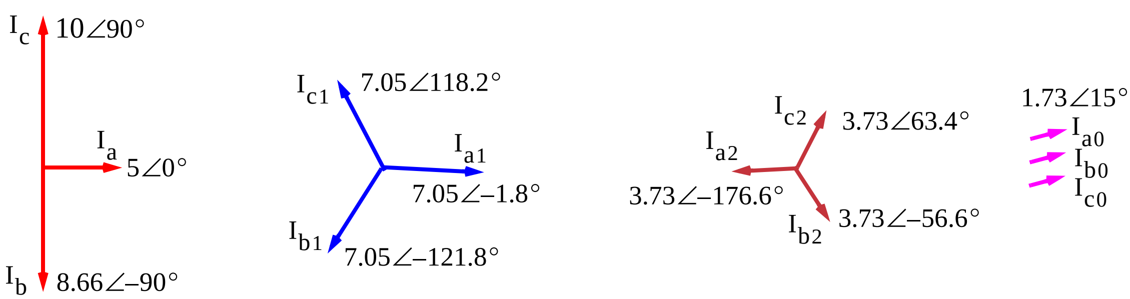
$$I_{a2} = \frac{1}{3}(I_a + a^2 I_b + aI_c) = \frac{1}{3}(I_a + I_b\angle -120° + I_c\angle 120°)$$
$$= \frac{1}{3}(5 + 8.66\angle 150° + 10\angle 210°)$$
$$= \frac{1}{3}(5 + 8.66(-\sqrt{3}/2) + j8.66(1/2) + 10(-\sqrt{3}/2) - j10(1/2))$$
$$= \frac{1}{3}(5 - 7.5 + j4.33 - 8.66 - j5) = \frac{1}{3}(-11.16 - j0.67)$$
$$= -3.72 - j0.22 = 3.73\angle -176.6°$$

$$I_{b2} = aI_{a2} = I_{a2}\angle 120° = 3.73\angle -56.6° = 2.05 - j3.11$$

$$I_{c2} = a^2 I_{a1} = I_{a2}\angle -120° = 3.73\angle 63.4° = 1.67 + j3.34$$

$$I_{a0} = \frac{1}{3}(I_a + I_b + I_c) = \frac{1}{3}(5 - j8.66 + j10) = 1.67 + j0.45 = 1.73\angle 15°$$

and the phasor diagrams are shown below.



Appendix A

Introduction to MATLAB®

This appendix serves as an introduction to the basic MATLAB commands and functions, procedures for naming and saving the user generated files, comment lines, access to MATLAB's Editor / Debugger, finding the roots of a polynomial, and making plots. Several examples are provided with detailed explanations.

A.1 MATLAB® and Simulink®

MATLAB and Simulink are products of The MathWorks,™ Inc. These are two outstanding software packages for scientific and engineering computations and are used in educational institutions and in industries including automotive, aerospace, electronics, telecommunications, and environmental applications. MATLAB enables us to solve many advanced numerical problems rapidly and efficiently.

A.2 Command Window

To distinguish the screen displays from the user commands, important terms, and MATLAB functions, we will use the following conventions:

Click: Click the left button of the mouse
`Courier Font:` Screen displays
Helvetica Font: User inputs at MATLAB's command window prompt >> or `EDU>>`*
Helvetica Bold: MATLAB functions

Times Bold Italic: Important terms and facts, notes and file names
When we first start MATLAB, we see various help topics and other information. Initially, we are interested in the *command screen* which can be selected from the Window drop menu. When the command screen, we see the prompt >> or `EDU>>`. This prompt is displayed also after execution of a command; MATLAB now waits for a new command from the user. It is highly recommended that we use the *Editor/Debugger* to write our program, save it, and return to the command screen to execute the program as explained below.

To use the Editor/Debugger:

1. From the *File* menu on the toolbar, we choose *New* and click on *M–File*. This takes us to the *Editor Window* where we can type our *script* (list of statements) for a new file, or open a previously saved file. We must save our program with a file name which starts with a letter.

* EDU>> *is the MATLAB prompt in the Student Version*

Important! MATLAB is *case sensitive*, that is, it distinguishes between upper– and lower–case letters. Thus, *t* and *T* are two different letters in MATLAB language. The files that we create are saved with the file name we use and the extension ***.m***; for example, ***myfile01.m***. It is a good practice to save the script in a file name that is descriptive of our script content. For instance, if the script performs some matrix operations, we ought to name and save that file as ***matrices01.m*** or any other similar name. We should also use a floppy disk or an external drive to backup our files.

2. Once the script is written and saved as an *m–file*, we may exit the *Editor/Debugger* window by clicking on *Exit Editor/Debugger* of the *File* menu. MATLAB then returns to the command window.

3. To execute a program, we type the file name ***without*** the ***.m*** extension at the >> prompt; then, we press <enter> and observe the execution and the values obtained from it. If we have saved our file in drive ***a*** or any other drive, we must make sure that it is added it to the desired directory in MATLAB's search path. The MATLAB User's Guide provides more information on this topic.

Henceforth, it will be understood that each input command is typed after the >> prompt and followed by the <enter> key.

The command **help matlab\iofun** will display input/output information. To get help with other MATLAB topics, we can type help followed by any topic from the displayed menu. For example, to get information on graphics, we type help matlab\graphics. The MATLAB User's Guide contains numerous help topics.

To appreciate MATLAB's capabilities, we type demo and we see the `MATLAB Demos` menu. We can do this periodically to become familiar with them. Whenever we want to return to the command window, we click on the `Close` button.

When we are done and want to leave MATLAB, we type **quit** or **exit**. But if we want to clear all previous values, variables, and equations without exiting, we should use the command **clear**. This command erases everything; it is like exiting MATLAB and starting it again. The command **clc** clears the screen but MATLAB still remembers all values, variables and equations that we have already used. In other words, if we want to clear all previously entered commands, leaving only the >> prompt on the upper left of the screen, we use the **clc** command.

All text after the **%** (percent) symbol is interpreted as a *comment line* by MATLAB, and thus it is ignored during the execution of a program. A comment can be typed on the same line as the function or command or as a separate line. For instance,

```
conv(p,q)    % performs multiplication of polynomials p and q

% The next statement performs partial fraction expansion of p(x) / q(x)
```

are both correct.

One of the most powerful features of MATLAB is the ability to do computations involving *complex numbers*. We can use either i, or j to denote the imaginary part of a complex number, such as 3-4i or 3-4j. For example, the statement

z=3–4j

displays

```
z = 3.0000-4.0000i
```

In the above example, a multiplication (*) sign between 4 and j was not necessary because the complex number consists of numerical constants. However, if the imaginary part is a function, or variable such as cos(x), we must use the multiplication sign, that is, we must type cos(x)*j or j*cos(x) for the imaginary part of the complex number.

A.3 Roots of Polynomials

In MATLAB, a polynomial is expressed as a *row vector* of the form $[a_n \; a_{n-1} \; \ldots \; a_2 \; a_1 \; a_0]$. These are the coefficients of the polynomial in descending order. ***We must include terms whose coefficients are zero.***

We find the roots of any polynomial with the **roots(p)** function; **p** is a row vector containing the polynomial coefficients in descending order.

Example A.1

Find the roots of the polynomial

$$p_1(x) = x^4 - 10x^3 + 35x^2 - 50x + 24$$

Solution:

The roots are found with the following two statements where we have denoted the polynomial as p1, and the roots as roots_ p1.

```
p1=[1 –10 35 –50 24]     % Specify and display the coefficients of p1(x)

p1 =
     1    -10     35    -50     24

roots_ p1=roots(p1)      % Find the roots of p1(x)

roots_p1 =
   4.0000
   3.0000
   2.0000
   1.0000
```

We observe that MATLAB displays the polynomial coefficients as a row vector, and the roots as a column vector.

Example A.2

Find the roots of the polynomial

$$p_2(x) = x^5 - 7x^4 + 16x^2 + 25x + 52$$

Solution:

There is no cube term; therefore, we must enter zero as its coefficient. The roots are found with the statements below, where we have defined the polynomial as p2, and the roots of this polynomial as roots_ p2. The result indicates that this polynomial has three real roots, and two complex roots. Of course, complex roots always occur in *complex conjugate** pairs.

```
p2=[1 -7  0 16 25 52]

p2 =
     1    -7     0    16    25    52

roots_ p2=roots(p2)

roots_p2 =
   6.5014
   2.7428
  -1.5711
  -0.3366 + 1.3202i
  -0.3366 - 1.3202i
```

A.4 Polynomial Construction from Known Roots

We can compute the coefficients of a polynomial, from a given set of roots, with the **poly(r)** function where **r** is a row vector containing the roots.

Example A.3

It is known that the roots of a polynomial are $1, 2, 3,$ and 4. Compute the coefficients of this polynomial.

* *By definition, the conjugate of a complex number* $A = a + jb$ *is* $A^* = a - jb$

Solution:

We first define a row vector, say $r3$, with the given roots as elements of this vector; then, we find the coefficients with the **poly(r)** function as shown below.

```
r3=[1 2 3 4]          % Specify the roots of the polynomial

r3 =
     1     2     3     4

poly_r3=poly(r3)      % Find the polynomial coefficients

poly_r3 =
     1   -10    35   -50    24
```

We observe that these are the coefficients of the polynomial $p_1(x)$ of Example A.1.

Example A.4

It is known that the roots of a polynomial are $-1,\ -2,\ -3,\ 4+j5,$ and $4-j5$. Find the coefficients of this polynomial.

Solution:

We form a row vector, say $r4$, with the given roots, and we find the polynomial coefficients with the **poly(r)** function as shown below.

```
r4=[ –1  –2  –3  4+5j  4–5j ]

r4 =
  Columns 1 through 4
  -1.0000   -2.0000   -3.0000   -4.0000+ 5.0000i
  Column 5
  -4.0000- 5.0000i

poly_r4=poly(r4)

poly_r4 =
     1    14   100   340   499   246
```

Therefore, the polynomial is

$$p_4(x) = x^5 + 14x^4 + 100x^3 + 340x^2 + 499x + 246$$

A.5 Evaluation of a Polynomial at Specified Values

The **polyval(p,x)** function evaluates a polynomial $p(x)$ at some specified value of the independent variable x.

Example A.5

Evaluate the polynomial

$$p_5(x) = x^6 - 3x^5 + 5x^3 - 4x^2 + 3x + 2 \quad (A.1)$$

at $x = -3$.

Solution:

```
p5=[1 –3  0  5 –4  3  2];   % These are the coefficients of the given polynomial
                            % The semicolon (;) after the right bracket suppresses the
                            %  display of the row vector that contains the coefficients of p5.
%
val_minus3=polyval(p5, –3)  % Evaluate p5 at x=–3; no semicolon is used here
                            % because we want the answer to be displayed

val_minus3 =
        1280
```

Other MATLAB functions used with polynomials are the following:

conv(a,b) – multiplies two polynomials **a** and **b**

[q,r]=deconv(c,d) –divides polynomial **c** by polynomial **d** and displays the quotient **q** and remainder **r**.

polyder(p) – produces the coefficients of the derivative of a polynomial **p**.

Example A.6

Let

$$p_1 = x^5 - 3x^4 + 5x^2 + 7x + 9$$

and

$$p_2 = 2x^6 - 8x^4 + 4x^2 + 10x + 12$$

Compute the product $p_1 \cdot p_2$ using the **conv(a,b)** function.

Solution:

```
p1=[1 –3  0 5 7 9];         % The coefficients of p1
p2=[2  0 –8 0 4 10 12];     % The coefficients of p2
p1p2=conv(p1,p2)            % Multiply p1 by p2 to compute coefficients of the product p1p2
```

```
p1p2 =
  2   -6   -8   34   18   -24   -74   -88   78   166   174   108
```

Therefore,

$$p_1 \cdot p_2 = 2x^{11} - 6x^{10} - 8x^{9} + 34x^{8} + 18x^{7} - 24x^{6}$$
$$-74x^{5} - 88x^{4} + 78x^{3} + 166x^{2} + 174x + 108$$

Example A.7

Let

$$p_3 = x^7 - 3x^5 + 5x^3 + 7x + 9$$

and

$$p_4 = 2x^6 - 8x^5 + 4x^2 + 10x + 12$$

Compute the quotient p_3 / p_4 using the **[q,r]=deconv(c,d)** function.

Solution:

```
% It is permissible to write two or more statements in one line separated by semicolons
p3=[1  0 –3   0  5  7   9];  p4=[2 –8  0   0  4  10  12];  [q,r]=deconv(p3,p4)
```

```
q =
    0.5000
r =
     0     4    -3     0     3     2     3
```

Therefore,

$$q = 0.5 \qquad r = 4x^5 - 3x^4 + 3x^2 + 2x + 3$$

Example A.8

Let

$$p_5 = 2x^6 - 8x^4 + 4x^2 + 10x + 12$$

Compute the derivative $\frac{d}{dx} p_5$ using the **polyder(p)** function.

Solution:

```
p5=[2  0  −8  0  4  10  12];     % The coefficients of p5
der_p5=polyder(p5)               % Compute the coefficients of the derivative of p5
```

```
der_p5 =
    12      0    -32      0      8     10
```

Therefore,

$$\frac{d}{dx}p_5 = 12x^5 - 32x^3 + 8x + 10$$

A.6 Rational Polynomials

Rational Polynomials are those which can be expressed in ratio form, that is, as

$$R(x) = \frac{Num(x)}{Den(x)} = \frac{b_n x^n + b_{n-1} x^{n-1} + b_{n-2} x^{n-2} + \dots + b_1 x + b_0}{a_m x^m + a_{m-1} x^{m-1} + a_{m-2} x^{m-2} + \dots + a_1 x + a_0} \tag{A.2}$$

where some of the terms in the numerator and/or denominator may be zero. We can find the roots of the numerator and denominator with the **roots(p)** function as before.

As noted in the comment line of Example A.7, we can write MATLAB statements in one line, if we separate them by commas or semicolons. ***Commas will display the results whereas semicolons will suppress the display.***

Example A.9

Let

$$R(x) = \frac{p_{num}}{p_{den}} = \frac{x^5 - 3x^4 + 5x^2 + 7x + 9}{x^6 - 4x^4 + 2x^2 + 5x + 6}$$

Express the numerator and denominator in factored form, using the **roots(p)** function.

Solution:

```
num=[1  −3  0  5  7  9]; den=[1  0  −4  0  2  5  6];    % Do not display num and den coefficients
roots_num=roots(num), roots_den=roots(den)             % Display num and den roots
```

```
roots_num =
   2.4186 + 1.0712i    2.4186 - 1.0712i   -1.1633
  -0.3370 + 0.9961i   -0.3370 - 0.9961i
```

```
roots_den =
   1.6760 + 0.4922i     1.6760 - 0.4922i   -1.9304
  -0.2108 + 0.9870i    -0.2108 - 0.9870i   -1.0000
```

As expected, the complex roots occur in complex conjugate pairs.

For the numerator, we have the factored form

$$p_{num} = (x-2.4186-j1.0712)(x-2.4186+j1.0712)(x+1.1633)$$
$$(x+0.3370-j0.9961)(x+0.3370+j0.9961)$$

and for the denominator, we have

$$p_{den} = (x-1.6760-j0.4922)(x-1.6760+j0.4922)(x+1.9304)$$
$$(x+0.2108-j0.9870)(x+0.2108+j0.9870)(x+1.0000)$$

We can also express the numerator and denominator of this rational function as a combination of *linear* and *quadratic* factors. We recall that, in a quadratic equation of the form $x^2+bx+c = 0$ whose roots are x_1 and x_2, the negative sum of the roots is equal to the coefficient b of the x term, that is, $-(x_1+x_2) = b$, while the product of the roots is equal to the constant term c, that is, $x_1 \cdot x_2 = c$. Accordingly, we form the coefficient b by addition of the complex conjugate roots and this is done by inspection; then we multiply the complex conjugate roots to obtain the constant term c using MATLAB as follows:

```
(2.4186 + 1.0712i)*(2.4186 –1.0712i)
ans = 6.9971
(–0.3370+ 0.9961i)*(–0.3370–0.9961i)
ans = 1.1058
(1.6760+ 0.4922i)*(1.6760–0.4922i)
ans = 3.0512
(–0.2108+ 0.9870i)*(–0.2108–0.9870i)
ans = 1.0186
```

Thus,

$$R(x) = \frac{p_{num}}{p_{den}} = \frac{(x^2-4.8372x+6.9971)(x^2+0.6740x+1.1058)(x+1.1633)}{(x^2-3.3520x+3.0512)(x^2+0.4216x+1.0186)(x+1.0000)(x+1.9304)}$$

We can check this result of Example A.9 above with MATLAB's *Symbolic Math Toolbox* which is a collection of tools (functions) used in solving symbolic expressions. They are discussed in detail in MATLAB's Users Manual. For the present, our interest is in using the **collect(s)** function that is used to multiply two or more symbolic expressions to obtain the result in polynomial form. We must remember that the **conv(p,q)** function is used with numeric expressions only, that is, polynomial coefficients.

Before using a symbolic expression, we must create one or more symbolic variables such as x, y, t, and so on. For our example, we use the following script:

```
syms x % Define a symbolic variable and use collect(s) to express numerator in polynomial form
collect((x^2–4.8372*x+6.9971)*(x^2+0.6740*x+1.1058)*(x+1.1633))
```

```
ans =
  x^5-29999/10000*x^4-1323/3125000*x^3+7813277909/
  1562500000*x^2+1750276323053/250000000000*x+4500454743147/
  500000000000
```

and if we simplify this, we find that is the same as the numerator of the given rational expression in polynomial form. We can use the same procedure to verify the denominator.

A.7 Using MATLAB to Make Plots

Quite often, we want to plot a set of ordered pairs. This is a very easy task with the MATLAB **plot(x,y)** command that plots *y* versus *x*, where *x* is the horizontal axis (abscissa) and *y* is the vertical axis (ordinate).

Example A.10

Consider the electric circuit of Figure A.1, where the radian frequency ω (radians/second) of the applied voltage was varied from 300 to 3000 in steps of 100 radians/second, while the amplitude was held constant.

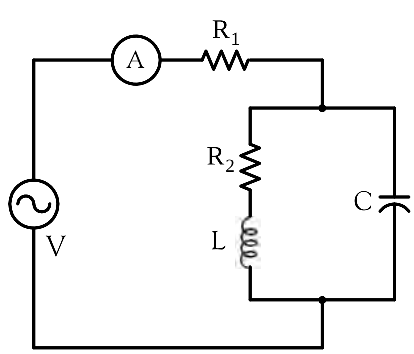


Figure A.1. Electric circuit for Example A.10

The ammeter readings were then recorded for each frequency. The magnitude of the impedance $|Z|$ was computed as $|Z| = |V/A|$ and the data were tabulated on Table A.1.

TABLE A.1 Table for Example A.10

ω (rads/s)	|Z| Ohms	ω (rads/s)	|Z| Ohms
300	39.339	1700	90.603
400	52.589	1800	81.088
500	71.184	1900	73.588
600	97.665	2000	67.513
700	140.437	2100	62.481
800	222.182	2200	58.240
900	436.056	2300	54.611
1000	1014.938	2400	51.428
1100	469.83	2500	48.717
1200	266.032	2600	46.286
1300	187.052	2700	44.122
1400	145.751	2800	42.182
1500	120.353	2900	40.432
1600	103.111	3000	38.845

Plot the magnitude of the impedance, that is, $|Z|$ versus radian frequency ω.

Solution:

We cannot type ω (omega) in the MATLAB Command prompt, so we will use the English letter w instead.

If a statement, or a row vector is too long to fit in one line, it can be continued to the next line by typing three or more periods, then pressing <*enter*> to start a new line, and continue to enter data. This is illustrated below for the data of w and z. Also, as mentioned before, we use the semicolon (;) to suppress the display of numbers that we do not care to see on the screen.

The data are entered as follows:

```
w=[300 400 500 600 700 800 900 1000 1100 1200 1300 1400 1500 1600 1700 1800 1900....
2000 2100 2200 2300 2400 2500 2600 2700 2800 2900 3000];
%
z=[39.339  52.789  71.104  97.665  140.437  222.182  436.056....
1014.938  469.830  266.032 187.052 145.751 120.353  103.111....
90.603  81.088  73.588  67.513  62.481  58.240  54.611  51.468....
48.717  46.286  44.122  42.182  40.432  38.845];
```

Of course, if we want to see the values of *w* or *z* or both, we simply type w or z, and we press

<*enter*>. To plot z (y–axis) versus w (x–axis), we use the **plot(x,y)** command. For this example, we use **plot(w,z)**. When this command is executed, MATLAB displays the plot on MATLAB's *graph screen* and MATLAB denotes this plot as Figure 1. This plot is shown in Figure A.2.

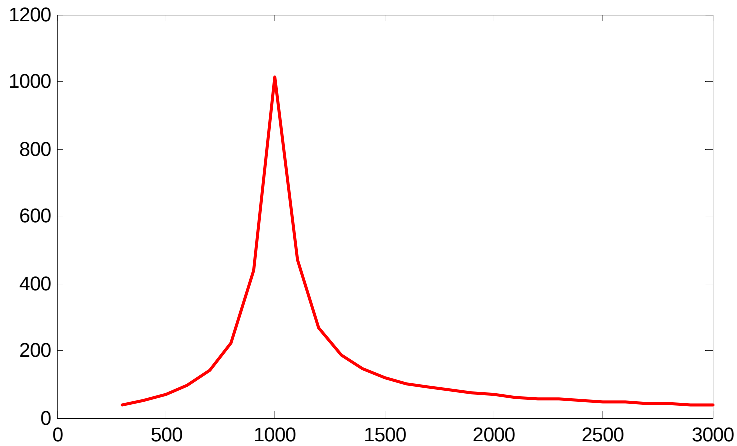


Figure A.2. Plot of impedance |z| *versus frequency* ω *for Example A.10*

This plot is referred to as the *magnitude frequency response* of the circuit.

To return to the command window, we press any key, or from the *Window* pull–down menu, we select *MATLAB Command Window*. To see the graph again, we click on the Window pull–down menu, and we choose *Figure 1*.

We can make the above, or any plot, more presentable with the following commands:

grid on: This command adds grid lines to the plot. The **grid off** command removes the grid. The command **grid** toggles them, that is, changes from off to on or vice versa. The default* is off.

box off: This command removes the box (the solid lines which enclose the plot), and **box on** restores the box. The command **box** toggles them. The default is on.

title('string'): This command adds a line of the text **string** (label) at the top of the plot.

xlabel('string') and **ylabel('string')** are used to label the x– and y–axis respectively.

The magnitude frequency response is usually represented with the x–axis in a logarithmic scale. We can use the **semilogx(x,y)** command which is similar to the **plot(x,y)** command, except that the x–axis is represented as a log scale, and the y–axis as a linear scale. Likewise, the **semilogy(x,y)** command is similar to the **plot(x,y)** command, except that the y–axis is represented as a

* *A default is a particular value for a variable that is assigned automatically by an operating system and remains in effect unless canceled or overridden by the operator.*

log scale, and the x–axis as a linear scale. The **loglog(x,y)** command uses logarithmic scales for both axes.

Throughout this text it will be understood that ***log*** is the common (base 10) logarithm, and ***ln*** is the natural (base e) logarithm. We must remember, however, the function **log(x)** in MATLAB is the natural logarithm, whereas the common logarithm is expressed as **log10(x)**, and the logarithm to the base 2 as **log2(x).**

Let us now redraw the plot with the above options by adding the following statements:

```
semilogx(w,z); grid;          % Replaces the plot(w,z) command
title('Magnitude of Impedance vs. Radian Frequency');
xlabel('w in rads/sec'); ylabel('|Z| in Ohms')
```

After execution of these commands, the plot is as shown in Figure A.3.

If the y–axis represents power, voltage or current, the x–axis of the frequency response is more often shown in a logarithmic scale, and the y–axis in dB (decibels).

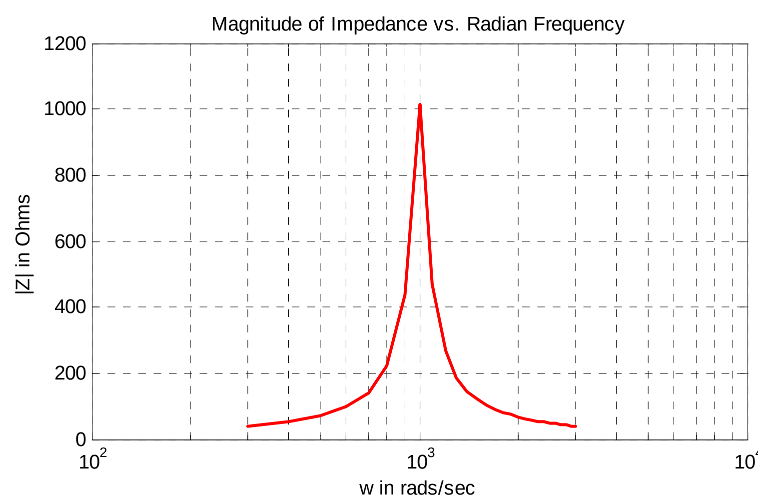


Figure A.3. Modified frequency response plot of Figure A.2.

To display the voltage v in a dB scale on the y–axis, we add the relation dB=20*log10(v), and we replace the **semilogx(w,z)** command with semilogx(w,dB) provided that v is predefined.

The command **gtext('string')*** switches to the current *Figure Window*, and displays a cross–hair that can be moved around with the mouse. For instance, we can use the command gtext('Impedance |Z| versus Frequency'), and this will place a cross–hair in the *Figure* window. Then, using

* *With the latest MATLAB Versions 6 and 7 (Student Editions 13 and 14), we can add text, lines and arrows directly into the graph using the tools provided on the Figure Window. For advanced MATLAB graphics, please refer to The MathWorks* ***Using MATLAB Graphics*** *documentation.*

the mouse, we can move the cross–hair to the position where we want our label to begin, and we press <enter>.

The command **text(x,y,'string')** is similar to **gtext('string')**. It places a label on a plot in some specific location specified by **x** and **y**, and **string** is the label which we want to place at that location. We will illustrate its use with the following example which plots a *3–phase* sinusoidal waveform.

The first line of the script below has the form

linspace(first_value, last_value, number_of_values)

This function specifies ***the number of data points*** but not the increments between data points. An alternate function is

x=first: increment: last

and this specifies ***the increments between points*** but not the number of data points.

The script for the 3–phase plot is as follows:

```
x=linspace(0, 2*pi, 60);        % pi is a built–in function in MATLAB;
% we could have used x=0:0.02*pi:2*pi or x = (0: 0.02: 2)*pi instead;
y=sin(x); u=sin(x+2*pi/3); v=sin(x+4*pi/3);
plot(x,y,x,u,x,v);              % The x–axis must be specified for each function
grid on, box on,                % turn grid and axes box on
text(0.75, 0.65, 'sin(x)'); text(2.85, 0.65, 'sin(x+2*pi/3)'); text(4.95, 0.65, 'sin(x+4*pi/3)')
```

These three waveforms are shown on the same plot of Figure A.4.

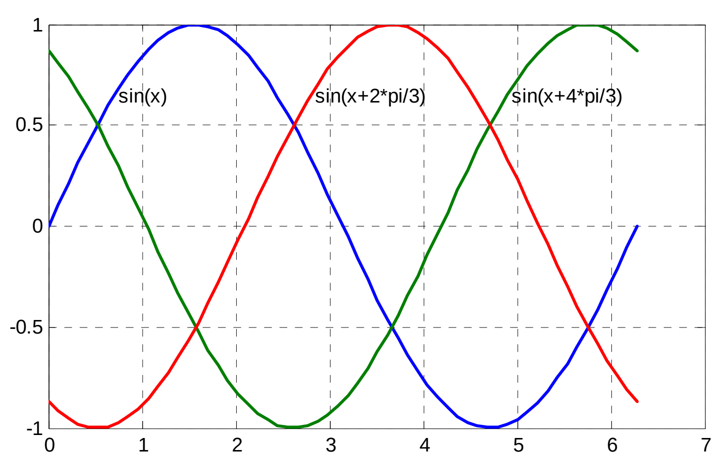


Figure A.4. Three–phase waveforms

In our previous examples, we did not specify line styles, markers, and colors for our plots. However, MATLAB allows us to specify various line types, plot symbols, and colors. These, or a combination of these, can be added with the **plot(x,y,s)** command, where **s** is a character string containing one or more characters shown on the three columns of Table A.2. MATLAB has no default color; it starts with blue and cycles through the first seven colors listed in Table A.2 for each additional line in the plot. Also, there is no default marker; no markers are drawn unless they are selected. The default line is the solid line. But with the latest MATLAB versions, we can select the line color, line width, and other options directly from the *Figure Window.*

TABLE A.2 Styles, colors, and markets used in MATLAB

Symbol	*Color*	*Symbol*	*Marker*	*Symbol*	*Line Style*
b	blue	.	point	–	solid line
g	green	o	circle	:	dotted line
r	red	x	x–mark	–.	dash–dot line
c	cyan	+	plus	––	dashed line
m	magenta	*	star		
y	yellow	s	square		
k	black	d	diamond		
w	white	v	triangle down		
		^	triangle up		
		<	triangle left		
		>	triangle right		
		p	pentagram		
		h	hexagram		

For example, **plot(x,y,'m*:')** plots a magenta dotted line with a star at each data point, and **plot(x,y,'rs')** plots a red square at each data point, but does not draw any line because no line was selected. If we want to connect the data points with a solid line, we must type **plot(x,y,'rs–')**. For additional information we can type **help plot** in MATLAB's command screen.

The plots we have discussed thus far are two–dimensional, that is, they are drawn on two axes. MATLAB has also a three–dimensional (three–axes) capability and this is discussed next.

The **plot3(x,y,z)** command plots a line in *3–space* through the points whose coordinates are the elements of *x*, *y* and *z*, where *x*, *y* and *z* are three vectors of the same length.

The general format is **plot3(x_1,y_1,z_1,s_1,x_2,y_2,z_2,s_2,x_3,y_3,z_3,s_3,...)** where x_n, y_n and z_n are vectors or matrices, and s_n are strings specifying color, marker symbol, or line style. These strings are the same as those of the two–dimensional plots.

Example A.11

Plot the function

$$z = -2x^3 + x + 3y^2 - 1 \tag{A.3}$$

Solution:

We arbitrarily choose the interval (length) shown on the script below.

```
x= –10: 0.5: 10;            % Length of vector x
y= x;                       % Length of vector y must be same as x
z= –2.*x.^3+x+3.*y.^2–1;    % Vector z is function of both x and y*
plot3(x,y,z); grid
```

The three–dimensional plot is shown in Figure A.5.

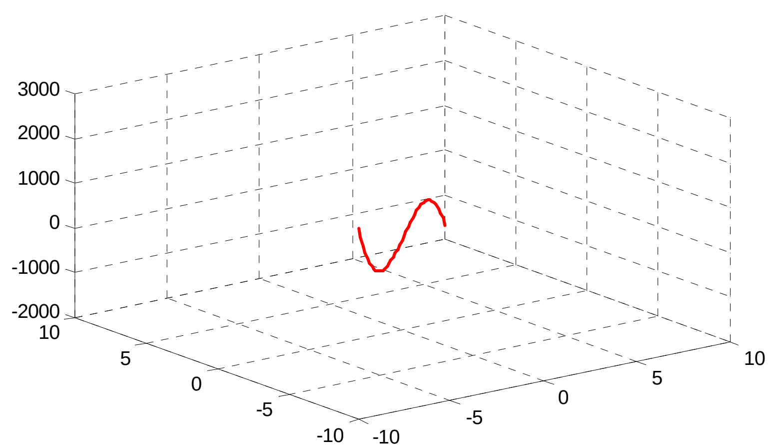


Figure A.5. Three dimensional plot for Example A.11

In a two–dimensional plot, we can set the limits of the *x*– and *y–axes* with the **axis([xmin xmax ymin ymax])** command. Likewise, in a three–dimensional plot we can set the limits of all three axes with the **axis([xmin xmax ymin ymax zmin zmax])** command. It must be placed after the **plot(x,y)** or **plot3(x,y,z)** commands, or on the same line without first executing the **plot** command. This must be done for each plot. The three–dimensional **text(x,y,z,’string’)** command will place **string** beginning at the co–ordinate (x,y,z) on the plot.

For three–dimensional plots, **grid on** and **box off** are the default states.

* *This statement uses the so called dot multiplication, dot division, and dot exponentiation where the multiplication, division, and exponential operators are preceded by a dot. These important operations will be explained in Section A.9.*

We can also use the **mesh(x,y,z)** command with two vector arguments. These must be defined as $length(x) = n$ and $length(y) = m$ where $[m, n] = size(Z)$. In this case, the vertices of the mesh lines are the triples $\{x(j), y(i), Z(i, j)\}$. We observe that **x** corresponds to the columns of *Z*, and **y** corresponds to the rows.

To produce a mesh plot of a function of two variables, say $z = f(x, y)$, we must first generate the X and Y matrices that consist of repeated rows and columns over the range of the variables *x* and *y*. We can generate the matrices X and Y with the **[X,Y]=meshgrid(x,y)** function that creates the matrix X whose rows are copies of the vector **x**, and the matrix Y whose columns are copies of the vector **y**.

Example A.12

The volume V of a right circular cone of radius r and height h is given by

$$V = \frac{1}{3}\pi r^2 h \tag{A.4}$$

Plot the volume of the cone as r and h vary on the intervals $0 \le r \le 4$ and $0 \le h \le 6$ meters.

Solution:

The volume of the cone is a function of both the radius *r* and the height *h*, that is,

$$V = f(r, h)$$

The three–dimensional plot is created with the following MATLAB script where, as in the previous example, in the second line we have used the dot multiplication, dot division, and dot exponentiation. This will be explained in Section A.9.

```
[R,H]=meshgrid(0: 4, 0: 6);    % Creates R and H matrices from vectors r and h;...
V=(pi .* R .^ 2 .* H) ./ 3;  mesh(R, H, V);...
xlabel('x–axis, radius r (meters)'); ylabel('y–axis, altitude h (meters)');...
zlabel('z–axis, volume (cubic meters)'); title('Volume of Right Circular Cone'); box on
```

The three–dimensional plot of Figure A.6 shows how the volume of the cone increases as the radius and height are increased.

The plots of Figure A.5 and A.6 are rudimentary; MATLAB can generate very sophisticated three–dimensional plots. The MATLAB User's Manual and the Using MATLAB Graphics Manual contain numerous examples.

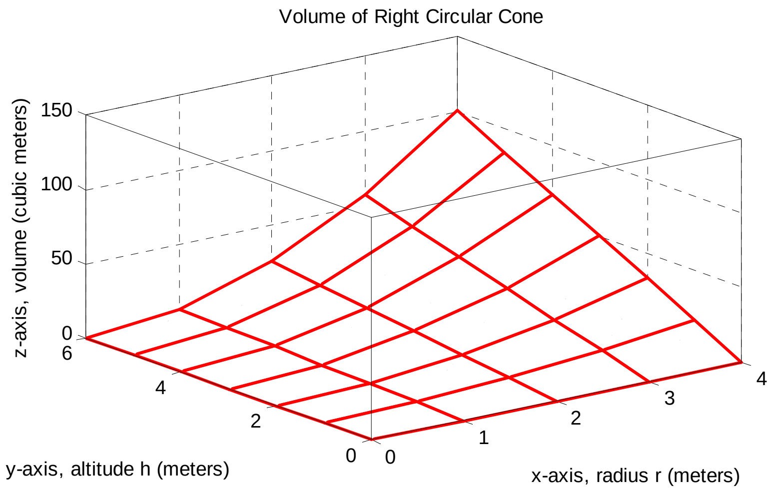


Figure A.6. Volume of a right circular cone.

A.8 Subplots

MATLAB can display up to four windows of different plots on the *Figure* window using the command **subplot(m,n,p)**. This command divides the window into an $m \times n$ matrix of plotting areas and chooses the *pth* area to be active. No spaces or commas are required between the three integers *m*, *n* and *p*. The possible combinations are shown in Figure A.7.

We will illustrate the use of the **subplot(m,n,p)** command following the discussion on multiplication, division and exponentiation that follows.

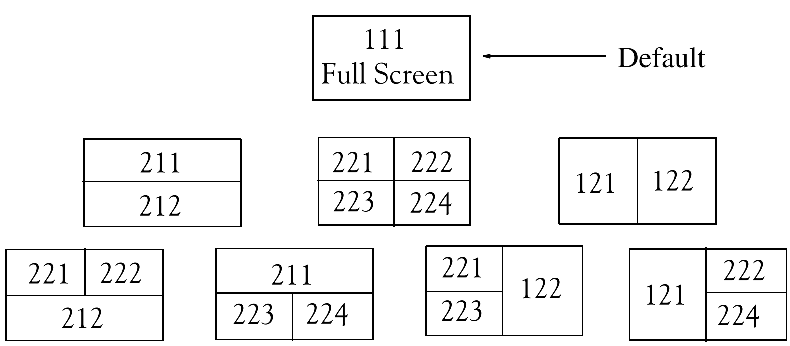


Figure A.7. Possible subplot arrangements in MATLAB

A.9 Multiplication, Division, and Exponentiation

MATLAB recognizes two types of multiplication, division, and exponentiation. These are the ***matrix*** multiplication, division, and exponentiation, and the ***element–by–element*** multiplication, division, and exponentiation. They are explained in the following paragraphs.

In Section A.2, the arrays [a b c ...], such a those that contained the coefficients of polynomials, consisted of one row and multiple columns, and thus are called ***row vectors***. If an array has one column and multiple rows, it is called a ***column vector***. We recall that the elements of a row vector are separated by spaces. To distinguish between row and column vectors, the elements of a column vector must be separated by semicolons. An easier way to construct a column vector, is to write it first as a row vector, and then transpose it into a column vector. MATLAB uses the single quotation character (′) to transpose a vector. Thus, a column vector can be written either as

```
b=[−1; 3; 6; 11]
```

or as

```
b=[−1  3  6  11]'
```

As shown below, MATLAB produces the same display with either format.

```
b=[−1; 3; 6; 11]

b  =
      -1
       3
       6
      11

b=[−1  3  6  11]'        % Observe the single quotation character (‘)

b  =
      -1
       3
       6
      11
```

We will now define Matrix Multiplication and Element−by−Element multiplication.

1. **Matrix Multiplication** (multiplication of row by column vectors)

   Let

   $$\mathbf{A} = [a_1 \quad a_2 \quad a_3 \quad \dots \quad a_n]$$

   and

   $$\mathbf{B} = [b_1 \quad b_2 \quad b_3 \quad \dots \quad b_n]'$$

   be two vectors. We observe that $\mathbf{A}$ is defined as a row vector whereas $\mathbf{B}$ is defined as a column vector, as indicated by the transpose operator (′). Here, multiplication of the row vector $\mathbf{A}$ by the column vector $\mathbf{B}$, is performed with the matrix multiplication operator (*). Then,

   $$\mathbf{A*B} = [a_1b_1 + a_2b_2 + a_3b_3 + \dots + a_nb_n] = \text{single value} \qquad (A.5)$$

For example, if

$$\mathbf{A} = [1 \quad 2 \quad 3 \quad 4 \quad 5]$$

and

$$\mathbf{B} = [-2 \quad 6 \quad -3 \quad 8 \quad 7]'$$

the matrix multiplication **A*B** produces the single value 68, that is,

$$\mathbf{A}^*\mathbf{B} = 1\times(-2)+2\times6+3\times(-3)+4\times8+5\times7 = 68$$

and this is verified with the MATLAB script

```
A=[1  2   3  4  5]; B=[ -2  6  -3  8  7]'; A*B   % Observe transpose operator (') in B
ans =
   68
```

Now, let us suppose that both **A** and **B** are row vectors, and we attempt to perform a row–by–row multiplication with the following MATLAB statements.

```
A=[1  2  3  4  5]; B=[-2  6  -3  8  7]; A*B        % No transpose operator (') here
```

When these statements are executed, MATLAB displays the following message:

```
??? Error using ==> *
Inner matrix dimensions must agree.
```

Here, because we have used the matrix multiplication operator (*) in A*B, MATLAB expects vector **B** to be a column vector, not a row vector. It recognizes that **B** is a row vector, and warns us that we cannot perform this multiplication using the matrix multiplication operator (*). Accordingly, we must perform this type of multiplication with a different operator. This operator is defined below.

2. **Element–by–Element Multiplication** (multiplication of a row vector by another row vector)

Let

$$\mathbf{C} = [c_1 \quad c_2 \quad c_3 \quad \dots \quad c_n]$$

and

$$\mathbf{D} = [d_1 \quad d_2 \quad d_3 \quad \dots \quad d_n]$$

be two row vectors. Here, multiplication of the row vector **C** by the row vector **D** is performed with the ***dot multiplication operator*** (**.***). There is no space between the dot and the multiplication symbol. Thus,

$$\mathbf{C}.^*\mathbf{D} = [c_1d_1 \quad c_2d_2 \quad c_3d_3 \quad \dots \quad c_nd_n] \tag{A.6}$$

This product is another row vector with the same number of elements, as the elements of **C** and **D**.

As an example, let

$$\mathbf{C} = [1 \quad 2 \quad 3 \quad 4 \quad 5]$$

and

$$\mathbf{D} = [-2 \quad 6 \quad -3 \quad 8 \quad 7]$$

Dot multiplication of these two row vectors produce the following result.

$$\mathbf{C}.*\mathbf{D} = 1\times(-2) \quad 2\times6 \quad 3\times(-3) \quad 4\times8 \quad 5\times7 = -2 \quad 12 \quad -9 \quad 32 \quad 35$$

Check with MATLAB:

```
C=[1  2   3   4  5];       %  Vectors C and D must have
D=[−2  6 −3   8  7];       %  same number of elements
C.*D                       % We observe that this is a dot multiplication
```

```
ans =
    -2      12      -9      32      35
```

Similarly, the division (/) and exponentiation (^) operators, are used for matrix division and exponentiation, whereas dot division (./) and dot exponentiation (.^) are used for element–by–element division and exponentiation, as illustrated in Examples A.11 and A.12 above.

We must remember that ***no space is allowed between the dot (.) and the multiplication, division, and exponentiation operators.***

Note: A dot (.) is never required with the plus (+) and minus (–) operators.

Example A.13

Write the MATLAB script that produces a simple plot for the waveform defined as

$$y = f(t) = 3e^{-4t}\cos 5t - 2e^{-3t}\sin 2t + \frac{t^2}{t+1} \tag{A.7}$$

in the $0 \le t \le 5$ seconds interval.

Solution:

The MATLAB script for this example is as follows:

```
t=0: 0.01: 5;                 %  Define t–axis in 0.01 increments
y=3 .* exp(–4 .* t) .* cos(5 .* t)–2 .* exp(–3 .* t) .* sin(2 .* t) + t .^2 ./ (t+1);
plot(t,y); grid; xlabel('t'); ylabel('y=f(t)'); title('Plot for Example A.13')
```

The plot for this example is shown in Figure A.8.

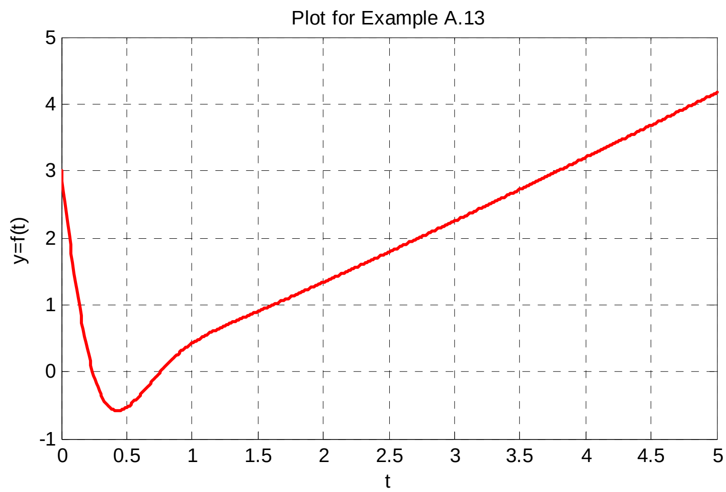


Figure A.8. Plot for Example A.13

Had we, in this example, defined the time interval starting with a negative value equal to or less than -1, say as $-3 \le t \le 3$, MATLAB would have displayed the following message:

```
Warning: Divide by zero.
```

This is because the last term (the rational fraction) of the given expression, is divided by zero when $t = -1$. To avoid division by zero, we use the special MATLAB function **eps,** which is a number approximately equal to 2.2×10^{-16}. It will be used with the next example.

The command **axis([xmin xmax ymin ymax])** scales the current plot to the values specified by the arguments **xmin, xmax, ymin and ymax.** There are no commas between these four arguments. This command must be placed ***after*** the plot command and must be repeated for each plot. The following example illustrates the use of the dot multiplication, division, and exponentiation, the **eps** number, the **axis([xmin xmax ymin ymax])** command, and also MATLAB's capability of displaying up to four windows of different plots.

Example A.14

Plot the functions

$$y = \sin^2 x, \quad z = \cos^2 x, \quad w = \sin^2 x \cdot \cos^2 x, \quad v = \sin^2 x / \cos^2 x$$

in the interval $0 \le x \le 2\pi$ using 100 data points. Use the **subplot** command to display these functions on four windows on the same graph.

Solution:

The MATLAB script to produce the four subplots is as follows:

```
x=linspace(0,2*pi,100);          % Interval with 100 data points
y=(sin(x).^ 2);  z=(cos(x).^ 2);
w=y.* z;
v=y./ (z+eps);%  add eps to avoid division by zero
subplot(221);% upper left of four subplots
plot(x,y);  axis([0 2*pi 0 1]);
title('y=(sinx) ^ 2');
subplot(222);                    % upper right of four subplots
plot(x,z);  axis([0 2*pi 0 1]);
title('z=(cosx) ^ 2');
subplot(223);                    % lower left of four subplots
plot(x,w);  axis([0 2*pi 0 0.3]);
title('w=(sinx) ^ 2*(cosx) ^ 2');
subplot(224);                    % lower right of four subplots
plot(x,v);  axis([0 2*pi 0 400]);
title('v=(sinx) ^ 2/(cosx) ^ 2');
```

These subplots are shown in Figure A.9.

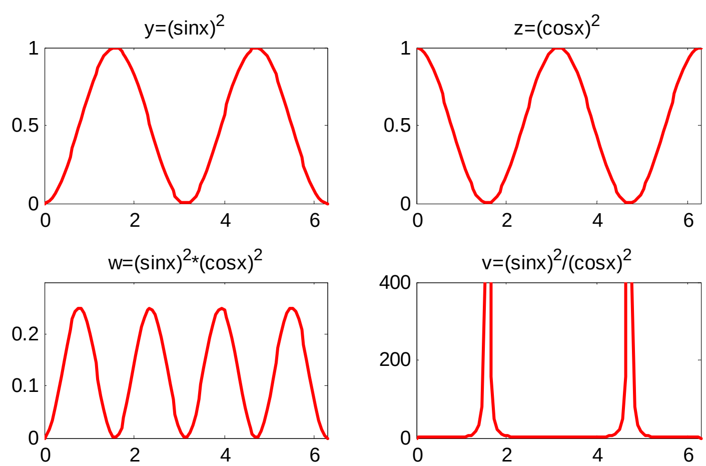


Figure A.9. Subplots for the functions of Example A.14

The next example illustrates MATLAB's capabilities with imaginary numbers. We will introduce the **real(z)** and **imag(z)** functions that display the real and imaginary parts of the complex quantity $z = x + iy$, the **abs(z)**, and the **angle(z)** functions that compute the absolute value (magnitude) and phase angle of the complex quantity $z = x + iy = r\angle\theta$. We will also use the **polar(theta,r)** function that produces a plot in polar coordinates, where **r** is the magnitude, **theta** is the angle in radians, and the **round(n)** function that rounds a number to its nearest integer.

Example A.15

Consider the electric circuit of Figure A.10.

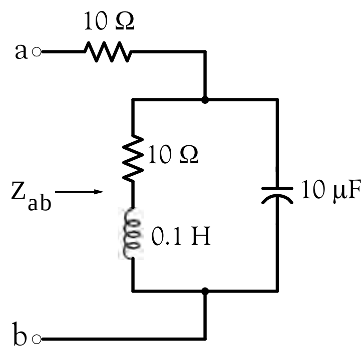


Figure A.10. Electric circuit for Example A.15

With the given values of resistance, inductance, and capacitance, the impedance Z_{ab} as a function of the radian frequency ω can be computed from the following expression:

$$Z_{ab} = Z = 10 + \frac{10^4 - j(10^6/\omega)}{10 + j(0.1\omega - 10^5/\omega)} \tag{A.8}$$

a. Plot Re{Z} (the real part of the impedance *Z*) versus frequency ω.

b. Plot Im{Z} (the imaginary part of the impedance *Z*) versus frequency ω.

c. Plot the impedance *Z* versus frequency ω in polar coordinates.

Solution:

The MATLAB script below computes the real and imaginary parts of Z_{ab} which, for simplicity, are denoted as z, and plots these as two separate graphs (parts a & b). It also produces a polar plot (part c).

```
w=0: 1: 2000;                    % Define interval with one radian interval;...
z=(10+(10 .^ 4 –j .* 10 .^ 6 ./ (w+eps)) ./ (10 + j .* (0.1 .* w –10.^5./ (w+eps))));...
%
% The first five statements (next two lines) compute and plot Re{z}
real_part=real(z); plot(w,real_part);...
xlabel('radian frequency w'); ylabel('Real part of Z'); grid
```

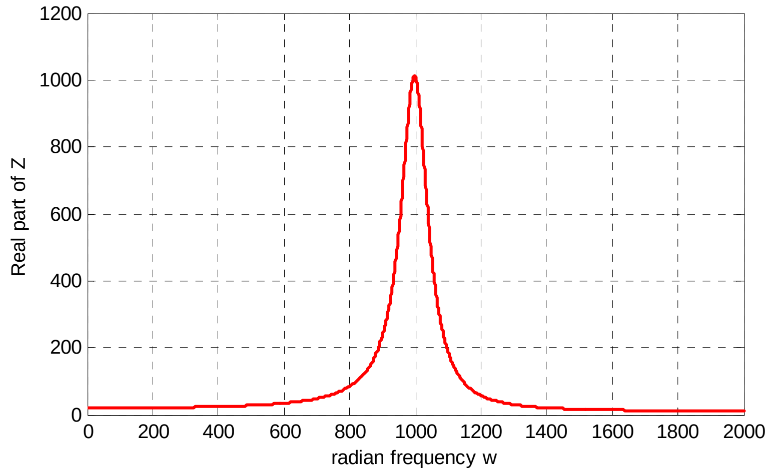


Figure A.11. Plot for the real part of the impedance in Example A.15

```
% The next five statements (next two lines) compute and plot Im{z}
imag_part=imag(z); plot(w,imag_part);...
xlabel('radian frequency w'); ylabel('Imaginary part of Z'); grid
```

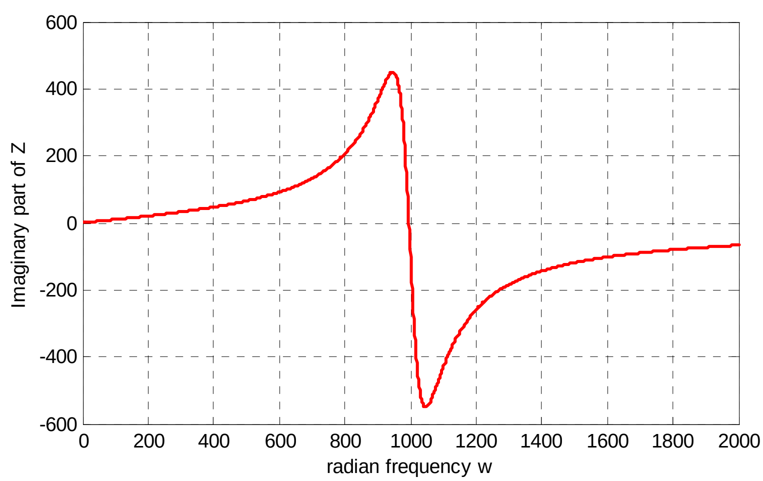


Figure A.12. Plot for the imaginary part of the impedance in Example A.15

```
% The last six statements (next five lines) below produce the polar plot of z
mag=abs(z);                    % Computes |Z|;...
rndz=round(abs(z));            % Rounds |Z| to read polar plot easier;...
theta=angle(z);                % Computes the phase angle of impedance Z;...
polar(theta,rndz);             % Angle is the first argument
ylabel('Polar Plot of Z'); grid
```

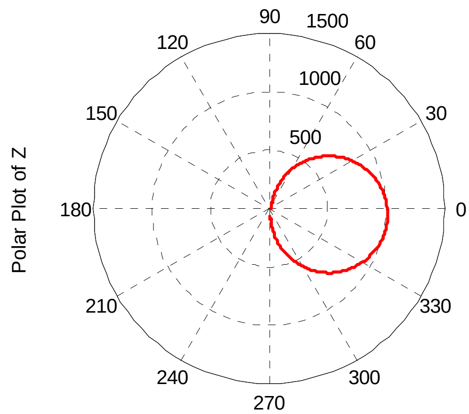


Figure A.13. Polar plot of the impedance in Example A.15

Example A.15 clearly illustrates how powerful, fast, accurate, and flexible MATLAB is.

A.10 Script and Function Files

MATLAB recognizes two types of files: ***script files*** and ***function files***. Both types are referred to as ***m–files*** since both require the *.m* extension.

A ***script file*** consists of two or more built–in functions such as those we have discussed thus far. Thus, the script for each of the examples we discussed earlier, make up a script file. Generally, a script file is one which was generated and saved as an m–file with an editor such as the MATLAB's Editor/Debugger.

A ***function file*** is a user–defined function using MATLAB. We use function files for repetitive tasks. The first line of a function file must contain the word *function*, followed by the output argument, the equal sign (=), and the input argument enclosed in parentheses. The function name and file name must be the same, but the file name must have the extension ***.m***. For example, the function file consisting of the two lines below

```
function y = myfunction(x)
y=x.^ 3 + cos(3.* x)
```

is a function file and must be saved as ***myfunction.m***

For the next example, we will use the following MATLAB functions:

fzero(f,x) – attempts to find a zero of a function of one variable, where **f** is a string containing the name of a real–valued function of a single real variable. MATLAB searches for a value near a point where the function **f** changes sign, and returns that value, or returns NaN if the search fails.

Important: We must remember that we use **roots(p)** to find the roots of polynomials only, such as those in Examples A.1 and A.2.

fplot(fcn,lims) – plots the function specified by the string **fcn** between the x–axis limits specified by **lims = [xmin xmax]**. Using **lims = [xmin xmax ymin ymax]** also controls the y–axis limits. The string **fcn** must be the name of an *m–file* function or a string with variable x.

NaN (Not–a–Number) is not a function; it is MATLAB's response to an undefined expression such as $0/0$, ∞/∞, or inability to produce a result as described on the next paragraph. We can avoid division by zero using the **eps** number, which we mentioned earlier.

Example A.16

Find the zeros, the minimum, and the maximum values of the function

$$f(x) = \frac{1}{(x-0.1)^2+0.01} - \frac{1}{(x-1.2)^2+0.04} - 10 \qquad \text{(A.9)}$$

in the interval $-1.5 \le x \le 1.5$

Solution:

We first plot this function to observe the approximate zeros, maxima, and minima using the following script.

```
x=–1.5: 0.01: 1.5;
y=1./ ((x–0.1).^ 2 + 0.01) –1./ ((x–1.2).^ 2 + 0.04) –10;
plot(x,y); grid
```

The plot is shown in Figure A.14.

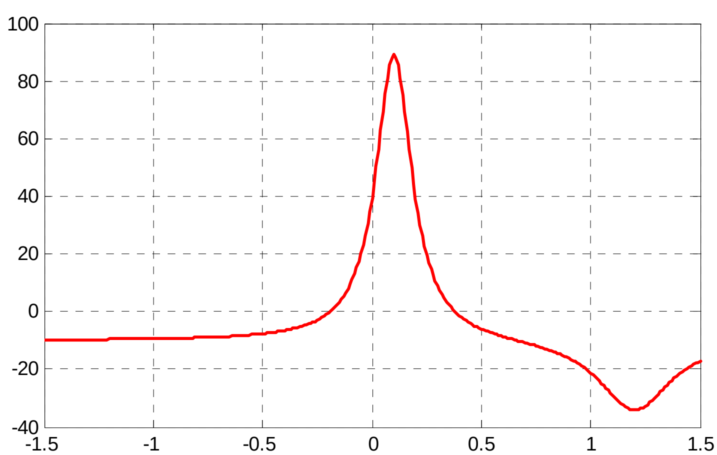


Figure A.14. Plot for Example A.16 using the plot command

The roots (zeros) of this function appear to be in the neighborhood of $x = -0.2$ and $x = 0.3$. The maximum occurs at approximately $x = 0.1$ where, approximately, $y_{max} = 90$, and the minimum occurs at approximately $x = 1.2$ where, approximately, $y_{min} = -34$.

Next, we define and save *f(x)* as the **funczero01.m** function m–file with the following script:

```
function y=funczero01(x)
% Finding the zeros of the function shown below
y=1/((x–0.1)^2+0.01)–1/((x–1.2)^2+0.04)–10;
```

To save this file, from the File drop menu on the Command Window, we choose New, and when the Editor Window appears, we type the script above and we save it as **funczero01.** MATLAB appends the extension **.m** to it.

Now, we can use the **fplot(fcn,lims)** command to plot $f(x)$ as follows:

```
fplot('funczero01', [–1.5  1.5]); grid
```

This plot is shown in Figure A.15. As expected, this plot is identical to the plot of Figure A.14 which was obtained with the **plot(x,y)** command as shown in Figure A.14.

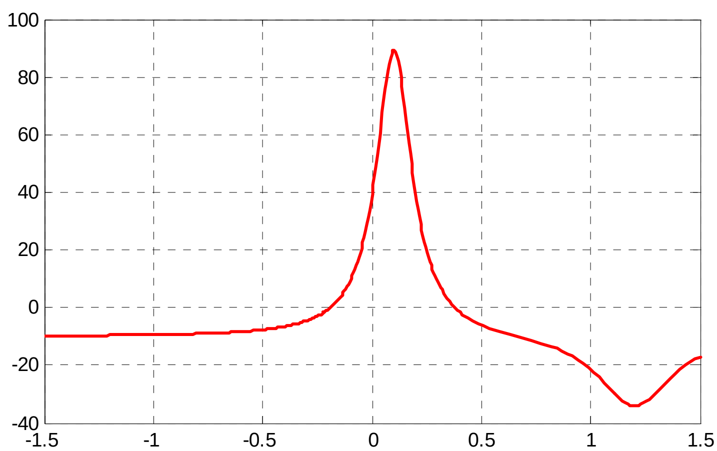


Figure A.15. Plot for Example A.16 using the fplot command

We will use the **fzero(f,x)** function to compute the roots of $f(x)$ in Equation (A.9) more precisely. The MATLAB script below will accomplish this.

```
x1= fzero('funczero01', –0.2);
x2= fzero('funczero01', 0.3);
fprintf('The roots (zeros) of this function are r1= %3.4f', x1);
fprintf(' and r2= %3.4f \n', x2)
```

MATLAB displays the following:

```
The roots (zeros) of this function are r1= -0.1919 and r2= 0.3788
```

The earlier MATLAB versions included the function **fmin(f,x1,x2)** and with this function we could compute both a minimum of some function $f(x)$ or a maximum of $f(x)$ since a maximum of $f(x)$ is equal to a minimum of $-f(x)$. This can be visualized by flipping the plot of a function $f(x)$ upside–down. This function is no longer used in MATLAB and thus we will compute the maxima and minima from the derivative of the given function.

From elementary calculus, we recall that the maxima or minima of a function $y = f(x)$ can be found by setting the first derivative of a function equal to zero and solving for the independent variable x. For this example we use the **diff(x)** function which produces the approximate derivative of a function. Thus, we use the following MATLAB script:

```
syms x ymin zmin; ymin=1/((x–0.1)^2+0.01)–1/((x–1.2)^2+0.04)–10;...
zmin=diff(ymin)
```

```
zmin =
-1/((x-1/10)^2+1/100)^2*(2*x-1/5)+1/((x-6/5)^2+1/25)^2*(2*x-12/5)
```

When the command

```
solve(zmin)
```

is executed, MATLAB displays a very long expression which when copied at the command prompt and executed, produces the following:

```
ans =
   0.6585 + 0.3437i
ans =
   0.6585 - 0.3437i
ans =
    1.2012
```

The real value 1.2012 above is the value of x at which the function y has its minimum value as we observe also in the plot of Figure A.15.

To find the value of y corresponding to this value of x, we substitute it into $f(x)$, that is,

```
x=1.2012; ymin=1 / ((x–0.1) ^ 2 + 0.01) –1 / ((x–1.2) ^ 2 + 0.04) –10
```

```
ymin = -34.1812
```

We can find the maximum value from $-f(x)$ whose plot is produced with the script

```
x=–1.5:0.01:1.5; ymax=–1./((x–0.1).^2+0.01)+1./((x–1.2).^2+0.04)+10; plot(x,ymax); grid
```

and the plot is shown in Figure A.16.

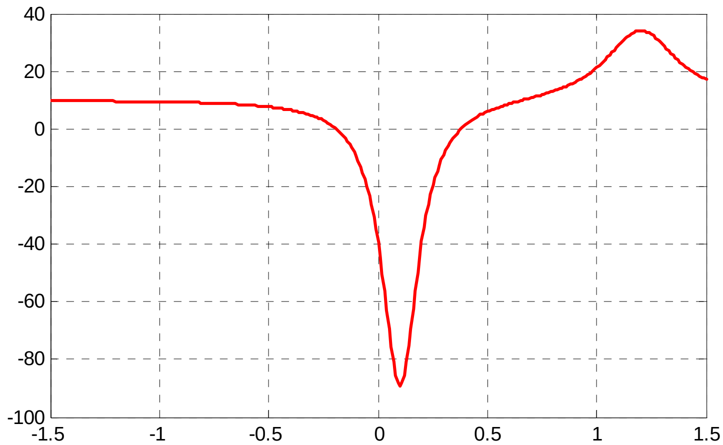


Figure A.16. Plot of $-f(x)$ *for Example A.16*

Next we compute the first derivative of $-f(x)$ and we solve for x to find the value where the maximum of ymax occurs. This is accomplished with the MATLAB script below.

```
syms x ymax zmax; ymax=–(1/((x–0.1)^2+0.01)–1/((x–1.2)^2+0.04)–10); zmax=diff(ymax)
```

```
zmax =
 1/((x-1/10)^2+1/100)^2*(2*x-1/5)-1/((x-6/5)^2+1/25)^2*(2*x-12/5)
```

```
solve(zmax)
```

When the command

```
solve(zmax)
```

is executed, MATLAB displays a very long expression which when copied at the command prompt and executed, produces the following:

```
ans =
   0.6585 + 0.3437i

ans =
   0.6585 - 0.3437i

ans =
    1.2012
ans =
    0.0999
```

From the values above we choose $x = 0.0999$ which is consistent with the plots of Figures A.15 and A.16. Accordingly, we execute the following script to obtain the value of ymin.

```
x=0.0999;                    % Using this value find the corresponding value of ymax
ymax=1 / ((x–0.1) ^ 2 + 0.01) –1 / ((x–1.2) ^ 2 + 0.04) –10
```

```
ymax = 89.2000
```

A.11 Display Formats

MATLAB displays the results on the screen in integer format without decimals if the result is an integer number, or in short floating point format with four decimals if it a fractional number. The format displayed has nothing to do with the accuracy in the computations. MATLAB performs all computations with accuracy up to 16 decimal places.

The output format can changed with the **format** command. The available MATLAB formats can be displayed with the **help format** command as follows:

```
help format
```

```
FORMAT Set output format.
All computations in MATLAB are done in double precision.
FORMAT may be used to switch between different output display formats
as follows:

FORMAT  Default. Same as SHORT.
FORMAT SHORT Scaled fixed point format with 5 digits.
FORMAT LONG Scaled fixed point format with 15 digits.
FORMAT SHORT E Floating point format with 5 digits.
FORMAT LONG E  Floating point format with 15 digits.
FORMAT SHORT G Best of fixed or floating point format with 5 digits.
FORMAT LONG G Best of fixed or floating point format with 15 digits.
FORMAT HEX Hexadecimal format.
FORMAT + The symbols +, - and blank are printed for positive, negative,
        and zero elements.Imaginary parts are ignored.
FORMAT BANK Fixed format for dollars and cents.
FORMAT RAT Approximation by ratio of small integers.

Spacing:

FORMAT COMPACT Suppress extra line-feeds.
FORMAT LOOSE  Puts the extra line-feeds back in.

Some examples with different format displays age given below.

format short  33.3335  Four decimal digits (default)
format long  33.33333333333334 16 digits
format short e  3.3333e+01  Four decimal digits plus exponent
format short g  33.333  Better of format short or format short e
format bank  33.33 two decimal digits
format +  only + or - or zero are printed
```

```
format rat 100/3 rational approximation
```

The **disp(X)** command displays the array **X** without printing the array name. If **X** is a string, the text is displayed.

The **fprintf(format,array)** command displays and prints both text and arrays. It uses specifiers to indicate where and in which format the values would be displayed and printed. Thus, if **%f** is used, the values will be displayed and printed in fixed decimal format, and if **%e** is used, the values will be displayed and printed in scientific notation format. With this command only the real part of each parameter is processed.

This appendix is just an introduction to MATLAB.* This outstanding software package consists of many applications known as *Toolboxes*. The MATLAB Student Version contains just a few of these Toolboxes. Others can be bought directly from The MathWorks,™ Inc., as add–ons.

* *For more MATLAB applications, please refer to Numerical Analysis Using MATLAB and Excel, ISBN 978–1–934404–03–4.*

Appendix B

Introduction to Simulink®

This appendix is a brief introduction to Simulink. This author feels that we can best introduce Simulink with a few examples. Some familiarity with MATLAB is essential in understanding Simulink, and for this purpose, Appendix A is included as an introduction to MATLAB.

B.1 Simulink and its Relation to MATLAB

The MATLAB® and Simulink® environments are integrated into one entity, and thus we can analyze, simulate, and revise our models in either environment at any point. We invoke Simulink from within MATLAB. We will introduce Simulink with a few illustrated examples.

Example B.1

For the circuit of Figure B.1, the initial conditions are $i_L(0^-) = 0$, and $v_c(0^-) = 0.5\ V$. We will compute $v_c(t)$.

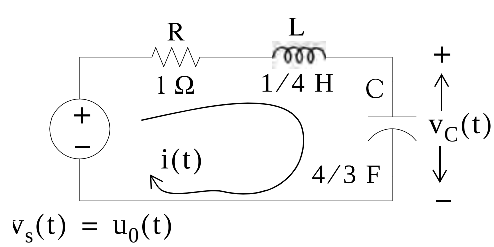


Figure B.1. Circuit for Example B.1

For this example,

$$i = i_L = i_C = C\frac{dv_C}{dt} \tag{B.1}$$

and by Kirchoff's voltage law (KVL),

$$Ri_L + L\frac{di_L}{dt} + v_C = u_0(t) \tag{B.2}$$

Substitution of (B.1) into (B.2) yields

$$RC\frac{dv_C}{dt} + LC\frac{d^2v_C}{dt^2} + v_C = u_0(t) \tag{B.3}$$

Substituting the values of the circuit constants and rearranging we obtain:

$$\frac{1}{3}\frac{d^2v_C}{dt^2} + \frac{4}{3}\frac{dv_C}{dt} + v_C = u_0(t)$$

$$\frac{d^2v_C}{dt^2} + 4\frac{dv_C}{dt} + 3v_C = 3u_0(t) \tag{B.4}$$

$$\frac{d^2v_C}{dt^2} + 4\frac{dv_C}{dt} + 3v_C = 3 \qquad t > 0 \tag{B.5}$$

To appreciate Simulink's capabilities, for comparison, three different methods of obtaining the solution are presented, and the solution using Simulink follows.

First Method – Assumed Solution

Equation (B.5) is a second–order, non–homogeneous differential equation with constant coefficients, and thus the complete solution will consist of the sum of the forced response and the natural response. It is obvious that the solution of this equation cannot be a constant since the derivatives of a constant are zero and thus the equation is not satisfied. Also, the solution cannot contain sinusoidal functions (sine and cosine) since the derivatives of these are also sinusoids. However, decaying exponentials of the form ke^{-at} where *k* and *a* are constants, are possible candidates since their derivatives have the same form but alternate in sign.

It is shown in Appendix H that if $k_1e^{-s_1t}$ and $k_2e^{-s_2t}$ where k_1 and k_2 are constants and s_1 and s_2 are the roots of the characteristic equation of the homogeneous part of the given differential equation, the natural response is the sum of the terms $k_1e^{-s_1t}$ and $k_2e^{-s_2t}$. Therefore, the total solution will be

$$v_c(t) = \text{natural response} + \text{forced response} = v_{cn}(t) + v_{cf}(t) = k_1e^{-s_1t} + k_2e^{-s_2t} + v_{cf}(t) \tag{B.6}$$

The values of s_1 and s_2 are the roots of the characteristic equation

$$s^2 + 4s + 3 = 0 \tag{B.7}$$

Solution of (B.7) yields of $s_1 = -1$ and $s_2 = -3$ and with these values (B.6) is written as

$$v_c(t) = k_1 e^{-t} + k_2 e^{-3t} + v_{cf}(t) \tag{B.8}$$

The forced component $v_{cf}(t)$ is found from (B.5), i.e.,

$$\frac{d^2 v_C}{dt^2} + 4\frac{dv_C}{dt} + 3v_C = 3 \qquad t > 0 \tag{B.9}$$

Since the right side of (B.9) is a constant, the forced response will also be a constant and we denote it as $v_{Cf} = k_3$. By substitution into (B.9) we obtain

$$0 + 0 + 3k_3 = 3$$

or

$$v_{Cf} = k_3 = 1 \tag{B.10}$$

Substitution of this value into (B.8), yields the total solution as

$$v_C(t) = v_{Cn}(t) + v_{Cf} = k_1 e^{-t} + k_2 e^{-3t} + 1 \tag{B.11}$$

The constants k_1 and k_2 will be evaluated from the initial conditions. First, using $v_C(0) = 0.5$ V and evaluating (B.11) at $t = 0$, we obtain

$$v_C(0) = k_1 e^0 + k_2 e^0 + 1 = 0.5$$

$$k_1 + k_2 = -0.5 \tag{B.12}$$

Also,

$$i_L = i_C = C\frac{dv_C}{dt}, \quad \frac{dv_C}{dt} = \frac{i_L}{C}$$

and

$$\left.\frac{dv_C}{dt}\right|_{t=0} = \frac{i_L(0)}{C} = \frac{0}{C} = 0 \tag{B.13}$$

Next, we differentiate (B.11), we evaluate it at $t = 0$, and equate it with (B.13). Thus,

$$\left.\frac{dv_C}{dt}\right|_{t=0} = -k_1 - 3k_2 \tag{B.14}$$

By equating the right sides of (B.13) and (B.14) we obtain

$$-k_1 - 3k_2 = 0 \tag{B.15}$$

Simultaneous solution of (B.12) and (B.15), gives $k_1 = -0.75$ and $k_2 = 0.25$. By substitution into (B.8), we obtain the total solution as

$$\mathbf{v_C(t) = (-0.75e^{-t} + 0.25e^{-3t} + 1)u_0(t)} \tag{B.16}$$

Check with MATLAB:

```
syms t                                          % Define symbolic variable t
y0=-0.75*exp(-t)+0.25*exp(-3*t)+1;              % The total solution y(t), for our example, vc(t)
y1=diff(y0)                                     % The first derivative of y(t)

y1 =
3/4*exp(-t)-3/4*exp(-3*t)

y2=diff(y0,2)                                   % The second derivative of y(t)

y2 =
-3/4*exp(-t)+9/4*exp(-3*t)

y=y2+4*y1+3*y0                                  % Summation of y and its derivatives

y =
3
```

Thus, the solution has been verified by MATLAB. Using the expression for $v_C(t)$ in (B.16), we find the expression for the current as

$$\mathbf{i = i_L = i_C = C\frac{dv_C}{dt} = \frac{4}{3}\left(\frac{3}{4}e^{-t} - \frac{3}{4}e^{-3t}\right) = e^{-t} - e^{-3t}\ A} \tag{B.17}$$

Second Method – Using the Laplace Transformation

The transformed circuit is shown in Figure B.2.

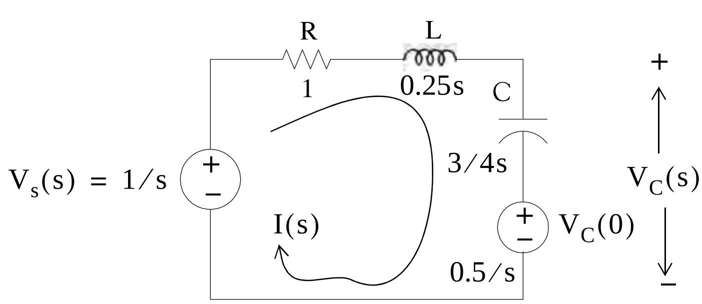


Figure B.2. Transformed Circuit for Example B.1

By the voltage division* expression,

$$V_C(s) = \frac{3/4s}{(1+0.25s+3/4s)} \cdot \left(\frac{1}{s} - \frac{0.5}{s}\right) + \frac{0.5}{s} = \frac{1.5}{s(s^2+4s+3)} + \frac{0.5}{s} = \frac{0.5s^2+2s+3}{s(s+1)(s+3)}$$

Using partial fraction expansion,† we let

$$\frac{0.5s^2+2s+3}{s(s+1)(s+3)} = \frac{r_1}{s} + \frac{r_2}{(s+1)} + \frac{r_3}{(s+3)} \tag{B.18}$$

$$r_1 = \left.\frac{0.5s^2+2s+3}{(s+1)(s+3)}\right|_{s=0} = 1$$

$$r_2 = \left.\frac{0.5s^2+2s+3}{s(s+3)}\right|_{s=-1} = -0.75$$

$$r_3 = \left.\frac{0.5s^2+2s+3}{s(s+1)}\right|_{s=-3} = 0.25$$

and by substitution into (B.18)

$$V_C(s) = \frac{0.5s^2+2s+3}{s(s+1)(s+3)} = \frac{1}{s} + \frac{-0.75}{(s+1)} + \frac{0.25}{(s+3)}$$

Taking the Inverse Laplace transform‡ we find that

$$\mathbf{v_C(t) = 1 - 0.75e^{-t} + 0.25e^{-3t}}$$

Third Method – Using State Variables

$$Ri_L + L\frac{di_L}{dt} + v_C = u_0(t)^{**}$$

---

* *For derivation of the voltage division and current division expressions, please refer to Circuit Analysis I with MATLAB Computing and Simulink / SimPowerSystems , ISBN 978−1−934404−17−1.*

† *Partial fraction expansion is discussed in Chapter 5, this text.*

‡ *For an introduction to Laplace Transform and Inverse Laplace Transform, please refer Chapters 4 and 5, this text.*

** *Usually, in State−Space and State Variables Analysis,* $u(t)$ *denotes any input. For distinction, we will denote the Unit Step Function as* $u_0(t)$*. For a detailed discussion on State−Space and State Variables Analysis, please refer to Chapter 7, this text.*

By substitution of given values and rearranging, we obtain

$$\frac{1}{4}\frac{di_L}{dt} = (-1)i_L - v_C + 1$$

or

$$\frac{di_L}{dt} = -4i_L - 4v_C + 4 \tag{B.19}$$

Next, we define the state variables $x_1 = i_L$ and $x_2 = v_C$. Then,

$$\dot{x}_1 = \frac{di_L}{dt}{}^* \tag{B.20}$$

and

$$\dot{x}_2 = \frac{dv_C}{dt} \tag{B.21}$$

Also,

$$i_L = C\frac{dv_C}{dt}$$

and thus,

$$x_1 = i_L = C\frac{dv_C}{dt} = C\dot{x}_2 = \frac{4}{3}\dot{x}_2$$

or

$$\dot{x}_2 = \frac{3}{4}x_1 \tag{B.22}$$

Therefore, from (B.19), (B.20), and (B.22), we obtain the state equations

$$\dot{x}_1 = -4x_1 - 4x_2 + 4$$

$$\dot{x}_2 = \frac{3}{4}x_1$$

and in matrix form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -4 & -4 \\ 3/4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \end{bmatrix}u_0(t) \tag{B.23}$$

Solution† of (B.23) yields

* *The notation* $\dot{x}$ *(x dot) is often used to denote the first derivative of the function* x, *that is,* $\dot{x} = dx/dt$.

† *The detailed solution of (B.23) is given in Chapter 7, Example 7.10, Page 7−23, this text.*

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} e^{-t} - e^{-3t} \\ 1 - 0.75e^{-t} + 0.25e^{-3t} \end{bmatrix}$$

Then,

$$x_1 = i_L = e^{-t} - e^{-3t} \tag{B.24}$$

and

$$x_2 = v_C = 1 - 0.75e^{-t} + 0.25e^{-3t} \tag{B.25}$$

Modeling the Differential Equation of Example B.1 with Simulink

To run Simulink, we must first invoke MATLAB. Make sure that Simulink is installed in your system. In the MATLAB Command prompt, we type:

```
simulink
```

Alternately, we can click on the Simulink icon shown in Figure B.3. It appears on the top bar on MATLAB's Command prompt.

Figure B.3. The Simulink icon

Upon execution of the Simulink command, the **Commonly Used Blocks** appear as shown in Figure B.4.

In Figure B.4, the left side is referred to as the **Tree Pane** and displays all Simulink libraries installed. The right side is referred to as the **Contents Pane** and displays the blocks that reside in the library currently selected in the Tree Pane.

Let us express the differential equation of Example B.1 as

$$\frac{d^2v_C}{dt^2} = -4\frac{dv_C}{dt} - 3v_C + 3u_0(t) \tag{B.26}$$

A block diagram representing relation (B.26) above is shown in Figure B.5. We will use Simulink to draw a similar block diagram.*

* *Henceforth, all Simulink block diagrams will be referred to as models.*

Figure B.4. The Simulink Library Browser

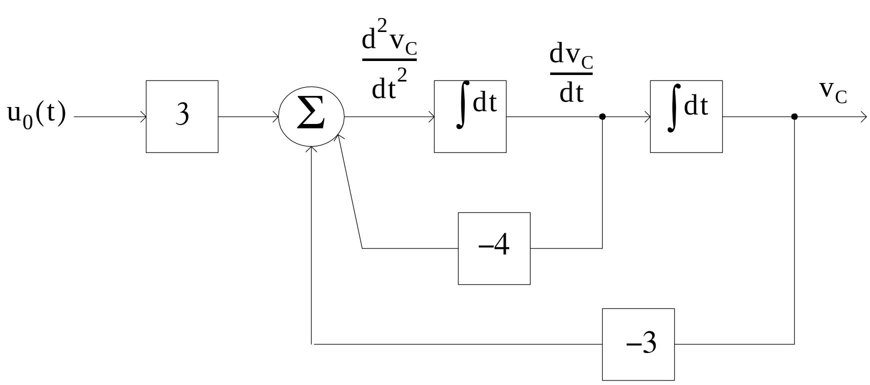


Figure B.5. Block diagram for equation (B.26)

To model the differential equation (B.26) using Simulink, we perform the following steps:

1. On the **Simulink Library Browser**, we click on the leftmost icon shown as a blank page on the top title bar. A new model window named **untitled** will appear as shown in Figure B.6.

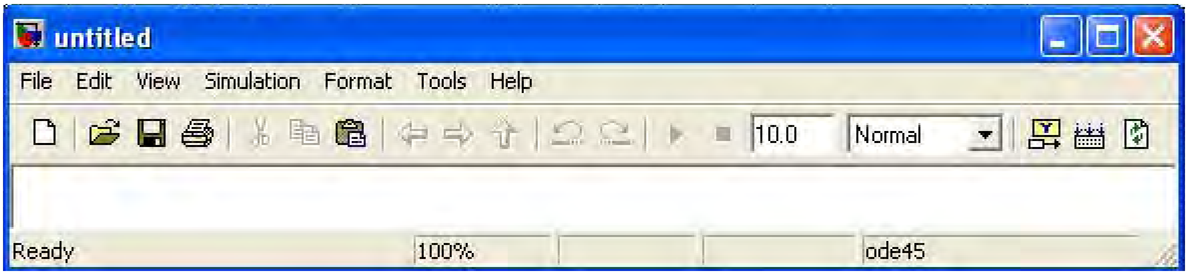


Figure B.6. The Untitled model window in Simulink.

The window of Figure B.6 is the model window where we enter our blocks to form a block diagram. We save this as model file name Equation_1_26. This is done from the File drop menu of Figure B.6 where we choose **Save as** and name the file as Equation_1_26. Simulink will add the extension .mdl. The new model window will now be shown as **Equation_1_26**, and all saved files will have this appearance. See Figure B.7.

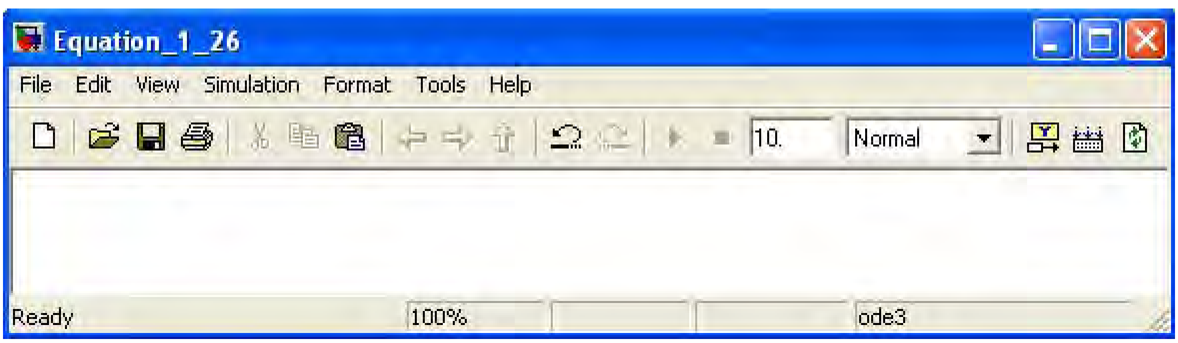


Figure B.7. Model window for Equation_1_26.mdl file

2. With the **Equation_1_26** model window and the **Simulink Library Browser** both visible, we click on the **Sources** appearing on the left side list, and on the right side we scroll down until we see the unit step function shown as **Step**. See Figure B.8. We select it, and we drag it into the **Equation_1_26** model window which now appears as shown in Figure B.8. We save file Equation_1_26 using the File drop menu on the **Equation_1_26** model window (right side of Figure B.8).

3. With reference to block diagram of Figure B.5, we observe that we need to connect an amplifier with Gain 3 to the unit step function block. The gain block in Simulink is under **Commonly Used Blocks** (first item under Simulink on the **Simulink Library Browser**). See Figure B.8. If the **Equation_1_26** model window is no longer visible, it can be recalled by clicking on the white page icon on the top bar of the **Simulink Library Browser**.

4. We choose the gain block and we drag it to the right of the unit step function. The triangle on the right side of the unit step function block and the > symbols on the left and right sides of the gain block are connection points. We point the mouse close to the connection point of the unit step function until is shows as a cross hair, and draw a straight line to connect the two

blocks.* We double–click on the gain block and on the **Function Block Parameters**, we change the gain from 1 to 3. See Figure B.9.

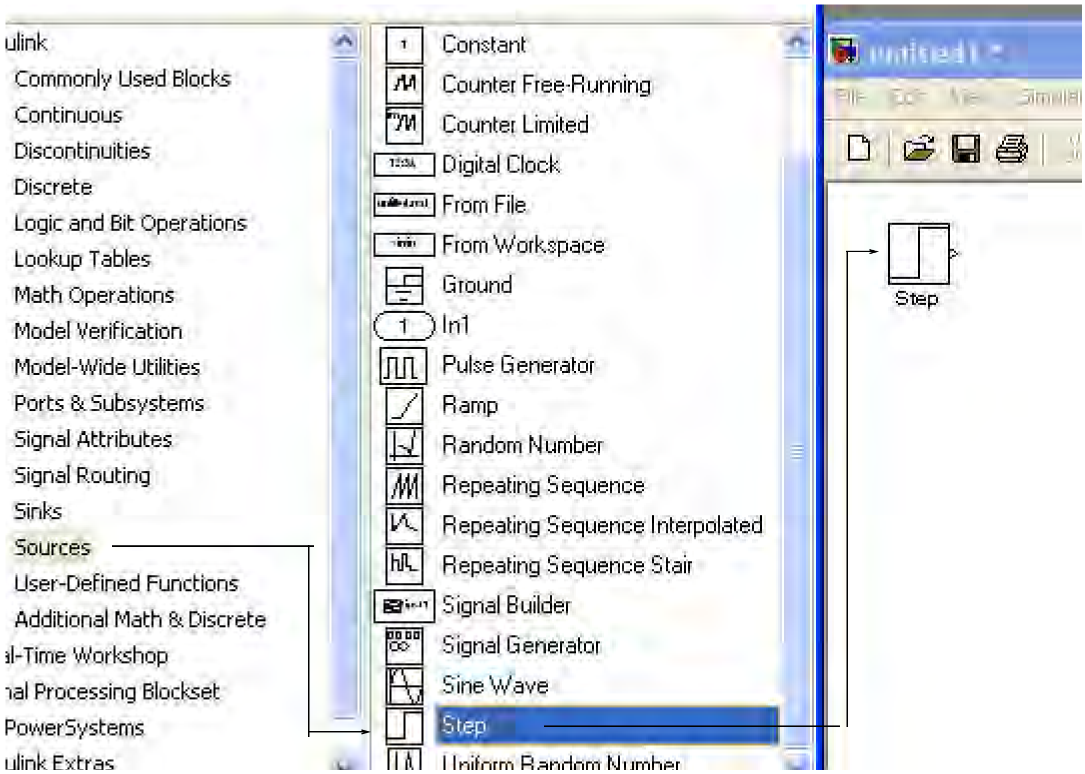


Figure B.8. Dragging the unit step function into File Equation_1_26

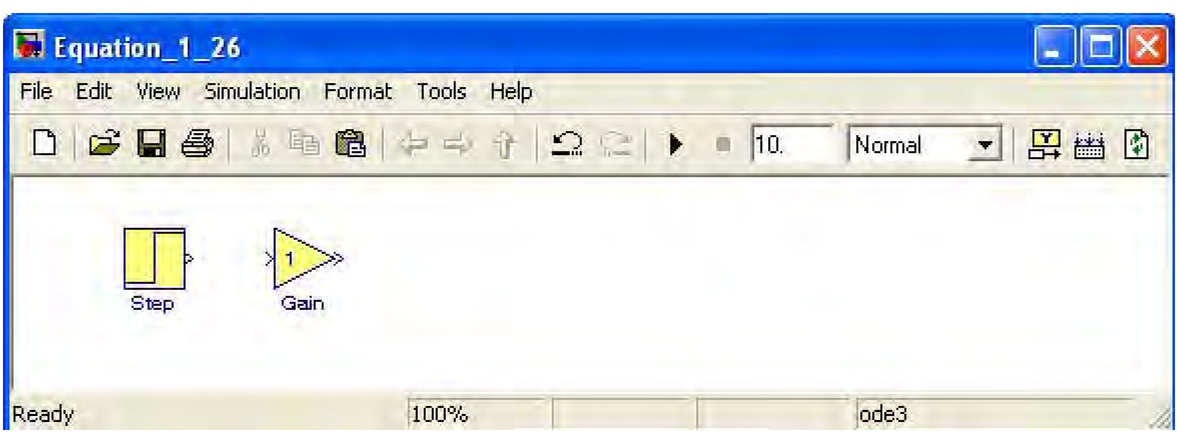


Figure B.9. File Equation_1_26 with added Step and Gain blocks

* *An easy method to interconnect two Simulink blocks by clicking on the source block to select it, then hold down the* **Ctrl** *key and left–click on the destination block.*

5. Next, we need to add a thee–input adder. The adder block appears on the right side of the **Simulink Library Browser** under **Math Operations**. We select it, and we drag it into the Equation_1_26 model window. We double click it, and on the **Function Block Parameters** window which appears**,** we specify 3 inputs. We then connect the output of the of the gain block to the first input of the adder block as shown in Figure B.10.

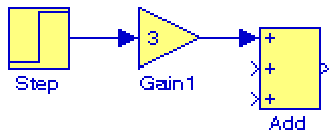


Figure B.10. File Equation_1_26 with added gain block

6. From the **Commonly Used Blocks** of the **Simulink Library Browser**, we choose the **Integrator** block, we drag it into the Equation_1_26 model window, and we connect it to the output of the **Add** block. We repeat this step and to add a second **Integrator** block. We click on the text "Integrator" under the first integrator block, and we change it to Integrator 1. Then, we change the text "Integrator 1" under the second Integrator to "Integrator 2" as shown in Figure B.11.

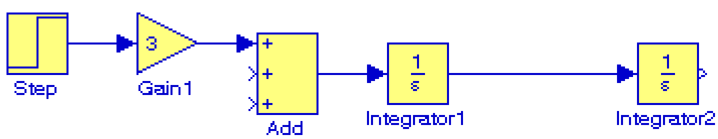


Figure B.11. File Equation_1_26 with the addition of two integrators

7. To complete the block diagram, we add the **Scope** block which is found in the **Commonly Used Blocks** on the **Simulink Library Browser**, we click on the Gain block, and we copy and paste it twice. We flip the pasted Gain blocks by using the **Flip Block** command from the Format drop menu, and we label these as Gain 2 and Gain 3. Finally, we double–click on these gain blocks and on the **Function Block Parameters** window, we change the gains from to −4 and −3 as shown in Figure B.12.

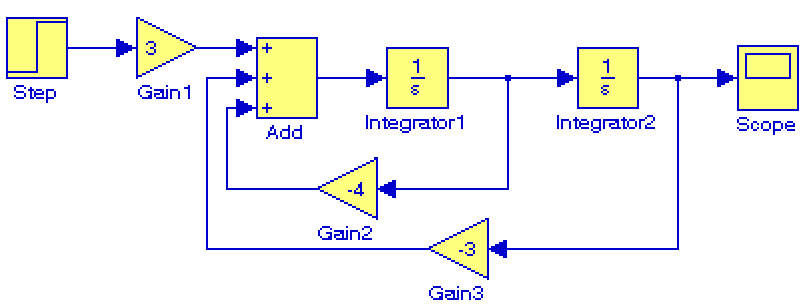


Figure B.12. File Equation_1_26 complete block diagram

8. The initial conditions $i_L(0^-) = C\left.\frac{dv_C}{dt}\right|_{t=0} = 0$, and $v_c(0^-) = 0.5\ V$ are entered by double clicking the Integrator blocks and entering the values 0 for the first integrator, and 0.5 for the second integrator. We also need to specify the simulation time. This is done by specifying the simulation time to be 10 seconds on the **Configuration Parameters** from the **Simulation** drop menu. We can start the simulation on **Start** from the **Simulation** drop menu or by clicking on the  icon.

9. To see the output waveform, we double click on the **Scope** block, and then clicking on the Autoscale  icon, we obtain the waveform shown in Figure B.13.

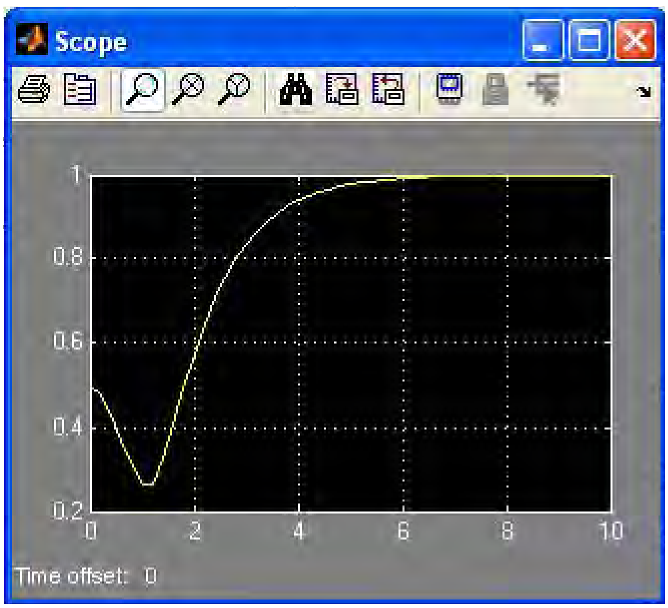


Figure B.13. The waveform for the function $v_C(t)$ for Example B.1

Another easier method to obtain and display the output $v_C(t)$ for Example B.1, is to use **State–Space** block from **Continuous** in the Simulink Library Browser, as shown in Figure B.14.

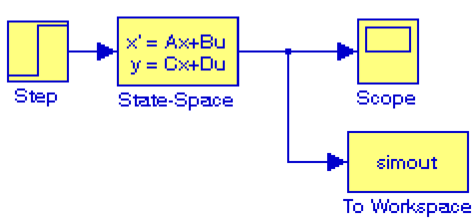


Figure B.14. Obtaining the function $v_C(t)$ for Example B.1 with the State–Space block.

The **simout To Workspace** block shown in Figure B.14 writes its input to the workspace. The data and variables created in the MATLAB Command window, reside in the MATLAB Workspace. This block writes its output to an array or structure that has the name specified by the block's Variable name parameter. This gives us the ability to delete or modify selected variables. We issue the command who to see those variables. From Equation B.23, Page B–6,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -4 & -4 \\ 3/4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \end{bmatrix} u_0(t)$$

The output equation is

$$y = Cx + du$$

or

$$y = [0 \;\; 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0]u$$

We double–click on the **State–Space** block, and in the **Functions Block Parameters** window we enter the constants shown in Figure B.15.

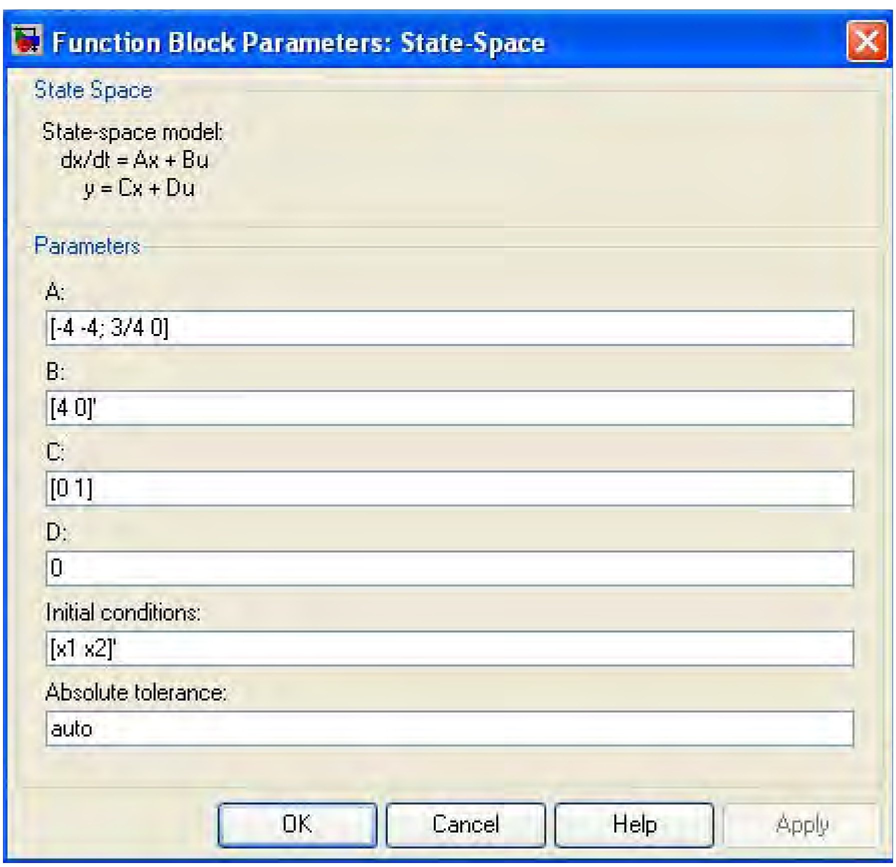
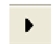


Figure B.15. The Function block parameters for the State–Space block.

The initials conditions [x1 x2]' are specified in MATLAB's Command prompt as

```
x1=0; x2=0.5;
```

As before, to start the simulation we click clicking on the  icon, and to see the output waveform, we double click on the **Scope** block, and then clicking on the Autoscale  icon, we obtain the waveform shown in Figure B.16.

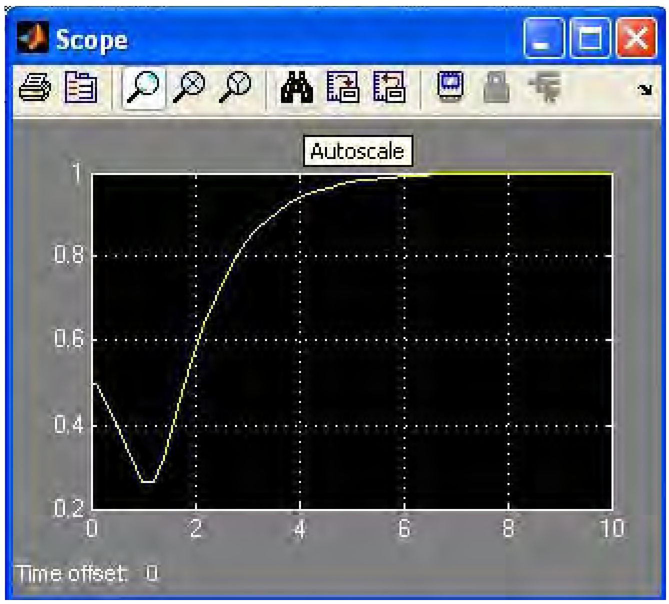


Figure B.16. The waveform for the function $v_C(t)$ for Example B.1 with the State–Space block.

The state–space block is the best choice when we need to display the output waveform of three or more variables as illustrated by the following example.

Example B.2

A fourth–order network is described by the differential equation

$$\frac{d^4y}{dt^4} + a_3\frac{d^3y}{dt^3} + a_2\frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_0\,y(t) = u(t) \tag{B.27}$$

where $y(t)$ is the output representing the voltage or current of the network, and $u(t)$ is any input, and the initial conditions are $y(0) = y'(0) = y''(0) = y'''(0) = 0$.

a. We will express (B.27) as a set of state equations

b. It is known that the solution of the differential equation

$$\frac{d^4y}{dt^4} + 2\frac{d^2y}{dt^2} + y(t) = \sin t \tag{B.28}$$

subject to the initial conditions $y(0) = y'(0) = y''(0) = y'''(0) = 0$, has the solution

$$y(t) = 0.125[(3 - t^2) - 3t\cos t] \tag{B.29}$$

In our set of state equations, we will select appropriate values for the coefficients $a_3, a_2, a_1,$ and a_0 so that the new set of the state equations will represent the differential equation of (B.28), and using Simulink, we will display the waveform of the output $y(t)$.

1. The differential equation of (B.28) is of fourth–order; therefore, we must define four state variables that will be used with the four first–order state equations.

   We denote the state variables as x_1, x_2, x_3, and x_4, and we relate them to the terms of the given differential equation as

$$x_1 = y(t) \qquad x_2 = \frac{dy}{dt} \qquad x_3 = \frac{d^2y}{dt^2} \qquad x_4 = \frac{d^3y}{dt^3} \tag{B.30}$$

   We observe that

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= x_4 \\ \frac{d^4y}{dt^4} = \dot{x}_4 &= -a_0x_1 - a_1x_2 - a_2x_3 - a_3x_4 + u(t) \end{aligned} \tag{B.31}$$

   and in matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t) \tag{B.32}$$

   In compact form, (B.32) is written as

$$\dot{x} = Ax + bu \tag{B.33}$$

   Also, the output is

$$y = Cx + du \tag{B.34}$$

   where

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \text{and } u = u(t) \tag{B.35}$$

and since the output is defined as

$$y(t) = x_1$$

relation (B.34) is expressed as

$$y = [1\ \ 0\ \ 0\ \ 0] \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + [0]u(t) \tag{B.36}$$

2. By inspection, the differential equation of (B.27) will be reduced to the differential equation of (B.28) if we let

$$a_3 = 0 \qquad a_2 = 2 \qquad a_1 = 0 \qquad a_0 = 1 \qquad u(t) = \sin t$$

and thus the differential equation of (B.28) can be expressed in state–space form as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \sin t \tag{B.37}$$

where

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & 0 & -2 & 0 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \text{and } u = \sin t \tag{B.38}$$

Since the output is defined as

$$y(t) = x_1$$

in matrix form it is expressed as

$$y = [1\ \ 0\ \ 0\ \ 0] \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + [0]\sin t \qquad (B.39)$$

We invoke MATLAB, we start Simulink by clicking on the Simulink icon, on the **Simulink Library Browser** we click on the **Create a new model** (blank page icon on the left of the top bar), and we save this model as Example_1_2. On the **Simulink Library Browser** we select **Sources**, we drag the **Signal Generator** block on the Example_1_2 model window, we click and drag the **State–Space** block from the **Continuous** on **Simulink Library Browser**, and we click and drag the **Scope** block from the **Commonly Used Blocks** on the **Simulink Library Browser**. We also add the **Display** block found under **Sinks** on the **Simulink Library Browser**. We connect these four blocks and the complete block diagram is as shown in Figure B.17.

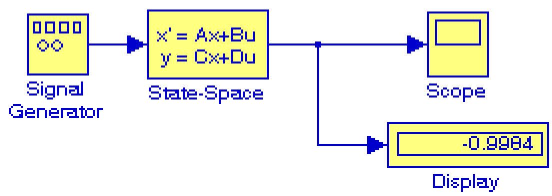


Figure B.17. Block diagram for Example B.2

We now double–click on the **Signal Generator** block and we enter the following in the **Function Block Parameters:**

Wave form: sine

Time (t): Use simulation time

Amplitude: 1

Frequency: 2

Units: Hertz

Next, we double–click on the **state–space** block and we enter the following parameter values in the **Function Block Parameters**:

A: [0 1 0 0; 0 0 1 0; 0 0 0 1; −a0 −a1 −a2 −a3]

B: [0 0 0 1]’

C: [1 0 0 0]

D: [0]

Initial conditions: x0

Absolute tolerance: auto

Now, we switch to the MATLAB Command prompt and we type the following:

```
>> a0=1; a1=0; a2=2; a3=0; x0=[0  0  0  0]';
```

We change the **Simulation Stop time** to 25, and we start the simulation by clicking on the icon. To see the output waveform, we double click on the **Scope** block, then clicking on the Autoscale icon, we obtain the waveform shown in Figure B.18.

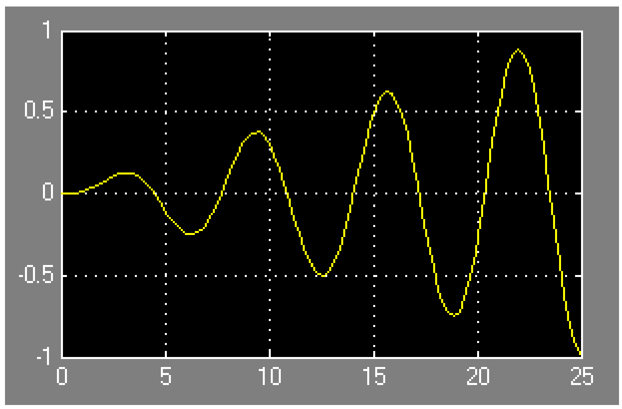


Figure B.18. Waveform for Example B.2

The **Display** block in Figure B.17 shows the value at the end of the simulation stop time.

Examples B.1 and B.2 have clearly illustrated that the State–Space is indeed a powerful block. We could have obtained the solution of Example B.2 using four Integrator blocks by this approach would have been more time consuming.

Example B.3

Using **Algebraic Constraint** blocks found in the **Math Operations** library, **Display** blocks found in the **Sinks** library, and **Gain** blocks found in the **Commonly Used Blocks** library, we will create a model that will produce the simultaneous solution of three equations with three unknowns.

The model will display the values for the unknowns z_1, z_2, and z_3 in the system of the equations

$$\begin{aligned} a_1 z_1 + a_2 z_2 + a_3 z_3 + k_1 &= 0 \\ a_4 z_1 + a_5 z_2 + a_6 z_3 + k_2 &= 0 \\ a_7 z_1 + a_8 z_2 + a_9 z_3 + k_3 &= 0 \end{aligned} \qquad \text{(B.40)}$$

The model is shown in Figure B.19.

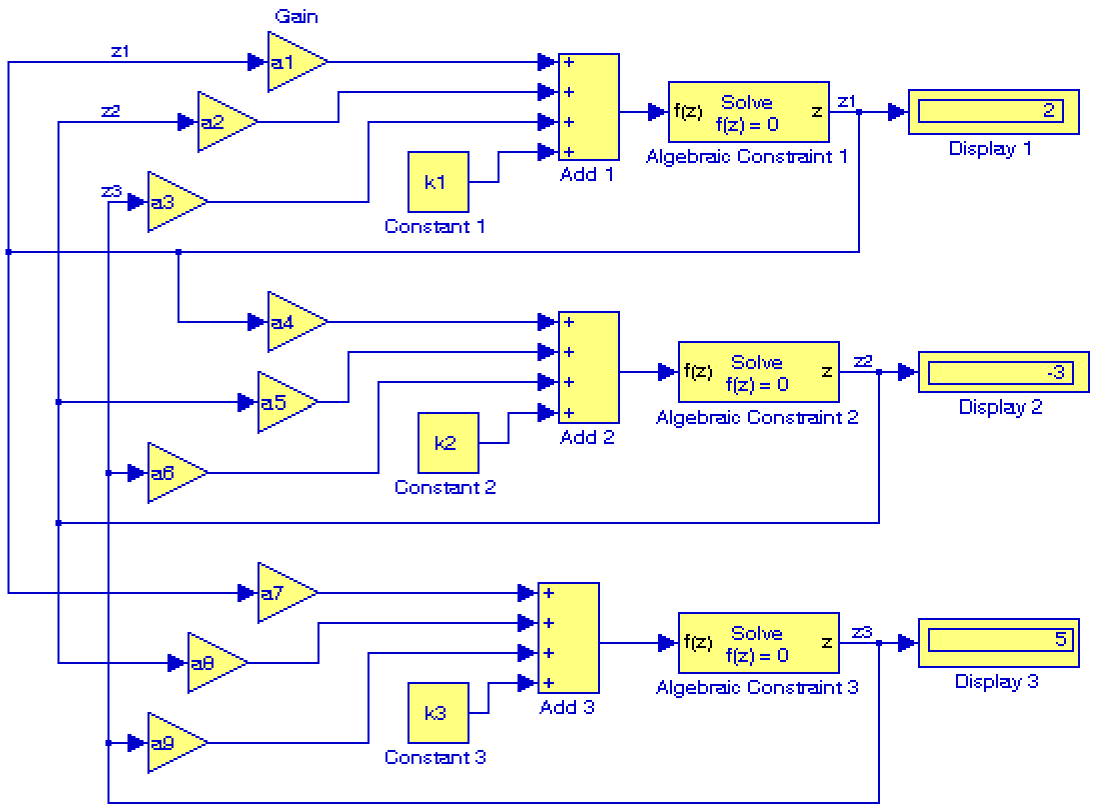


Figure B.19. Model for Example B.3

Next, we go to MATLAB's Command prompt and we enter the following values:

```
a1=2; a2=–3; a3=–1; a4=1; a5=5; a6=4; a7=–6; a8=1; a9=2;...
k1=–8; k2=–7; k3=5;
```

After clicking on the simulation icon, we observe the values of the unknowns as $z_1 = 2$, $z_2 = -3$, and $z_3 = 5$. These values are shown in the Display blocks of Figure B.19.

The **Algebraic Constraint** block constrains the input signal $f(z)$ to zero and outputs an algebraic state z. The block outputs the value necessary to produce a zero at the input. The output must affect the input through some feedback path. This enables us to specify algebraic equations for index 1 differential/algebraic systems (DAEs). By default, the Initial guess parameter is zero. We can improve the efficiency of the algebraic loop solver by providing an Initial guess for the algebraic state z that is close to the solution value.

An outstanding feature in Simulink is the representation of a large model consisting of many blocks and lines, to be shown as a single Subsystem block.* For instance, we can group all blocks and lines in the model of Figure B.19 except the display blocks, we choose **Create Subsystem** from the **Edit** menu, and this model will be shown as in Figure B.20† where in MATLAB's Command prompt we have entered:

```
a1=5; a2=–1; a3=4; a4=11; a5=6; a6=9; a7=–8; a8=4; a9=15;...
k1=14; k2=–6; k3=9;
```

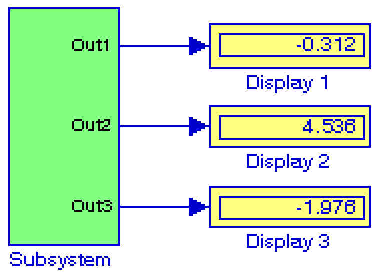


Figure B.20. The model of Figure B.19 represented as a subsystem

The Display blocks in Figure B.20 show the values of z_1, z_2, and z_3 for the values specified at the MATLAB command prompt.

B.2 Simulink Demos

At this time, the reader with no prior knowledge of Simulink, should be ready to learn Simulink's additional capabilities. It is highly recommended that the reader becomes familiar with the block libraries found in the Simulink Library Browser. Then, the reader can follow the steps delineated in The MathWorks Simulink User's Manual to run the Demo Models beginning with the thermo model. This model can be seen by typing

```
thermo
```

at the MATLAB command prompt.

* *The Subsystem block is described in detail in Chapter 2, Section 2.1, Page 2–2, Introduction to Simulink with Engineering Applications, 978–1–934404–09–6.*

† *The contents of the Subsystem block are not lost. We can double–click on the Subsystem block to see its contents. The Subsystem block replaces the inputs and outputs of the model with Inport and Outport blocks. These blocks are described in Section 2.1, Chapter 2, Page 2–2, Introduction to Simulink with Engineering Applications, ISBN 978–1–934404–09–6.*

Appendix C

Introduction to SimPowerSystems®

This appendix is a brief introduction to **SimPowerSystems®** blockset that operates in the **Simulink®** environment. An introduction to **Simulink** is presented in Appendix B. For additional help with Simulink, please refer to the Simulink documentation.

C.1 Simulation of Electric Circuits with SimPowerSystems

As stated in Appendix B, the MATLAB® and Simulink® environments are integrated into one entity, and thus we can analyze, simulate, and revise our models in either environment at any point. We can invoke **Simulink** from within MATLAB or by typing simulink at the MATLAB command prompt, and we can invoke **SimPowerSystems** from within Simulink or by typing powerlib at the MATLAB command prompt. We will introduce **SimPowerSystems** with two illustrated examples, a DC electric circuit, and an AC electric circuit

Example C.1

For the simple resistive circuit in Figure C.1, $v_S = 12v$, $R_1 = 7\Omega$, and $R_2 = 5\Omega$. From the voltage division expression, $v_{R2} = R_2 \times v_S / (R_1 + R_2) = 5 \times 12/12 = 5v$ and from Ohm's law, $i = v_S / (R_1 + R_2) = 1A$.

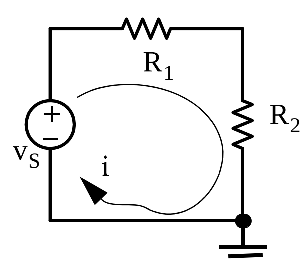


Figure C.1. Circuit for Example C.1

To model the circuit in Figure C.1, we enter the following command at the MATLAB prompt.

```
powerlib
```

and upon execution of this command, the powerlib window shown in Figure C.2 is displayed.

From the **File** menu in Figure C.2, we open a new window and we name it Sim_Fig_C3 as shown in Figure C.3.

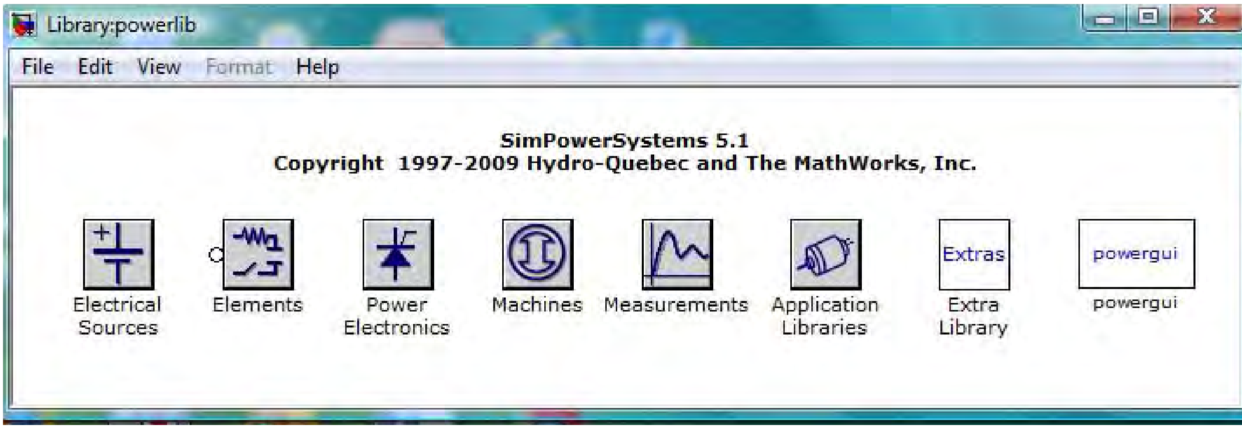


Figure C.2. Library blocks for SimPowerSystems

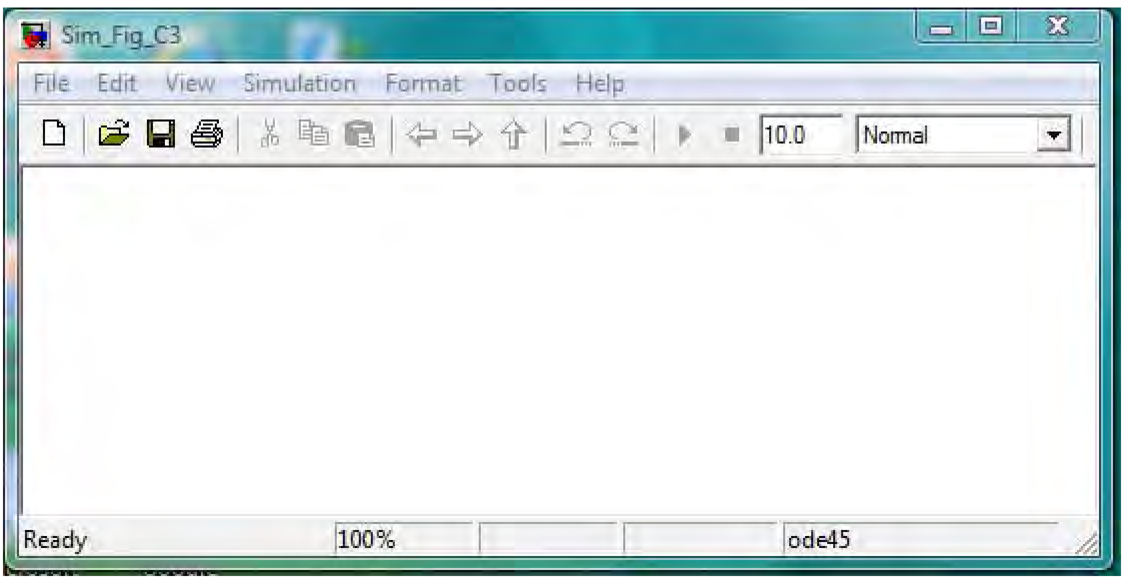


Figure C.3. New window for modeling the circuit shown in Figure C.1

The powergui block in Figure C.2 is referred to as the **Environmental** block for **SimPowerSystems** models and it must be included in every model containing **SimPowerSystems** blocks. Accordingly, we begin our model by adding this block as shown in Figure C.4.

We observe that in Figure C.4, the powergui block is named Continuous. This is the default method of solving an electric circuit and uses a variable step Simulink solver. Other methods are the Discrete method used when the discretization of the system at fixed time steps is desired, and the Phasors method which performs phasor simulation at the frequency specified by the Phasor frequency parameter. These methods are described in detail in the **SimPowerSystems** documentation.

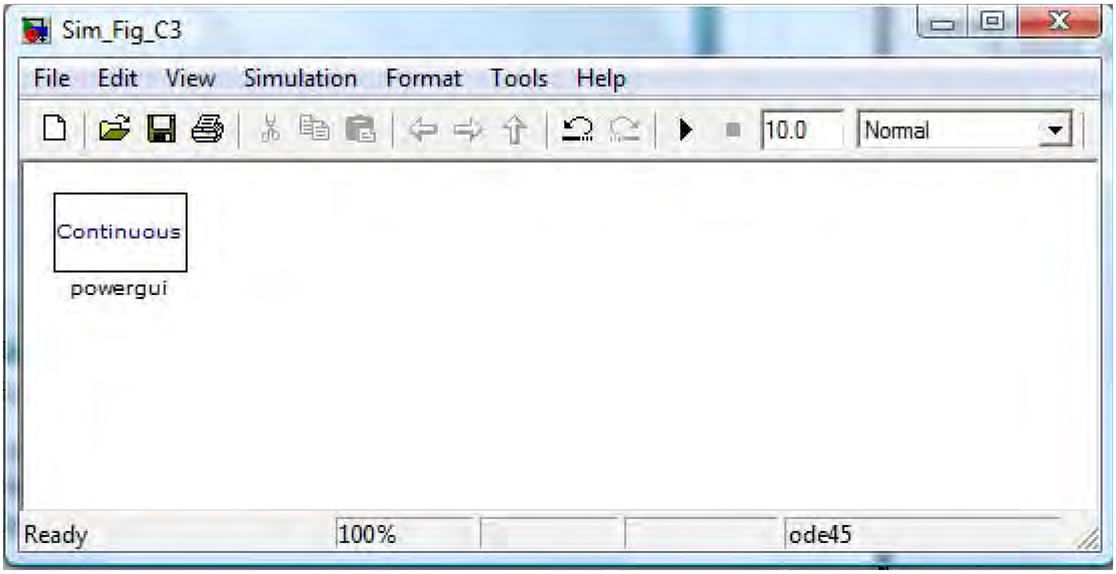


Figure C.4. Window with the addition of the powergui block

Next, we need to the components of the electric circuit shown in Figure C.1. From the Electrical Sources library in Figure C.2 we select the **DC Voltage Source** block and drag it into the model, from the Elements library we select and drag the **Series RLC Branch** block and the **Ground** block, from the Measurements library we select the **Current Measurement** and the **Voltage Measurement** blocks, and from the Simulink Sinks library we select and drag the **Display** block. The model now appears as shown in Figure C.5.

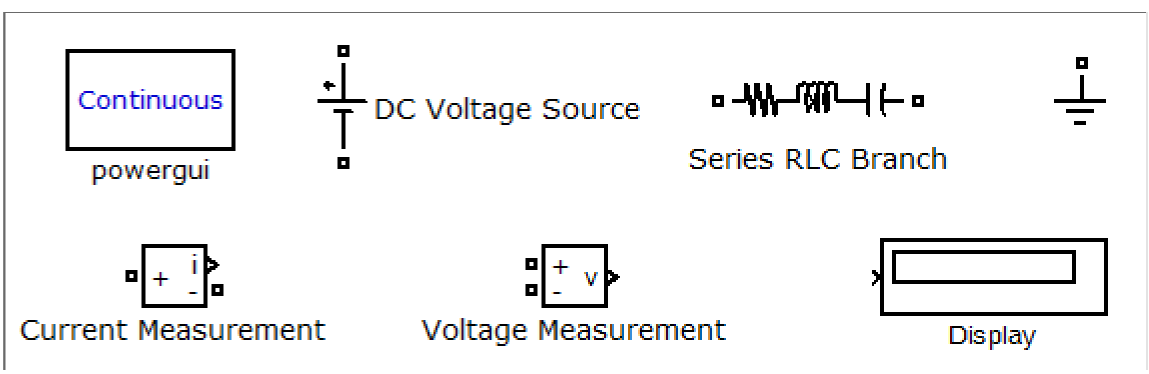


Figure C.5. The circuit components for our model

From the **Series RLC Branch** block we only need the resistor, and to eliminate the inductor and the capacitor, we double click it and from the **Block Parameters** window we select the R component with value set at $7\ \Omega$ as shown in Figure C.6.

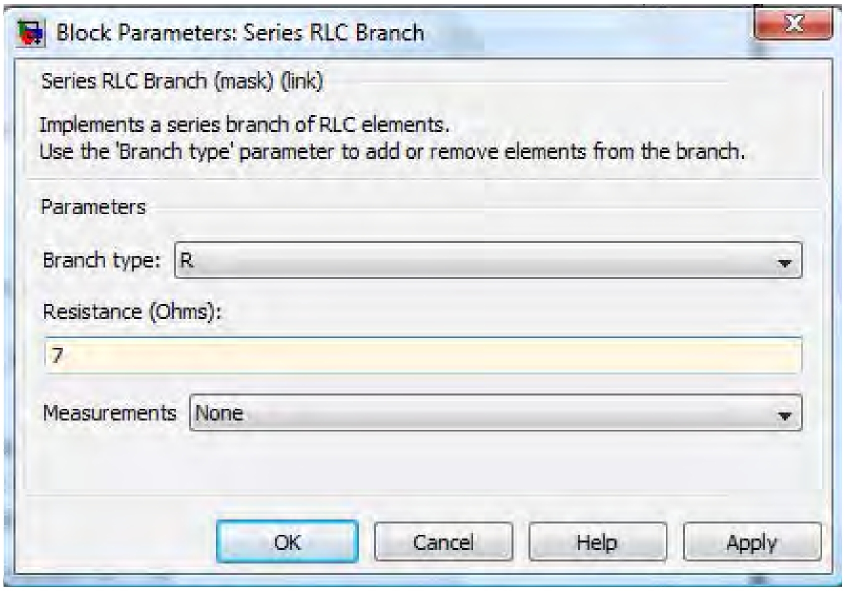


Figure C.6. The Block Parameters window for the Series RLC Branch

We need two resistors for our model and thus we copy and paste the resistor into the model, using the **Block Parameters** window we change its value to $5\ \Omega$, and from the **Format** drop window we click the **Rotate** block option and we rotate it **clockwise**. We also need two **Display** blocks, one for the current measurement and the second for the voltage measurement and thus we copy and paste the **Display** block into the model. We also copy and paste twice the **Ground** block and the model is now as shown in Figure C.7 where we also have renamed the blocks to shorter names.

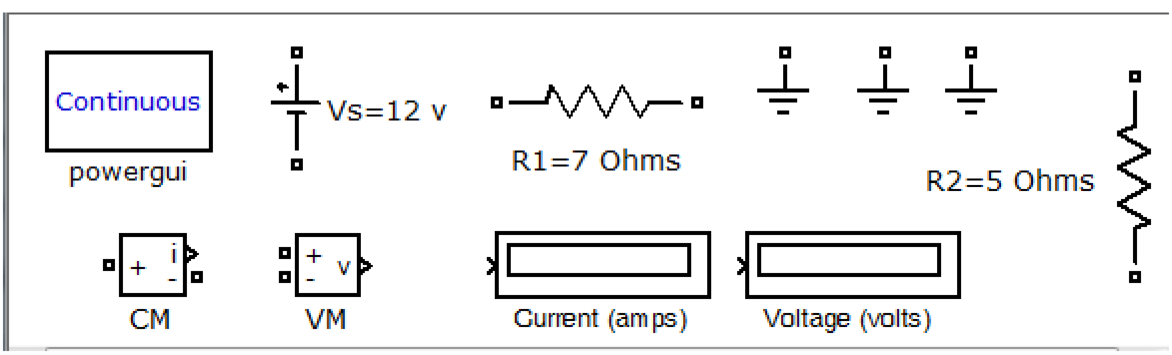


Figure C.7. Model with blocks renamed

From Figure C.7 above, we observe that both terminals of the voltage source and the resistors are shown with small square ( ) ports, the left ports of the CM (Current Measurement), and VM (Voltage Measurement) are also shown with ports, but the terminals on the right are shown with the **Simulink** output ports as >. The rules for the **SimPowerSystems** electrical terminal ports and connection lines are as follows:

1. We can connect **Simulink** ports (>) only to other **Simulink** ports.

2. We can connect **SimPowerSystems** ports ( ) only to other **SimPowerSystems** ports.*

3. If it is necessary to connect Simulink ports (>) to SimPowerSystems ports ( ), we can use SimPowerSystems blocks that contain both Simulink and SimPowerSystems ports such as the Current Measurement (CM) block and the Voltage Measurement (VM) block shown in Figure C.7.

The model for the electric circuit in Figure C.1 is shown in Figure C.8.

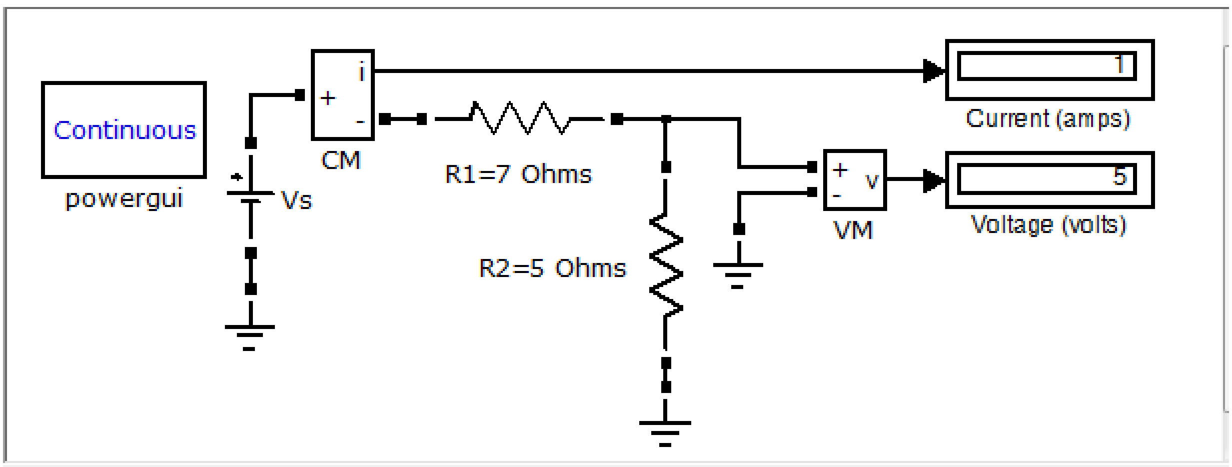


Figure C.8. The final form of the SimPowerSystems model for the electric circuit in Figure C.1

For the model in Figure C.8 we used the **DC Voltage Source** block. The **SimPowerSystems** documentation states that we can also use the **AC Voltage Source** block as a **DC Voltage Source** block provided that we set the frequency at 0 Hz and the phase at 90 degrees in the **Block Parameters** window as shown in Figure C.9.

* *As in Simulink, we can autoconnect two **SimPowerSystems** blocks by selecting the source block, then holding down the **Ctrl** key, and left-clicking the destination block.*

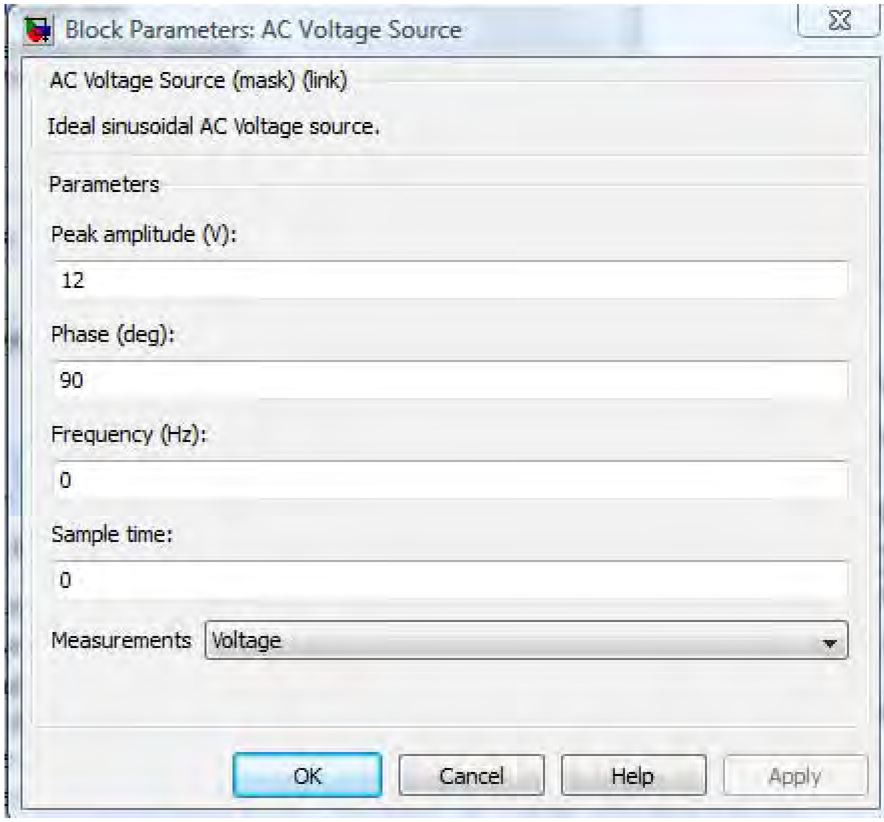


Figure C.9. Block parameter settings when using an **AC Voltage Source** block as a **DC Voltage Source**

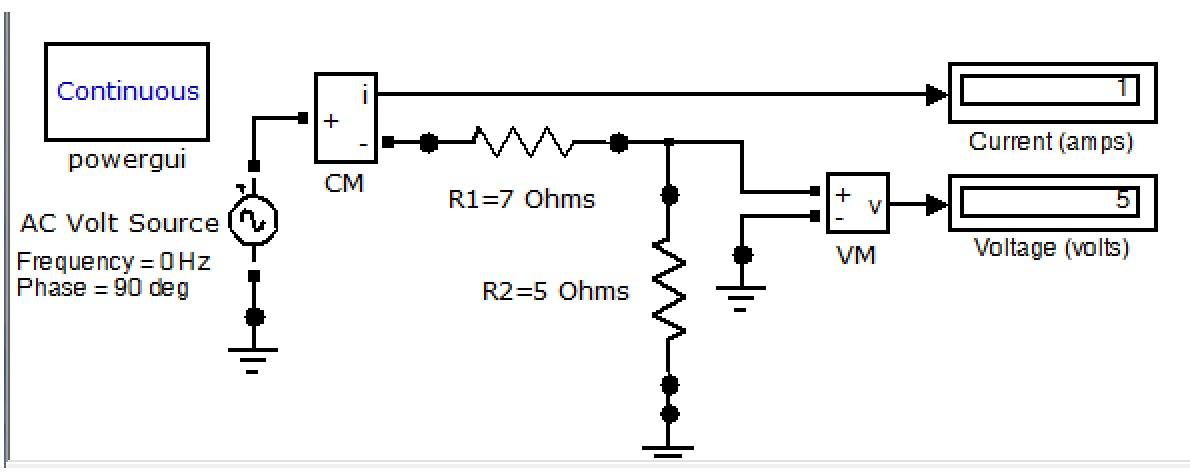


Figure C.10. Model with AC Voltage Source used as DC Voltage Source

A third option is to use a **Controlled Voltage Source** block with a **Constant** block set to the numerical value of the **DC voltage Source** as shown in the model of Figure C.11.

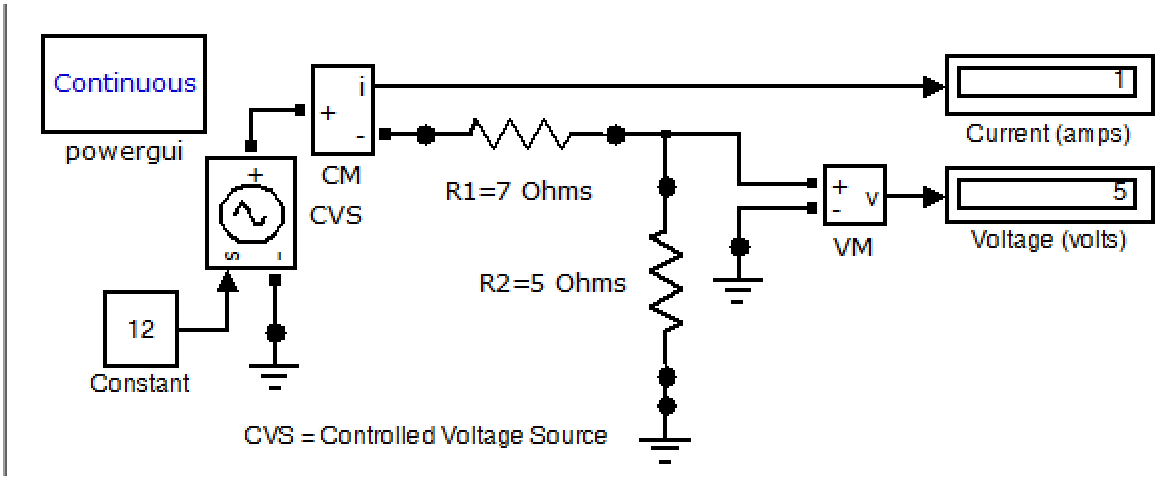


Figure C.11. Model with **Controlled Voltage Source** block

Example C.2

Consider the AC electric circuit in Figure C.12

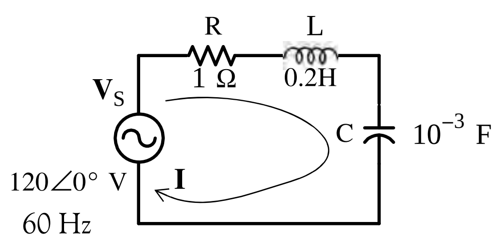


Figure C.12. Electric circuit for Example C.2

The current **I** and the voltage **Vc** across the capacitor are computed with MATLAB as follows:

```
Vs=120; f=60; R=1; L=0.2; C=10^(–3); XL=2*pi*f*L; XC=1/(2*pi*f*C);...
Z=sqrt(R^2+(XL–XC)^2); I=Vs/Z, Vc=XC*I
```

```
I =
    1.6494
Vc =
    4.3752
```

The **SimPowerSystems** model and the waveforms for the current **I** and the voltage **Vc** are shown in Figures C.13 and C.14 respectively.

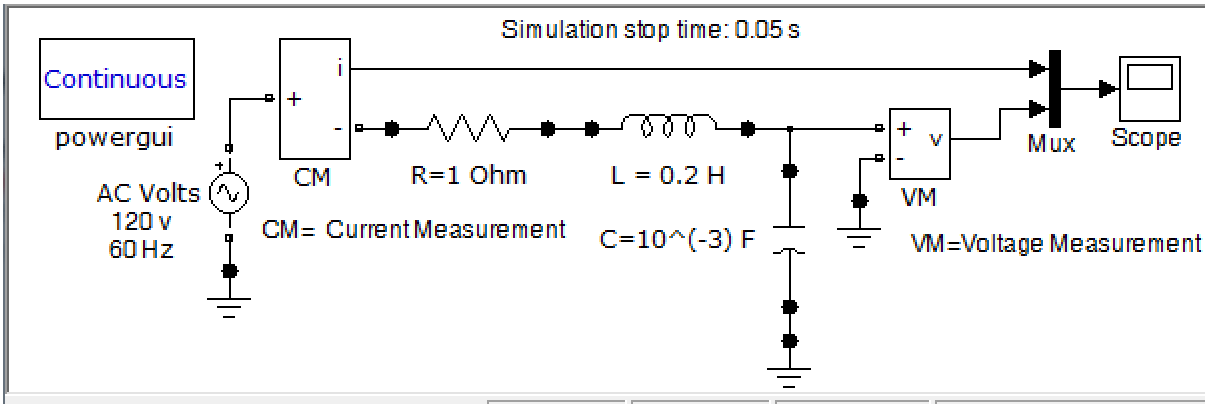


Figure C.13. **SimPowerSystems** model for the electric circuit in Figure C.12

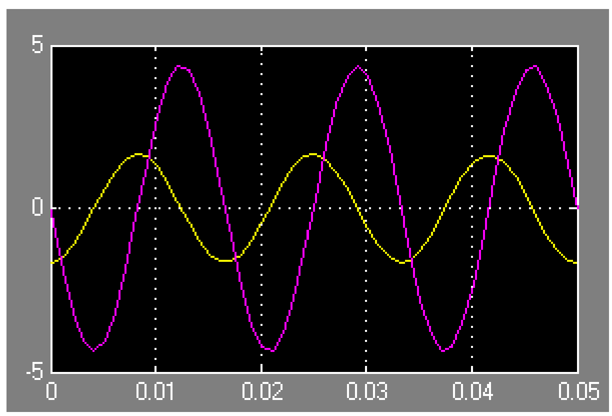


Figure C.14. Waveforms for the current I and voltage Vc across the capacitor in Figure C.12

The same results are obtained if we replace the applied AC voltage source block in the model of Figure C.13 with a **Controlled Voltage Source** (CVS) block as shown in Figure C.15.

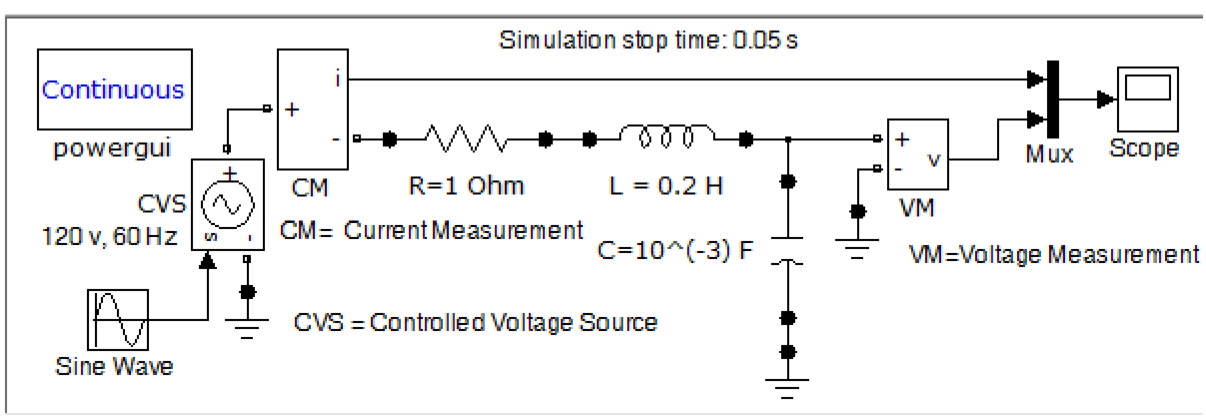


Figure C.15. The model in Figure C.13 with the AC Voltage Source block replaced with a CVS block

Appendix D

Review of Complex Numbers

This appendix is a review of the algebra of complex numbers. The basic operations are defined and illustrated by several examples. Applications using Euler's identities are presented, and the exponential and polar forms are discussed and illustrated with examples.

D.1 Definition of a Complex Number

In the language of mathematics, the square root of minus one is denoted as i, that is, $i = \sqrt{-1}$. In the electrical engineering field, we denote i as j to avoid confusion with current i. Essentially, j is an operator that produces a 90–degree counterclockwise rotation to any vector to which it is applied as a multiplying factor. Thus, if it is given that a vector A has the direction along the right side of the x–axis as shown in Figure D.1, multiplication of this vector by the operator j will result in a new vector jA whose magnitude remains the same, but it has been rotated counterclockwise by $90°$.

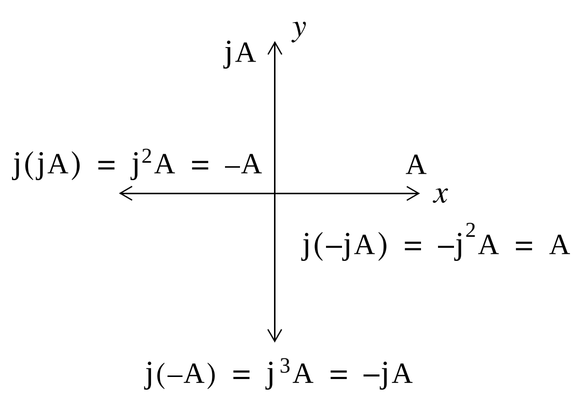


Figure D.1. The j operator

Also, another multiplication of the new vector jA by j will produce another $90°$ counterclockwise direction. In this case, the vector A has rotated $180°$ and its new value now is $-A$. When this vector is rotated by another $90°$ for a total of $270°$, its value becomes $j(-A) = -jA$. A fourth $90°$ rotation returns the vector to its original position, and thus its value is again A. Therefore, we conclude that $j^2 = -1$, $j^3 = -j$, and $j^4 = 1$.

Note: In our subsequent discussion, we will denote the x–axis (abscissa) as the *real axis*, and the y–axis (ordinate) as the *imaginary axis* with the understanding that the "imaginary" axis is just as "real" as the real axis. In other words, the imaginary axis is just as important as the real axis.*

An *imaginary number* is the product of a real number, say r, by the operator j. Thus, r is a real number and jr is an imaginary number.

A *complex number* is the sum (or difference) of a real number and an imaginary number. For example, the number $A = a + jb$ where a and b are both real numbers, is a complex number. Then, $a = Re\{A\}$ and $b = Im\{A\}$ where $Re\{A\}$ denotes real part of A, and $b = Im\{A\}$ the imaginary part of A.

By definition, two complex numbers A and B where $A = a + jb$ and $B = c + jd$, are equal if and only if their real parts are equal, and also their imaginary parts are equal. Thus, $A = B$ if and only if $a = c$ and $b = d$.

D.2 Addition and Subtraction of Complex Numbers

The sum of two complex numbers has a real component equal to the sum of the real components, and an imaginary component equal to the sum of the imaginary components. For subtraction, we change the signs of the components of the subtrahend and we perform addition. Thus, if

$$A = a + jb \text{ and } B = c + jd$$

then

$$A + B = (a + c) + j(b + d)$$

and

$$A - B = (a - c) + j(b - d)$$

Example D.1

It is given that $A = 3 + j4$, and $B = 4 - j2$. Find $A + B$ and $A - B$

Solution:

$$A + B = (3 + j4) + (4 - j2) = (3 + 4) + j(4 - 2) = 7 + j2$$

and

$$A - B = (3 + j4) - (4 - j2) = (3 - 4) + j(4 + 2) = -1 + j6$$

* *We may think the real axis as the cosine axis and the imaginary axis as the sine axis.*

D.3 Multiplication of Complex Numbers

Complex numbers are multiplied using the rules of elementary algebra, and making use of the fact that $j^2 = -1$. Thus, if

$$A = a + jb \text{ and } B = c + jd$$

then

$$A \cdot B = (a + jb) \cdot (c + jd) = ac + jad + jbc + j^2 bd$$

and since $j^2 = -1$, it follows that

$$\begin{aligned} A \cdot B &= ac + jad + jbc - bd \\ &= (ac - bd) + j(ad + bc) \end{aligned} \tag{D.1}$$

Example D.2

It is given that $A = 3 + j4$ and $B = 4 - j2$. Find $A \cdot B$

Solution:

$$A \cdot B = (3 + j4) \cdot (4 - j2) = 12 - j6 + j16 - j^2 8 = 20 + j10$$

The *conjugate* of a complex number, denoted as A^*, is another complex number with the same real component, and with an imaginary component of opposite sign. Thus, if $A = a + jb$, then $A^* = a - jb$.

Example D.3

It is given that $A = 3 + j5$. Find A^*

Solution:

The conjugate of the complex number A has the same real component, but the imaginary component has opposite sign. Then, $A^* = 3 - j5$

If a complex number A is multiplied by its conjugate, the result is a real number. Thus, if $A = a + jb$, then

$$A \cdot A^* = (a + jb)(a - jb) = a^2 - jab + jab - j^2 b^2 = a^2 + b^2$$

Example D.4

It is given that $A = 3 + j5$. Find $A \cdot A^*$

Solution:

$$A \cdot A^* = (3 + j5)(3 - j5) = 3^2 + 5^2 = 9 + 25 = 34$$

D.4 Division of Complex Numbers

When performing division of complex numbers, it is desirable to obtain the quotient separated into a real part and an imaginary part. This procedure is called *rationalization of the quotient*, and it is done by multiplying the denominator by its conjugate. Thus, if $A = a + jb$ and $B = c + jd$, then,

$$\frac{A}{B} = \frac{a + jb}{c + jd} = \frac{(a + jb)(c - jd)}{(c + jd)(c - jd)} = \frac{A}{B} \cdot \frac{B^*}{B^*} = \frac{(ac + bd) + j(bc - ad)}{c^2 + d^2}$$
$$= \frac{(ac + bd)}{c^2 + d^2} + j\frac{(bc - ad)}{c^2 + d^2} \qquad \text{(D.2)}$$

In (D.2), we multiplied both the numerator and denominator by the conjugate of the denominator to eliminate the *j* operator from the denominator of the quotient. Using this procedure, we see that the quotient is easily separated into a real and an imaginary part.

Example D.5

It is given that $A = 3 + j4$, and $B = 4 + j3$. Find A/B

Solution:

Using the procedure of (D.2), we obtain

$$\frac{A}{B} = \frac{3 + j4}{4 + j3} = \frac{(3 + j4)(4 - j3)}{(4 + j3)(4 - j3)} = \frac{12 - j9 + j16 + 12}{4^2 + 3^2} = \frac{24 + j7}{25} = \frac{24}{25} + j\frac{7}{25} = 0.96 + j0.28$$

D.5 Exponential and Polar Forms of Complex Numbers

The relations

$$\boxed{e^{j\theta} = \cos\theta + j\sin\theta} \qquad \text{(D.3)}$$

and

$$\boxed{e^{-j\theta} = \cos\theta - j\sin\theta} \tag{D.4}$$

are known as the *Euler's identities*.

Multiplying (D.3) by the *real* positive constant C we obtain:

$$Ce^{j\theta} = C\cos\theta + jC\sin\theta \tag{D.5}$$

This expression represents a complex number, say $a + jb$, and thus

$$Ce^{j\theta} = a + jb \tag{D.6}$$

where the left side of (D.6) is the *exponential form*, and the right side is the *rectangular form*.

Equating real and imaginary parts in (D.5) and (D.6), we obtain

$$a = C\cos\theta \quad \text{and} \quad b = C\sin\theta \tag{D.7}$$

Squaring and adding the expressions in (D.7), we obtain

$$a^2 + b^2 = (C\cos\theta)^2 + (C\sin\theta)^2 = C^2(\cos^2\theta + \sin^2\theta) = C^2$$

Then,

$$C^2 = a^2 + b^2$$

or

$$\boxed{C = \sqrt{a^2 + b^2}} \tag{D.8}$$

Also, from (D.7)

$$\frac{b}{a} = \frac{C\sin\theta}{C\cos\theta} = \tan\theta$$

or

$$\boxed{\theta = \tan^{-1}\left(\frac{b}{a}\right)} \tag{D.9}$$

To convert a complex number from rectangular to exponential form, we use the expression

$$\boxed{a + jb = \sqrt{a^2 + b^2}\, e^{j\left(\tan^{-1}\frac{b}{a}\right)}} \tag{D.10}$$

To convert a complex number from exponential to rectangular form, we use the expressions

$$\boxed{\begin{aligned} Ce^{j\theta} &= C\cos\theta + jC\sin\theta \\ Ce^{-j\theta} &= C\cos\theta - jC\sin\theta \end{aligned}} \tag{D.11}$$

The *polar form* is essentially the same as the exponential form but the notation is different, that is,

$$\boxed{Ce^{j\theta} = C\angle\theta} \tag{D.12}$$

where the left side of (D.12) is the exponential form, and the right side is the polar form.

We must remember that *the phase angle* θ *is always measured with respect to the positive real axis, and rotates in the counterclockwise direction.*

Example D.6

Convert the following complex numbers from rectangular* to exponential and polar forms:

a. $3 + j4$

b. $-1 + j2$

c. $-2 - j$

d. $4 - j3$

Solution:

a. The real and imaginary components of this complex number are shown in Figure D.2.

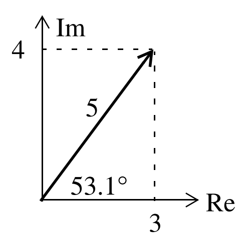


Figure D.2. The components of $3 + j4$

Then,

$$3 + j4 = \sqrt{3^2 + 4^2}\, e^{j\left(\tan^{-1}\frac{4}{3}\right)} = 5e^{j53.1°} = 5\angle 53.1°$$

Check with MATLAB:

```
x=3+j*4; magx=abs(x); thetax=angle(x)*180/pi;  disp(magx); disp(thetax)
     5
```

* *The rectangular form is also known as Cartesian form.*

```
53.1301
```

Check with the Simulink **Complex to Magnitude−Angle** block* shown in the Simulink model of Figure D.3.

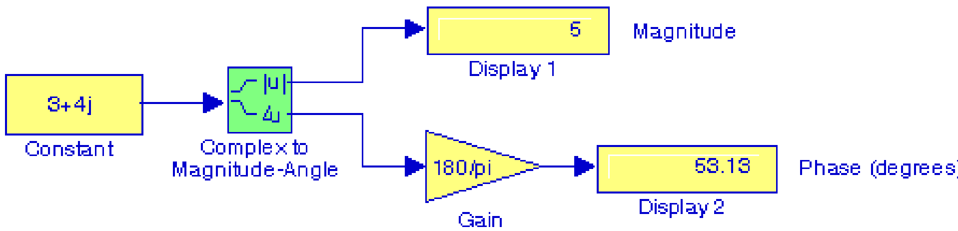


Figure D.3. Simulink model for Example D.6a

b. The real and imaginary components of this complex number are shown in Figure D.4.

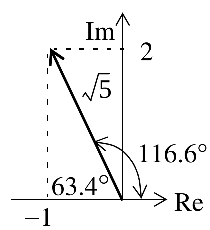


Figure D.4. The components of $-1 + j2$

Then,

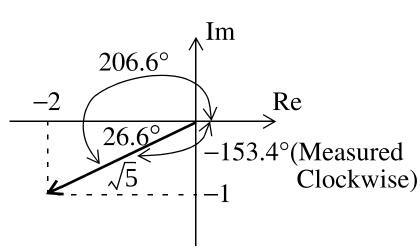
$$-1 + j2 = \sqrt{1^2 + 2^2}\, e^{j\left(\tan^{-1}\frac{2}{-1}\right)} = \sqrt{5}e^{j116.6°} = \sqrt{5}\angle 116.6°$$

Check with MATLAB:

```
y=−1+j*2; magy=abs(y); thetay=angle(y)*180/pi;  disp(magy); disp(thetay)
```

```
  2.2361
116.5651
```

c. The real and imaginary components of this complex number are shown in Figure D.5.



* *For a detailed description and examples with this and other related transformation blocks, please refer to Introduction to Simulink with Engineering Applications, ISBN 978−1−934404−09−6.*

Figure D.5. The components of $-2-j$

Then,

$$-2-j1 = \sqrt{2^2+1^2}\,e^{j\left(\tan^{-1}\frac{-1}{-2}\right)} = \sqrt{5}e^{j206.6°} = \sqrt{5}\angle 206.6° = \sqrt{5}e^{j(-153.4)°} = \sqrt{5}\angle -153.4°$$

Check with MATLAB:

```
v=–2–j*1; magv=abs(v); thetav=angle(v)*180/pi;  disp(magv); disp(thetav)
```

```
    2.2361
 -153.4349
```

d. The real and imaginary components of this complex number are shown in Figure D.6.

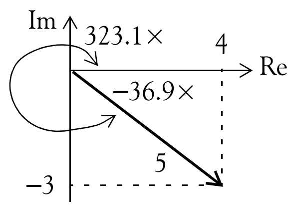


Figure D.6. The components of $4-j3$

Then,

$$4-j3 = \sqrt{4^2+3^2}\,e^{j\left(\tan^{-1}\frac{-3}{4}\right)} = 5e^{j323.1°} = 5\angle 323.1° = 5e^{-j36.9°} = 5\angle -36.9°$$

Check with MATLAB:

```
w=4–j*3; magw=abs(w); thetaw=angle(w)*180/pi;  disp(magw); disp(thetaw)
```

```
     5
 -36.8699
```

Example D.7

Express the complex number $-2\angle 30°$ in exponential and in rectangular forms.

Solution:

We recall that $-1 = j^2$. Since each j rotates a vector by $90°$ counterclockwise, then $-2\angle 30°$ is the same as $2\angle 30°$ rotated counterclockwise by $180°$. Therefore,

$$-2\angle 30° = 2\angle(30°+180°) = 2\angle 210° = 2\angle -150°$$

The components of this complex number are shown in Figure D.7.

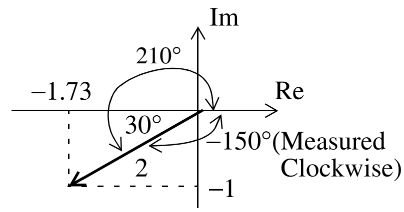


Figure D.7. The components of $2\angle{-150°}$

Then,

$$2\angle{-150°} = 2e^{-j150°} = 2(\cos 150° - j\sin 150°) = 2(-0.866 - j0.5) = -1.73 - j$$

Note: The rectangular form is most useful when we add or subtract complex numbers; however, the exponential and polar forms are most convenient when we multiply or divide complex numbers.

To multiply two complex numbers in exponential (or polar) form, we multiply the magnitudes and we add the phase angles, that is, if

$$A = M\angle\theta \quad \text{and} \quad B = N\angle\phi$$

then,

$$AB = MN\angle(\theta + \phi) = Me^{j\theta}Ne^{j\phi} = MNe^{j(\theta + \phi)} \qquad \text{(D.13)}$$

Example D.8

Multiply $A = 10\angle 53.1°$ by $B = 5\angle{-36.9°}$

Solution:

Multiplication in polar form yields

$$AB = (10 \times 5)\angle[53.1° + (-36.9°)] = 50\angle 16.2°$$

and multiplication in exponential form yields

$$AB = (10e^{j53.1°})(5e^{-j36.9°}) = 50e^{j(53.1° - 36.9°)} = 50e^{j16.2°}$$

To divide one complex number by another when both are expressed in exponential or polar form, we divide the magnitude of the dividend by the magnitude of the divisor, and we subtract the phase angle of the divisor from the phase angle of the dividend, that is, if

$$A = M\angle\theta \quad \text{and} \quad B = N\angle\phi$$

then,

$$\frac{A}{B} = \frac{M}{N}\angle(\theta-\phi) = \frac{Me^{j\theta}}{Ne^{j\phi}} = \frac{M}{N}e^{j(\theta-\phi)} \tag{D.14}$$

Example D.9

Divide $A = 10\angle 53.1°$ by $B = 5\angle -36.9°$

Solution:

Division in polar form yields

$$\frac{A}{B} = \frac{10\angle 53.1°}{5\angle -36.9°} = 2\angle[53.1° - (-36.9°)] = 2\angle 90°$$

Division in exponential form yields

$$\frac{A}{B} = \frac{10e^{j53.1°}}{5e^{-j36.9°}} = 2e^{j53.1°}e^{j36.9°} = 2e^{j90°}$$

Appendix E

Matrices and Determinants

This appendix is an introduction to matrices and matrix operations. Determinants, Cramer's rule, and Gauss's elimination method are reviewed. Some definitions and examples are not applicable to the material presented in this text, but are included for subject continuity, and academic interest. They are discussed in detail in matrix theory textbooks. These are denoted with a dagger (†) and may be skipped.

E.1 Matrix Definition

A *matrix* is a rectangular array of numbers such as those shown below.

$$\begin{bmatrix} 2 & 3 & 7 \\ 1 & -1 & 5 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 3 & 1 \\ -2 & 1 & -5 \\ 4 & -7 & 6 \end{bmatrix}$$

In general form, a matrix A is denoted as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix} \tag{E.1}$$

The numbers a_{ij} are the *elements* of the matrix where the index i indicates the row, and j indicates the column in which each element is positioned. For instance, a_{43} indicates the element positioned in the fourth row and third column.

A matrix of m rows and n columns is said to be of $m \times n$ *order matrix*.

If $m = n$, the matrix is said to be a *square matrix of order* m (or n). Thus, if a matrix has five rows and five columns, it is said to be a square matrix of order 5.

In a square matrix, the elements $a_{11}, a_{22}, a_{33}, \ldots, a_{nn}$ are called the *main diagonal elements*. Alternately, we say that the matrix elements $a_{11}, a_{22}, a_{33}, \ldots, a_{nn}$, are located on the *main diagonal*.

† The sum of the diagonal elements of a square matrix A is called the *trace** *of* A.

† A matrix in which every element is zero, is called a *zero matrix*.

E.2 Matrix Operations

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are equal, that is, $A = B$, if and only if

$$a_{ij} = b_{ij} \qquad i = 1, 2, 3, \ldots, m \qquad j = 1, 2, 3, \ldots, n \tag{E.2}$$

Two matrices are said to be *conformable for addition* (*subtraction*), if they are of the same order $m \times n$.

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are conformable for addition (subtraction), their sum (difference) will be another matrix C with the same order as A and B, where each element of C is the sum (difference) of the corresponding elements of A and B, that is,

$$C = A \pm B = [a_{ij} \pm b_{ij}] \tag{E.3}$$

Example E.1

Compute $A + B$ and $A - B$ given that

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 3 & 0 \\ -1 & 2 & 5 \end{bmatrix}$$

Solution:

$$A + B = \begin{bmatrix} 1+2 & 2+3 & 3+0 \\ 0-1 & 1+2 & 4+5 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 3 \\ -1 & 3 & 9 \end{bmatrix}$$

and

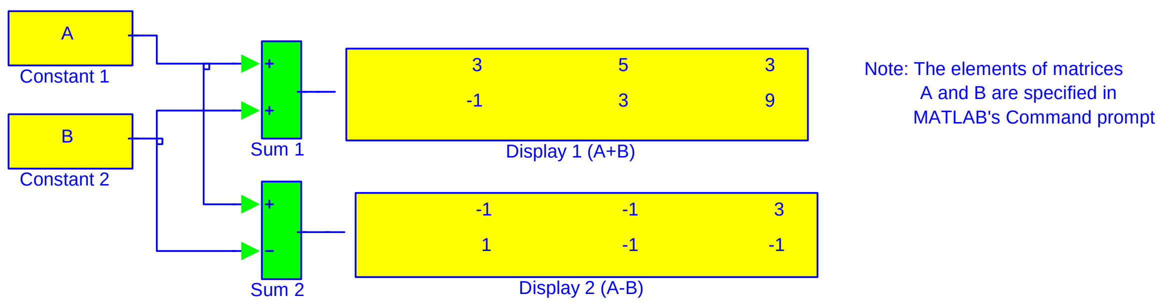
$$A - B = \begin{bmatrix} 1-2 & 2-3 & 3-0 \\ 0+1 & 1-2 & 4-5 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 3 \\ 1 & -1 & -1 \end{bmatrix}$$

* *Henceforth, all paragraphs and topics preceded by a dagger (†) may be skipped. These are discussed in matrix theory textbooks.*

Check with MATLAB:

```
A=[1  2  3;   0  1  4];  B=[2  3  0; −1  2  5];     % Define matrices A and B
A+B, A−B                                             % Add A and B, then Subtract B from A

ans =
     3     5     3
    -1     3     9

ans =
    -1    -1     3
     1    -1    -1
```

Check with Simulink:



If k is any scalar (a positive or negative number), and not $[k]$ which is a 1×1 matrix, then multiplication of a matrix A by the scalar k is the multiplication of every element of A by k.

Example E.2

Multiply the matrix

$$A = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}$$

by

a. $k_1 = 5$

b. $k_2 = -3 + j2$

Solution:

a.

$$k_1 \cdot A = 5 \times \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 5 \times 1 & 5 \times (-2) \\ 5 \times 2 & 5 \times 3 \end{bmatrix} = \begin{bmatrix} 5 & -10 \\ 10 & 15 \end{bmatrix}$$

b.

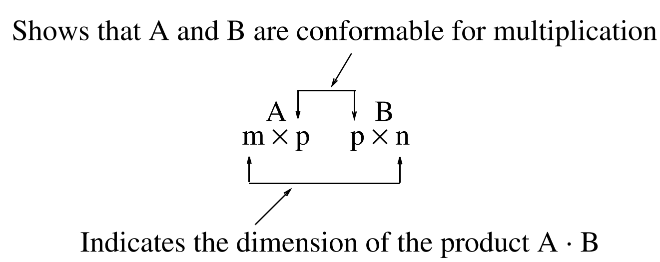
$$k_2 \cdot A = (-3 + j2) \times \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} (-3+j2) \times 1 & (-3+j2) \times (-2) \\ (-3+j2) \times 2 & (-3+j2) \times 3 \end{bmatrix} = \begin{bmatrix} -3+j2 & 6-j4 \\ -6+j4 & -9+j6 \end{bmatrix}$$

Check with MATLAB:

```
k1=5; k2=(–3 + 2*j);          % Define scalars k1 and k2
A=[1 –2; 2  3];               % Define matrix A
k1*A, k2*A                    % Multiply matrix A by scalars k1 and k2
```

```
ans =
     5   -10
    10    15

ans =
   -3.0000+ 2.0000i   6.0000- 4.0000i
   -6.0000+ 4.0000i  -9.0000+ 6.0000i
```

Two matrices A and B are said to be *conformable for multiplication* $A \cdot B$ in that order, only when the number of columns of matrix A is equal to the number of rows of matrix B. That is, the product $A \cdot B$ (but not $B \cdot A$) is conformable for multiplication only if A is an $m \times p$ matrix and matrix B is an $p \times n$ matrix. The product $A \cdot B$ will then be an $m \times n$ matrix. A convenient way to determine if two matrices are conformable for multiplication is to write the dimensions of the two matrices side–by–side as shown below.



For the product $B \cdot A$ we have:

Here, B and A are not conformable for multiplication

$$\underset{p \times n}{B} \quad \underset{m \times p}{A}$$

For matrix multiplication, the operation is row by column. Thus, to obtain the product $A \cdot B$, we multiply each element of a row of A by the corresponding element of a column of B; then, we add these products.

Example E.3

Matrices C and D are defined as

$$C = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

Compute the products $C \cdot D$ and $D \cdot C$

Solution:

The dimensions of matrices C and D are respectively 1×3 3×1; therefore the product $C \cdot D$ is feasible, and will result in a 1×1, that is,

$$C \cdot D = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} (2) \cdot (1) + (3) \cdot (-1) + (4) \cdot (2) \end{bmatrix} = \begin{bmatrix} 7 \end{bmatrix}$$

The dimensions for D and C are respectively 3×1 1×3 and therefore, the product $D \cdot C$ is also feasible. Multiplication of these will produce a 3×3 matrix as follows:

$$D \cdot C = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} (1) \cdot (2) & (1) \cdot (3) & (1) \cdot (4) \\ (-1) \cdot (2) & (-1) \cdot (3) & (-1) \cdot (4) \\ (2) \cdot (2) & (2) \cdot (3) & (2) \cdot (4) \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 \\ -2 & -3 & -4 \\ 4 & 6 & 8 \end{bmatrix}$$

Check with MATLAB:

```
C=[2  3  4];  D=[1  –1  2]';        % Define matrices C and D. Observe that D is a column vector
C*D, D*C                            % Multiply C by D, then multiply D by C

   ans  =
         7
```

```
ans =
     2     3     4
    -2    -3    -4
     4     6     8
```

Division of one matrix by another, is not defined. However, an analogous operation exists, and it will become apparent later in this chapter when we discuss the inverse of a matrix.

E.3 Special Forms of Matrices

† A square matrix is said to be *upper triangular* when all the elements below the diagonal are zero. The matrix A of (E.4) is an upper triangular matrix. In an upper triangular matrix, not all elements above the diagonal need to be non–zero.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & \dots & \dots & \dots \\ \dots & \dots & 0 & \dots & \dots \\ 0 & 0 & 0 & \dots & a_{mn} \end{bmatrix} \tag{E.4}$$

† A square matrix is said to be *lower triangular*, when all the elements above the diagonal are zero. The matrix B of (E.5) is a lower triangular matrix. In a lower triangular matrix, not all elements below the diagonal need to be non–zero.

$$B = \begin{bmatrix} a_{11} & 0 & 0 & \dots & 0 \\ a_{21} & a_{22} & 0 & \dots & 0 \\ \dots & \dots & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & 0 \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix} \tag{E.5}$$

† A square matrix is said to be *diagonal*, if all elements are zero, except those in the diagonal. The matrix C of (E.6) is a diagonal matrix.

$$C = \begin{bmatrix} a_{11} & 0 & 0 & \dots & 0 \\ 0 & a_{22} & 0 & \dots & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \dots & a_{nn} \end{bmatrix} \qquad (E.6)$$

† A diagonal matrix is called a *scalar matrix*, if $a_{11} = a_{22} = a_{33} = \dots = a_{nn} = k$ where k is a scalar. The matrix D of (E.7) is a scalar matrix with $k = 4$.

$$D = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \qquad (E.7)$$

A scalar matrix with $k = 1$, is called an *identity matrix* I. Shown below are 2×2, 3×3, and 4×4 identity matrices.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (E.8)$$

The MATLAB **eye(n)** function displays an $n \times n$ identity matrix. For example,

```
eye(4)     % Display a 4 by 4 identity matrix
```

```
ans =
     1     0     0     0
     0     1     0     0
     0     0     1     0
     0     0     0     1
```

Likewise, the **eye(size(A))** function, produces an identity matrix whose size is the same as matrix A. For example, let matrix A be defined as

```
A=[1  3  1; –2  1 –5; 4 –7  6]      % Define matrix A
```

```
A =
     1     3     1
    -2     1    -5
     4    -7     6
```

Then,

```
eye(size(A))
```

displays

```
ans =
     1     0     0
     0     1     0
     0     0     1
```

† The *transpose of a matrix* A, denoted as A^T, is the matrix that is obtained when the rows and columns of matrix A are interchangeE. For example, if

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \text{then} \quad A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \tag{E.9}$$

In MATLAB, we use the apostrophe (′) symbol to denote and obtain the transpose of a matrix. Thus, for the above example,

```
A=[1 2 3; 4 5 6]      % Define matrix A

A =
     1     2     3
     4     5     6

A'                    % Display the transpose of A

ans =
     1     4
     2     5
     3     6
```

† A *symmetric matrix* A is a matrix such that $A^T = A$, that is, the transpose of a matrix A is the same as A. An example of a symmetric matrix is shown below.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & -5 \\ 3 & -5 & 6 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & -5 \\ 3 & -5 & 6 \end{bmatrix} = A \tag{E.10}$$

† If a matrix A has complex numbers as elements, the matrix obtained from A by replacing each element by its conjugate, is called the *conjugate of* A, and it is denoted as A^*, for example,

$$A = \begin{bmatrix} 1+j2 & j \\ 3 & 2-j3 \end{bmatrix} \qquad A^* = \begin{bmatrix} 1-j2 & -j \\ 3 & 2+j3 \end{bmatrix}$$

MATLAB has two built–in functions which compute the complex conjugate of a number. The first, **conj(x)**, computes the complex conjugate of any complex number, and the second, **conj(A)**, computes the conjugate of a matrix A. Using MATLAB with the matrix A defined as above, we obtain

```
A = [1+2j   j;  3   2–3j]  % Define and display matrix A

   A =
     1.0000 + 2.0000i          0 + 1.0000i
     3.0000               2.0000 - 3.0000i

conj_A=conj(A)           % Compute and display the conjugate of A

   conj_A =
     1.0000 - 2.0000i          0 - 1.0000i
     3.0000               2.0000 + 3.0000i
```

† A square matrix A such that $A^T = -A$ is called *skew-symmetric*. For example,

$$A = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & -4 \\ 3 & 4 & 0 \end{bmatrix} \quad A^T = \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix} = -A$$

Therefore, matrix A above is skew symmetric.

† A square matrix A such that $A^{T*} = A$ is called *Hermitian*. For example,

$$A = \begin{bmatrix} 1 & 1-j & 2 \\ 1+j & 3 & j \\ 2 & -j & 0 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & 1+j & 2 \\ 1-j & 3 & -j \\ 2 & j & 0 \end{bmatrix} \quad A^{T*} = \begin{bmatrix} 1 & 1+j & 2 \\ 1-j & 3 & -j \\ 2 & j & 0 \end{bmatrix} = A$$

Therefore, matrix A above is Hermitian.

† A square matrix A such that $A^{T*} = -A$ is called *skew–Hermitian*. For example,

$$A = \begin{bmatrix} j & 1-j & 2 \\ -1-j & 3j & j \\ -2 & j & 0 \end{bmatrix} \quad A^T = \begin{bmatrix} j & -1-j & -2 \\ 1-j & 3j & j \\ 2 & j & 0 \end{bmatrix} \quad A^{T*} = \begin{bmatrix} -j & -1+j & -2 \\ 1+j & -3j & -j \\ 2 & -j & 0 \end{bmatrix} = -A$$

Therefore, matrix A above is skew–Hermitian.

E.4 Determinants

Let matrix A be defined as the square matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix} \tag{E.11}$$

then, the *determinant of* A, denoted as $\det A$, is defined as

$$\begin{aligned} \det A = {} & a_{11}a_{22}a_{33}\dots a_{nn} + a_{12}a_{23}a_{34}\dots a_{n1} + a_{13}a_{24}a_{35}\dots a_{n2} + \dots \\ & - a_{n1}\dots a_{22}a_{13}\dots - a_{n2}\dots a_{23}a_{14} - a_{n3}\dots a_{24}a_{15} - \dots \end{aligned} \tag{E.12}$$

The determinant of a square matrix of order n is referred to as *determinant of order n.*

Let A be a determinant of order 2, that is,

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{E.13}$$

Then,

$$\det A = a_{11}a_{22} - a_{21}a_{12} \tag{E.14}$$

Example E.4

Matrices A and B are defined as

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix}$$

Compute $\det A$ and $\det B$.

Solution:

$$\det A = 1 \cdot 4 - 3 \cdot 2 = 4 - 6 = -2$$
$$\det B = 2 \cdot 0 - 2 \cdot (-1) = 0 - (-2) = 2$$

Check with MATLAB:

```
A=[1  2; 3  4]; B=[2  −1; 2  0];          % Define matrices A and B
det(A), det(B)                            % Compute the determinants of A and B
```

```
ans =
    -2
ans =
     2
```

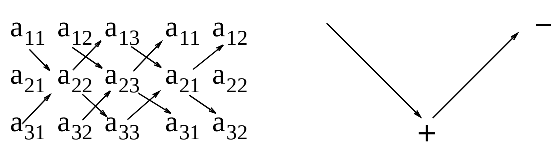
Let A be a matrix of order 3, that is,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{E.15}$$

then, $\det A$ is found from

$$\begin{aligned} \det A &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{11}a_{22}a_{33} \\ &\quad -a_{11}a_{22}a_{33} - a_{11}a_{22}a_{33} - a_{11}a_{22}a_{33} \end{aligned} \tag{E.16}$$

A convenient method to evaluate the determinant of order 3, is to write the first two columns to the right of the 3×3 matrix, and add the products formed by the diagonals from upper left to lower right; then subtract the products formed by the diagonals from lower left to upper right as shown on the diagram of the next page. When this is done properly, we obtain (E.16) above.



This method works only with second and third order determinants. To evaluate higher order determinants, we must first compute the *cofactors*; these will be defined shortly.

Example E.5

Compute $\det A$ and $\det B$ if matrices A and B are defined as

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & -3 & -4 \\ 1 & 0 & -2 \\ 0 & -5 & -6 \end{bmatrix}$$

Solution:

$$\det A = \begin{matrix} 2 & 3 & 5 & 2 & 3 \\ 1 & 0 & 1 & 1 & 0 \\ 2 & 1 & 0 & 2 & 1 \end{matrix}$$

or

$$\det A = (2 \times 0 \times 0) + (3 \times 1 \times 1) + (5 \times 1 \times 1)$$
$$- (2 \times 0 \times 5) - (1 \times 1 \times 2) - (0 \times 1 \times 3) = 11 - 2 = 9$$

Likewise,

$$\det B = \begin{matrix} 2 & -3 & -4 & 2 & -3 \\ 1 & 0 & -2 & 1 & -2 \\ 0 & -5 & -6 & 2 & -6 \end{matrix}$$

or

$$\det B = [2 \times 0 \times (-6)] + [(-3) \times (-2) \times 0] + [(-4) \times 1 \times (-5)]$$
$$- [0 \times 0 \times (-4)] - [(-5) \times (-2) \times 2] - [(-6) \times 1 \times (-3)] = 20 - 38 = -18$$

Check with MATLAB:

```
A=[2  3  5;  1  0  1;  2  1  0]; det(A)      % Define matrix A and compute detA
```

```
ans =
     9
```

```
B=[2  –3  –4;  1  0  –2;  0  –5  –6];det(B) % Define matrix B and compute detB
```

```
ans =
   -18
```

E.5 Minors and Cofactors

Let matrix A be defined as the square matrix of order n as shown below.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix} \qquad \text{(E.17)}$$

If we remove the elements of its ith row, and jth column, the remaining $n-1$ square matrix is called the *minor of* A, and it is denoted as $[M_{ij}]$.

The signed minor $(-1)^{i+j}[M_{ij}]$ is called the *cofactor* of a_{ij} and it is denoted as α_{ij}.

Example E.6

Matrix A is defined as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{E.18}$$

Compute the minors $[M_{11}]$, $[M_{12}]$, $[M_{13}]$ and the cofactors α_{11}, α_{12} and α_{13}.

Solution:

$$[M_{11}] = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} \qquad [M_{12}] = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} \qquad [M_{11}] = \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

and

$$\alpha_{11} = (-1)^{1+1}[M_{11}] = [M_{11}] \qquad \alpha_{12} = (-1)^{1+2}[M_{12}] = -[M_{12}] \qquad \alpha_{13} = [M_{13}] = (-1)^{1+3}[M_{13}]$$

The remaining minors

$$[M_{21}],\ [M_{22}],\ [M_{23}],\ [M_{31}],\ [M_{32}],\ [M_{33}]$$

and cofactors

$$\alpha_{21}, \alpha_{22}, \alpha_{23}, \alpha_{31}, \alpha_{32}, \text{and } \alpha_{33}$$

are defined similarly.

Example E.7

Compute the cofactors of matrix A defined as

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & -4 & 2 \\ -1 & 2 & -6 \end{bmatrix} \tag{E.19}$$

Solution:

$$\alpha_{11} = (-1)^{1+1}\begin{bmatrix} -4 & 2 \\ 2 & -6 \end{bmatrix} = 20 \qquad \alpha_{12} = (-1)^{1+2}\begin{bmatrix} 2 & 2 \\ -1 & -6 \end{bmatrix} = 10 \tag{E.20}$$

$$\alpha_{13} = (-1)^{1+3}\begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix} = 0 \qquad \alpha_{21} = (-1)^{2+1}\begin{bmatrix} 2 & -3 \\ 2 & -6 \end{bmatrix} = 6 \tag{E.21}$$

$$\alpha_{22} = (-1)^{2+2}\begin{bmatrix} 1 & -3 \\ -1 & -6 \end{bmatrix} = -9 \qquad \alpha_{23} = (-1)^{2+3}\begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} = -4 \tag{E.22}$$

$$\alpha_{31} = (-1)^{3+1}\begin{bmatrix} 2 & -3 \\ -4 & 2 \end{bmatrix} = -8, \qquad \alpha_{32} = (-1)^{3+2}\begin{bmatrix} 1 & -3 \\ 2 & 2 \end{bmatrix} = -8 \tag{E.23}$$

$$\alpha_{33} = (-1)^{3+3}\begin{bmatrix} 1 & 2 \\ 2 & -4 \end{bmatrix} = -8 \tag{E.24}$$

It is useful to remember that the signs of the cofactors follow the pattern below

$$\begin{matrix} + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \end{matrix}$$

that is, the cofactors on the diagonals have the same sign as their minors.

Let A be a square matrix of any size; the value of the determinant of A is the sum of the products obtained by multiplying each element of *any* row or *any* column by its cofactor.

Example E.8

Matrix A is defined as

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & -4 & 2 \\ -1 & 2 & -6 \end{bmatrix} \tag{E.25}$$

Compute the determinant of A using the elements of the first row.

Solution:

$$\det A = 1\begin{bmatrix} -4 & 2 \\ 2 & -6 \end{bmatrix} - 2\begin{bmatrix} 2 & 2 \\ -1 & -6 \end{bmatrix} - 3\begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix} = 1\times 20 - 2\times(-10) - 3\times 0 = 40$$

Check with MATLAB:

```
A=[1  2  -3; 2  -4  2; -1  2  -6]; det(A)        % Define matrix A and compute detA
ans =
    40
```

We must use the above procedure to find the determinant of a matrix A of order 4 or higher. Thus, a fourth-order determinant can first be expressed as the sum of the products of the elements of its first row by its cofactor as shown below.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = a_{11}\begin{bmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{bmatrix} - a_{21}\begin{bmatrix} a_{12} & a_{13} & a_{14} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{bmatrix}$$

$$+a_{31}\begin{bmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{42} & a_{43} & a_{44} \end{bmatrix} - a_{41}\begin{bmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \end{bmatrix} \tag{E.26}$$

Determinants of order five or higher can be evaluated similarly.

Example E.9

Compute the value of the determinant of the matrix A defined as

$$A = \begin{bmatrix} 2 & -1 & 0 & -3 \\ -1 & 1 & 0 & -1 \\ 4 & 0 & 3 & -2 \\ -3 & 0 & 0 & 1 \end{bmatrix} \tag{E.27}$$

Solution:

Using the above procedure, we will multiply each element of the first column by its cofactor. Then,

$$A = \underbrace{2\begin{bmatrix} 1 & 0 & -1 \\ 0 & 3 & -2 \\ 0 & 0 & 1 \end{bmatrix}}_{[a]} \underbrace{-(-1)\begin{bmatrix} -1 & 0 & -3 \\ 0 & 3 & -2 \\ 0 & 0 & 1 \end{bmatrix}}_{[b]} \underbrace{+4\begin{bmatrix} -1 & 0 & -3 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}}_{[c]} \underbrace{-(-3)\begin{bmatrix} -1 & 0 & -3 \\ 1 & 0 & -1 \\ 0 & 3 & -2 \end{bmatrix}}_{[d]}$$

Next, using the procedure of Example E.5 or Example E.8, we find

$$[a] = 6\ ,\ [b] = -3\ ,\ [c] = 0\ ,\ [d] = -36$$

and thus

$$\det A = [a] + [b] + [c] + [d] = 6 - 3 + 0 - 36 = -33$$

We can verify our answer with MATLAB as follows:

```
A=[ 2 -1 0 -3; -1 1 0 -1; 4 0 3 -2; -3 0 0 1]; delta = det(A)
```

```
delta =
   -33
```

Some useful properties of determinants are given below.

Property 1: *If all elements of one row or one column are zero, the determinant is zero.* An example of this is the determinant of the cofactor $[c]$ above.

Property 2: *If all the elements of one row or column are m times the corresponding elements of another row or column, the determinant is zero.* For example, if

$$A = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 6 & 1 \\ 1 & 2 & 1 \end{bmatrix} \qquad \text{(E.28)}$$

then,

$$\det A = \left| \begin{matrix} 2 & 4 & 1 \\ 3 & 6 & 1 \\ 1 & 2 & 1 \end{matrix} \right| \begin{matrix} 2 & 4 \\ 3 & 6 \\ 1 & 2 \end{matrix} = 12 + 4 + 6 - 6 - 4 - 12 = 0 \qquad \text{(E.29)}$$

Here, $\det A$ is zero because the second column in A is 2 times the first column.

Check with MATLAB:

```
A=[2 4 1; 3 6 1; 1 2 1]; det(A)
```

```
ans =
     0
```

Property 3: *If two rows or two columns of a matrix are identical, the determinant is zero.* This follows from Property 2 with $m = 1$.

E.6 Cramer's Rule

Let us consider the systems of the three equations below:

$$\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= A \\ a_{21}x + a_{22}y + a_{23}z &= B \\ a_{31}x + a_{32}y + a_{33}z &= C \end{aligned} \qquad (E.30)$$

and let

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad D_1 = \begin{vmatrix} A & a_{11} & a_{13} \\ B & a_{21} & a_{23} \\ C & a_{31} & a_{33} \end{vmatrix} \quad D_2 = \begin{vmatrix} a_{11} & A & a_{13} \\ a_{21} & B & a_{23} \\ a_{31} & C & a_{33} \end{vmatrix} \quad D_3 = \begin{vmatrix} a_{11} & a_{12} & A \\ a_{21} & a_{22} & B \\ a_{31} & a_{32} & C \end{vmatrix}$$

Cramer's rule states that the unknowns *x, y,* and *z* can be found from the relations

$$x = \frac{D_1}{\Delta} \qquad y = \frac{D_2}{\Delta} \qquad z = \frac{D_3}{\Delta} \qquad (E.31)$$

provided that the determinant Δ (delta) is not zero.

We observe that the numerators of (E.31) are determinants that are formed from Δ by the substitution of the known values A, B, and C, for the coefficients of the desired unknown.

Cramer's rule applies to systems of two or more equations.

If (E.30) is a homogeneous set of equations, that is, if $A = B = C = 0$, then, D_1, D_2, and D_3 are all zero as we found in Property 1 above. Then, $x = y = z = 0$ also.

Example E.10

Use Cramer's rule to find v_1, v_2, and v_3 if

$$\begin{aligned} 2v_1 - 5 - v_2 + 3v_3 &= 0 \\ -2v_3 - 3v_2 - 4v_1 &= 8 \\ v_2 + 3v_1 - 4 - v_3 &= 0 \end{aligned} \qquad (E.32)$$

and verify your answers with MATLAB.

Solution:

Rearranging the unknowns v, and transferring known values to the right side, we obtain

$$\begin{aligned} 2v_1 - v_2 + 3v_3 &= 5 \\ -4v_1 - 3v_2 - 2v_3 &= 8 \\ 3v_1 + v_2 - v_3 &= 4 \end{aligned} \qquad (E.33)$$

By Cramer's rule,

$$\Delta = \begin{vmatrix} 2 & -1 & 3 \\ -4 & -3 & -2 \\ 3 & 1 & -1 \end{vmatrix}\begin{matrix} 2 & -1 \\ -4 & -3 \\ 3 & 1 \end{matrix} = 6+6-12+27+4+4 = 35$$

$$D_1 = \begin{vmatrix} 5 & -1 & 3 \\ 8 & -3 & -2 \\ 4 & 1 & -1 \end{vmatrix}\begin{matrix} 5 & -1 \\ 8 & -3 \\ 4 & 1 \end{matrix} = 15+8+24+36+10-8 = 85$$

$$D_2 = \begin{vmatrix} 2 & 5 & 3 \\ -4 & 8 & -2 \\ 3 & 4 & -1 \end{vmatrix}\begin{matrix} 2 & 5 \\ -4 & 8 \\ 3 & 4 \end{matrix} = -16-30-48-72+16-20 = -170$$

$$D_3 = \begin{vmatrix} 2 & -1 & 5 \\ -4 & -3 & 8 \\ 3 & 1 & 4 \end{vmatrix}\begin{matrix} 2 & -1 \\ -4 & -3 \\ 3 & 1 \end{matrix} = -24-24-20+45-16-16 = -55$$

Using relation (E.31) we obtain

$$x_1 = \frac{D_1}{\Delta} = \frac{85}{35} = \frac{17}{7} \qquad x_2 = \frac{D_2}{\Delta} = -\frac{170}{35} = -\frac{34}{7} \qquad x_3 = \frac{D_3}{\Delta} = -\frac{55}{35} = -\frac{11}{7} \tag{E.34}$$

We will verify with MATLAB as follows:

```
% The following script will compute and display the values of v1, v2 and v3.
format rat                            % Express answers in ratio form
B=[2 –1 3; –4 –3 –2; 3 1 –1];         % The elements of the determinant D of matrix B
delta=det(B);                         % Compute the determinant D of matrix B
d1=[5 –1 3; 8 –3 –2; 4 1 –1];         % The elements of D1
detd1=det(d1);                        % Compute the determinant of D1
d2=[2 5 3; –4 8 –2; 3 4 –1];          % The elements of D2
detd2=det(d2);                        % Compute the determinant of D2
d3=[2 –1 5; –4 –3 8; 3 1 4];          % The elements of D3
detd3=det(d3);                        % Compute he determinant of D3
v1=detd1/delta;                       % Compute the value of v1
v2=detd2/delta;                       % Compute the value of v2
v3=detd3/delta;                       % Compute the value of v3
                                      %
disp('v1=');disp(v1);                 % Display the value of v1
disp('v2=');disp(v2);                 % Display the value of v2
disp('v3=');disp(v3);                 % Display the value of v3
```

```
v1=
    17/7
v2=
   -34/7
v3=
   -11/7
```

These are the same values as in (E.34)

E.7 Gaussian Elimination Method

We can find the unknowns in a system of two or more equations also by the *Gaussian elimination method*. With this method, the objective is to eliminate one unknown at a time. This can be done by multiplying the terms of any of the equations of the system by a number such that we can add (or subtract) this equation to another equation in the system so that one of the unknowns will be eliminated. Then, by substitution to another equation with two unknowns, we can find the second unknown. Subsequently, substitution of the two values found can be made into an equation with three unknowns from which we can find the value of the third unknown. This procedure is repeated until all unknowns are found. This method is best illustrated with the following example which consists of the same equations as the previous example.

Example E.11

Use the Gaussian elimination method to find v_1, v_2, and v_3 of the system of equations

$$\begin{aligned} 2v_1 - v_2 + 3v_3 &= 5 \\ -4v_1 - 3v_2 - 2v_3 &= 8 \\ 3v_1 + v_2 - v_3 &= 4 \end{aligned} \tag{E.35}$$

Solution:

As a first step, we add the first equation of (E.35) with the third to eliminate the unknown v_2 and we obtain the equation

$$5v_1 + 2v_3 = 9 \tag{E.36}$$

Next, we multiply the third equation of (E.35) by 3, and we add it with the second to eliminate v_2, and we obtain the equation

$$5v_1 - 5v_3 = 20 \tag{E.37}$$

Subtraction of (E.37) from (E.36) yields

$$7v_3 = -11 \;\text{ or }\; v_3 = -\frac{11}{7} \tag{E.38}$$

Now, we can find the unknown v_1 from either (E.36) or (E.37). By substitution of (D.38) into (E.36) we obtain

$$5v_1 + 2 \cdot \left(-\frac{11}{7}\right) = 9 \;\text{ or }\; v_1 = \frac{17}{7} \tag{E.39}$$

Finally, we can find the last unknown v_2 from any of the three equations of (E.35). By substitution into the first equation we obtain

$$v_2 = 2v_1 + 3v_3 - 5 = \frac{34}{7} - \frac{33}{7} - \frac{35}{7} = -\frac{34}{7} \tag{E.40}$$

These are the same values as those we found in Example E.10.

The Gaussian elimination method works well if the coefficients of the unknowns are small integers, as in Example E.11. However, it becomes impractical if the coefficients are large or fractional numbers.

E.8 The Adjoint of a Matrix

Let us assume that A is an n square matrix and α_{ij} is the cofactor of a_{ij}. Then *the adjoint of* A, denoted as $\mathrm{adj}A$, is defined as the n square matrix below.

$$\mathrm{adj}A = \begin{bmatrix} \alpha_{11} & \alpha_{21} & \alpha_{31} & \dots & \alpha_{n1} \\ \alpha_{12} & \alpha_{22} & \alpha_{32} & \dots & \alpha_{n2} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} & \dots & \alpha_{n3} \\ \dots & \dots & \dots & \dots & \dots \\ \alpha_{1n} & \alpha_{2n} & \alpha_{3n} & \dots & \alpha_{nn} \end{bmatrix} \tag{E.41}$$

We observe that the cofactors of the elements of the ith row (column) of A are the elements of the ith column (row) of $\mathrm{adj}A$.

Example E.12

Compute $\mathrm{adj}A$ if Matrix A is defined as

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{bmatrix} \qquad (E.42)$$

Solution:

$$adjA = \begin{bmatrix} \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix} & -\begin{bmatrix} 2 & 3 \\ 4 & 3 \end{bmatrix} & \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} \\ -\begin{bmatrix} 1 & 4 \\ 1 & 3 \end{bmatrix} & \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix} & -\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} \\ \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} & -\begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix} & \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} -7 & 6 & -1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix}$$

E.9 Singular and Non–Singular Matrices

An n square matrix A is called *singular* if $\det A = 0$; if $\det A \neq 0$, A is called *non–singular*.

Example E.13

Matrix A is defined as

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix} \qquad (E.43)$$

Determine whether this matrix is singular or non–singular.

Solution:

$$\det A = \left| \begin{array}{ccc|cc} 1 & 2 & 3 & 1 & 2 \\ 2 & 3 & 4 & 2 & 3 \\ 3 & 5 & 7 & 3 & 5 \end{array} \right. = 21 + 24 + 30 - 27 - 20 - 28 = 0$$

Therefore, matrix A is singular.

E.10 The Inverse of a Matrix

If A and B are n square matrices such that $AB = BA = I$, where I is the identity matrix, B is called the *inverse* of A, denoted as $B = A^{-1}$, and likewise, A is called the inverse of B, that is, $A = B^{-1}$

If a matrix A is non-singular, we can compute its inverse A^{-1} from the relation

$$A^{-1} = \frac{1}{\det A}\operatorname{adj}A \tag{E.44}$$

Example E.14

Matrix A is defined as

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{bmatrix} \tag{E.45}$$

Compute its inverse, that is, find A^{-1}

Solution:

Here, $\det A = 9 + 8 + 12 - 9 - 16 - 6 = -2$, and since this is a non-zero value, it is possible to compute the inverse of A using (E.44).

From Example E.12,

$$\operatorname{adj}A = \begin{bmatrix} -7 & 6 & -1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix}$$

Then,

$$A^{-1} = \frac{1}{\det A}\operatorname{adj}A = \frac{1}{-2}\begin{bmatrix} -7 & 6 & -1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 3.5 & -3 & 0.5 \\ -0.5 & 0 & 0.5 \\ -0.5 & 1 & -0.5 \end{bmatrix} \tag{E.46}$$

Check with MATLAB:

```
A=[1  2  3;  1  3  4;  1  4  3],  invA=inv(A)      % Define matrix A and compute its inverse
```

```
A =
     1     2     3
     1     3     4
     1     4     3
```

```
invA =
    3.5000   -3.0000    0.5000
   -0.5000         0    0.5000
   -0.5000    1.0000   -0.5000
```

Multiplication of a matrix A by its inverse A^{-1} produces the identity matrix I, that is,

$$AA^{-1} = I \quad \text{or} \quad A^{-1}A = I \tag{E.47}$$

Example E.15

Prove the validity of (E.47) for the Matrix A defined as

$$A = \begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix}$$

Proof:

$$\det A = 8 - 6 = 2 \quad \text{and} \quad \operatorname{adj} A = \begin{bmatrix} 2 & -3 \\ -2 & 4 \end{bmatrix}$$

Then,

$$A^{-1} = \frac{1}{\det A}\operatorname{adj} A = \frac{1}{2}\begin{bmatrix} 2 & -3 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -3/2 \\ -1 & 2 \end{bmatrix}$$

and

$$AA^{-1} = \begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & -3/2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 4-3 & -6+6 \\ 2-2 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

E.11 Solution of Simultaneous Equations with Matrices

Consider the relation

$$AX = B \tag{E.48}$$

where A and B are matrices whose elements are known, and X is a matrix (a column vector) whose elements are the unknowns. We assume that A and X are conformable for multiplication.

Multiplication of both sides of (E.48) by A^{-1} yields:

$$A^{-1}AX = A^{-1}B = IX = A^{-1}B \tag{E.49}$$

or

$$\boxed{X = A^{-1}B} \tag{E.50}$$

Therefore, we can use (E.50) to solve any set of simultaneous equations that have solutions. We will refer to this method as the *inverse matrix method of solution* of simultaneous equations.

Example E.16

For the system of the equations

$$\begin{Bmatrix} 2x_1 + 3x_2 + x_3 = 9 \\ x_1 + 2x_2 + 3x_3 = 6 \\ 3x_1 + x_2 + 2x_3 = 8 \end{Bmatrix} \tag{E.51}$$

compute the unknowns $x_1, x_2,$ and x_3 using the inverse matrix method.

Solution:

In matrix form, the given set of equations is $AX = B$ where

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad B = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix} \tag{E.52}$$

Then,

$$X = A^{-1}B \tag{E.53}$$

or

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix} \tag{E.54}$$

Next, we find the determinant $\det A$, and the adjoint $\mathrm{adj}A$.

$$\det A = 18 \quad \text{and} \quad \mathrm{adj}A = \begin{bmatrix} 1 & -5 & 7 \\ 7 & 1 & -5 \\ -5 & 7 & 1 \end{bmatrix}$$

Therefore,

$$A^{-1} = \frac{1}{\det A}\,\mathrm{adj}A = \frac{1}{18}\begin{bmatrix} 1 & -5 & 7 \\ 7 & 1 & -5 \\ -5 & 7 & 1 \end{bmatrix}$$

and with relation (E.53) we obtain the solution as follows:

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 1 & -5 & 7 \\ 7 & 1 & -5 \\ -5 & 7 & 1 \end{bmatrix}\begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 35 \\ 29 \\ 5 \end{bmatrix} = \begin{bmatrix} 35/18 \\ 29/18 \\ 5/18 \end{bmatrix} = \begin{bmatrix} 1.94 \\ 1.61 \\ 0.28 \end{bmatrix} \quad \text{(E.55)}$$

To verify our results, we could use the MATLAB's **inv(A)** function, and then multiply A^{-1} by B. However, it is easier to use the *matrix left division* operation $X = A \backslash B$; this is MATLAB's solution of $A^{-1}B$ for the matrix equation $A \cdot X = B$, where matrix X is the same size as matrix B. For this example,

```
A=[2  3  1; 1  2  3; 3  1  2]; B=[9  6  8]'; X=A \ B
```

```
X =
    1.9444
    1.6111
    0.2778
```

Example E.17

For the electric circuit of Figure E.1,

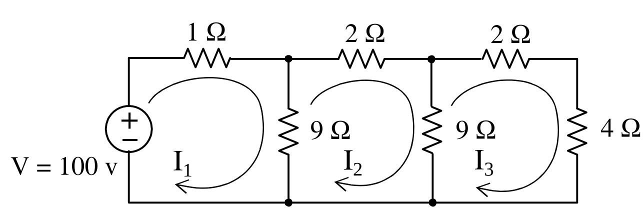


Figure E.1. Electric circuit for Example E.17

the loop equations are

$$\begin{aligned} 10I_1 - 9I_2 \qquad\quad &= 100 \\ -9I_1 + 20I_2 - 9I_3 &= 0 \\ -9I_2 + 15I_3 &= 0 \end{aligned} \quad \text{(E.56)}$$

Use the inverse matrix method to compute the values of the currents I_1, I_2, and I_3

Solution:

For this example, the matrix equation is $RI = V$ or $I = R^{-1}V$, where

$$R = \begin{bmatrix} 10 & -9 & 0 \\ -9 & 20 & -9 \\ 0 & -9 & 15 \end{bmatrix}, \quad V = \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad I = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}$$

The next step is to find R^{-1}. It is found from the relation

$$R^{-1} = \frac{1}{\det R}\,\mathrm{adj}R \tag{E.57}$$

Therefore, we must find the determinant and the adjoint of R. For this example, we find that

$$\det R = 975, \quad \mathrm{adj}R = \begin{bmatrix} 219 & 135 & 81 \\ 135 & 150 & 90 \\ 81 & 90 & 119 \end{bmatrix} \tag{E.58}$$

Then,

$$R^{-1} = \frac{1}{\det R}\mathrm{adj}R = \frac{1}{975}\begin{bmatrix} 219 & 135 & 81 \\ 135 & 150 & 90 \\ 81 & 90 & 119 \end{bmatrix}$$

and

$$I = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \frac{1}{975}\begin{bmatrix} 219 & 135 & 81 \\ 135 & 150 & 90 \\ 81 & 90 & 119 \end{bmatrix}\begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix} = \frac{100}{975}\begin{bmatrix} 219 \\ 135 \\ 81 \end{bmatrix} = \begin{bmatrix} 22.46 \\ 13.85 \\ 8.31 \end{bmatrix}$$

Check with MATLAB:

```
R=[10 –9  0; –9  20 –9;  0 –9  15]; V=[100  0  0]'; I=R\V; fprintf(' \n');...
fprintf('I1 = %4.2f \t', I(1)); fprintf('I2 = %4.2f \t', I(2)); fprintf('I3 = %4.2f \t', I(3)); fprintf(' \n')
```

```
I1 = 22.46    I2 = 13.85    I3 = 8.31
```

We can also use subscripts to address the individual elements of the matrix. Accordingly, the MATLAB script above could also have been written as:

```
R(1,1)=10; R(1,2)=–9;     % No need to make entry for A(1,3) since it is zero.
R(2,1)=–9; R(2,2)=20; R(2,3)=–9; R(3,2)=–9; R(3,3)=15; V=[100 0 0]'; I=R\V; fprintf(' \n');...
fprintf('I1 = %4.2f \t', I(1)); fprintf('I2 = %4.2f \t', I(2)); fprintf('I3 = %4.2f \t', I(3)); fprintf(' \n')
```

```
I1 = 22.46    I2 = 13.85    I3 = 8.31
```

Spreadsheets also have the capability of solving simultaneous equations with real coefficients using the inverse matrix method. For instance, we can use Microsoft Excel's MINVERSE (Matrix Inversion) and MMULT (Matrix Multiplication) functions, to obtain the values of the three currents in Example E.17.

The procedure is as follows:

1. We begin with a blank spreadsheet and in a block of cells, say B3:D5, we enter the elements of matrix R as shown in Figure D.2. Then, we enter the elements of matrix V in G3:G5.

2. Next, we compute and display the inverse of R, that is, R^{-1}. We choose B7:D9 for the elements of this inverted matrix. We format this block for number display with three decimal places. With this range highlighted and making sure that the cell marker is in B7, we type the formula

   =MININVERSE(B3:D5)

   and we press the *Crtl-Shift-Enter* keys simultaneously. We observe that R^{-1} appears in these cells.

3. Now, we choose the block of cells G7:G9 for the values of the current I. As before, we highlight them, and with the cell marker positioned in G7, we type the formula

   =MMULT(B7:D9,G3:G5)

   and we press the Crtl-Shift-Enter keys simultaneously. The values of I then appear in G7:G9.

	A	B	C	D	E	F	G	H
1	Spreadsheet for Matrix Inversion and Matrix Multiplication							
2								
3		10	-9	0			100	
4	R=	-9	20	-9		V=	0	
5		0	-9	15			0	
6								
7		0.225	0.138	0.083			22.462	
8	R^{-1}=	0.138	0.154	0.092		I=	13.846	
9		0.083	0.092	0.122			8.3077	
10								

Figure E.2. Solution of Example E.17 with a spreadsheet

Example E.18

For the phasor circuit of Figure E.18

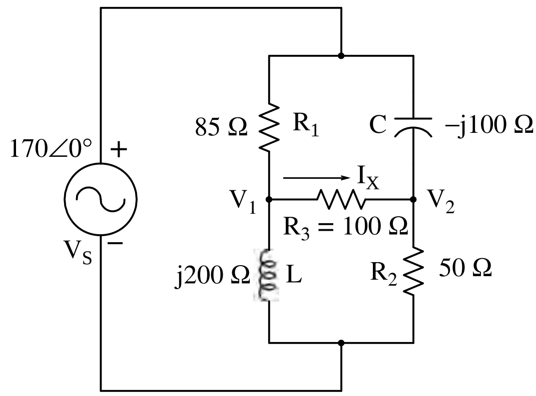


Figure E.3. Circuit for Example E.18

the current I_X can be found from the relation

$$I_X = \frac{V_1 - V_2}{R_3} \tag{E.59}$$

and the voltages V_1 and V_2 can be computed from the nodal equations

$$\frac{V_1 - 170\angle 0°}{85} + \frac{V_1 - V_2}{100} + \frac{V_1 - 0}{j200} = 0 \tag{E.60}$$

and

$$\frac{V_2 - 170\angle 0°}{-j100} + \frac{V_2 - V_1}{100} + \frac{V_2 - 0}{50} = 0 \tag{E.61}$$

Compute, and express the current I_x in both rectangular and polar forms by first simplifying like terms, collecting, and then writing the above relations in matrix form as $YV = I$, where $Y = \text{Admittance}$, $V = \text{Voltage}$, and $I = \text{Current}$

Solution:

The Y matrix elements are the coefficients of V_1 and V_2. Simplifying and rearranging the nodal equations of (E.60) and (E.61), we obtain

$$\begin{aligned}(0.0218 - j0.005)V_1 - 0.01V_2 &= 2\\ -0.01V_1 + (0.03 + j0.01)V_2 &= j1.7\end{aligned} \tag{E.62}$$

Next, we write (E.62) in matrix form as

$$\underbrace{\begin{bmatrix} 0.0218 - j0.005 & -0.01 \\ -0.01 & 0.03 + j0.01 \end{bmatrix}}_{Y} \underbrace{\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}}_{V} = \underbrace{\begin{bmatrix} 2 \\ j1.7 \end{bmatrix}}_{I} \tag{E.63}$$

where the matrices Y, V, and I are as indicated.

We will use MATLAB to compute the voltages V_1 and V_2, and to do all other computations. The script is shown below.

```
Y=[0.0218–0.005j  –0.01;  –0.01  0.03+0.01j]; I=[2; 1.7j]; V=Y\I;  % Define Y, I, and find V
fprintf('\n');                                                       % Insert a line
disp('V1 = '); disp(V(1)); disp('V2 = '); disp(V(2));               % Display values of V1 and V2
```

```
V1 =
 1.0490e+002 + 4.9448e+001i
V2 =
  53.4162 + 55.3439i
```

Next, we find I_X from

```
R3=100; IX=(V(1)–V(2))/R3                    % Compute the value of IX
```

```
IX =
   0.5149 - 0.0590i
```

This is the rectangular form of I_X. For the polar form we use the MATLAB script

```
magIX=abs(IX), thetaIX=angle(IX)*180/pi  % Compute the magnitude and the angle in degrees
```

```
magIX =
    0.5183

thetaIX =
   -6.5326
```

Therefore, in polar form,

$$I_X = 0.518\angle -6.53°$$

Spreadsheets have limited capabilities with complex numbers, and thus we cannot use them to compute matrices that include complex numbers in their elements as in Example E.18.

E.12 Exercises

For Exercises 1, 2, and 3 below, the matrices A, B, C, and D are defined as:

$$A = \begin{bmatrix} 1 & -1 & -4 \\ 5 & 7 & -2 \\ 3 & -5 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 5 & 9 & -3 \\ -2 & 8 & 2 \\ 7 & -4 & 6 \end{bmatrix} \quad C = \begin{bmatrix} 4 & 6 \\ -3 & 8 \\ 5 & -2 \end{bmatrix} \quad D = \begin{bmatrix} 1 & -2 & 3 \\ -3 & 6 & -4 \end{bmatrix}$$

1. Perform the following computations, if possible. Verify your answers with MATLAB.

a. $A + B$ b. $A + C$ c. $B + D$ d. $C + D$

e. $A - B$ f. $A - C$ g. $B - D$ h. $C - D$

2. Perform the following computations, if possible. Verify your answers with MATLAB.

a. $A \cdot B$ b. $A \cdot C$ c. $B \cdot D$ d. $C \cdot D$

e. $B \cdot A$ f. $C \cdot A$ g. $D \cdot A$ h. $\dot{D} \cdot C$

3. Perform the following computations, if possible. Verify your answers with MATLAB.

a. $\det A$ b. $\det B$ c. $\det C$ d. $\det D$ e. $\det(A \cdot B)$ f. $\det(A \cdot C)$

4. Solve the following systems of equations using Cramer's rule. Verify your answers with MATLAB.

a.
$$\begin{aligned} x_1 - 2x_2 + x_3 &= -4 \\ -2x_1 + 3x_2 + x_3 &= 9 \\ 3x_1 + 4x_2 - 5x_3 &= 0 \end{aligned}$$

b.
$$\begin{aligned} -x_1 + 2x_2 - 3x_3 + 5x_4 &= 14 \\ x_1 + 3x_2 + 2x_3 - x_4 &= 9 \\ 3x_1 - 3x_2 + 2x_3 + 4x_4 &= 19 \\ 4x_1 + 2x_2 + 5x_3 + x_4 &= 27 \end{aligned}$$

5. Repeat Exercise 4 using the Gaussian elimination method.

6. Solve the following systems of equations using the inverse matrix method. Verify your answers with MATLAB.

a.
$$\begin{bmatrix} 1 & 3 & 4 \\ 3 & 1 & -2 \\ 2 & 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ 0 \end{bmatrix}$$

b.
$$\begin{bmatrix} 2 & 4 & 3 & -2 \\ 2 & -4 & 1 & 3 \\ -1 & 3 & -4 & 2 \\ 2 & -2 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -14 \\ 7 \end{bmatrix}$$

Appendix F

Scaling

This chapter discusses magnitude and frequency scaling procedures that allow us to transform circuits that contain passive devices with unrealistic values to equivalent circuits with realistic values.

F.1 Magnitude Scaling

Magnitude scaling is the process by which the impedance of a two terminal network is changed by a factor k_m which is a real positive number greater or smaller than unity.

If we increase the input impedance by a factor k_m, we must increase the impedance of each device of the network by the same factor. Thus, if a network consists of R, L, and C devices and we wish to scale this network by this factor, the magnitude scaling process entails the following transformations where the subscript *m* denotes magnitude scaling.

$$\begin{aligned} R_m &\rightarrow k_m R \\ L_m &\rightarrow k_m L \\ C_m &\rightarrow \frac{C}{k_m} \end{aligned} \tag{F.1}$$

These transformations are consistent with the time–domain to frequency domain transformations

$$\begin{aligned} R &\rightarrow R \\ L &\rightarrow j\omega L \\ C &\rightarrow \frac{1}{j\omega C} \end{aligned} \tag{F.2}$$

and the t–domain to s–domain transformations

$$\begin{aligned} R &\rightarrow R \\ L &\rightarrow sL \\ C &\rightarrow \frac{1}{sC} \end{aligned} \tag{F.3}$$

F.2 Frequency Scaling

Frequency scaling is the process in which we change the values of the network devices so that at the new frequency the impedance of each device has the same value as at the original frequency.

The frequency scaling factor is denoted as k_f. This factor is also a real positive number and can be greater or smaller than unity.

The resistance value is independent of the frequency. However, the complex impedance of any inductor is sL, and in order to maintain the same impedance at a frequency k_f times as great, we must replace the inductor value by another which is equal to L/k_f. Similarly, a capacitor with value C must be replaced with another having a capacitance value equal to C/k_f. For frequency scaling then, the following transformations are necessary where the subscript f denotes magnitude scaling.

$$R_f \rightarrow R$$
$$L_f \rightarrow \frac{L}{k_f} \qquad (F.4)$$
$$C_f \rightarrow \frac{C}{k_f}$$

A circuit can be scaled simultaneously in both magnitude and frequency using the scales values below where the subscript mf denotes simultaneous magnitude and frequency scaling.

$$R_{mf} \rightarrow k_m R$$
$$L_{mf} \rightarrow \frac{k_m}{k_f} L \qquad (F.5)$$
$$C_{mf} \rightarrow \frac{1}{k_m k_f} C$$

Example F.1

For the network of Figure F.6 compute

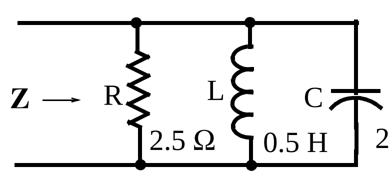


Figure F.6. Network for Example F.1

a. the resonant frequency ω_0.

b. the maximum impedance Z_{max}.

c. the quality factor Q_{0P}.

d. the bandwidth *BW*.

e. the magnitude of the input impedance Z, and using MATLAB sketch it as a function of frequency.

f. Scale this circuit so that the impedance will have a maximum value of $5\ K\Omega$ at a resonant frequency of 5×10^6 rad/s

Solution:

a. The resonant frequency of the given circuit is

$$\omega_0 = \frac{1}{\sqrt{LC}} = 1 \text{ rad/s}$$

and thus the circuit is parallel resonant.

b. The impedance is maximum at parallel resonance. Therefore,

$$Z_{max} = 2.5\ \Omega$$

c. The quality factor at parallel resonance is

$$Q_{0P} = \frac{\omega_0 C}{G} = \omega_0 CR = 1 \times 2 \times 2.5 = 5$$

d. The bandwidth of this circuit is

$$BW = \frac{\omega_0}{Q_{0P}} = \frac{1}{5} = 0.2$$

e. The magnitude of the input impedance versus radian frequency ω is shown in Figure F.7 and was generated with the MATLAB script below.

```
w=0.01: 0.005: 5; R=2.5; G=1/R; C=2; L=0.5; Y=G+j.*(w.*C–1./(w.*L));...
magY=abs(Y); magZ=1./magY; plot(w,magZ); grid
```

f. Using (F.1), we obtain

$$k_m = \frac{R_m}{R} = \frac{5000}{2.5} = 2000$$

Then,

$$L_m = k_m L = 2000 \times 0.5 = 1000 \text{ H}$$

and

$$C_m = \frac{C}{k_m} = \frac{2}{2000} = 10^{-3} \text{ F}$$

After being scaled in magnitude by the factor $k_m = 2000$, the network constants are as shown in Figure F.8, and the plot is shown in Figure F.9.

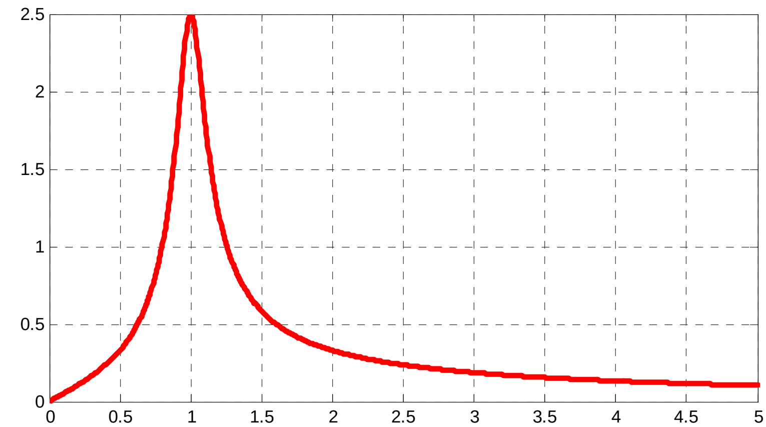


Figure F.7. Plot for Example F.1

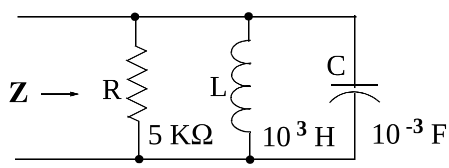


Figure F.8. The network in Figure F.6 scaled by the factor $k_m = 2000$

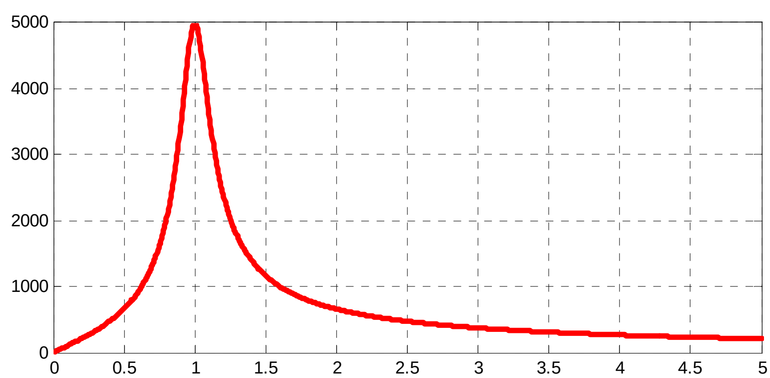


Figure F.9. Plot for the network of Figure F.6 after being scaled by the factor $k_m = 2000$

The final step is to scale the above circuit to 5×10^6 rad/s. Using (F.4), we obtain:

$$R_f = R = 5\ k\Omega$$

$$L_f = L/k_f = 1000/(5 \times 10^6) = 200\ \mu H$$

$$C_f = C/k_f = 10^{-3}/5 \times 10^6 = 200\ pF$$

The network constants and its response, in final form, are as shown in Figures F.10 and F.11 respectively.

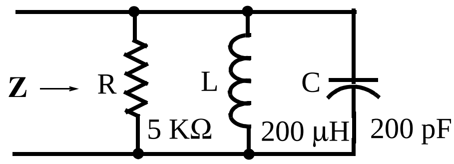


Figure F.10. The network in Figure F.6 scaled to its final form

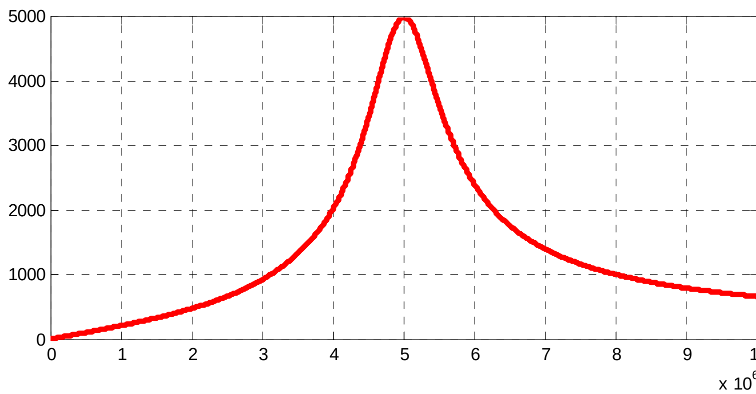


Figure F.11. Plot for Example F.1 scaled to its final form

The plot of Figure F.11 was generated with the following MATLAB script:

```
w=1: 10^3: 10^7; R=5000; G=1/R; C=200.*10.^(–12); L=200.*10.^(–6); ...
magY=sqrt(G.^2+(w.*C–1./(w.*L)).^2); magZ=1./magY; plot(w,magZ); grid
```

Check:

The resonant frequency of the scaled circuit is

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.2 \times 10^{-3} \times 0.2 \times 10^{-9}}} = \frac{1}{0.2 \times 10^{-6}} = 5 \times 10^6 \text{ rad/s}$$

and thus the circuit is parallel resonant at this frequency.

The impedance is maximum at parallel resonance. Therefore,

$$Z_{max} = 5 \text{ K}\Omega$$

The quality factor at parallel resonance is

$$Q_{0P} = \frac{\omega_0 C}{G} = \omega_0 CR = 5 \times 10^6 \times 2 \times 10^{-10} \times 5 \times 10^3 = 5$$

and the bandwidth is

$$BW = \frac{\omega_0}{Q_{0P}} = \frac{5 \times 10^6}{5} = 10^6$$

The values of the circuit devices could have been obtained also by direct application of (F.5), that is,

$$R_{mf} \rightarrow k_m R$$

$$L_{mf} \rightarrow \frac{k_m}{k_f} L$$

$$C_{mf} \rightarrow \frac{k_m}{k_f} C$$

$$R_{mf} = k_m R = 2000 \times 2.5 = 5\ K\Omega$$

$$L_{mf} = \frac{k_m}{k_f} L = \frac{2000}{5 \times 10^6} \times 0.5 = 200\ \mu H$$

$$C_{mf} = \frac{1}{k_m k_f} C = \frac{1}{2 \times 10^3 \times 5 \times 10^6} \times 2 = 200\ pf$$

and these values are the same as obtained before.

Example F.2

A series RLC circuit has resistance $R = 1\ \Omega$, inductance $L = 1\ H$, and capacitance $C = 1\ F$. Use scaling to compute the new values of R and L which will result in a circuit with the same quality factor Q_{0S}, resonant frequency at $500\ Hz$ and the new value of the capacitor to be $2\ \mu F$.

Solution:

The resonant frequency of the circuit before scaling is

$$\omega_0 = \frac{1}{\sqrt{LC}} = 1\ rad/s$$

and we want the resonant frequency of the scaled circuit to be $500\ Hz$ or $2\pi \times 500 = 3142\ rad/s$. Therefore, the frequency scaling factor must be

$$k_f = \frac{3142}{1} = 3142$$

Now, we must compute the magnitude scale factor, and since we want the capacitor value to be $2\ \mu F$, we use (F.5), that is,

$$C_{mf} = \frac{1}{k_m k_f} C$$

or

$$k_m = \frac{C}{k_f C_{mf}} = \frac{1}{3142 \times 2 \times 10^{-6}} = 159$$

Then, the scaled values for the resistance and inductance are

$$R_m = k_m R = 159 \times 1 = 159\ \Omega$$

and

$$L_{mf} = \frac{k_m}{k_f} L = \frac{159}{3142} \times 1 = 50.6\ \text{mH}$$

F.3 Exercises

1. A series resonant circuit has a bandwidth of $100\ \mathrm{rad/s}$, $Q_{0s} = 20$ and $C = 50\ \mu F$. Compute the new resonant frequency and inductance if the circuit is scaled

   a. in magnitude by a factor of 5

   b. in frequency by a factor of 5

   c. in both magnitude and frequency by factors of 5

2. A scaled parallel resonant circuit consists of $R = 4\ K\Omega$, $L = 0.1\ H$, and $C = 0.3\ \mu F$. Compute k_m and k_f if the original circuit had the following values before scaling.

   a. $R = 10\ \Omega$ and $L = 1\ H$

   b. $R = 10\ \Omega$ and $C = 5\ F$

   c. $L = 1\ H$ and $C = 5\ F$

F.4 Solutions to End–of–Appendix Exercises

1. **a.** It is given that $BW = \omega_0/Q_{OS} = 100$ and $Q_{OS} = 20$; then,

$$\omega_0 = BW \cdot Q_{OS} = 100 \times 20 = 2000 \text{ rad/s}$$

Since $\omega_0^2 = 1/LC$, $L_{OLD} = 1/\omega_0^2 C = 1/(4 \times 10^6 \times 50 \times 10^{-6}) = 5 \text{ mH}$, and with $k_m = 5$,

$L_{NEW} = k_m L_{OLD} = 5 \times 5 \text{ mH} = 25 \text{ mH}$. Also, $C_{NEW} = C_{OLD}/k_m = 50 \times 10^{-6}/5 = 10 \ \mu F$ and

$$\omega_{0\ NEW}^2 = 1/L_{NEW}C_{NEW} = 1/(25 \times 10^{-3} \times 10 \times 10^{-6}) = 10^8/25 \text{ or } \omega_{0\ NEW} = 2000 \text{ r/s}$$

b. It is given that $C_{OLD} = 50 \times 10^{-6}$ and from (a) $L_{OLD} = 5 \text{ mH}$. Then, with $k_f = 5$,

$$L_{NEW} = L_{OLD}/k_f = 5 \times 10^{-3}/5 = 1 \text{ mH}$$

Also,

$$C_{NEW} = C_{OLD}/k_f = 50 \times 10^{-6}/5 = 10 \ \mu F$$

and

$$\omega_{0\ NEW}^2 = 1/L_{NEW}C_{NEW} = 1/(10^{-3} \times 10 \times 10^{-6}) = 10^8$$

or

$$\omega_{0\ NEW} = 10000 \text{ r/s}$$

c. $L_{OLD} = 5 \text{ mH}$ and $C_{OLD} = 50 \times 10^{-6}$. Then, from (F.5)

$$L_{NEW} = (k_m/k_f) \cdot L_{OLD} = (5/5) \cdot 5 \text{ mH} = 5 \text{ mH}$$

Also from (F.5)

$$C_{NEW} = (1/(k_m k_f)) \cdot C_{OLD} = 50 \ \mu F/(5 \times 5) = 2 \ \mu F$$

and

$$\omega_{0\ NEW}^2 = 1/L_{NEW}C_{NEW} = 1/(5 \times 10^{-3} \times 2 \times 10^{-6}) = 10^8 \text{ or } \omega_{0\ NEW} = 10000 \text{ r/s}$$

2. **a.** From (F.1), $k_m = R_{NEW}/R_{OLD} = 4000/10 = 400$ and from (F.5)

$$k_f = (L_{OLD}/L_{NEW}) \cdot k_m = (1/0.1) \times 400 = 4000$$

b. From (a) $k_m = 400$ and from (F.5),

$$k_f = (1/k_m) \cdot (C_{OLD}/C_{NEW}) = (1/400) \cdot (5/0.3 \times 10^{-6}) = 41677$$

c. From (F.5) $k_f/k_m = L_{OLD}/L_{NEW} = 1/0.1 = 10$ and thus $k_f = 10k_m$ (1)

Also from (F.5), $k_m \cdot k_f = C_{OLD}/C_{NEW} = 5/0.3 \times 10^{-6} = 5 \times 10^6/0.3$ (2)

Substitution of (1) into (2) yields $10k_m \cdot k_m = 5 \times 10^6/0.3$, $k_m^2 = 5 \times 10^6/3$, or $k_m = 1291$,

and from (1) $k_f = 1291 \times 10 = 12910$

Appendix G

Per Unit System

This chapter introduces the per unit system. This system allows us to work with normalized power, voltage current, impedance, and admittance values known as per unit (pu) values. The relationship between units in a per-unit system depends on whether the system is single–phase or three–phase. Three-phase systems are discussed in Chapters 11 and 12.

G.1 Per Unit Defined

By definition,

$$\text{Per Unit Value} = \frac{\text{Actual Value}}{\text{Base Value}} \tag{G.1}$$

A per unit (pu) system defines per unit values for volt–ampere (VA) power, voltage, current, impedance, and admittance, and of these only two of these are independent. It is customary to choose VA (or KVA) power and nominal voltage as the independent base values, and others are specified as multiples of selected base values.

For single-phase systems, the pu values are based on rated VA (or KVA) rated power and on the nominal voltage of the equipment, e.g., single–phase transformer, single–phase motor.

Example G.1

A single–phase transformer is rated $10\ \text{KVA}$ and the nominal voltage on the primary winding is $480\ \text{V RMS}$. Compute its pu impedance.

Solution:

$$\begin{aligned}
\text{Base Current (amperes)} &= \frac{\text{Base KVA}}{\text{Base Volts}} = \frac{10000\ \text{VA}}{480\ \text{V}} = 20.83\ \text{A RMS} \\
\text{Base Impedance (Ohms)} &= \frac{\text{Base Volts}}{\text{Base Current}} = \frac{480\ \text{V}}{20.83\ \text{A}} = 23.04\ \Omega
\end{aligned} \tag{G.2}$$

and assuming that the actual primary winding voltage, current, and impedance are $436\ \text{Volts RMS}$, $15\ \text{A RMS}$, and $5\ \Omega$, respectively, the per unit values are computed as follows:

$$\text{Voltage}_{pu} = \frac{\text{Actual Volts}}{\text{Base Volts}} = \frac{436\ \text{V}}{480\ \text{V}} \approx 0.91\ \text{pu}$$

$$\text{Current}_{pu} = \frac{\text{Actual Current}}{\text{Base Current}} = \frac{15\ \text{A}}{20.83\ \text{A}} \approx 0.72\ \text{pu} \qquad (G.3)$$

$$\text{Impedance}_{pu} = \frac{\text{Actual Impedance}}{\text{Base Impedance}} = \frac{5\ \Omega}{23.04\ \Omega} \approx 0.22\ \text{pu}$$

The base impedance in (G.2) is also expressed as

$$\text{Base Impedance (Ohms)} = \frac{\text{Base Volts}}{\text{Base Current}} = \frac{\text{Base Volts}}{(\text{Base KVA})/(\text{Base Volts})} = \frac{(\text{Base Volts})^2}{(\text{Base KVA})} \qquad (G.4)$$

Thus, the pu impedance can also be expressed as

$$\text{Impedance}_{pu} = \frac{\text{Actual Impedance}}{\text{Base Impedance}} = \frac{\text{Actual Impedance}}{(\text{Base Volts})^2/(\text{Base KVA})} \qquad (G.5)$$

$$= \text{Actual Impedance} \times \frac{(\text{Base KVA})}{(\text{Base Volts})^2}$$

and using the values above we obtain

$$\text{Impedance}_{pu}(\Omega) = 5 \times \frac{10000}{480^2} \approx 0.22\ \text{pu}$$

as before.

The pu values allow us to express quantities in percentages, that is,

$$\% = \text{pu} \times 100 \qquad (G.6)$$

and thus $0.22\ \text{pu} = 22\%$

The per unit values in three−phase systems are based on

$$\text{Base VA} = \text{3-phase VA}$$
$$\text{Base Volts} = \text{Line-to-Line Volts RMS} \qquad (G.7)$$

Example G.2

A three−phase Y−connected transformer is rated $7.5\ \text{KVA}$ and the line−to−line voltage is $480\ \text{V RMS}$. Compute its per phase (line−to−neutral) pu impedance.

Solution:

The per phase (line−to−neutral) pu values are computed as follows:

$$\text{Per phase Base Current (A)} = \frac{\text{Per phase Base KVA}}{\text{Per phase Base Volts}} = \frac{7.5/3\ \text{KVA}}{480/\sqrt{3}\ \text{V}} = 9.02\ \text{A RMS}$$

$$\text{Base Impedance }(\Omega) = \frac{\text{Per phase Base Volts}}{\text{Per phase Base Current}} = \frac{480/\sqrt{3}\ \text{KV}}{9.02\ \text{A}} = 30.73\ \Omega \qquad \text{(G.8)}$$

and assuming that the per phase (line–to–neutral) actual primary winding voltage, current, and impedance are $472/\sqrt{3}$ Volts RMS, 12.2 A RMS, and $5\ \Omega$ respectively, the per phase (line–to–neutral) per unit values are computed as follows:

$$\text{Voltage}_{pu} = \frac{\text{Actual Volts}}{\text{Base Volts}} = \frac{472/\sqrt{3}\ \text{V}}{480/\sqrt{3}\ \text{V}} \approx 0.98\ \text{pu}$$

$$\text{Current}_{pu} = \frac{\text{Actual Current}}{\text{Base Current}} = \frac{9.02\ \text{A}}{12.2\ \text{A}} \approx 0.74\ \text{pu} \qquad \text{(G.9)}$$

$$\text{Impedance}_{pu} = \frac{\text{Actual Impedance}}{\text{Base Impedance}} = \frac{5\ \Omega}{30.73\ \Omega} \approx 0.16\ \text{pu}$$

G.2 Impedance Transformation from One Base to Another Base

Often, we need to change the base values from one base to another, and thus we must change the original pu values to the new base pu values. Denoting the original pu as pu_1 and the new pu as pu_2, and using relation (G.5) we obtain:

$$\frac{\text{Impedance}_{pu1}}{\text{Impedance}_{pu2}} = \frac{\text{Actual Impedance} \times (\text{Base KVA}_1)/(\text{Base Volts}_1)^2}{\text{Actual Impedance} \times (\text{Base KVA}_2)/(\text{Base Volts}_2)^2} \qquad \text{(G.10)}$$

from which,

$$\text{Impedance}_{pu2} = \text{Impedance}_{pu1} \cdot \frac{(\text{Base KVA}_2)}{(\text{Base KVA}_1)} \cdot \left(\frac{\text{Base Volts}_1}{\text{Base Volts}_2}\right)^2 \qquad \text{(G.11)}$$

Example G.3

A three–phase AC motor rated 500 hp, 2.0 KV, 60 Hz, pu impedance $= 0.26$, full–load efficiency 88 %, power factor 0.85, is connected to a $10,000$ KVA, $4,160$ V system. Compute its pu impedance on the system base values.

Solution:

First, we must find the rated KVA of the motor. It is computed from the equation

$$\text{Motor Rated KVA} = \frac{(\text{Rated hp}) \times 0.746\ \text{Kw/hp}}{\text{Full Load Efficiency} \times \text{Rated Power Factor}} \tag{G.12}$$

Thus,

$$\text{Motor Rated KVA} = \frac{500 \times 0.746}{0.88 \times 0.85} = \frac{500 \times 0.746}{0.88 \times 0.85} \approx 500 = \text{KVA}_1$$

and with (G.11) we obtain

$$\text{Impedance}_{pu2} = 0.26 \cdot \frac{10000}{500} \cdot \left(\frac{2}{4.16}\right)^2 = 1.2$$

Example G.4

A step–down three–phase transformer is rated $1,000$ KVA, $13,200/480$ V, with 0.0575 pu impedance. It is proposed to use this transformer on a 750 KVA, $12,000$ V system. Compute:

a. The pu impedance of the 750 KVA, $12,000$ V system.

b. If the $12,000$ V is to be used as the new base voltage on the high voltage side, what would the base voltage be on the low voltage side?

c. What would the base current values be on the high voltage side and the low voltage side on the 750 KVA, $12,000$ V system?

Solution:

a.

$$\text{Impedance}_{pu2} = \text{Impedance}_{pu1} \cdot \frac{(\text{Base KVA}_2)}{(\text{Base KVA}_1)} \cdot \left(\frac{\text{Base Volts}_1}{\text{Base Volts}_2}\right)^2$$

$$= 0.0575 \cdot \frac{750}{1,000} \cdot \left(\frac{13.2}{12}\right)^2 = 0.052$$

b.

By proportion,

$$\text{Low voltage side} = 480 \cdot \frac{12}{13.2} = 436$$

c.

$$\text{High voltage side base current} = \frac{750}{\sqrt{3} \cdot 12} = 36\ \text{A}$$

$$\text{Low voltage side base current} = \frac{750}{\sqrt{3} \cdot 0.436} = 993\ \text{A}$$

Appendix H

Review of Differential Equations

This appendix is a review of ordinary differential equations. Some definitions, topics, and examples are not applicable to introductory circuit analysis but are included for continuity of the subject, and for reference to more advance topics in electrical engineering such as state variables. These are denoted with an asterisk and may be skipped.

H.1 Simple Differential Equations

In this section we present two simple examples to show the importance of differential equations in engineering applications.

Example H.1

A $1\ F$ capacitor is being charged by a constant current I. Find the voltage v_C across this capacitor as a function of time given that the voltage at some reference time $t = 0$ is V_0.

Solution:

It is given that the current, as a function of time, is constant, that is,

$$i_C(t) = I = constant \tag{H.1}$$

We know that the current and voltage in a capacitor are related by

$$i_C(t) = C\frac{dv_C}{dt} \tag{H.2}$$

and for our example, $C = 1$. Then, by substitution of (H.2) into (H.1) we obtain

$$\frac{dv_C}{dt} = I$$

By separation of the variables,

$$dv_C = Idt \tag{H.3}$$

and by integrating both sides of (H.3) we obtain

$$v_C(t) = It + k \tag{H.4}$$

where k represents the constants of integration of both sides.

We can find the value of the constant k by making use of the initial condition, i.e., at $t = 0$, $v_C = V_0$ and (H.4) then becomes

$$V_0 = 0 + k \tag{H.5}$$

or $k = V_0$, and by substitution into (H.4),

$$\boxed{v_C(t) = It + V_0} \tag{H.6}$$

This example shows that *when a capacitor is charged with a constant current, a linear voltage is produced across the terminals of the capacitor*.

Example H.2

Find the current $i_L(t)$ through an inductor whose slope at the coordinate (t, i_L) is $\cos t$ and the current i_L passes through the point $(\pi/2, 1)$.

Solution:

We are given that

$$\frac{di_L}{dt} = \cos t \tag{H.7}$$

By separating the variables we obtain

$$di_L = \cos t\,dt \tag{H.8}$$

and integrating both sides we obtain

$$i_L(t) = \sin t + k \tag{H.9}$$

where k represents the constants of integration of both sides.

We find the value of the constant k by making use of the initial condition. For this example, $\omega = 1$ and thus at $\omega t = t = \pi/2$, $i_L = 1$. With these values (H.9) becomes

$$1 = \sin\frac{\pi}{2} + k \tag{H.10}$$

or $k = 0$, and by substitution into (H.9),

$$\boxed{i_L(t) = \sin t} \tag{H.11}$$

H.2 Classification

Differential equations are classified by:

1. *Type* – Ordinary or Partial
2. *Order* – The highest order derivative which is included in the differential equation
3. *Degree* – The exponent of the highest power of the highest order derivative after the differential equation has been cleared of any fractions or radicals in the dependent variable and its derivatives

For example, the differential equation

$$\left(\frac{d^4y}{dx^4}\right)^2 + 5\left(\frac{d^3y}{dx^3}\right)^4 + 6\left(\frac{d^2y}{dx^2}\right)^6 + 3\left(\frac{dy}{dx}\right)^8 + \frac{y^2}{x^3+1} = ye^{-2x}$$

is an ordinary differential equation of order 4 and degree 2.

If the dependent variable y is a function of only a single variable x, that is, if $y = f(x)$, the differential equation which relates y and x is said to be an *ordinary differential equation* and it is abbreviated as ODE.

The differential equation

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2 = 5\cos 4t$$

is an ODE with constant coefficients.

The differential equation

$$x^2\frac{d^2y}{dt^2} + x\frac{dy}{dt} + (x^2 - n^2) = 0$$

is an ODE with variable coefficients.

If the dependent variable y is a function of two or more variables such as $y = f(x, t)$, where x and t are independent variables, the differential equation that relates y, x, and t is said to be a *partial differential equation* and it is abbreviated as *PDE*.

An example of a partial differential equation is the well–known *one–dimensional wave equation* shown below.

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$$

Most of the electrical engineering problems are solved with ordinary differential equations with constant coefficients; however, partial differential equations provide often quick solutions to some practical applications as illustrated with the following three examples.

Example H.3

The equivalent resistance R_T of three resistors R_1, R_2, and R_3 in parallel is given by

$$\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

Given that initially $R_1 = 5\ \Omega$, $R_2 = 20\ \Omega$, and $R_3 = 4\ \Omega$ compute the change in R_T if R_2 is increased by 10% and R_3 is decreased by 5% while R_1 does not change.

Solution:

The initial value of the equivalent resistance is $R_T = 5 \parallel 20 \parallel 4 = 2\ \Omega$.

We begin by treating R_2 and R_3 as constants and differentiating R_T with respect to R_1 we obtain

$$-\frac{1}{R_T^2}\frac{\partial R_T}{\partial R_1} = -\frac{1}{R_1^2} \quad \text{or} \quad \frac{\partial R_T}{\partial R_1} = \left(\frac{R_T}{R_1}\right)^2$$

Similarly,

$$\frac{\partial R_T}{\partial R_2} = \left(\frac{R_T}{R_2}\right)^2 \quad \text{and} \quad \frac{\partial R_T}{\partial R_3} = \left(\frac{R_T}{R_3}\right)^2$$

and the total differential dR_T is

$$lR_T = \frac{\partial R_T}{\partial R_1}dR_1 + \frac{\partial R_T}{\partial R_2}dR_2 + \frac{\partial R_T}{\partial R_3}dR_3 = \left(\frac{R_T}{R_1}\right)^2 dR_1 + \left(\frac{R_T}{R_2}\right)^2 dR_2 + \left(\frac{R_T}{R_3}\right)^2 dR$$

By substitution of the given numerical values we obtain

$$dR_T = \left(\frac{2}{5}\right)^2(0) + \left(\frac{2}{20}\right)^2(2) + \left(\frac{2}{4}\right)^2(-0.2) = 0.02 - 0.05 = -0.03$$

Therefore, the eequivalent resistance decreases by 3%.

Example H.4

In a series RC circuit that is excited by a sinusoidal voltage, the magnitude of the impedance Z is computed from $Z = \sqrt{R^2 + X_C^2}$. Initially, $R = 4\ \Omega$ and $X_C = 3\ \Omega$. Find the change in the impedance Z if the resistance R is increased by $0.25\ \Omega$ (6.25%) and the capacitive reactance X_C is decreased by $0.125\ \Omega$ (-4.167%).

Solution:

We will first find the partial derivatives $\frac{\partial Z}{\partial R}$ and $\frac{\partial Z}{\partial X_C}$; then we compute the change in impedance from the total differential dZ. Thus,

$$\frac{\partial Z}{\partial R} = \frac{R}{\sqrt{R^2 + X_C^2}} \quad \text{and} \quad \frac{\partial Z}{\partial X_C} = \frac{X_C}{\sqrt{R^2 + X_C^2}}$$

and

$$dZ = \frac{\partial Z}{\partial R} dR + \frac{\partial Z}{\partial X_C} dX_C = \frac{R\ dR + X_C\ dX_C}{\sqrt{R^2 + X_C^2}}$$

and by substitution of the given values

$$dZ = \frac{4\ (0.25) + 3\ (-0.125)}{\sqrt{4^2 + 3^2}} = \frac{1\ - 0.375}{5} = 0.125$$

Therefore, if R increases by 6.25% and X_C decreases by 4.167%, the impedance Z increases by 4.167%.

Example H.5

A light bulb is rated at 120 volts and 75 watts. If the voltage decreases by 5 volts and the resistance of the bulb is increased by $8\ \Omega$, by how much will the power change?

Solution:

At $V = 120$ volts and $P = 75$ watts, the bulb resistance is

$$R = \frac{V^2}{P} = \frac{120^2}{75} = 192\ \Omega$$

and since

$$P = \frac{V^2}{R} \quad \text{then} \quad \frac{\partial P}{\partial V} = \frac{2V}{R} \quad \text{and} \quad \frac{\partial P}{\partial R} = -\frac{V^2}{R^2}$$

and the total differential is

$$dP = \frac{\partial P}{\partial V} dV + \frac{\partial P}{\partial R} dR = \frac{2V}{R} dV - \frac{V^2}{R^2} dR$$

$$= \frac{2(120)}{192}(-5) - \frac{120^2}{192^2}(8) = -9.375$$

That is, the power will decrease by 9.375 watts.

H.3 Solutions of Ordinary Differential Equations (ODE)

A function $y = f(x)$ is a solution of a differential equation if the latter is satisfied when y and its derivatives are replaced throughout by $f(x)$ and its corresponding derivatives. Also, the initial conditions must be satisfied.

For example a solution of the differential equation

$$\frac{d^2y}{dx^2} + y = 0$$

is

$$y = k_1 \sin x + k_2 \cos x$$

since y and its second derivative satisfy the given differential equation.

Any linear, time–invariant electric circuit can be described by an ODE which has the form

$$a_n \frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \dots + a_1\frac{dy}{dt} + a_0 y = \underbrace{b_m\frac{d^m x}{dt^m} + b_{m-1}\frac{d^{m-1}x}{dt^{n-1}} + \dots + b_1\frac{dx}{dt} + b_0 x}_{\text{Excitation (Forcing) Function } x(t)} \tag{H.12}$$

NON – HOMOGENEOUS DIFFERENTIAL EQUATION

If the excitation in (B12) is not zero, that is, if $x(t) \neq 0$, the ODE is called a *non–homogeneous ODE*. If $x(t) = 0$, it reduces to:

$$a_n \frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \dots + a_1\frac{dy}{dt} + a_0 y = 0 \tag{H.13}$$

HOMOGENEOUS DIFFERENTIAL EQUATION

The differential equation of (H.13) above is called a *homogeneous ODE* and has n different linearly independent solutions denoted as $y_1(t), y_2(t), y_3(t), \dots, y_n(t)$.

We will now prove that the *most general solution* of (H.13) is:

$$y_H(t) = k_1 y_1(t) + k_2 y_2(t) + k_3 y_3(t) + \dots + k_n y_n(t) \tag{H.14}$$

where the subscript H on the left side is used to emphasize that this is the form of the solution of the homogeneous ODE and $k_1, k_2, k_3, \dots, k_n$ are arbitrary constants.

Proof:

Let us assume that $y_1(t)$ is a solution of (H.13); then by substitution,

$$a_n\frac{d^n y_1}{dt^n}+a_{n-1}\frac{d^{n-1}y_1}{dt^{n-1}}+\dots+a_1\frac{dy_1}{dt}+a_0 y_1 = 0 \tag{H.15}$$

A solution of the form $k_1 y_1(t)$ will also satisfy (H.13) since

$$\begin{aligned}&a_n\frac{d^n}{dt^n}(k_1 y_1)+a_{n-1}\frac{d^{n-1}}{dt^{n-1}}(k_1 y_1)+\dots+a_1\frac{d}{dt}(k_1 y_1)+a_0(k_1 y_1)\\&= k_1\left(a_n\frac{d^n y_1}{dt^n}+a_{n-1}\frac{d^{n-1}y_1}{dt^{n-1}}+\dots+a_1\frac{dy_1}{dt}+a_0 y_1\right) = 0\end{aligned} \tag{H.16}$$

If $y = y_1(t)$ and $y = y_2(t)$ are any two solutions, then $y = y_1(t)+y_2(t)$ will also be a solution since

$$a_n\frac{d^n y_1}{dt^n}+a_{n-1}\frac{d^{n-1}y_1}{dt^{n-1}}+\dots+a_1\frac{dy_1}{dt}+a_0 y_1 = 0$$

and

$$a_n\frac{d^n y_2}{dt^n}+a_{n-1}\frac{d^{n-1}y_2}{dt^{n-1}}+\dots+a_1\frac{d\,y_2}{dt}+a_0 y_2 = 0$$

Therefore,

$$\begin{aligned}&a_n\frac{d^n}{dt^n}(y_1+y_2)+a_{n-1}\frac{d^{n-1}}{dt^{n-1}}(y_1+y_2)+\dots+a_1\frac{d}{dt}(y_1+y_2)+a_0(y_1+y_2)\\&= a_n\frac{d^n}{dt^n}y_1+a_{n-1}\frac{d^{n-1}}{dt^{n-1}}y_1+\dots+a_1\frac{d}{dt}y_1+a_0 y_1\\&+a_n\frac{d^n}{dt^n}y_2+a_{n-1}\frac{d^{n-1}}{dt^{n-1}}y_2+\dots+a_1\frac{d}{dt}y_2+a_0 y_2 = 0\end{aligned} \tag{H.17}$$

In general, if

$$y= k_1y_1(t), k_2y_1(t), k_3y_3(t), \dots, k_ny_n(t)$$

are the n solutions of the homogeneous ODE of (H.13), the linear combination

$$y= k_1y_1(t)+k_2y_1(t)+k_3y_3(t)+\dots+k_ny_n(t)$$

is also a solution.

In our subsequent discussion, the solution of the homogeneous ODE, i.e., the complementary solution, will be referred to as the *natural response*, and will be denoted as $y_N(t)$ or simply y_N. The particular solution of a non–homogeneous ODE will be referred to as the *forced response*, and will be denoted as $y_F(t)$ or simply y_F. Accordingly, we express the total solution of the non–homogeneous ODE of (H.12) as:

$$y(t) = \underbrace{y}_{\text{Natural Response}} + \underbrace{y}_{\text{Forced Response}} = y_N + y_F \qquad \text{(H.18)}$$

The natural response y_N contains arbitrary constants and these can be evaluated from the given initial conditions. The forced response y_F, however, contains no arbitrary constants. It is imperative to remember that the arbitrary constants of the natural response must be evaluated from the total response.

H.4 Solution of the Homogeneous ODE

Let the solutions of the homogeneous ODE

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_1 \frac{dy}{dt} + a_0 y = 0 \qquad \text{(H.19)}$$

be of the form

$$y = ke^{st} \qquad \text{(H.20)}$$

Then, by substitution of (H.20) into (H.19) we obtain

$$a_n ks^n e^{st} + a_{n-1} ks^{n-1} e^{st} + \ldots + a_1 kse^{st} + a_0 ke^{st} = 0$$

or

$$(a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0) ke^{st} = 0 \qquad \text{(H.21)}$$

We observe that (H.21) can be satisfied when

$$(a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0) = 0 \quad \text{or} \quad k = 0 \quad \text{or} \quad s = -\infty \qquad \text{(H.22)}$$

but the only meaningful solution is the quantity enclosed in parentheses since the latter two yield trivial (meaningless) solutions. We, therefore, accept the expression inside the parentheses as the only meaningful solution and this is referred to as the *characteristic (auxiliary) equation*, that is,

$$\underbrace{(a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0) = 0}_{\text{Characteristic Equation}} \qquad \text{(H.23)}$$

Since the characteristic equation is an algebraic equation of an *nth–power* polynomial, its solutions are $s_1, s_2, s_3, \ldots, s_n$, and thus the solutions of the homogeneous ODE are:

$$y_1 = k_1 e^{s_1 t}, \quad y_2 = k_2 e^{s_2 t}, \quad y_3 = k_3 e^{s_3 t}, \ldots, \quad y_n = k_n e^{s_n t} \qquad \text{(H.24)}$$

Case I – Distinct Roots

If the roots of the characteristic equation are *distinct* (different from each another), the n solutions of (H.23) are independent and the most general solution is:

$$y_N = k_1 e^{s_1 t} + k_2 e^{s_2 t} + \ldots + k_n e^{s_n t}$$
FOR DISTINCT ROOTS (H.25)

Case II – Repeated Roots

If two or more roots of the characteristic equation are *repeated* (same roots), then some of the terms of (H.24) are not independent and therefore (H.25) does not represent the most general solution. If, for example, $s_1 = s_2$, then,

$$k_1 e^{s_1 t} + k_2 e^{s_2 t} = k_1 e^{s_1 t} + k_2 e^{s_1 t} = (k_1 + k_2) e^{s_1 t} = k_3 e^{s_1 t}$$

and we see that one term of (H.25) is lost. In this case, we express one of the terms of (H.25), say $k_2 e^{s_1 t}$ as $k_2 t e^{s_1 t}$. These two represent two independent solutions and therefore the most general solution has the form:

$$y_N = (k_1 + k_2 t) e^{s_1 t} + k_3 e^{s_3 t} + \ldots + k_n e^{s_n t} \qquad \text{(H.26)}$$

If there are *m* equal roots the most general solution has the form:

$$y_N = (k_1 + k_2 t + \ldots + k_m t^{m-1}) e^{s_1 t} + k_{n-i} e^{s_2 t} + \ldots + k_n e^{s_n t}$$
FOR M EQUAL ROOTS (H.27)

Case III – Complex Roots

If the characteristic equation contains complex roots, these occur as complex conjugate pairs. Thus, if one root is $s_1 = -\alpha + j\beta$ where α and β are real numbers, then another root is $s_1 = -\alpha - j\beta$. Then,

$$\begin{aligned} k_1 e^{s_1 t} + k_2 e^{s_2 t} &= k_1 e^{-\alpha t + j\beta t} + k_2 e^{-\alpha t - j\beta t} = e^{-\alpha t}(k_1 e^{j\beta t} + k_2 e^{-j\beta t}) \\ &= e^{-\alpha t}(k_1 \cos\beta t + jk_1 \sin\beta t + k_2 \cos\beta t - jk_2 \sin\beta t) \\ &= e^{-\alpha t}[(k_1 + k_2)\cos\beta t + j(k_1 - k_2)\sin\beta t] \\ &= e^{-\alpha t}(k_3 \cos\beta t + k_4 \sin\beta t) = e^{-\alpha t} k_5 \cos(\beta t + \varphi) \end{aligned}$$
FOR *TWO COMPLEX CONJUGATE ROOTS* (H.28)

If (H.28) is to be a real function of time, the constants k_1 and k_2 must be complex conjugates. The other constants k_3, k_4, k_5, and the phase angle φ are real constants.

The forced response can be found by

a. *The Method of Undetermined Coefficients* or

b. *The Method of Variation of Parameters*

We will study the Method of Undetermined Coefficients first.

H.5 Using the Method of Undetermined Coefficients for the Forced Response

For simplicity, we will only consider ODEs of order 2. Higher order ODEs are discussed in differential equations textbooks.

Consider the non–homogeneous ODE

$$a\frac{d^2y}{dt^2} + b\frac{d}{dt}y + cy = f(x) \tag{H.29}$$

where a, b, and c are real constants.

We have learned that the total (complete) solution consists of the summation of the natural and forced responses.

For the natural response, if y_1 and y_2 are any two solutions of (H.29), the linear combination $y_3 = k_1y_1 + k_2y_2$, where k_1 and k_2 are arbitrary constants, is also a solution, that is, if we know the two solutions, we can obtain the most general solution by forming the linear combination of y_1 and y_2. To be certain that there exist no other solutions, we examine the Wronskian Determinant defined below.

$$W(y_1, y_2) \equiv \begin{bmatrix} y_1 & y_2 \\ \frac{d}{dx}y_1 & \frac{d}{dx}y_2 \end{bmatrix} = y_1\frac{d}{dx}y_2 - y_2\frac{d}{dx}y_1 \neq 0 \tag{H.30}$$

WRONSKIAN DETERMINANT

If (H.30) is true, we can be assured that all solutions of (H.29) are indeed the linear combination of y_1 and y_2.

The forced response is, in most circuit analysis problems, obtained by observation of the right side of the given ODE as it is illustrated by the examples that follow.

Example H.6

Find the total solution of the ODE

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = 0 \tag{H.31}$$

subject to the initial conditions $y(0) = 3$ and $y'(0) = 4$ where $y' = dy/dt$

Solution:

This is a homogeneous ODE and its total solution is just the natural response found from the characteristic equation $s^2 + 4s + 3 = 0$ whose roots are $s_1 = -1$ and $s_2 = -3$. The total response is:

$$y(t) = y_N(t) = k_1e^{-t} + k_2e^{-3t} \tag{H.32}$$

The constants k_1 and k_2 are evaluated from the given initial conditions. For this example,

$$y(0) = 3 = k_1e^0 + k_2e^0$$

or

$$k_1 + k_2 = 3 \tag{H.33}$$

Also,

$$y'(0) = 4 = \left.\frac{dy}{dt}\right|_{t=0} = -k_1e^{-t} - 3k_2e^{-3t}\Big|_{t=0}$$

or

$$-k_1 - 3k_2 = 4 \tag{H.34}$$

Simultaneous solution of (H.33) and (H.34) yields $k_1 = 6.5$ and $k_2 = -3.5$. By substitution into (H.32), we obtain

$$y(t) = y_N(t) = 6.5e^{-t} - 3.5e^{-3t} \tag{H.35}$$

Check with MATLAB:

```
y=dsolve('D2y+4*Dy+3*y=0', 'y(0)=3', 'Dy(0)=4')  % Must have Symbolic Math Tool box installed
```

```
y =
13/(2*exp(t)) - 7/(2*exp(3*t))
```

```
pretty(y)
```

```
  13 exp(-t)   7 exp(-3 t)
  ---------- - -----------
       2            2
```

The function $y = f(t)$ is shown in Figure H.1 plotted with the MATLAB command

```
ezplot(y,[0 10])
```

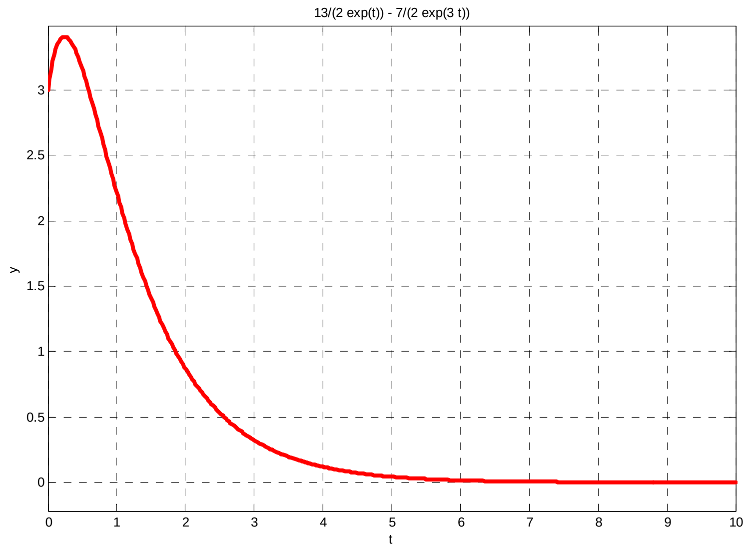


Figure H.1. Plot for the function $y = f(t)$ *of Example H.6.*

Example H.7

Find the total solution of the ODE

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = 3e^{-2t} \tag{H.36}$$

subject to the initial conditions $y(0) = 1$ and $y'(0) = -1$

Solution:

The left side of (H.36) is the same as that of Example H.6.Therefore,

$$y_N(t) = k_1e^{-t} + k_2e^{-3t} \tag{H.37}$$

(We must remember that the constants k_1 and k_2 must be evaluated from the total response).

To find the forced response, we assume a solution of the form

$$y_F = Ae^{-2t} \tag{H.38}$$

We can find out whether our assumption is correct by substituting (H.38) into the given ODE of (H.36). Then,

$$4Ae^{-2t} - 8Ae^{-2t} + 3Ae^{-2t} = 3e^{-2t} \tag{H.39}$$

from which $A = -3$ and the total solution is

$$y(t) = y_N + y_F = k_1e^{-t} + k_2e^{-3t} - 3e^{-2t} \quad (H.40)$$

The constants k_1 and k_2 are evaluated from the given initial conditions. For this example,

$$y(0) = 1 = k_1e^0 + k_2e^0 - 3e^0$$

or

$$k_1 + k_2 = 4 \quad (H.41)$$

Also,

$$y'(0) = -1 = \left.\frac{dy}{dt}\right|_{t=0} = -k_1e^{-t} - 3k_2e^{-3t} + 6e^{-2t}\Big|_{t=0}$$

or

$$-k_1 - 3k_2 = -7$$

Simultaneous solution of (H.41) and (H.42) yields $k_1 = 2.5$ and $k_2 = 1.5$. By substitution into (H.40), we obtain

$$y(t) = y_N + y_F = 2.5e^{-t} + 1.5e^{-3t} - 3e^{-2t} \quad (H.42)$$

Check with MATLAB:

```
% Must have Symbolic Math Tool box installed
y=dsolve('D2y+4*Dy+3*y=3*exp(–2*t)', 'y(0)=1', 'Dy(0)=–1')

y=
    5/(2*exp(t)) - 3/exp(2*t) + 3/(2*exp(3*t))

pretty(y)

  5 exp(-t)                   3 exp(-3 t)
  --------- - 3 exp(-2 t) + -----------
      2                          2

ezplot(y,[0 8])
```

The plot is shown in Figure H.2

Example H.8

Find the total solution of the ODE

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 0 \quad (H.43)$$

subject to the initial conditions $y(0) = -1$ and $y'(0) = 1$

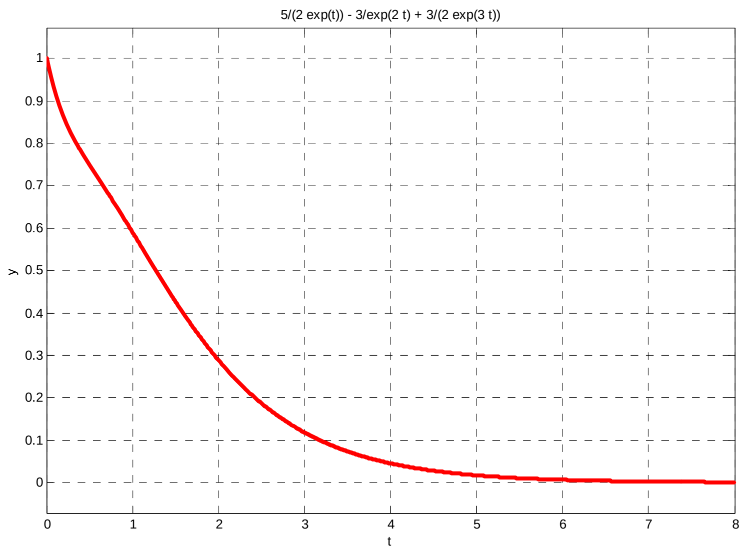


Figure H.2. Plot for the function $y = f(t)$ *of Example H.7*

Solution:

This is a homogeneous ODE and therefore its total solution is just the natural response found from the characteristic equation $s^2 + 6s + 9 = 0$ whose roots are $s_1 = s_2 = -3$ (repeated roots). Thus, the total response is

$$y(t) = y_N = k_1 e^{-3t} + k_2 t e^{-3t} \tag{H.44}$$

Next, we evaluate the constants k_1 and k_2 from the given initial conditions. For this example,

$$y(0) = -1 = k_1 e^0 + k_2 (0) e^0$$

or

$$k_1 = -1 \tag{H.45}$$

Also,

$$y'(0) = 1 = \left.\frac{dy}{dt}\right|_{t=0} = -3k_1 e^{-3t} + k_2 e^{-3t} - 3k_2 t e^{-3t}\Big|_{t=0}$$

or

$$-3k_1 + k_2 = 1 \tag{H.46}$$

From (H.45) and (H.46) we obtain $k_1 = -1$ and $k_2 = -2$. By substitution into (H.44),

$$y(t) = -e^{-3t} - 2te^{-3t} \tag{H.47}$$

Check with MATLAB:

```
% Must have Symbolic Math Tool box installed
y=dsolve('D2y+6*Dy+9*y=0', 'y(0)=–1', 'Dy(0)=1')
```

```
y =
- 1/exp(3*t) - (2*t)/exp(3*t)
```

```
ezplot(y,[0 4])
```

The plot is shown in Figure H.3.

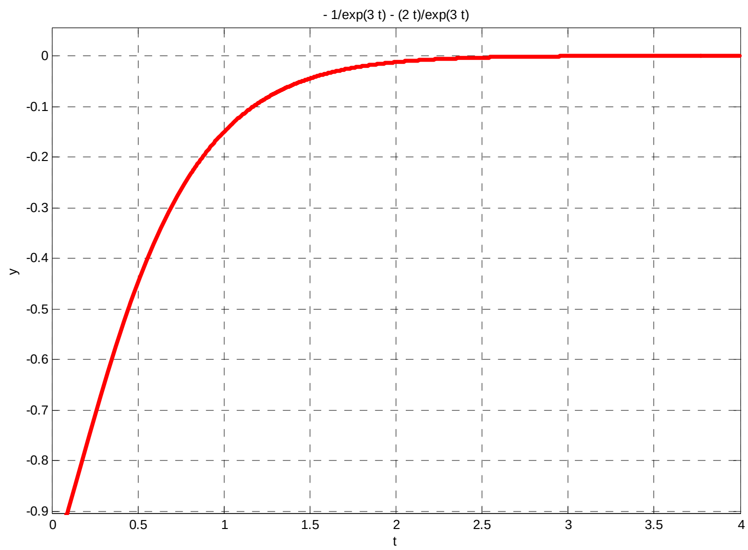


Figure H.3. Plot for the function $y = f(t)$ *of Example H.8.*

Example H.9

Find the total solution of the ODE

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 3e^{-2t} \qquad \text{(H.48)}$$

Solution:

No initial conditions are given; therefore, we will express the solution in terms of the constants k_1 and k_2. By inspection, the roots of the characteristic equation of (H.48) are $s_1 = -2$ and $s_2 = -3$ and thus the natural response has the form

$$y_N = k_1 e^{-2t} + k_2 e^{-3t} \qquad \text{(H.49)}$$

Next, we find the forced response by assuming a solution of the form

$$y_F = Ae^{-2t} \qquad \text{(H.50)}$$

We can find out whether our assumption is correct by substitution of (H.50) into the given ODE of (H.48). Then,

$$4Ae^{-2t} - 10Ae^{-2t} + 6Ae^{-2t} = 3e^{-2t} \tag{H.51}$$

but the sum of the three terms on the left side of (H.52) is zero whereas the right side can never be zero unless we let $t \to \infty$ and this produces a meaningless result.

The problem here is that the right side of the given ODE of (H.48) has the same form as one of the terms of the natural response of (H.49), namely the term $k_1 e^{-2t}$.

To work around this problem, we assume that the forced response has the form

$$y_F = Ate^{-2t} \tag{H.52}$$

that is, we multiply (H.50) by t in order to eliminate the duplication of terms in the total response. Then, by substitution of (H.52) into (H.48) and equating like terms, we find that $A = 3$. Therefore, the total response is

$$y(t) = y_N + y_F = k_1e^{-2t} + k_2e^{-3t} + 3te^{-2t} \tag{H.53}$$

Check with MATLAB:

```
% Must have Symbolic Math Tool box installed
y=dsolve('D2y+5*Dy+6*y=3*exp(–2*t)')
```

```
y =
-3*exp(-2*t)+3*t*exp(-2*t)+C1*exp(-3*t)+C2*exp(-2*t)
```

Example H.10

Find the total solution of the ODE

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 4\cos 5t \tag{H.54}$$

Solution:

No initial conditions are given; therefore, we will express solution in terms of the constants k_1 and k_2. We observe that the left side of (H.54) is the same of that of Example H.9. Therefore, the natural response is the same, that is, it has the form

$$y_N = k_1e^{-2t} + k_2e^{-3t} \tag{H.55}$$

Next, to find the forced response and we assume a solution of the form

$$y_F = A\cos 5t \tag{H.56}$$

We can find out whether our assumption is correct by substitution of the assumed solution of (H.56) into the given ODE of (H.55). Then,

$$-25A\cos 5t - 25A\sin 5t + 6A\cos 5t = -19A\cos 5t - 25A\sin 5t = 4\cos 5t$$

but this relation is invalid since by equating cosine and sine terms, we find that $A = -4/19$ and also $A = 0$. This inconsistency is a result of our failure to recognize that the derivatives of $A\cos 5t$ produce new terms of the form $B\sin 5t$ and these terms must be included in the forced response. Accordingly, we let

$$y_F = k_3\sin 5t + k_4\cos 5t \tag{H.57}$$

and by substitution into (H.54) we obtain

$$\begin{aligned}-25k_3\sin 5t - 25k_4\cos 5t + 25k_3\cos 5t - 25k_4\sin 5t \\ + 6k_3\sin 5t + 6k_4\cos 5t = 4\cos 5t\end{aligned}$$

Collecting like terms and equating sine and cosine terms, we obtain the following set of equations

$$\begin{aligned}19k_3 + 25k_4 &= 0 \\ 25k_3 - 19k_4 &= 4\end{aligned} \tag{H.58}$$

We use MATLAB to solve (H.58)

```
% Must have Symbolic Math Tool box installed
format rat;  [k3 k4]=solve(19*x+25*y, 25*x–19*y–4)
```

```
k3 =
 50/493
k4 =
 -38/493
```

Therefore, the total solution is

$$y(t) = y_N + y_F(t) = k_1e^{-2t} + k_2e^{-3t} + \frac{50}{493}\sin 5t + \frac{-38}{493}\cos 5t \tag{H.59}$$

Check with MATLAB:

```
% Must have Symbolic Math Tool box installed
y=dsolve('D2y+5*Dy+6*y=4*cos(5*t)'); y=simple(y)
```

```
y =
  -38/493*cos(5*t)+50/493*sin(5*t)+C1*exp(-3*t)+C2*exp(-2*t)
```

In most engineering problems the right side of the non–homogeneous ODE consists of elementary functions such as k (constant), x^n where n is a positive integer, e^{kx}, $\cos kx$, $\sin kx$, and linear combinations of these. Table H.1 summarizes the forms of the forced response for a second order ODE with constant coefficients.

TABLE H.1 Form of the forced response for 2nd order differential equations

Forced Response of the ODE $a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = f(t)$	
$\mathbf{f(t)}$	*Form of Forced Response* $\mathbf{y_F(t)}$
k (constant)	K (constant)
kt^n $(n =$ positive integer)	$K_0t^n + K_1t^{n-1} + \dots + K_{n-1}t + K_n$
ke^{rt} $(r =$real or complex)	Ke^{rt}
$k\cos\alpha t$ or $k\sin\alpha t$ $(\alpha =$constant)	$K_1\mathrm{coa}\alpha t + K_2\sin\alpha t$
$kt^ne^{rt}\cos\alpha t$ or $kt^ne^{rt}\sin\alpha t$	$(K_0t^n + K_1t^{n-1} + \dots + K_{n-1}t + K_n)e^{rt}\cos\alpha t$ $+ (K_0t^n + K_1t^{n-1} + \dots + K_{n-1}t + K_n)e^{rt}\sin\alpha t$

We must remember that if $f(t)$ is the sum of several terms, the most general form of the forced response $y_F(t)$ is the linear combination of these terms. Also, if a term in $y_F(t)$ is a duplicate of a term in the natural response $y_N(t)$, we must multiply $y_F(t)$ by the lowest power of t that will eliminate the duplication.

Example H.11

Find the total solution of the ODE

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = te^{-2t} - e^{-2t} \tag{H.60}$$

Solution:

No initial conditions are given; therefore we will express solution in terms of the constants k_1 and k_2. The roots of the characteristic equation are equal, that is, $s_1 = s_2 = -2$, and thus the natural response has the form

$$y_N = k_1e^{-2t} + k_2te^{-2t} \tag{H.61}$$

To find the forced response (particular solution), we refer to Table H.1 and from the last row we choose the term $kt^ne^{rt}\cos\alpha t$. This term with $n = 1$, $r = -2$, and $\alpha = 0$, reduces to kte^{-2t}.

Therefore the forced response will have the form

$$y_F = (k_3 t + k_4)e^{-2t} \quad \text{(H.62)}$$

But the terms e^{-2t} and te^{-2t} are also present in (H.61); therefore, we multiply (H.62) by t^2 to obtain a suitable form for the forced response which now is

$$y_F = (k_3 t^3 + k_4 t^2)e^{-2t} \quad \text{(H.63)}$$

Now, we need to evaluate the constants k_3 and k_4. This is done by substituting (H.63) into the given ODE of (H.60) and equating with the right side. We use MATLAB do the computations as shown below.

```
syms t k3 k4                       % Define symbolic variables
f0=(k3*t^3+k4*t^2)*exp(−2*t);  % Forced response (H.64)
f1=diff(f0); f1=simple(f1)         % Compute and simplify first derivative
```

```
f1 =
 -t*exp(-2*t)*(-3*k3*t-2*k4+2*k3*t^2+2*k4*t)
```

```
f2=diff(f0,2); f2=simple(f2)     % Compute and simplify second derivative
```

```
f2 =
 2*exp(-2*t)*(3*k3*t+k4-6*k3*t^2-4*k4*t+2*k3*t^3+2*k4*t^2)
```

```
f=f2+4*f1+4*f0; f=simple(f)% Form and simplify the left side of the given ODE
```

```
f = 2*(3*k3*t+k4)*exp(-2*t)
```

Finally, we equate `f` above with the right side of the given ODE, that is

$$2(3k_3 t + k_4)e^{-2t} = te^{-2t} - e^{-2t} \quad \text{(H.64)}$$

and we find $k_3 = 1/6$ and $k_4 = -1/2$. By substitution of these values into (H.64) and combining the forced response with the natural response, we obtain the total solution

$$y(t) = k_1 e^{-2t} + k_2 te^{-2t} + \frac{1}{6}t^3 e^{-2t} - \frac{1}{2}t^2 e^{-2t} \quad \text{(H.65)}$$

We verify this solution with MATLAB.

```
% Must have Symbolic Math Tool box installed
z=dsolve('D2y+4*Dy+4*y=t*exp(–2*t)–exp(–2*t)')
```

```
z =
1/6*exp(–2*t)*t^3–1/2*exp(–2*t)*t^2
+C1*exp(–2*t)+C2*t*exp(–2*t)
```

H.6 Using the Method of Variation of Parameters for the Forced Response

In certain non–homogeneous ODEs, the right side $f(t)$ cannot be determined by the method of undetermined coefficients. For these ODEs we must use the method of variation of parameters. This method will work with all linear equations including those with variable coefficients such as

$$\frac{d^2y}{dt^2} + \alpha(t)\frac{dy}{dt} + \beta(t)y = f(t) \tag{H.66}$$

provided that the general form of the natural response is known.

Our discussion will be restricted to second order ODEs with constant coefficients.

The method of variation of parameters replaces the constants k_1 and k_2 by two variables u_1 and u_2 that satisfy the following three relations:

$$\boxed{y = u_1 y_1 + u_2 y_2} \tag{H.67}$$

$$\boxed{\frac{du_1}{dt} y_1 + \frac{du_2}{dt} y_2 = 0} \tag{H.68}$$

$$\boxed{\frac{du_1}{dt} \cdot \frac{dy_1}{dt} + \frac{du_2}{dt} \cdot \frac{dy_2}{dt} = f(t)} \tag{H.69}$$

Simultaneous solution of (H.68) and (H.69) will yield the values of du_1/dt and du_2/dt; then, integration of these will produce u_1 and u_2, which when substituted into (H.67) will yield the total solution.

Example H.12

Find the total solution of

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = 12 \tag{H.70}$$

in terms of the constants k_1 and k_2 by the

a. method of undetermined coefficients

b. method of variation of parameters

Solution:

With either method, we must first find the natural response. The characteristic equation yields the roots $s_1 = -1$ and $s_2 = -3$. Therefore, the natural response is

$$y_N = k_1 e^{-t} + k_2 e^{-3t} \tag{H.71}$$

a. Using the method of undetermined coefficients we let $y_F = k_3$ (a constant). Then, by substitution into (H.70) we obtain $k_3 = 4$ and thus the total solution is

$$y(t) = y_N + y_F = k_1 e^{-t} + k_2 e^{-3t} + 4 \tag{H.72}$$

b. With the method of variation of parameters we start with the natural response found above as (H.71) and we let the solutions y_1 and y_2 be represented as

$$y_1 = e^{-t} \text{ and } y_2 = e^{-3t} \tag{H.73}$$

Then by (H.67), the total solution is

$$y = u_1 y_1 + u_2 y_2$$

or

$$y = u_1 e^{-t} + u_2 e^{-3t} \tag{H.74}$$

Also, from (H.68),

$$\frac{du_1}{dt} y_1 + \frac{du_2}{dt} y_2 = 0$$

or

$$\frac{du_1}{dt} e^{-t} + \frac{du_2}{dt} e^{-3t} = 0 \tag{H.75}$$

and from (H.69),

$$\frac{du_1}{dt} \cdot \frac{dy_1}{dt} + \frac{du_2}{dt} \cdot \frac{dy_2}{dt} = f(t)$$

or

$$\frac{du_1}{dt}(-e^{-t}) + \frac{du_2}{dt}(-3e^{-3t}) = 12 \tag{H.76}$$

Next, we find du_1/dt and du_2/dt by Cramer's rule as follows:

$$\frac{du_1}{dt} = \frac{\begin{vmatrix} 0 & e^{-3t} \\ 12 & -3e^{-3t} \end{vmatrix}}{\begin{vmatrix} e^{-t} & e^{-3t} \\ -e^{-t} & -3e^{-3t} \end{vmatrix}} = \frac{-12e^{-3t}}{-3e^{-4t} + e^{-4t}} = \frac{-12e^{-3t}}{-2e^{-4t}} = 6e^{t} \tag{H.77}$$

and

$$\frac{du_2}{dt} = \frac{\begin{vmatrix} e^{-t} & 0 \\ -e^{-t} & 12 \end{vmatrix}}{-2e^{-4t}} = \frac{12e^{-t}}{-2e^{-4t}} = -6e^{3t} \tag{H.78}$$

Now, integration of (H.77) and (H.78) and substitution into (H.75) yields

$$u_1 = 6\int e^t dt = 6e^t + k_1 \qquad u_2 = -6\int e^{3t} dt = -2e^{3t} + k_2 \tag{H.79}$$

$$\begin{aligned} y &= u_1 e^{-t} + u_2 e^{-3t} \\ &= ((6e^t + k_1)e^{-t} + (-2e^{3t} + k_2)e^{-3t}) \\ &= (6 + k_1 e^{-t} - 2 + k_2 e^{-3t}) \\ &= (k_1 e^{-t} + k_2 e^{-3t} + 4) \end{aligned} \tag{H.80}$$

We observe that the last expression in (H.80) is the same as (H.72) of part (a).

Check with MATLAB:

```
% Must have Symbolic Math Tool box installed
y=dsolve('D2y+4*Dy+3*y=12')
```

```
y =
(4*exp(t)+C1*exp(-3*t)*exp(t)+C2)/exp(t)
```

Example H.13

Find the total solution of

$$\frac{d^2y}{dt^2} + 4y = \tan 2t \tag{H.81}$$

in terms of the constants k_1 and k_2 by any method.

Solution:

This ODE cannot be solved by the method of undetermined coefficients; therefore, we will use the method of variation of parameters.

The characteristic equation is $s^2 + 4 = 0$ from which $s = \pm j2$ and thus the natural response is

$$y_N = k_1 e^{j2t} + k_2 e^{-j2t} \tag{H.82}$$

We let

$$y_1 = \cos 2t \ \text{ and } \ y_2 = \sin 2t \tag{H.83}$$

Then, by (H.67) the solution is

$$y = u_1 y_1 + u_2 y_2 = u_1 \cos 2t + u_2 \sin 2t \tag{H.84}$$

Also, from (H.68),

$$\frac{du_1}{dt} y_1 + \frac{du_2}{dt} y_2 = 0$$

or

$$\frac{du_1}{dt} \cos 2t + \frac{du_2}{dt} \sin 2t = 0 \tag{H.85}$$

and from (H.69),

$$\frac{du_1}{dt}\cdot\frac{dy_1}{dt}+\frac{du_2}{dt}\cdot\frac{dy_2}{dt} = f(t) = \frac{du_1}{dt}(-2\sin 2t)+\frac{du_2}{dt}(2\cos 2t) = \tan 2t \tag{H.86}$$

Next, we find du_1/dt and du_2/dt by Cramer's rule as follows:

$$\frac{du_1}{dt} = \frac{\begin{vmatrix} 0 & \sin 2t \\ \tan 2t & 2\cos 2t \end{vmatrix}}{\begin{vmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{vmatrix}} = \frac{-\dfrac{\sin^2 2t}{\cos 2t}}{2\cos^2 2t + 2\sin^2 2t} = \frac{-\sin^2 2t}{2\cos 2t} \tag{H.87}$$

and

$$\frac{du_2}{dt} = \frac{\begin{vmatrix} \cos 2t & 0 \\ -2\sin 2t & \tan 2t \end{vmatrix}}{2} = \frac{\sin 2t}{2} \tag{H.88}$$

Now, integration of (H.87) and (H.88) and substitution into (H.84) yields

$$u_1 = -\frac{1}{2}\int\frac{\sin^2 2t}{\cos 2t}dt = \frac{\sin 2t}{4} - \frac{1}{4}\ln(\sec 2t + \tan 2t) + k_1 \tag{H.89}$$

$$u_2 = \frac{1}{2}\int \sin 2t dt = -\frac{\cos 2t}{4} + k_2 \tag{H.90}$$

$$\begin{aligned} y = u_1y_1 + u_2y_2 &= \frac{\sin 2t\cos 2t}{4} - \frac{1}{4}\cos 2t\ln(\sec 2t + \tan 2t) + k_1\cos 2t - \frac{\sin 2t\cos 2t}{4} + k_2\sin 2t \\ &= -\frac{1}{4}\cos 2t\ln(\sec 2t + \tan 2t) + k_1\cos 2t + k_2\sin 2t \end{aligned} \tag{H.91}$$

Check with MATLAB:

```
% Must have Symbolic Math Tool box installed
y=dsolve('D2y+4*y=tan(2*t)')

y =
-1/4*cos(2*t)*log((1+sin(2*t))/cos(2*t))+C1*cos(2*t)+C2*sin(2*t)
```

H.7 Exercises

Solve the following ODEs by any method.

1.

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = t - 1$$

Answer: $y = k_1 e^{-t} + k_2 e^{-3t} + \frac{1}{3}t - \frac{7}{9}$

2.

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = 4e^{-t}$$

Answer: $y = k_1 e^{-t} + k_2 e^{-3t} + 2te^{-t}$

3.

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = \cos^2 t$$ Hint: Use $\cos^2 t = \frac{1}{2}(\cos 2t + 1)$

Answer: $y = k_1 e^{-t} + k_2 te^{-t} + \frac{1}{2} - \frac{3\cos 2t - 4\sin 2t}{50}$

4.

$$\frac{d^2y}{dt^2} + y = \sec t$$

Answer: $y = k_1 \cos t + k_2 \sin t + t\sin t + \cos t(\ln \cos t)$

Appendix I

Constructing Semilog Paper with Excel® and with MATLAB®

This appendix contains instructions for constructing semilog plots with the Microsoft Excel spreadsheet. Semilog, short for semilogarithmic, paper is graph paper having one logarithmic and one linear scale. It is used in many scientific and engineering applications including frequency response illustrations and Bode Plots.

I.1 Instructions for Constructing Semilog Paper with Excel

Figure I.1 shows the Excel spreadsheet workspace and identifies the different parts of the Excel window when we first start Excel.

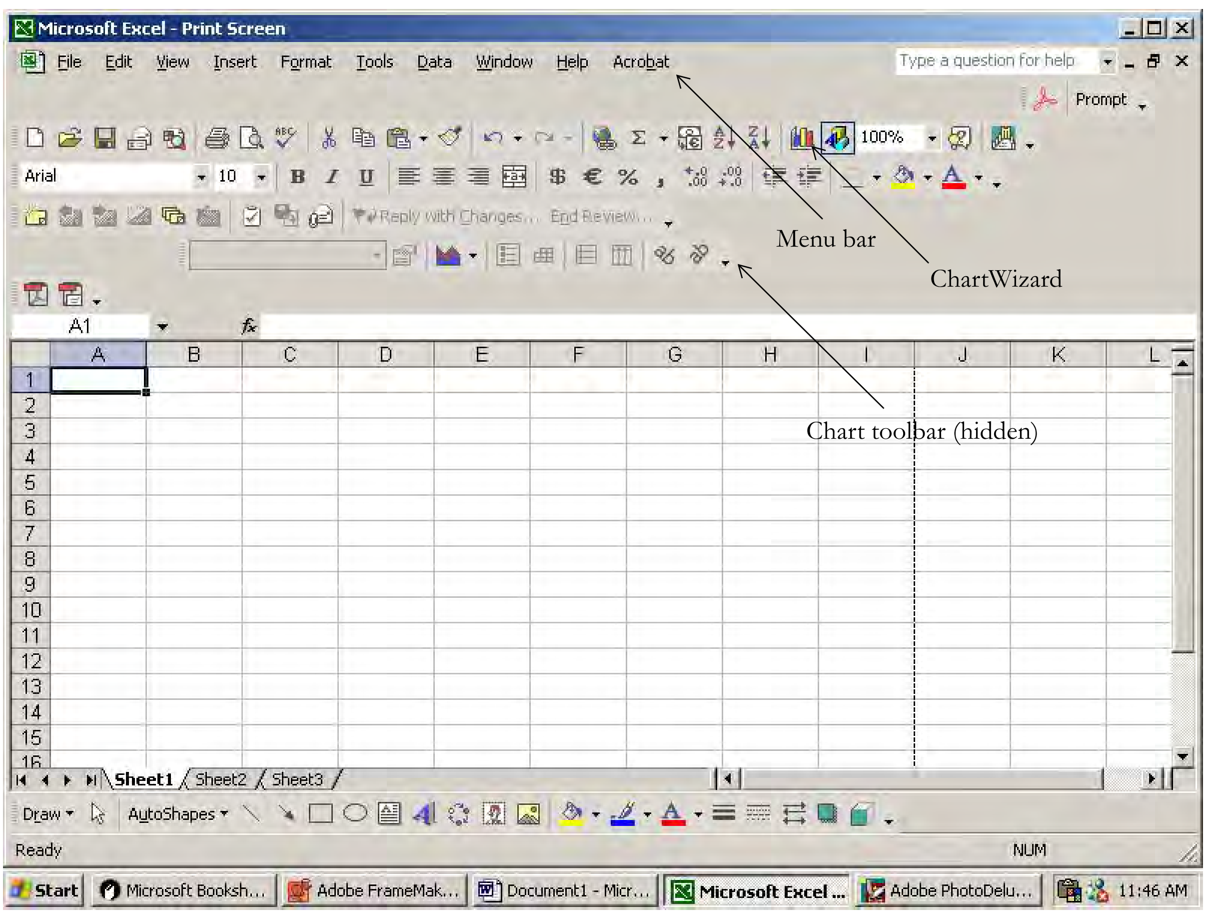


Figure I.1. The Excel Spreadsheet Workspace

Figure I.2 shows that whenever a chart is selected, as shown by the visible handles around the selected chart, the Chart drop menu appears on the Menu bar and that the Chart toolbar now is visible. We can now use the Chart Objects Edit Box and Format Chart Area tools to edit our chart.

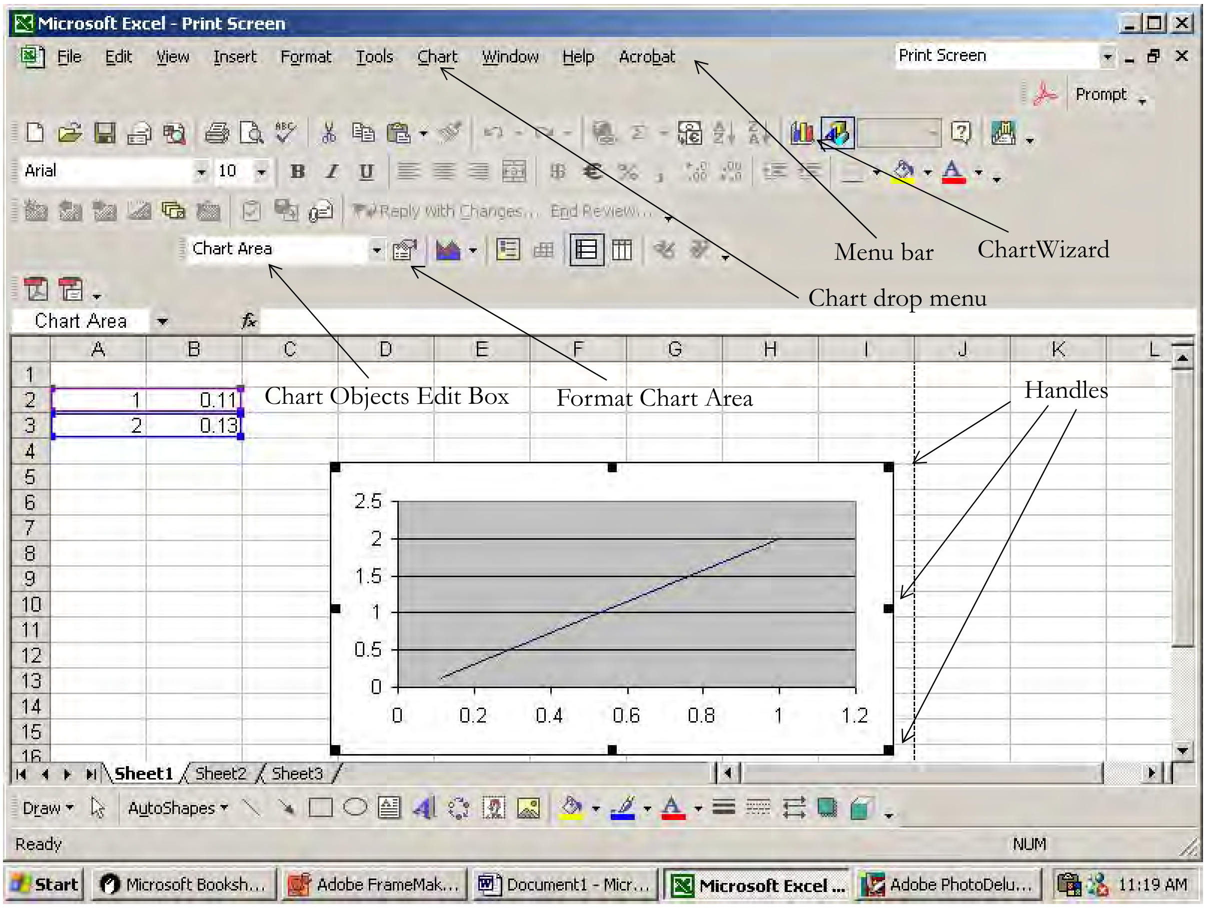


Figure I.2. The Excel Spreadsheet with Chart selected

1. Begin with a blank spreadsheet as shown in Figure I.1.
2. Click **Chart Wizard**.
3. Click **X–Y (Scatter)** Chart type under the Standard Types tab on the *Chart Wizard* menu.
4. The Chart sub–type shows five different sub–types. Click the upper right (the one showing two continuous curves without square points.)
5. Click **Next**, **Series** tab, **Add**, **Next**.
6. Click **Gridlines** tab and click all square boxes under **Value X–axis** and **Value Y–axis** to place check marks on **Major** and **Minor** gridlines.

7. Click **Next**, **Finish**, click **Series 1** box to select it, and press the **Delete** key on the keyboard to delete it.

8. The plot area normally appears in gray color. To change it to white, first make sure that the chart is selected, that is, the handles (black squares) around the plot are visible. Point the mouse on the **Chart Objects Edit Box** tool (refer to Figure I.2), scroll down, click **Plot Area**, then click **Format Plot Area** (shown as **Format Chart Area** tool in Figure I.2).

9. The Area section on the **Patterns** tab shows several squares with different colors. Click the white square, fifth row, right−most column, and click **OK** to return to the Chart display. You will observe that the Plot Area has now a white background.

10. Click anywhere near the x−axis (lowest horizontal line on the plot) and observe that the **Chart Objects Edit** box now displays **Value (X)** axis. Click the **Format Chart Area** tool which now displays **Format Axis**, click the *Scale* tab and make the following entries:

Minimum: 1 **Maximum**: 100000 **Major Unit**: 10 **Minor Unit**: 10

Make sure that the squares to the left of these values are not checked.

Click **Logarithmic scale** to place a check mark, and click **OK** to return to the plot.

11. Click anywhere near the y−axis (left−most vertical line on the plot) and observe that the **Chart Objects Edit** box now displays **Value (Y)** axis. Click the **Format Chart Area** tool which now displays **Format Axis**, click he **Scale** tab and make the following entries:

Minimum: −80 **Maximum**: 80 **Major Unit**: 20 **Minor Unit**: 20

Make sure that the squares to the left of these values are not checked. Also, make sure that the Logarithmic scale is not checked. Click **OK** to return to the plot.

12. You will observe that the x−axis values appear at the middle of the plot. To move them below the plot, click **Format Chart Area** tool, click **Patterns** tab, click **Tick** mark labels (lower right section), and click **OK** to return to the plot area.

13. To expand the plot so that it will look more useful and presentable, make sure that the chart is selected (the handles are visible). This is done by clicking anywhere in the chart area. Bring the mouse close to the lower center handle until a bidirectional arrow appears and stretch downwards. Repeat with the right center handle to stretch the plot to the right. Alternately, you may bring the mouse near the lower right handle and stretch the plot diagonally.

14. You may wish to display the x−axis values in exponential (scientific) format. To do that, click anywhere near the x−axis (zero point), and observe that the **Chart Objects Edit** box now displays **Value (X)** axis. Click the **Format Chart Area** tool which now displays **Format Axis**, click the **Number** tab and under **Category** click **Scientific** with zero decimal places.

15. If you wish to enter title and labels for the x− and y−axes, with the chart selected, click **Chart** (on the Menu bar), click **Chart Options**, and on the **Titles** tab enter the Title and the x− and y−axis labels. Remember that the Chart drop menu on the Menu bar and the Chart toolbar are hidden when the chart is deselected.

16. With the values used for this example, your semilog plot should look like the one in Figure I.3, and it can be printed for creating Bode plots.

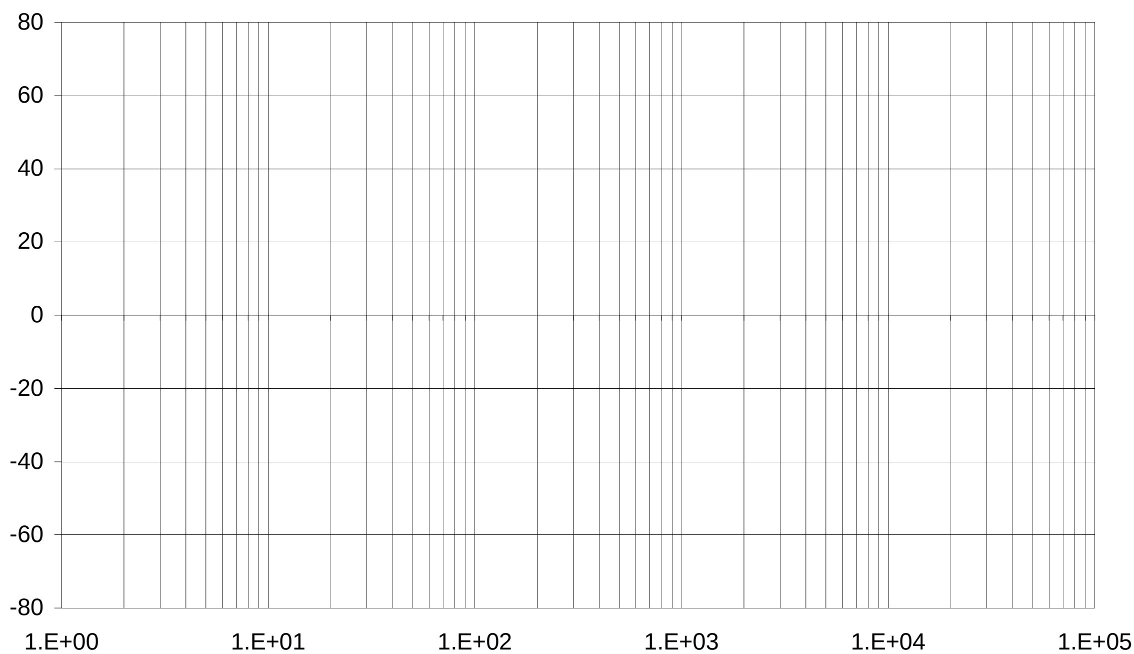


Figure I.3. Semilog paper created with Excel

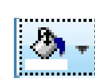
I.2 Instructions for Constructing Semilog Paper with MATLAB

It is much easier to construct semilog paper with MATLAB. The procedure is as follows:

1. Begin with the MATLAB script below.

```
x=linspace(1,10^6,7); y=linspace(-40,90,7); semilogx(x,y);...
grid; xlabel('Frequency (log scale)'); ylabel('Gain (linear scale)')
```

With this script, MATLAB creates the plot shown in Figure I.4.

2. To change the background from gray to white, scroll down the **Figure Color** icon  and select the white (blank) square by clicking it.

3. To erase the unwanted line segment, click it, and now the plot appears as shown in Figure I.5.

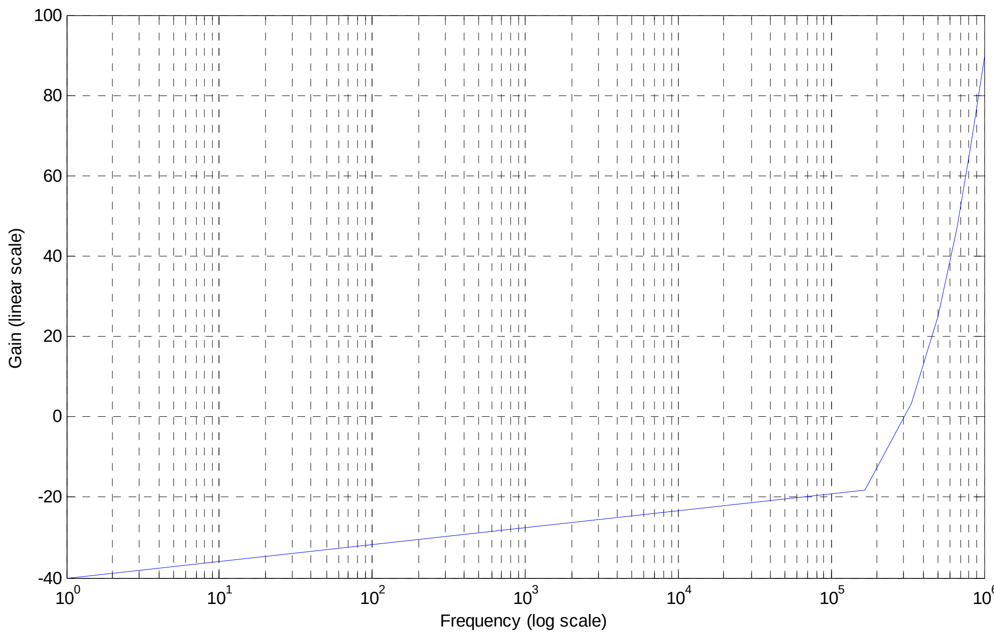


Figure I.4. MATLAB plot generated with the script above

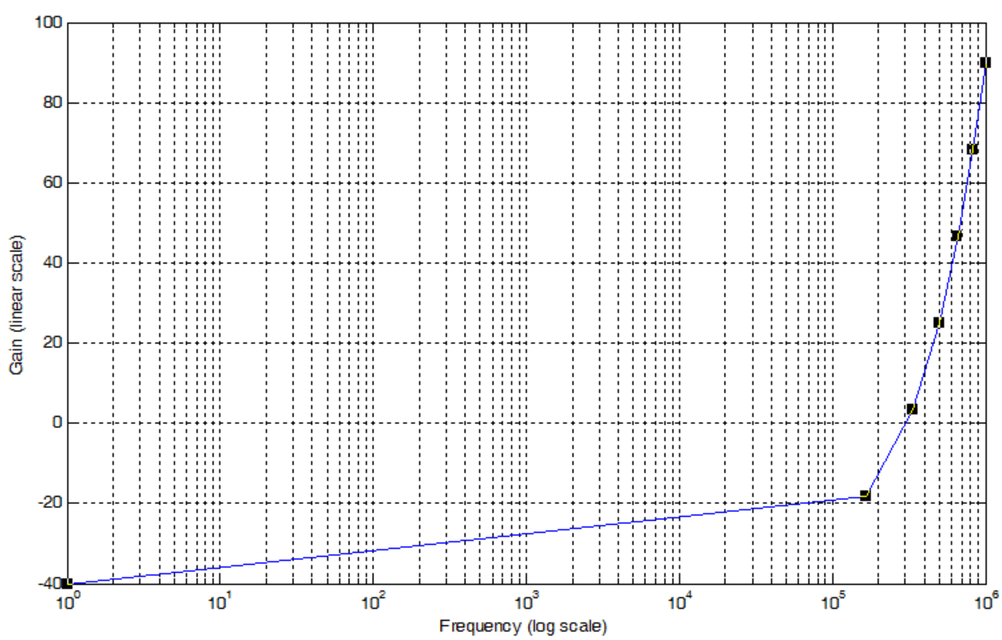


Figure I.5. Selecting the unwanted line segment to erase it

4. Change the Line parameter shown in Figure 1.6 to no line*. The plot now appears as shown in Figure 1.7, and can be printed for use with Bode plots.

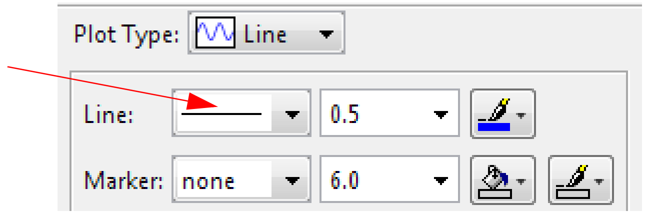


Figure I.6. Changing line to no line

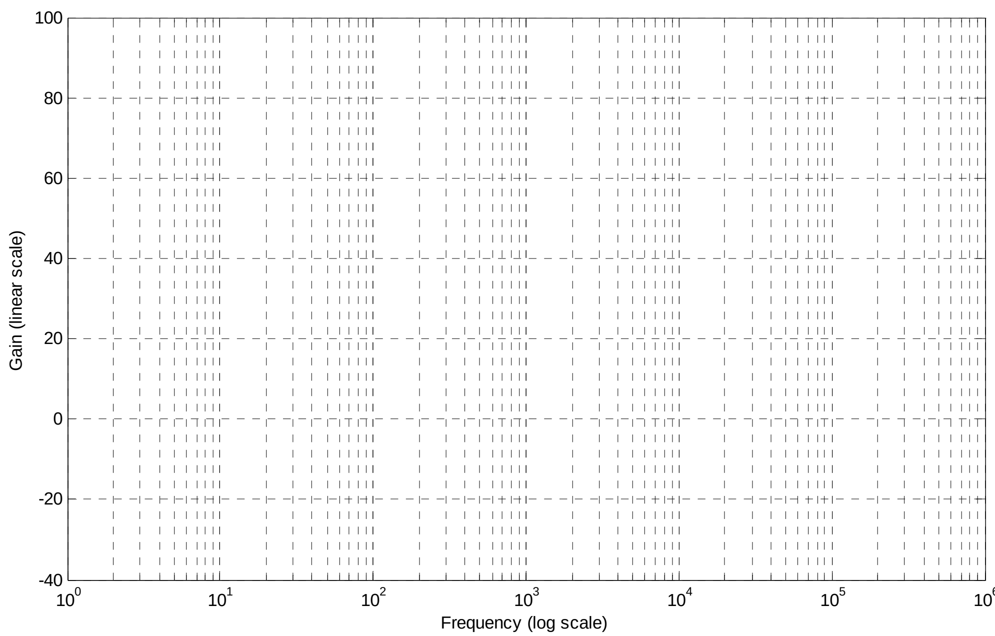


Figure I.7. Semilog paper created with MATLAB

*. *The unwanted line segment can also be erased with the Delete key.*

References and Suggestions for Further Study

A. The following publications by The MathWorks, are highly recommended for further study. They are available from The MathWorks, 3 Apple Hill Drive, Natick, MA, 01760, www.mathworks.com.

1. *Getting Started with MATLAB®*
2. *Using MATLAB®*
3. *Using MATLAB® Graphics*
4. *Using Simulink®*
5. *SimPowerSystems® for Use with Simulink®*
6. *Fixed–Point Toolbox*
7. *Simulink® Fixed–Point*
8. *Real–Time Workshop*
9. *Signal Processing Toolbox*

10. *Getting Started with Signal Processing Blockset*

10. *Signal Processing Blockset*

11. *Control System Toolbox*

12. *Stateflow®*

B. Other references indicated in text pages and footnotes throughout this text, are listed below.

1. *Mathematics for Business, Science, and Technology*, ISBN 978–1–934404–01–0
2. *Numerical Analysis Using MATLAB® and Excel®*, ISBN 978–1–934404–03–4
3. *Circuit Analysis I with MATLAB®* Computing and Simulink / SimPoweStems Modeling, ISBN 978–1–934404–17–1
4. *Signals and Systems with MATLAB® Computing and Simulink® Modeling*, ISBN 978–1–934404–11–9
5. *Electronic Devices and Amplifier Circuits with MATLAB® Applications*, ISBN 978–1–934404–13–3
6. *Digital Circuit Analysis and Design with Simulink Modeling and Introduction to CPLDs and FPGAs*, ISBN 978–1–934404–05–8

7. *Introduction to Simulink® with Engineering Applications*, ISBN 978–1–934404–09–6

8. Introduction to Stateflow® with Applications, ISBN 978–1–934404–07–2

9. *Reference Data for Radio Engineers*, ISBN 0–672–21218–8, Howard W. Sams & Co.

10. *Electronic Engineers' Handbook*, ISBN 0–07–020981–2, McGraw–Hill

Index

N

O

P

Q

R

S

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