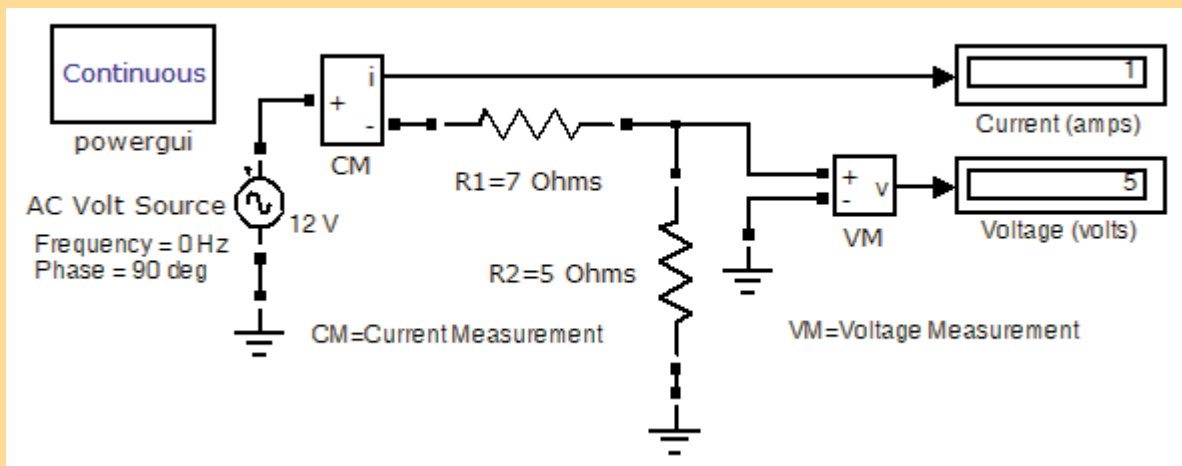


# Circuit Analysis I

with MATLAB® Computing and  
Simulink®/SimPowerSystems® Modeling

Steven T. Karris



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# Preface

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This text is an introduction to the basic principles of electrical engineering. It is the outgrowth of lecture notes prepared by this author while employed by the electrical engineering and computer engineering departments as adjunct instructor at various colleges and universities. Many of the examples and problems are based on the author's industrial experience. The text is an expansion of our previous publication, *Circuit Analysis I with MATLAB® Applications*, ISBN 978-0-9709511-2-0, and this text, in addition to MATLAB scripts for problem solution, includes several Simulink® and SimPowerSystems® models. The pages where these models appear are indicated in the Table of Contents.

The book is intended for students of college grade, both community colleges and universities. It presumes knowledge of first year differential and integral calculus and physics. While some knowledge of differential equations would be helpful, it is not absolutely necessary. Chapters 9 and 10 include step-by-step procedures for the solutions of simple differential equations used in the derivation of the natural and forced responses. Appendices D and E provide a thorough review of complex numbers and matrices respectively.

In addition to several problems provided at the end of each chapter, this text includes multiple-choice questions to test and enhance the reader's knowledge of this subject. Moreover, the answers to these questions and detailed solutions of all problems are provided at the end of each chapter. The rationale is to encourage the reader to solve all problems and check his effort for correct solutions and appropriate steps in obtaining the correct solution. And since this text was written to serve as a self-study, primary, or supplementary textbook, it provides the reader with a resource to test the reader's knowledge.

A previous knowledge of MATLAB® would be very helpful. However the material of this text can be learned without MATLAB, Simulink and SimPowerSystems. This author highly recommends that the reader studies this material in conjunction with the inexpensive Student Versions of The MathWorks™ Inc., the developers of these outstanding products, available from:

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Natick, MA, 01760  
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[www.mathworks.com](http://www.mathworks.com)  
[info@mathworks.com](mailto:info@mathworks.com).

Appendix A of this text provides a practical introduction to MATLAB, Appendix B is an introduction to Simulink, and Appendix C is an introduction to SimPowerSystems. These packages will be invaluable in later studies such as the design of analog and digital filters.

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## Preface

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Like any other new book, this text may contain some grammar and typographical errors; accordingly, all feedback for errors, advice and comments will be most welcomed and greatly appreciated.

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# Chapter 1

---

## Basic Concepts and Definitions

This chapter begins with the basic definitions in electric circuit analysis. It introduces the concepts and conventions used in introductory circuit analysis, the unit and quantities used in circuit analysis, and includes several practical examples to illustrate these concepts. Throughout this text, a left justified horizontal bar will denote the beginning of an example, and a right justified horizontal bar will denote the end of the example. These bars will not be shown whenever an example begins at the top of a page or at the bottom of a page. Also, when one example follows immediately after a previous example, the right justified bar will be omitted.

### 1.1 The Coulomb

Two identically charged (both positive or both negative) particles possess a charge of one *coulomb* when being separated by one meter in a vacuum, repel each other with a force of  $10^{-7}c^2$  newton where  $c$  = velocity of light  $\approx 3 \times 10^8$  m/s. The definition of coulomb is illustrated in Figure 1.1.

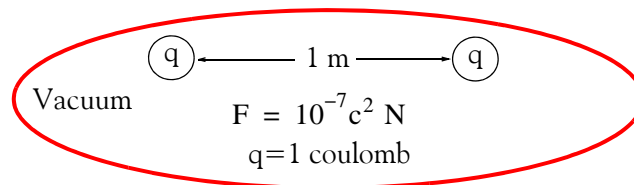


Figure 1.1. Definition of the coulomb

The coulomb, abbreviated as C, is the fundamental unit of charge. In terms of this unit, the charge of an electron is  $1.6 \times 10^{-19}$  C and one negative coulomb is equal to  $6.24 \times 10^{18}$  electrons. Charge, positive or negative, is denoted by the letter  $q$  or  $Q$ .

### 1.2 Electric Current and Ampere

Electric *current*  $i$  at a specified point and flowing in a specified direction is defined as the instantaneous rate at which net positive charge is moving past this point in that specified direction, that is,

$$i = \frac{dq}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\Delta q}{\Delta t} \quad (1.1)$$

---

## Chapter 1 Basic Concepts and Definitions

---

The unit of current is the *ampere* abbreviated as A and corresponds to charge  $q$  moving at the rate of one coulomb per second. In other words,

$$\boxed{1 \text{ ampere} = \frac{1 \text{ coulomb}}{1 \text{ second}}} \quad (1.2)$$

**Note:** Although it is known that current flow results from electron motion, it is customary to think of current as the motion of positive charge; this is known as *conventional current flow*.

To find an expression of the charge  $q$  in terms of the current  $i$ , let us consider the charge  $q$  transferred from some reference time  $t_0$  to some future time  $t$ . Then, since

$$i = \frac{dq}{dt}$$

the charge  $q$  is

$$q|_{t_0}^t = \int_{t_0}^t i dt$$

or

$$q(t) - q(t_0) = \int_{t_0}^t i dt$$

or

$$\boxed{q(t) = \int_{t_0}^t i dt + q(t_0)} \quad (1.3)$$

---

### Example 1.1

For the waveform of current  $i$  shown in Figure 1.2, compute the total charge  $q$  transferred between

- $t = 0$  and  $t = 3$  s
- $t = 0$  and  $t = 9$  s

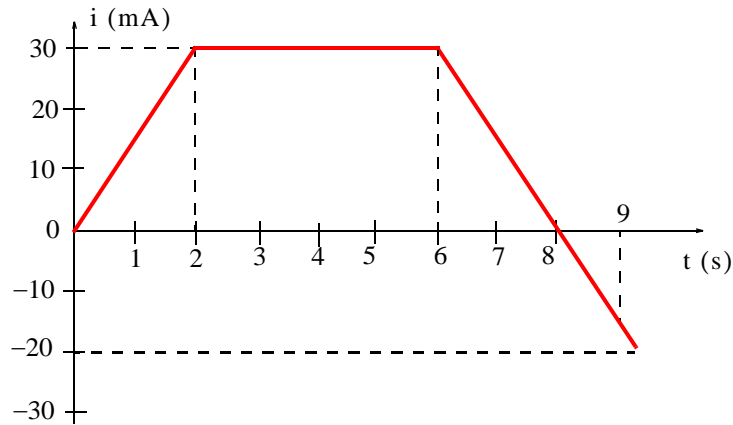


Figure 1.2. Waveform for Example 1.1

**Solution:**

We know that

$$q|_{t=0}^t = \int_0^t i dt = \text{Area}|_0^t$$

Then, by calculating the areas, we find that:

a. For  $0 < t < 2$  s, area =  $\frac{1}{2} \times (2 \times 30 \text{ mA}) = 30 \text{ mC}$

For  $2 < t < 6$  s, area =  $4 \times 30 = 120 \text{ mC}$

Therefore, for  $0 < t < 6$  s, total charge = total area =  $30 \text{ mC} + 120 \text{ mC} = 150 \text{ mC}$ .

b. For  $0 < t < 2$  s, area =  $\frac{1}{2} \times (2 \times 30 \text{ mA}) = 30 \text{ mC}$

For  $2 < t < 6$  s, area =  $4 \times 30 = 120 \text{ mC}$

For  $6 < t < 8$  s, area =  $\frac{1}{2} \times (2 \times 30 \text{ mA}) = 30 \text{ mC}$

For  $8 < t < 9$  s, we observe that the slope of the straight line for  $t > 6$  s is  $-30 \text{ mA} / 2 \text{ s}$ , or  $-15 \text{ mA} / \text{s}$ . Then, for  $8 < t < 9$  s, area =  $\frac{1}{2} \times \{1 \times (-15)\} = -7.5 \text{ mC}$ .

Therefore, for  $0 < t < 9$  s, total charge = total area =  $30 + 120 + 30 - 7.5 = 172.5 \text{ mC}$ .

**Convention:** We denote the current  $i$  by placing an arrow with the numerical value of the current next to the device in which the current flows. For example, the designation shown in Figure 1.3 indicates either a current of 2 A is flowing from left to right, or that a current of -2 A is moving from right to left.

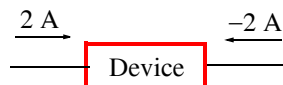


Figure 1.3. Direction of conventional current flow

**Caution:** The arrow may or may not indicate the *actual* conventional current flow. We will see later in Chapters 2 and 3 that in some circuits (to be defined shortly), the actual direction of the current cannot be determined by inspection. In such a case, we assume a direction with an arrow for said current  $i$ ; then, if the current with the assumed direction turns out to be negative, we conclude that the actual direction of the current flow is opposite to the direction of the arrow. Obviously, reversing the direction reverses the algebraic sign of the current as shown in Figure 1.3.

In the case of time-varying currents which change direction from time-to-time, it is convenient to think or consider the instantaneous current, that is, the direction of the current which flows at some particular instant. As before, we assume a direction by placing an arrow next to the device in which the current flows, and if a negative value for the current  $i$  is obtained, we conclude that the actual direction is opposite of that of the arrow.

### 1.3 Two Terminal Devices

In this text we will only consider two-terminal devices. In a two-terminal device the current entering one terminal is the same as the current leaving the other terminal\* as shown in Figure 1.4.

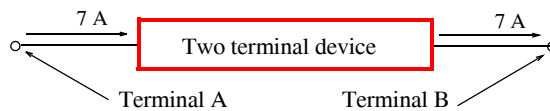


Figure 1.4. Current entering and leaving a two-terminal device

Let us assume that a constant value current (commonly known as *Direct Current* and abbreviated as DC) enters terminal A and leaves the device through terminal B in Figure 1.4. The passage of current (or charge) through the device requires some expenditure of energy, and thus we say that a *potential difference* or *voltage* exists “across” the device. This voltage across the terminals of the device is a measure of the work required to move the current (or charge) through the device.

---

#### Example 1.2

In a two-terminal device, a current  $i(t) = 20 \cos 100\pi t$  mA enters the left (first) terminal.

- What is the amount of current which enters that terminal in the time interval  $-10 \leq t \leq 20$  ms?
- What is the current at  $t = 40$  ms?
- What is the charge  $q$  at  $t = 5$  ms given that  $q(0) = 0$ ?

---

\* We will see in Chapter 5 that a two terminal device known as capacitor is capable of storing energy.

**Solution:**

a.

$$i|_{t_0}^t = 20 \cos 100\pi t \Big|_{-10 \times 10^{-3}}^{20 \times 10^{-3}} = 20 \cos 100\pi(20 \times 10^{-3}) - 20 \cos 100\pi(-10 \times 10^{-3})$$

$$= 20 \cos 2\pi - 20 \cos(-\pi) = 40 \text{ mA}$$

b.

$$i|_{t=0.4 \text{ ms}} = 20 \cos 100\pi t|_{t=0.4 \text{ ms}} = 20 \cos 40\pi = 20 \text{ mA}$$

c.

$$q(t) = \int_0^{5 \times 10^{-3}} i dt + q(0) = \int_0^{5 \times 10^{-3}} 20 \cos 100\pi t dt + 0$$

$$= \frac{0.2}{\pi} \sin 100\pi t \Big|_0^{5 \times 10^{-3}} = \frac{0.2}{\pi} \sin \frac{\pi}{2} - 0 = \frac{0.2}{\pi} \text{ C}$$

## 1.4 Voltage (Potential Difference)

The voltage (potential difference) across a two-terminal device is defined as the work required to move a positive charge of one coulomb from one terminal of the device to the other terminal.

The unit of voltage is the *volt* (abbreviated as V or v) and it is defined as

$$1 \text{ volt} = \frac{1 \text{ joule}}{1 \text{ coulomb}} \tag{1.4}$$

**Convention:** We denote the voltage  $v$  by a plus (+) minus (-) pair. For example, in Figure 1.5, we say that terminal A is 10 V positive with respect to terminal B or there is a potential difference of 10 V between points A and B. We can also say that there is a *voltage drop* of 10 V in going from point A to point B. Alternately, we can say that there is a *voltage rise* of 10 V in going from B to A.

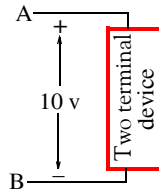


Figure 1.5. Illustration of voltage polarity for a two-terminal device

**Caution:** The (+) and (-) pair may or may not indicate the actual voltage drop or voltage rise. As in the case with the current, in some circuits the actual polarity cannot be determined by



inspection. In such a case, again we assume a voltage reference polarity for the voltage; if this reference polarity turns out to be negative, this means that the potential at the (+) sign terminal is at a lower potential than the potential at the (-) sign terminal.

In the case of time-varying voltages which change (+) and (-) polarity from time-to-time, it is convenient to think the *instantaneous voltage*, that is, the voltage reference polarity at some particular instance. As before, we assume a voltage reference polarity by placing (+) and (-) polarity signs at the terminals of the device, and if a negative value of the voltage is obtained, we conclude that the actual polarity is opposite to that of the assumed reference polarity. We must remember that reversing the reference polarity reverses the algebraic sign of the voltage as shown in Figure 1.6.

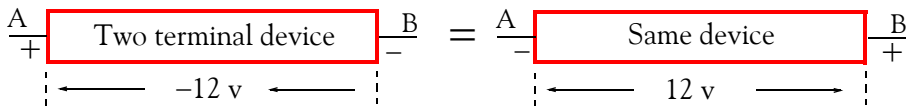


Figure 1.6. Alternate ways of denoting voltage polarity in a two-terminal device

---

### Example 1.3

The  $i-v$  (current-voltage) relation of a non-linear electrical device is given by

$$i(t) = 0.1(e^{0.2\sin 3t} - 1) \quad (10.5)$$

- Use MATLAB®\* to sketch this function for the interval  $0 \leq t \leq 10$  s
- Use the MATLAB **quad** function to find the charge at  $t = 5$  s given that  $q(0) = 0$

#### Solution:

- We use the following script to sketch  $i(t)$ .

```
t=0: 0.1: 10; it=0.1.*(exp(0.2.*sin(3.*t))-1);  
plot(t,it), grid, xlabel('time in sec.'), ylabel('current in amp.')
```

The plot for  $i(t)$  is shown in Figure 1.7.

---

\* MATLAB® and Simulink® are registered marks of The MathWorks,™ Inc., 3 Apple Hill Drive, Natick, MA, 01760, [www.mathworks.com](http://www.mathworks.com). An introduction to MATLAB is given in Appendix A, and an introduction to Simulink is given in Appendix B. Simulink operates in the MATLAB environment. The SimPowerSystems® is another product of The MathWorks and operates in the Simulink environment.

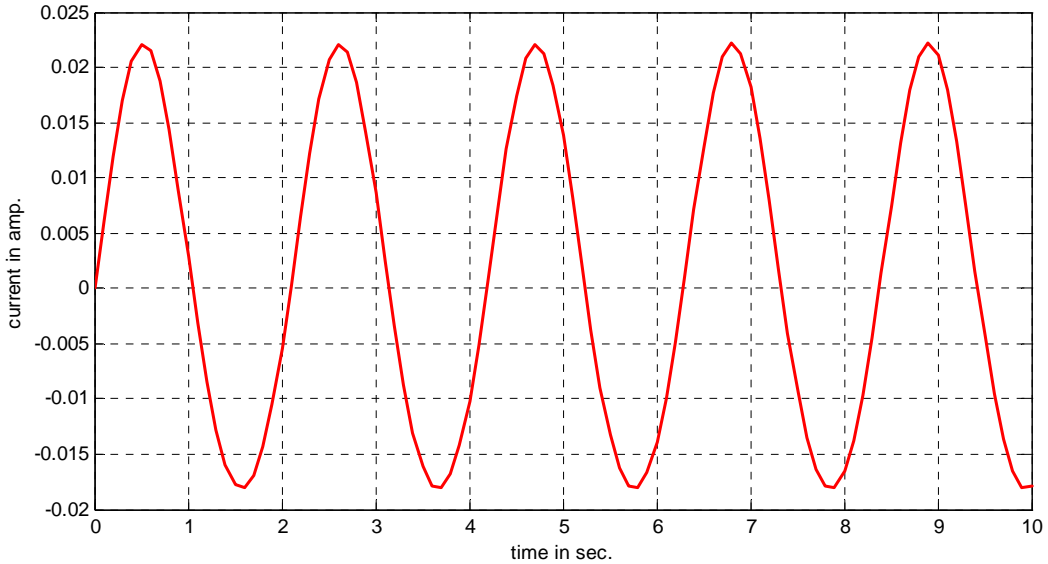


Figure 1.7. Plot of  $i(t)$  for Example 1.3

b. The charge  $q(t)$  is the integral of the current  $i(t)$ , that is,

$$q(t) = \int_{t_0}^{t_1} i(t)dt = 0.1 \int_0^{t_1} (e^{0.2 \sin 3t} - 1)dt \quad (1.6)$$

We will use the MATLAB **int(f,a,b)** integration function where **f** is a symbolic expression, and **a** and **b** are the lower and upper limits of integration respectively.

**Note:**

When MATLAB cannot find a solution, it returns a warning. For this example, MATLAB returns the following message when integration is attempted with the symbolic expression of (1.6).

```
t=sym('t'); % Refer to Appendix A, Page A-10, for a discussion on symbolic expressions
s=int(0.1*(exp(0.2*sin(3*t))-1),0,10)
```

When this script is executed, MATLAB displays the following message:

```
Warning: Explicit integral could not be found.
In C:\MATLAB 12\toolbox\symbolic\@sym\int.m at line 58
s = int(1/10*exp(1/5*sin(3*t))-1/10,t = 0 . . 10)
```

We will use numerical integration with Simpson's rule. MATLAB has two quadrature functions for performing numerical integration, the **quad**\* and **quad8**. The description of these can be seen by typing `help quad` or `help quad8`. at the MATLAB command prompt. Both of these functions use *adaptive quadrature methods*; this means that these methods can handle irregularities such as singularities. When such irregularities occur, MATLAB displays a warning message but still provides an answer.

For this example, we will use the **quad** function. It has the syntax **q=quad('f',a,b,tol)**, and performs an integration to a relative error **tol** which we must specify. If **tol** is omitted, it is understood to be the standard tolerance of  $10^{-3}$ . The string '**f**' is the name of a user defined function, and **a** and **b** are the lower and upper limits of integration respectively.

First, we need to create and save a function m-file.† We define it as shown below, and we save it as `CA_1_Ex_1_3.m`. This is a mnemonic for Circuit Analysis I, Example 1.3.

```
function t = fcn_example_1_3(t); t = 0.1*(exp(0.2*sin(3*t))-1);
```

With this file saved as `CA_1_Ex_1_3.m`, we write and execute the following script.

```
charge=quad('CA_1_Ex_1_3',0,5)
```

and MATLAB returns

```
charge =  
    0.0170
```

---

### 1.5 Power and Energy

Power  $p$  is the rate at which *energy* (or *work*)  $W$  is expended. That is,

$$\boxed{\text{Power} = p = \frac{dW}{dt}} \quad (1.7)$$

*Absorbed power* is proportional both to the current and the voltage needed to transfer one coulomb through the device. The unit of power is the watt. Then,

$$\text{Power} = p = \text{volts} \times \text{amperes} = vi = \frac{\text{joul}}{\text{coul}} \times \frac{\text{coul}}{\text{sec}} = \frac{\text{joul}}{\text{sec}} = \text{watts} \quad (1.8)$$

and

---

\* For a detailed discussion on numerical analysis and the MATLAB functions **quad** and **quad8**, the reader may refer to *Numerical Analysis Using MATLAB® and Excel*, ISBN 978-1-934404-03-4.

† For more information on function m-files, please refer to Appendix A, Page A-26.

$$1 \text{ watt} = 1 \text{ volt} \times 1 \text{ ampere} \quad (1.9)$$

**Passive Sign Convention:** Consider the two-terminal device shown in Figure 1.8.

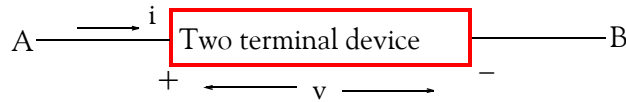


Figure 1.8. Illustration of the passive sign convention

In Figure 1.8, terminal A is  $v$  volts positive with respect to terminal B and current  $i$  enters the device through the positive terminal A. In this case, we satisfy the *passive sign convention* and power  $= p = vi$  is said to be *absorbed* by the device.

The passive sign convention states that if the arrow representing the current  $i$  and the (+) (-) pair are placed at the device terminals in such a way that the current enters the device terminal marked with the (+) sign, and if both the arrow and the sign pair are labeled with the appropriate algebraic quantities, the power absorbed or delivered to the device can be expressed as  $p = vi$ . If the numerical value of this product is positive, we say that the device is *absorbing power* which is equivalent to saying that power is delivered to the device. If, on the other hand, the numerical value of the product  $p = vi$  is negative, we say that the device delivers power to some other device. The passive sign convention is illustrated with the examples in Figures 1.9 and 1.10.

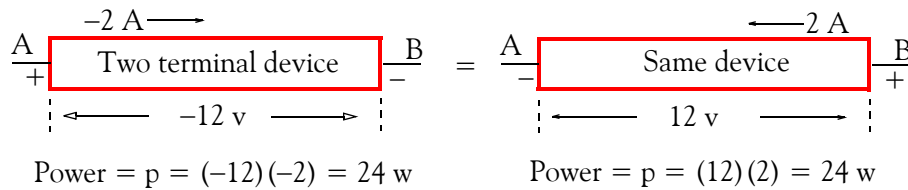


Figure 1.9. Examples where power is absorbed by a two-terminal device

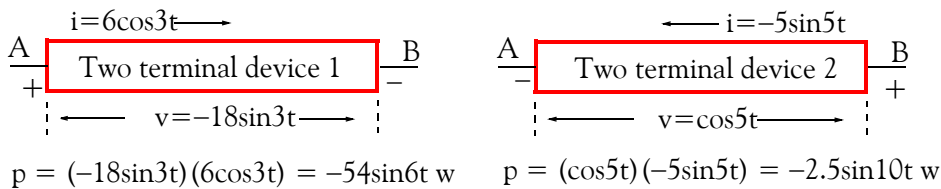


Figure 1.10. Examples where power is delivered to a two-terminal device

In Figure 1.9, power is absorbed by the device, whereas in Figure 1.10, power is delivered to the device.

### Example 1.4

It is assumed a 12-volt automotive battery is completely discharged and at some reference time  $t = 0$ , is connected to a battery charger to trickle charge it for the next 8 hours. It is also assumed that the charging rate is

$$i(t) = \begin{cases} 8e^{-t/3600} \text{ A} & 0 \leq t \leq 8 \text{ hr} \\ 0 & \text{otherwise} \end{cases}$$

For this 8-hour interval compute:

- the total charge delivered to the battery
- the *maximum* power (in watts) absorbed by the battery
- the total energy (in joules) supplied
- the *average* power (in watts) absorbed by the battery

#### Solution:

The current entering the positive terminal of the battery is the decaying exponential shown in Figure 1.11 where the time has been converted to seconds.

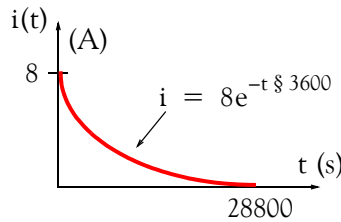


Figure 1.11. Decaying exponential for Example 1.4

Then,

a.

$$\begin{aligned} q|_{t=0}^{15000} &= \int_0^{15000} i dt = \int_0^{28800} 8e^{-t/3600} dt = \frac{8}{-1/3600} e^{-t/3600} \Big|_0^{28800} \\ &= -8 \times 3600(e^{-8} - 1) \approx 28800 \text{ C or } 28.8 \text{ kC} \end{aligned}$$

b.

$$i_{\max} = 8 \text{ A (occurs at } t=0)$$

Therefore,

$$P_{\max} = v i_{\max} = 12 \times 8 = 96 \text{ w}$$

c.

$$\begin{aligned} W &= \int p dt = \int_0^{28800} v i dt = \int_0^{28800} 12 \times 8e^{-t/3600} dt = \frac{96}{-1/3600} e^{-t/3600} \Big|_0^{28800} \\ &= 3.456 \times 10^5 (1 - e^{-8}) \approx 345.6 \text{ KJ.} \end{aligned}$$

d.

$$P_{\text{ave}} = \frac{1}{T} \int_0^T p dt = \frac{1}{28800} \int_0^{28800} 12 \times 8e^{-t/3600} dt = \frac{345.6 \times 10^3}{28.8 \times 10^3} = 12 \text{ w.}$$

### Example 1.5

The power absorbed by a non-linear device is  $p = 9(e^{0.16t^2} - 1)$ . If  $v = 3(e^{0.4t} + 1)$ , how much charge goes through this device in two seconds?

**Solution:**

The power is

$$p = vi, \quad i = \frac{p}{v} = \frac{9(e^{0.16t^2} - 1)}{3(e^{0.4t} + 1)} = \frac{9(e^{0.4t} + 1)(e^{0.4t} - 1)}{3(e^{0.4t} + 1)} = 3(e^{0.4t} - 1) \text{ A}$$

then, the charge for 2 seconds is

$$q|_{t_0}^t = \int_{t_0}^t i dt = 3 \int_0^2 (e^{0.4t} - 1) dt = \frac{3}{0.4} e^{0.4t} \Big|_0^2 - 3t \Big|_0^2 = 7.5(e^{0.8} - 1) - 6 = 3.19 \text{ C}$$

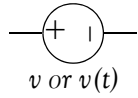
The two-terminal devices which we will be concerned with in this text are shown in Figure 1.12.

*Linear devices* are those in which there is a linear relationship between the voltage across that device and the current that flows through that device. Diodes and Transistors are *non-linear* devices, that is, their voltage-current relationship is non-linear. These will not be discussed in this text. A simple circuit with a diode is presented in Chapter 3.

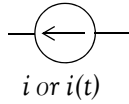
## 1.6 Active and Passive Devices

Independent and dependent voltage and current sources are *active devices*; they normally (but not always) deliver power to some external device. Resistors, inductors and capacitors are *passive devices*; they normally receive (absorb) power from an active device.

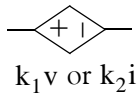
## Independent and Dependent Sources



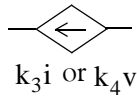
**Ideal Independent Voltage Source** – Maintains same voltage regardless of the amount of current that flows through it. Its value is either constant (DC) or sinusoidal (AC).



**Ideal Independent Current Source** – Maintains same current regardless of the voltage that appears across its terminals. Its value is either constant (DC) or sinusoidal (AC).



**Dependent Voltage Source** – Its value depends on another voltage or current elsewhere in the circuit. Here,  $k_1$  is a constant and  $k_2$  is a resistance as defined in linear devices below. When denoted as  $k_1 v$  it is referred to as *voltage controlled voltage source* and when denoted as  $k_2 i$  it is referred to as *current controlled voltage source*.



**Dependent Current Source** – Its value depends on another current or voltage elsewhere in the circuit. Here,  $k_3$  is a constant and  $k_4$  is a conductance as defined in linear devices below. When denoted as  $k_3 i$  it is referred to as *current controlled current source* and when denoted as  $k_4 v$  it is referred to as *voltage controlled current source*.

## Linear Devices

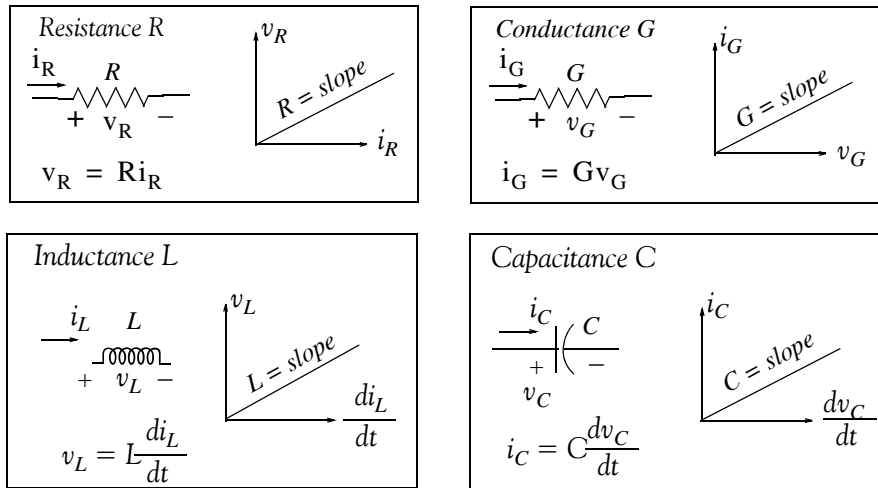


Figure 1.12. Voltage and current sources and linear devices

## 1.7 Circuits and Networks

A *network* is the interconnection of two or more simple devices as shown in Figure 1.13.

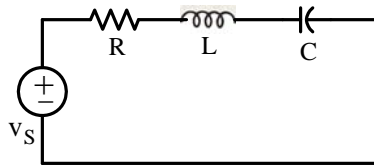


Figure 1.13. A network but not a circuit

A *circuit* is a network which contains at least one closed path. Thus every circuit is a network but not all networks are circuits. An example is shown in Figure 1.14.

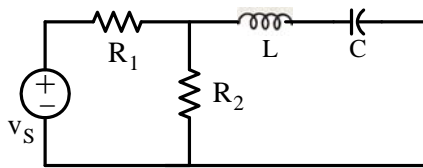


Figure 1.14. A network and a circuit

## 1.8 Active and Passive Networks

*Active Network* is a network which contains at least one active device (voltage or current source).

*Passive Network* is a network which does not contain any active device.

## 1.9 Necessary Conditions for Current Flow

There are two conditions which are necessary to set up and maintain a flow of current in a network or circuit. These are:

1. There must be a voltage source (potential difference) present to provide the electrical work which will force current to flow.
2. The circuit must be closed.

These conditions are illustrated in Figures 1.15 through 1.17.

Figure 1.15 shows a network which contains a voltage source but it is not closed and therefore, current will not flow.



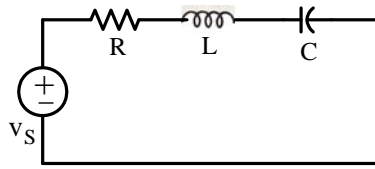


Figure 1.15. A network in which there is no current flow

Figure 1.16 shows a closed circuit but there is no voltage present to provide the electrical work for current to flow.

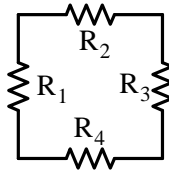


Figure 1.16. A closed circuit in which there is no current flow

Figure 1.17 shows a voltage source present and the circuit is closed. Therefore, both conditions are satisfied and current will flow.

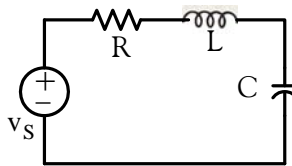


Figure 1.17. A circuit in which current flows

### 1.10 International System of Units

The *International System of Units* (abbreviated *SI* in all languages) was adopted by the General Conference on Weights and Measures in 1960. It is used extensively by the international scientific community. It was formerly known as the *Metric System*. The basic units of the *SI* system are listed in Table 1.1.

TABLE 1.1 SI Base Units

Unit of	Name	Abbreviation
Length	Metre	m
Mass	Kilogram	kg
Time	Second	s
Electric Current	Ampere	A
Temperature	Degrees Kelvin	°K
Amount of Substance	Mole	mol
Luminous Intensity	Candela	cd
Plane Angle	Radian	rad
Solid Angle	Steradian	sr

The SI uses larger and smaller units by various powers of 10 known as *standard prefixes*. The *common prefixes* are listed in Table 1.2 and the less frequently in Table 1.3. Table 1.4 shows some conversion factors between the SI and the English system. Table 1.5 shows typical temperature values in degrees Fahrenheit and the equivalent temperature values in degrees Celsius and degrees Kelvin. Other units used in physical sciences and electronics are derived from the SI base units and the most common are listed in Table 1.6.

TABLE 1.2 Most Commonly Used SI Prefixes

Value	Prefix	Symbol	Example
$10^9$	Giga	G	12 GHz (Gigahertz) = $12 \times 10^9$ Hz
$10^6$	Mega	M	25 MW (Megaohms) = $25 \times 10^6$ W (ohms)
$10^3$	Kilo	K	13.2 KV (Kilovolts) = $13.2 \times 10^3$ volts
$10^{-2}$	centi	c	2.8 cm (centimeters) = $2.8 \times 10^{-2}$ meter
$10^{-3}$	milli	m	4 mH (millihenries) = $4 \times 10^{-3}$ henry
$10^{-6}$	micro	$\mu$	$6 \mu\text{w}$ (microwatts) = $6 \times 10^{-6}$ watt
$10^{-9}$	nano	n	2 ns (nanoseconds) = $2 \times 10^{-9}$ second
$10^{-12}$	pico	p	3 pF (picofarads) = $3 \times 10^{-12}$ Farad

TABLE 1.3 Less Frequently Used SI Prefixes

Value	Prefix	Symbol	Example
$10^{18}$	Exa	E	1 Em (Exameter) = $10^{18}$ meters
$10^{15}$	Peta	P	5 Pyrs (Petayears) = $5 \times 10^{15}$ years
$10^{12}$	Tera	T	3 T\$ (Teradollars) = $3 \times 10^{12}$ dollars
$10^{-15}$	femto	f	7 fA (femtoamperes) = $7 \times 10^{-15}$ ampere
$10^{-18}$	atto	a	9 aC (attocoulombs) = $9 \times 10^{-18}$ coulomb

TABLE 1.4 Conversion Factors

1 in. (inch)	2.54 cm (centimeters)
1 mi. (mile)	1.609 Km (Kilometers)
1 lb. (pound)	0.4536 Kg (Kilograms)
1 qt. (quart)	946 cm <sup>3</sup> (cubic centimeters)
1 cm (centimeter)	0.3937 in. (inch)
1 Km (Kilometer)	0.6214 mi. (mile)
1 Kg (Kilogram)	2.2046 lbs (pounds)
1 lt. (liter) = 1000 cm <sup>3</sup>	1.057 quarts
1 Å (Angstrom)	$10^{-10}$ meter
1 mm (micron)	$10^{-6}$ meter

TABLE 1.5 Temperature Scale Equivalents

°F	°C	°K
-523.4	-273	0
32	0	273
0	-17.8	255.2
77	25	298
98.6	37	310
212	100	373

TABLE 1.6 SI Derived Units

Unit of	Name	Formula
Force	Newton (N)	$N = \text{kg} \cdot \text{m}/\text{s}^2$
Pressure or Stress	Pascal (Pa)	$\text{Pa} = \text{N}/\text{m}^2$
Work or Energy	Joule (J)	$J = \text{N} \cdot \text{m}$
Power	Watt (W)	$W = \text{J}/\text{s}$
Voltage	Volt (V)	$V = \text{W}/\text{A}$
Resistance	Ohm ( $\Omega$ )	$\Omega = \text{V}/\text{A}$
Conductance	Siemens (S) or ( $\Omega^{-1}$ )	$S = \text{A}/\text{V}$
Capacitance	Farad (F)	$F = \text{A} \cdot \text{s}/\text{V}$
Inductance	Henry (H)	$H = \text{V} \cdot \text{s}/\text{A}$
Frequency	Hertz (Hz)	$\text{Hz} = 1/\text{s}$
Quantity of Electricity	Coulomb (C)	$C = \text{A} \cdot \text{s}$
Magnetic Flux	Weber (Wb)	$\text{Wb} = \text{V} \cdot \text{s}$
Magnetic Flux Density	Tesla (T)	$T = \text{Wb}/\text{m}^2$
Luminous Flux	Lumen (lm)	$\text{lm} = \text{cd} \cdot \text{sr}$
Illuminance	Lux (lx)	$\text{lx} = \text{lm}/\text{m}^2$
Radioactivity	Becquerel (Bq)	$\text{Bq} = \text{s}^{-1}$
Radiation Dose	Gray (Gy)	$\text{S} = \text{J}/\text{kg}$
Volume	Litre (L)	$L = \text{m}^3 \times 10^{-3}$

## 1.11 Sources of Energy

The principal sources of energy are from chemical processes (coal, fuel oil, natural gas, wood etc.) and from mechanical forms (water falls, wind, etc.). Other sources include nuclear and solar energy.

### Example 1.6

A certain type of wood used in the generation of electric energy and we can get 12,000 BTUs from a pound (lb) of that wood when burned. Suppose that a computer system that includes a

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## Chapter 1 Basic Concepts and Definitions

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monitor, a printer, and other peripherals absorbs an average power of 500 w gets its energy from that burned wood and it is turned on for 8 hours. It is known that 1 BTU is equivalent to 778.3 ft-lb of energy, and 1 joule is equivalent to 0.7376 ft-lb.

Compute:

- the energy consumption during this 8-hour interval
- the cost for this energy consumption if the rate is \$0.15 per kw-hr
- the amount of wood in lbs burned during this time interval.

**Solution:**

- Energy consumption for 8 hours is

$$\text{Energy } W = P_{\text{ave}}t = 500 \text{ w} \times 8 \text{ hrs} \times \frac{3600 \text{ s}}{1 \text{ hr}} = 14.4 \text{ Mjoules}$$

- Since 1 kilowatt-hour =  $3.6 \times 10^6$  joules,

$$\text{Cost} = \frac{\$0.15}{\text{kw-hr}} \times \frac{1 \text{ kw-hr}}{3.6 \times 10^6 \text{ joules}} \times 14.4 \times 10^6 = \$0.60$$

- Wood burned in 8 hours,

$$14.4 \times 10^6 \text{ joules} \times 0.7376 \frac{\text{ft-lb}}{\text{joule}} \times \frac{1 \text{ BTU}}{778.3 \text{ ft-lb}} \times \frac{1 \text{ lb}}{12000 \text{ BTU}} = 1.137 \text{ lb}$$

---

## 1.12 Summary

- Two identically charged (both positive or both negative) particles possess a charge of one coulomb when being separated by one meter in a vacuum, repel each other with a force of  $10^{-7} c^2$  newton where  $c =$  velocity of light  $\approx 3 \times 10^8$  m/s. Thus, the force with which two electrically charged bodies attract or repel one another depends on the product of the charges (in coulombs) in both objects, and also on the distance between the objects. If the polarities are the same (negative/negative or positive/positive), the so-called coulomb force is repulsive; if the polarities are opposite (negative/positive or positive/negative), the force is attractive. For any two charged bodies, the coulomb force decreases in proportion to the square of the distance between their charge centers.
- Electric current is defined as the instantaneous rate at which net positive charge is moving past this point in that specified direction, that is,

$$i = \frac{dq}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\Delta q}{\Delta t}$$

- The unit of current is the ampere, abbreviated as A, and corresponds to charge  $q$  moving at the rate of one coulomb per second.
- In a two-terminal device the current entering one terminal is the same as the current leaving the other terminal.
- The voltage (potential difference) across a two-terminal device is defined as the work required to move a positive charge of one coulomb from one terminal of the device to the other terminal.
- The unit of voltage is the volt (abbreviated as V or  $v$ ) and it is defined as

$$1 \text{ volt} = \frac{1 \text{ joule}}{1 \text{ coulomb}}$$

- Power  $p$  is the rate at which *energy* (or *work*)  $W$  is expended. That is,

$$\text{Power} = p = \frac{dW}{dt}$$

- Absorbed power is proportional both to the current and the voltage needed to transfer one coulomb through the device. The unit of power is the watt and

$$1 \text{ watt} = 1 \text{ volt} \times 1 \text{ ampere}$$

- The passive sign convention states that if the arrow representing the current  $i$  and the plus (+) minus (-) pair are placed at the device terminals in such a way that the current enters the device terminal marked with the plus (+) sign, and if both the arrow and the sign pair are labeled with the appropriate algebraic quantities, the power absorbed or delivered to the

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## Chapter 1 Basic Concepts and Definitions

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device can be expressed as  $p = vi$ . If the numerical value of this product is positive, we say that the device is absorbing power which is equivalent to saying that power is delivered to the device. If, on the other hand, the numerical value of the product  $p = vi$  is negative, we say that the device delivers power to some other device.

- An ideal independent voltage source maintains the same voltage regardless of the amount of current that flows through it.
- An ideal independent current source maintains the same current regardless of the amount of voltage that appears across its terminals.
- The value of an dependent voltage source depends on another voltage or current elsewhere in the circuit.
- The value of an dependent current source depends on another current or voltage elsewhere in the circuit.
- Ideal voltage and current sources are just mathematical models. We will discuss practical voltage and current sources in Chapter 3.
- Independent and Dependent voltage and current sources are active devices; they normally (but not always) deliver power to some external device.
- Resistors, inductors, and capacitors are passive devices; they normally receive (absorb) power from an active device.
- A network is the interconnection of two or more simple devices.
- A circuit is a network which contains at least one closed path. Thus every circuit is a network but not all networks are circuits.
- An active network is a network which contains at least one active device (voltage or current source).
- A passive network is a network which does not contain any active device.
- To set up and maintain a flow of current in a network or circuit there must be a voltage source (potential difference) present to provide the electrical work which will force current to flow and the circuit must be closed.
- Linear devices are those in which there is a linear relationship between the voltage across that device and the current that flows through that device.
- The International System of Units is used extensively by the international scientific community. It was formerly known as the Metric System.
- The principal sources of energy are from chemical processes (coal, fuel oil, natural gas, wood etc.) and from mechanical forms (water falls, wind, etc.). Other sources include nuclear and solar energy.

## 1.13 Exercises

### Multiple choice

1. The unit of charge is the
  - A. ampere
  - B. volt
  - C. watt
  - D. coulomb
  - E. none of the above
2. The unit of current is the
  - A. ampere
  - B. coulomb
  - C. watt
  - D. joule
  - E. none of the above
3. The unit of electric power is the
  - A. ampere
  - B. coulomb
  - C. watt
  - D. joule
  - E. none of the above
4. The unit of energy is the
  - A. ampere
  - B. volt
  - C. watt
  - D. joule
  - E. none of the above
5. Power is
  - A. the integral of energy
  - B. the derivative of energy
  - C. current times some constant  $k$
  - D. voltage times some constant  $k$
  - E. none of the above
6. Active voltage and current sources
  - A. always deliver power to other external devices
  - B. normally deliver power to other external devices
  - C. neither deliver or absorb power to or from other devices
  - D. are just mathematical models
  - E. none of the above

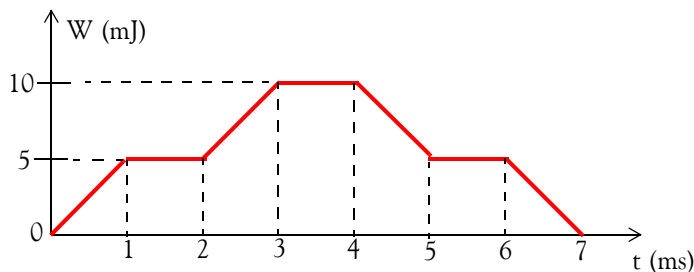


## Chapter 1 Basic Concepts and Definitions

7. An ideal independent voltage source
  - A. maintains the same voltage regardless of the amount of current that flows through it
  - B. maintains the same current regardless of the voltage rating of that voltage source
  - C. always delivers the same amount of power to other devices
  - D. is a source where both voltage and current can be variable
  - E. none of the above
8. An ideal independent current source
  - A. maintains the same voltage regardless of the amount of current that flows through it
  - B. maintains the same current regardless of the voltage that appears across its terminals
  - C. always delivers the same amount of power to other devices
  - D. is a source where both voltage and current can be variable
  - E. none of the above
9. The value of a dependent voltage source can be denoted as
  - A.  $kV$  where  $k$  is a conductance value
  - B.  $kI$  where  $k$  is a resistance value
  - C.  $kV$  where  $k$  is an inductance value
  - D.  $kI$  where  $k$  is a capacitance value
  - E. none of the above
10. The value of a dependent current source can be denoted as
  - A.  $kV$  where  $k$  is a conductance value
  - B.  $kI$  where  $k$  is a resistance value
  - C.  $kV$  where  $k$  is an inductance value
  - D.  $kI$  where  $k$  is a capacitance value
  - E. none of the above

### Problems

1. A two terminal device consumes energy as shown by the waveform below, and the current through this device is  $i(t) = 2 \cos 4000\pi t$  A. Find the voltage across this device at  $t = 0.5, 1.5, 4.75$  and  $6.5$  ms. Answers: 2.5 V, 0 V, 2.5 V, -2.5 V



- 
2. A household light bulb is rated 75 watts at 120 volts. Compute the number of electrons per second that flow through this bulb when it is connected to a 120 volt source.  
Answer:  $3.9 \times 10^{18}$  electrons/s
3. An airplane, whose total mass is 50,000 metric tons, reaches a height of 32,808 feet in 20 minutes after takeoff.
- Compute the potential energy that the airplane has gained at this height.  
Answer: 1, 736 MJ
  - If this energy could be converted to electric energy with a conversion loss of 10%, how much would this energy be worth at \$0.15 per kilowatt-hour? Answer: \$65.10
  - If this energy were converted into electric energy during the period of 20 minutes, what average number of kilowatts would be generated? Answer: 1, 450 Kw
4. The power input to a television station transmitter is 125 kw and the output is 100 kw which is transmitted as radio frequency power. The remaining 25 kw of power is converted into heat.
- How many BTUs per hour does this transmitter release as heat? 1 BTU = 1054.8 J  
Answer: 85, 234 BTU/hr
  - How many electron-volts per second is this heat equivalent to?  
1 electron – volt =  $1.6 \times 10^{-19}$  J  
Answer:  $1.56 \times 10^{23} \frac{\text{electron – volts}}{\text{sec.}}$
-

### 1.14 Answers / Solutions to End-of-Chapter Exercises

Dear Reader:

The remaining pages on this chapter contain answers to the multiple-choice questions and solutions to the exercises.

You must, for your benefit, make an honest effort to answer the multiple-choice questions and solve the problems without first looking at the solutions that follow. It is recommended that first you go through and answer those you feel that you know. For the multiple-choice questions and problems that you are uncertain, review this chapter and try again. If your answers to the problems do not agree with those provided, look over your procedures for inconsistencies and computational errors. Refer to the solutions as a last resort and rework those problems at a later date.

You should follow this practice with the multiple-choice and problems on all chapters of this book.

Multiple choice

1. D
2. A
3. C
4. D
5. B
6. B
7. A
8. B
9. B
10. A

Problems

1.

$$v = \frac{p}{i} = \frac{dW/dt}{i} = \frac{\text{slope}}{i}$$

a.

$$\text{slope}\Big|_0^{1 \text{ ms}} = \frac{5 \text{ mJ}}{1 \text{ ms}} = 5 \text{ J/s}$$

$$v\Big|_{t=0.5 \text{ ms}} = \frac{5 \text{ J/s}}{2 \cos 4000\pi(0.5 \times 10^{-3}) \text{ A}} = \frac{5 \text{ J/s}}{2 \cos 2\pi \text{ A}} = \frac{5 \text{ J/s}}{2 \text{ A}} = 2.5 \text{ V}$$

b.

$$\text{slope}\Big|_1^{2 \text{ ms}} = 0$$

$$v\Big|_{t=1.5 \text{ ms}} = \frac{0}{i} = 0 \text{ V}$$

c.

$$\text{slope}\Big|_4^{5 \text{ ms}} = \frac{-5 \text{ mJ}}{1 \text{ ms}} = -5 \text{ J/s}$$

$$v\Big|_{t=4.75 \text{ ms}} = \frac{-5 \text{ J/s}}{2 \cos 4000\pi(4.75 \times 10^{-3}) \text{ A}} = \frac{-5 \text{ J/s}}{2 \cos 19\pi \text{ A}} = \frac{-5 \text{ J/s}}{2 \cos \pi \text{ A}} = \frac{-5 \text{ J/s}}{-2 \text{ A}} = 2.5 \text{ V}$$

d.

$$\text{slope}|_6^{7 \text{ ms}} = \frac{-5 \text{ mJ}}{1 \text{ ms}} = -5 \text{ J/s}$$

$$v|_{t=6.5 \text{ ms}} = \frac{-5 \text{ J/s}}{2 \cos 4000\pi(6.5 \times 10^{-3}) \text{ A}} = \frac{-5 \text{ J/s}}{2 \cos 26\pi \text{ A}} = \frac{-5 \text{ J/s}}{2 \text{ A}} = -2.5 \text{ V}$$

2.

$$i = \frac{p}{v} = \frac{75 \text{ W}}{120 \text{ V}} = \frac{5}{8} \text{ A}$$

$$q = \int_{t_0}^t i dt$$

$$q|_{t=1 \text{ s}} = \int_0^1 \frac{5}{8} dt = \left. \frac{5}{8}t \right|_0^1 = \frac{5}{8} \text{ C/s}$$

$$\frac{5}{8} \text{ C/s} \times \frac{6.24 \times 10^{18} \text{ electrons}}{1 \text{ C}} = 3.9 \times 10^{18} \text{ electrons/s}$$

3.

$$W_p = W_k = \frac{1}{2}mv^2$$

where  $m$  = mass in kg and  $v$  = velocity in meters/sec.

$$33,808 \text{ ft} \times \frac{0.3048 \text{ m}}{\text{ft}} = 10,000 \text{ m} = 10 \text{ Km}$$

$$20 \text{ minutes} \times \frac{60 \text{ sec.}}{\text{min}} = 1,200 \text{ sec.}$$

$$v = \frac{10,000 \text{ m}}{1,200 \text{ sec.}} = \frac{25}{3} \text{ m/s}$$

$$50,000 \text{ metric tons} \times \frac{1,000 \text{ Kg}}{\text{metric ton}} = 5 \times 10^7 \text{ Kg}$$

Then,

a.

$$W_p = W_k = \frac{1}{2}(5 \times 10^7) \left( \frac{25}{3} \right)^2 = 173.61 \times 10^7 \text{ J} \approx 1,736 \text{ MJ}$$

b.

$$1 \text{ joule} = 1 \text{ watt-sec}$$

$$1,736 \times 10^6 \text{ J} \times \frac{1 \text{ watt-sec}}{1 \text{ joule}} \times \frac{1 \text{ Kw}}{1,000 \text{ w}} \times \frac{1 \text{ hr}}{3,600 \text{ sec.}} = 482.22 \text{ Kw-hr}$$

and with 10% conversion loss, the useful energy is

$$482.22 \times 0.9 = 482.22 \times 0.9 = 434 \text{ Kw-hr}$$

$$\text{Cost of Energy} = \frac{\$0.15}{\text{Kw-hr}} \times 434 \text{ Kw-hr} = \$65.10$$

c.

$$P_{\text{ave}} = \frac{W}{t} = \frac{1,736 \text{ MJ}}{20 \text{ min} \times \frac{60 \text{ sec}}{\text{min}}} = 1.45 \text{ Mw} = 1450 \text{ Kw}$$

4.

a.

$$1 \text{ BTU} = 1054.8 \text{ J}$$

$$25,000 \text{ watts} \times \frac{1 \text{ joule/sec.}}{\text{watt}} \times \frac{1 \text{ BTU}}{1054.8 \text{ J}} \times \frac{3600 \text{ sec.}}{1 \text{ hr}} = 85,234 \text{ BTU/hr}$$

b.

$$1 \text{ electron-volt} = 1.6 \times 10^{-19} \text{ J}$$

$$\frac{1 \text{ electron-volt}}{\text{sec.}} = \frac{1.6 \times 10^{-19} \text{ J}}{\text{sec.}} = 1.6 \times 10^{-19} \text{ watt}$$

$$25,000 \text{ watts} \times \frac{1 \text{ electron-volt/sec.}}{1.6 \times 10^{-19} \text{ watt}} = 1.56 \times 10^{23} \frac{\text{electron-volts}}{\text{sec.}}$$

---

# Chapter 2

---

## Analysis of Simple Circuits

This chapter defines constant and instantaneous values, Ohm's law, and Kirchhoff's Current and Voltage laws. Series and parallel circuits are also defined and nodal, mesh, and loop analyses are introduced. Combinations of voltage and current sources and resistance combinations are discussed, and the voltage and current division formulas are derived.

### 2.1 Conventions

We will use lower case letters such as  $v$ ,  $i$ , and  $p$  to denote *instantaneous values* of voltage, current, and power respectively, and we will use subscripts to denote specific voltages, currents, resistances, etc. For example,  $v_S$  and  $i_S$  will be used to denote voltage and current sources respectively. Notations like  $v_{R_1}$  and  $i_{R_2}$  will be used to denote the voltage across resistance  $R_1$  and the current through resistance  $R_2$  respectively. Other notations like  $v_A$  or  $v_1$  will represent the voltage (potential difference) between point A or point 1 *with respect* to some arbitrarily chosen reference point taken as “zero” volts or “ground”.

The designations  $v_{AB}$  or  $v_{12}$  will be used to denote the voltage between point A or point 1 *with respect* to point B or 2 respectively. We will denote voltages as  $v(t)$  and  $i(t)$  whenever we wish to emphasize that these quantities are time dependent. Thus, sinusoidal (AC) voltages and currents will be denoted as  $v(t)$  and  $i(t)$  respectively. Phasor quantities, to be introduced in Chapter 6, will be represented with bold capital letters,  $\mathbf{V}$  for phasor voltage and  $\mathbf{I}$  for phasor current.

### 2.2 Ohm's Law

We recall from Chapter 1 that *resistance*  $R$  is a *constant* that relates the voltage and the current as:

$$\boxed{v_R = R i_R} \quad (2.1)$$

This relation is known as *Ohm's law*.

The unit of resistance is the *Ohm* and its symbol is the Greek capital letter  $\Omega$ . One ohm is the resistance of a conductor such that a constant current of one ampere through it produces a voltage of one volt between its ends. Thus,

## Chapter 2 Analysis of Simple Circuits

$$1 \Omega = \frac{1 \text{ V}}{1 \text{ A}} \quad (2.2)$$

Physically, a *resistor* is a device that opposes current flow. Resistors are used as a *current limiting devices* and as *voltage dividers*.

In the previous chapter we defined *conductance*  $G$  as the constant that relates the current and the voltage as

$$i_G = Gv_G \quad (2.3)$$

This is another form of Ohm's law since by letting  $i_G = i_R$  and  $v_G = v_R$ , we obtain

$$G = \frac{1}{R} \quad (2.4)$$

The unit of conductance is the *siemens* or *mho* (ohm spelled backwards) and its symbol is  $S$  or  $\Omega^{-1}$ . Thus,

$$1 \Omega^{-1} = \frac{1 \text{ A}}{1 \text{ V}} \quad (2.5)$$

Resistances (or conductances) are commonly used to define an "open circuit" or a "short circuit". An *open circuit* is an adjective describing the "open space" between a pair of terminals, and can be thought of as an "infinite resistance" or "zero conductance". In contrast, a *short circuit* is an adjective describing the connection of a pair of terminals by a piece of wire of "infinite conductance" or a piece of wire of "zero" resistance.

The current through an "open circuit" is always zero but the voltage across the open circuit terminals may or may not be zero. Likewise, the voltage across a short circuit terminals is always zero but the current through it may or may not be zero. The open and short circuit concepts and their equivalent resistances or conductances are shown in Figure 2.1.

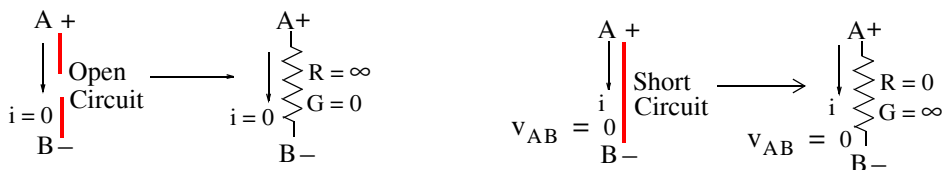


Figure 2.1. The concepts of open and short circuits

The fact that current does not flow through an open circuit and that zero voltage exists across the terminals of a short circuit, can also be observed from the expressions  $v_R = Ri_R$  and  $i_G = Gv_G$ .



That is, since  $G = \frac{1}{R}$ , infinite  $R$  means zero  $G$  and zero  $R$  means infinite  $G$ . Then, for a finite voltage, say  $v_G$ , and an open circuit,

$$\lim_{G \rightarrow 0} i_G = \lim_{G \rightarrow 0} Gv_G = 0 \quad (2.6)$$

Likewise, for a finite current, say  $i_R$ , and a short circuit,

$$\lim_{R \rightarrow 0} v_R = \lim_{R \rightarrow 0} Ri_R = 0 \quad (2.7)$$

**Reminder:**

We must remember that the expressions

$$v_R = Ri_R$$

and

$$i_G = Gv_G$$

are true only when the passive sign convention is observed. This is consistent with our classification of  $R$  and  $G$  being passive devices and thus  $v_R = Ri_R$  implies the current direction and voltage polarity are as shown in Figure 2.2.



Figure 2.2. Voltage polarity and current direction in accordance with the passive sign convention

But if the voltage polarities and current directions are as shown in Figure 2.3, then,

$$v_R = -Ri_R \quad (2.8)$$



Figure 2.3. Voltage polarity and current direction not in accordance to passive sign convention

**Note:** “Negative resistance,” as shown in (2.8), can be thought of as being a math model that supplies energy.

### 2.3 Power Absorbed by a Resistor

A resistor, being a passive device, absorbs power. This absorbed power can be found from Ohm’s law, that is,

## Chapter 2 Analysis of Simple Circuits

$$v_R = R i_R$$

and the power relation

$$p_R = v_R i_R$$

Then,

$$p_R = v_R i_R = (R i_R) i_R = R i_R^2 = v_R \left( \frac{v_R}{R} \right) = \frac{v_R^2}{R} \quad (2.9)$$

The voltage, current, resistance and power relations are arranged in the pie chart shown in Figure 2.4.

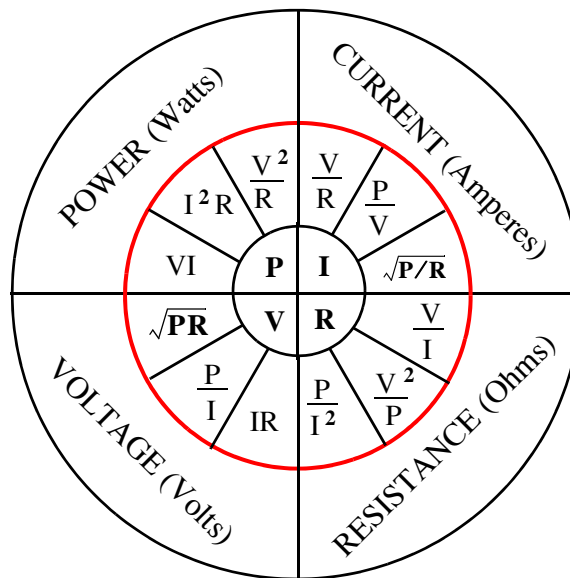


Figure 2.4. Pie chart for showing relations among voltage, current, resistance, and power

### Note:

A resistor, besides its resistance rating (ohms) has a power rating in watts commonly referred to as the *wattage* of the resistor. Common resistor wattage values are  $\frac{1}{4}$  watt,  $\frac{1}{2}$  watt, 1 watt, 2 watts, 5 watts and so on. This topic will be discussed in Section 2.16.

## 2.4 Energy Dissipated in a Resistor

A resistor, by its own nature, dissipates energy in the form of heat; it never stores energy. The energy dissipated in a resistor during a time interval, say from  $t_1$  to  $t_2$ , is given by the integral of the instantaneous power  $p_R$ . Thus,

$$W_{R \text{ diss}} = \int_{t_1}^{t_2} p_R dt \quad (2.10)$$

If the power is constant, say  $P$ , then (2.10) reduces to

$$W_{R \text{ diss}} = Pt \tag{2.11}$$

Alternately, if the energy is known, we can find the power by taking the derivative of the energy, that is,

$$P_R = \frac{d}{dt}W_{R \text{ diss}} \tag{2.12}$$

**Reminder:**

When using all formulas, we must express the quantities involved in their primary units. For instance in (2.11) above, the energy is in joules when the power is in *watts* and the time is in *seconds*.

## 2.5 Nodes, Branches, Loops and Meshes

### Definition 2.1

A *node* is the common point at which two or more devices (passive or active) are connected. An example of a node is shown in Figure 2.5.

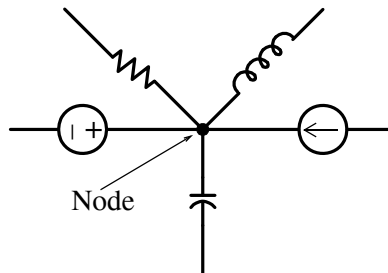


Figure 2.5. Definition of node

### Definition 2.2

A *branch* is a simple path composed of one single device as shown in Figure 2.6.

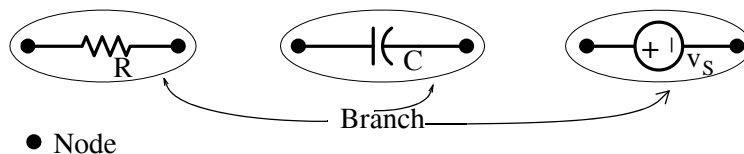


Figure 2.6. Definition of branch

### Definition 2.3

A *loop* is a closed path formed by the interconnection of simple devices. For example, the network shown in Figure 2.7 is a loop.

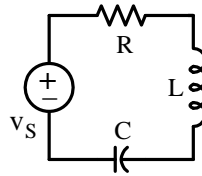


Figure 2.7. Definition of a loop

### Definition 2.4

A *mesh* is a loop which does not enclose any other loops. For example, in the circuit shown in Figure 2.8, ABEF is both a loop and a mesh, but ABCDEF is a loop but not a mesh.

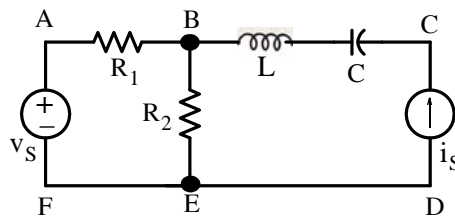


Figure 2.8. Example showing the difference between mesh and loop

## 2.6 Kirchhoff's Current Law (KCL)

KCL states that *the algebraic sum of all currents leaving (or entering) a node is equal to zero*. For example, in Figure 2.9, if we assign a plus (+) sign to the currents *leaving* the node, we must assign a minus (-) sign to the currents *entering* the node. Then by KCL,

$$-i_1 - i_2 + i_3 + i_4 = 0 \quad (2.13)$$

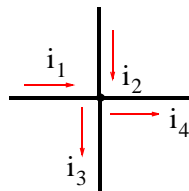


Figure 2.9. Node to illustrate KCL

But if we assign a plus (+) sign to the currents *entering* the node and minus (-) sign to the currents *leaving* the node, then by KCL,

$$i_1 + i_2 - i_3 - i_4 = 0 \quad (2.14)$$

or

$$-i_1 - i_2 + i_3 + i_4 = 0 \quad (2.15)$$

We observe that (2.13) and (2.15) are the same; therefore, it does not matter which we choose as plus (+).

**Convention:**

In our subsequent discussion we will assign plus (+) signs to the currents *leaving* the node.

### 2.7 Kirchhoff's Voltage Law (KVL)

KVL states that *the algebraic sum of the voltage drops (voltages from + to -) or voltage rises (voltages from - to +) around any closed path (mesh or loop) in a circuit is equal to zero*. For example, in the circuit shown in Figure 2.10, voltages  $v_1$ ,  $v_2$ ,  $v_3$ , and  $v_4$  represent the voltages across devices 1, 2, 3, and 4 respectively, and have the polarities shown.

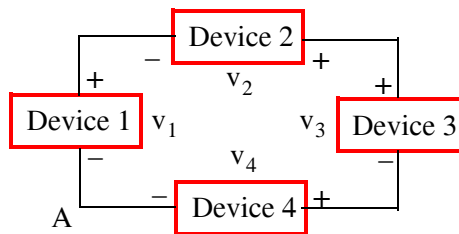


Figure 2.10. Circuit to illustrate KVL

Now, if we assign a (+) sign to the voltage drops, we must assign a (-) sign to the voltage rises. Then, by KVL starting at node A and going *clockwise* we obtain:

$$-v_1 - v_2 + v_3 + v_4 = 0 \tag{2.16}$$

or going *counterclockwise*, we obtain:

$$-v_4 - v_3 + v_2 + v_1 = 0 \tag{2.17}$$

Alternately, if we assign a (+) sign to the voltage rises, we must assign a (-) sign to the voltage drops. Then, by KVL starting again at node A and going *clockwise* we obtain:

$$v_1 + v_2 - v_3 - v_4 = 0 \tag{2.18}$$

or going *counterclockwise*, we obtain:

$$v_4 + v_3 - v_2 - v_1 = 0 \tag{2.19}$$

We observe that expressions (2.16) through (2.19) are the same.

**Convention:**

In our subsequent discussion we will assign plus (+) signs to voltage drops.

### Definition 2.5

Two or more devices are said to be *connected in series* if and only if the same current flows through them. For example, in the circuit of Figure 2.11, the same current  $i$  flows through the voltage source, the resistance, the inductance and the capacitance. Accordingly, this is classified as a series circuit.

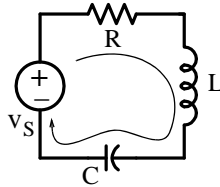


Figure 2.11. A simple series circuit

### Definition 2.6

Two or more devices are said to be *connected in parallel* if and only if the same voltage exists across each of the devices. For example, in the circuit of Figure 2.12, the same voltage  $v_{AB}$  exists across the current source, the conductance, the inductance, and the capacitance and therefore it is classified as a parallel circuit

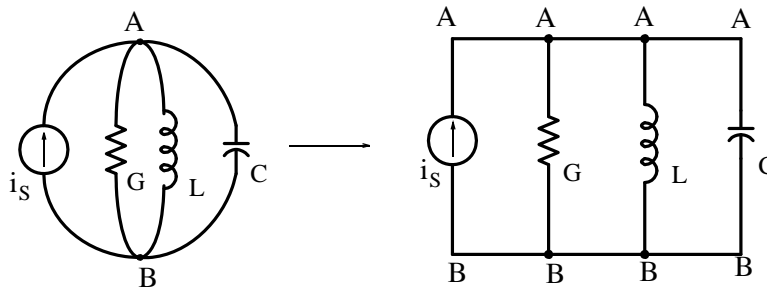


Figure 2.12. A simple parallel circuit

### Convention:

In our subsequent discussion we will adopt the conventional current flow, i.e., the current that flows from a higher (+) to a lower (-) potential. For example, if in Figure 2.13 we are given the indicated polarity,

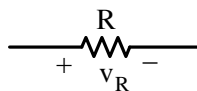


Figure 2.13. Device with established voltage polarity

then, the current arrow will be pointing to the right direction as shown in Figure 2.14.

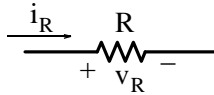


Figure 2.14. Direction of conventional current flow in device with established voltage polarity

Alternately, if current flows in an assumed specific direction through a device thus producing a voltage, we will assign a (+) sign at the terminal of the device at which the current enters. For example, if we are given this designation a device in which the current direction has been established as shown in Figure 2.15,

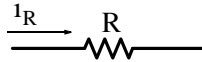


Figure 2.15. Device with established conventional current direction

then we assign (+) and (-) as shown in Figure 2.16.

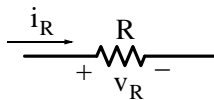


Figure 2.16. Voltage polarity in a device with established conventional current flow

**Note:**

Active devices, such as voltage and current sources, have their voltage polarity and current direction respectively, established as part of their notation. The current through and the voltage across these devices can easily be determined *if these devices deliver power to the rest of the circuit*. Thus with the voltage polarity as given in the circuit of Figure 2.17 (a), we assign a clockwise direction to the current as shown in Figure 2.17 (b). This is consistent with the passive sign convention since we have assumed that the voltage source delivers power to the rest of the circuit.

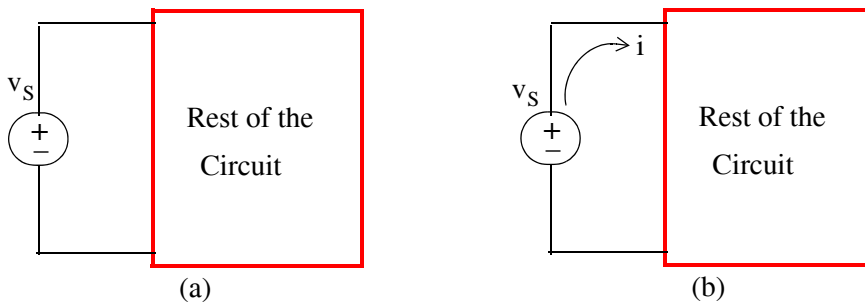


Figure 2.17. Direction of conventional current flow produced by voltage sources

Likewise, in the circuit of Figure 2.18 (a) below, the direction of the current source is clockwise, and assuming that this source delivers power to the rest of the circuit, we assign the voltage polarity shown in Figure 2.18 (b) to be consistent with the passive sign convention.

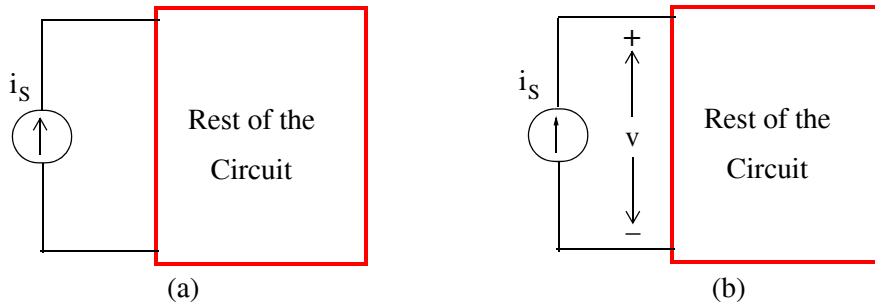


Figure 2.18. Voltage polarity across current sources

The following facts were discussed in the previous chapter but they are repeated here for emphasis.

There are two conditions required to setup and maintain the flow of an electric current:

1. *There must be some voltage (potential difference) to provide the energy (work) which will force electric current to flow in a specific direction in accordance with the conventional current flow (from a higher to a lower potential).*
2. *There must be a continuous (closed) external path for current to flow around this path (mesh or loop).*

The external path is usually made of two parts: (a) the *metallic wires* and (b) the *load* to which the electric power is to be delivered in order to accomplish some useful purpose or effect. The load may be a resistive, an inductive, or a capacitive circuit, or a combination of these.

### 2.8 Single Mesh Circuit Analysis

We will use the following example to develop a step-by-step procedure for analyzing (finding current, voltage drops and power) in a circuit with a single mesh.

---

#### Example 2.1

For the series circuit shown in Figure 2.19, we want to find:

- a. The current  $i$  which flows through each device
- b. The voltage drop across each resistor
- c. The power absorbed or delivered by each device



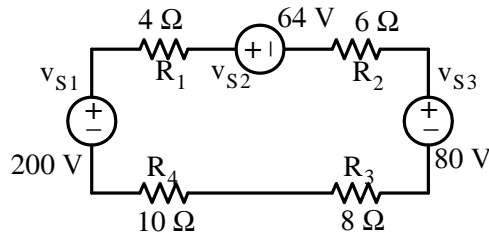


Figure 2.19. Circuit for Example 2.1

**Solution:**

- a. Step 1: We do not know which voltage source(s) deliver power to the other sources, so let us assume that the current  $i$  flows in the clockwise direction\* as shown in Figure 2.20.

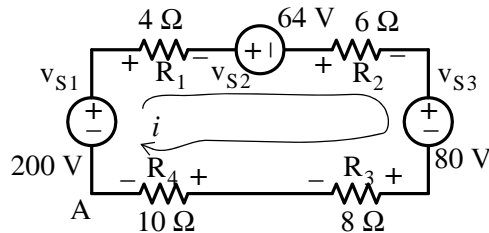


Figure 2.20. Circuit for Example 2.1 with assumed current direction

Step 2: We assign (+) and (-) polarities at each resistor's terminal in accordance with the established passive sign convention.

Step 3: By application of KVL and the adopted conventions, starting at node A and going clockwise, we obtain:

$$-v_{S1} + v_{R1} + v_{S2} + v_{R2} + v_{S3} + v_{R3} + v_{R4} = 0 \tag{2.20}$$

and by Ohm's law,

$$v_{R1} = R_1 i \quad v_{R2} = R_2 i \quad v_{R3} = R_3 i \quad v_{R4} = R_4 i$$

Then, by substitution of given values into (2.20), we obtain

$$-200 + 4i + 64 + 6i + 80 + 8i + 10i = 0$$

or

$$28i = 56$$

or

$$i = 2 \text{ A} \tag{2.21}$$

\* Henceforth, the current direction will be assumed to be that of the conventional current flow.

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## Chapter 2 Analysis of Simple Circuits

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- b. Knowing the current  $i$  from part (a), we can now compute the voltage drop across each resistor using Ohm's law  $v = Ri$ .

$$\begin{aligned}V_{R1} &= 4 \times 2 = 8 \text{ V} & V_{R2} &= 6 \times 2 = 12 \text{ V} \\V_{R3} &= 8 \times 2 = 16 \text{ V} & V_{R4} &= 10 \times 2 = 20 \text{ V}\end{aligned}\tag{2.22}$$

- c. The power absorbed (or delivered) by each device can be found from the power relation  $p = vi$ . Then, the power absorbed by each resistor is

$$\begin{aligned}p_{R1} &= 8 \times 2 = 16 \text{ w} & p_{R2} &= 12 \times 2 = 24 \text{ w} \\p_{R3} &= 16 \times 2 = 32 \text{ w} & p_{R4} &= 20 \times 2 = 40 \text{ w}\end{aligned}\tag{2.23}$$

and the power delivered (or absorbed) by each voltage source is

$$p_{V_{S1}} = -200 \times 2 = -400 \text{ w} \quad p_{V_{S2}} = 64 \times 2 = 128 \text{ w} \quad p_{V_{S3}} = 80 \times 2 = 160 \text{ w}\tag{2.24}$$

From (2.24), we observe that the 200 volt source absorbs  $-400$  watts of power. This means that this source delivers (supplies) 400 watts to the rest of the circuit. However, the other two voltage sources receive (absorb) power from the 200 volt source. Table 2.1 shows that the conservation of energy principle is satisfied since the total absorbed power is equal to the power delivered.

TABLE 2.1 Power delivered or absorbed by each device on the circuit of Figure 2.19

Device	Power Delivered (watts)	Power Absorbed (watts)
200 V Source	400	
64 V Source		128
80 V Source		160
4 W Resistor		16
6 W Resistor		24
8 W Resistor		32
10 W Resistor		40
Total	400	400

---

### Example 2.2

Repeat Example 2.1 with the assumption that the current  $i$  flows counterclockwise.

**Solution:**

We denote the current as  $i'$  ( $i$  prime) for this example. Then, starting at Node A and going counterclockwise, the voltage drops across each resistor are as indicated in Figure 2.21.

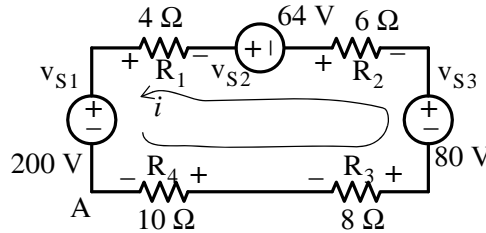


Figure 2.21. Circuit for Example 2.2

Repeating Steps 2 and 3 of Example 2.1, we obtain:

$$V_{R4} + V_{R3} - V_{S3} + V_{R2} - V_{S2} + V_{R1} + V_{S1} = 0 \quad (2.25)$$

Next, by Ohm's law,

$$V_{R1} = R_1 i' \quad V_{R2} = R_2 i' \quad V_{R3} = R_3 i' \quad V_{R4} = R_4 i'$$

By substitution of given values, we obtain

$$200 + 4i' - 64 + 6i' - 80 + 8i' + 10i' = 0$$

or

$$28i' = -56$$

or

$$i' = -2\text{A} \quad (2.26)$$

Comparing (2.21) with (2.26) we observe that  $i' = -i$  as expected.

**Definition 2.7**

A single node–pair circuit is one in which any number of simple elements are connected between the same pair of nodes. For example, the circuit of Figure 2.22 (a), which is more conveniently shown as Figure 2.22 (b), is a single node–pair circuit.

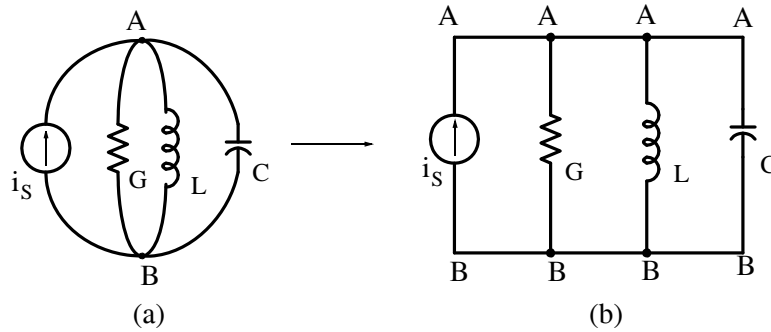


Figure 2.22. Circuit with a single node-pair

### 2.9 Single Node-Pair Circuit Analysis

We will use the following example to develop a step-by-step procedure for analyzing (finding currents, voltage drop and power) in a circuit with a single node-pair.

#### Example 2.3

For the parallel circuit shown in Figure 2.23, find:

- The voltage drop across each device
- The current  $i$  which flows through each conductance
- The power absorbed or delivered by each device

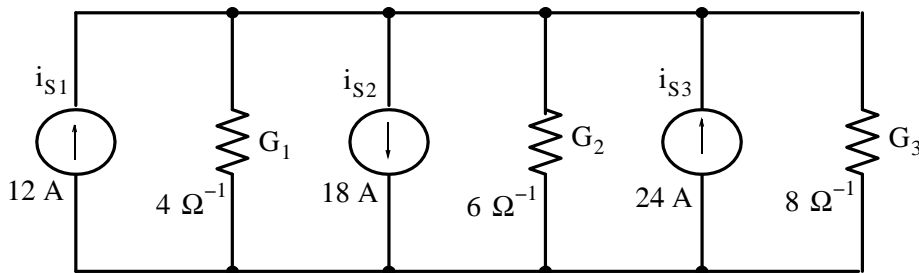


Figure 2.23. Circuit for Example 2.3

**Solution:**

- Step 1: We denote the single node-pair with the letters A and B as shown in Figure 2.24. It is important to observe that the same voltage (or potential difference) exists across each device. Node B is chosen as our reference node and it is convenient to assume that this reference node is at zero potential (ground) as indicated by the symbol  $\perp$

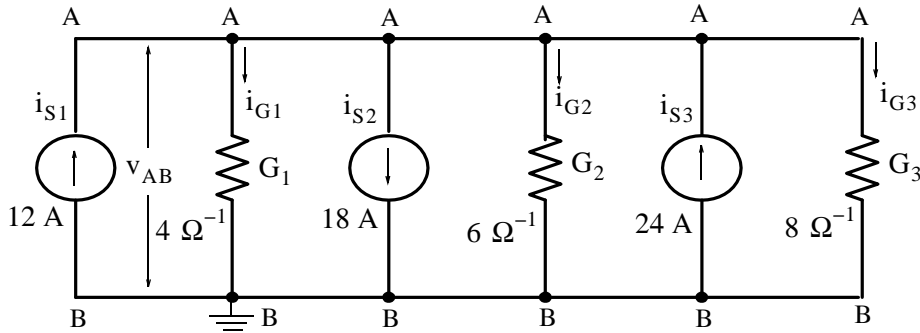


Figure 2.24. Circuit for Example 2.3 with assumed current directions

Step 2: We assign currents through each of the conductances  $G_1$ ,  $G_2$ , and  $G_3$  in accordance with the conventional current flow. These currents are shown as  $i_{G1}$ ,  $i_{G2}$ , and  $i_{G3}$ .

Step 3: By application of KCL and in accordance with our established convention, we choose node A which is the plus (+) reference point and we form the algebraic sum of the currents leaving (or entering) this node. Then, with plus (+) assigned to the currents leaving this node and with minus (–) entering this node we obtain

$$-i_{S1} + i_{G1} + i_{S2} + i_{G2} - i_{S3} + i_{G3} = 0 \quad (2.27)$$

and since

$$i_{G1} = G_1 v_{AB} \quad i_{G2} = G_2 v_{AB} \quad i_{G3} = G_3 v_{AB} \quad (2.28)$$

by substitution into (2.27),

$$-i_{S1} + G_1 v_{AB} + i_{S2} + G_2 v_{AB} - i_{S3} + G_3 v_{AB} = 0 \quad (2.29)$$

Solving for  $v_{AB}$ , we obtain

$$v_{AB} = \frac{i_{S1} - i_{S2} + i_{S3}}{G_1 + G_2 + G_3} \quad (2.30)$$

and by substitution of the given values, we obtain

$$v_{AB} = \frac{12 - 18 + 24}{4 + 6 + 8} \quad (2.31)$$

or

$$v_{AB} = 1 \text{ V} \quad (2.32)$$

b. From (2.28),

$$i_{G1} = 4 \quad i_{G2} = 6 \quad i_{G3} = 8 \quad (2.33)$$

and we observe that with these values, (2.27) is satisfied.

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## Chapter 2 Analysis of Simple Circuits

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- c. The power absorbed (or delivered) by each device can be found from the power relation  $p = vi$ . Then, the power absorbed by each conductance is

$$\begin{aligned}p_{G1} &= 1 \times 4 = 4 \text{ w} \\p_{G2} &= 1 \times 6 = 6 \text{ w} \\p_{G3} &= 1 \times 8 = 8 \text{ w}\end{aligned}\tag{2.34}$$

and the power delivered (or absorbed) by each current source is

$$\begin{aligned}p_{I1} &= 1 \times (-12) = -12 \text{ w} \\p_{I2} &= 1 \times 18 = 18 \text{ w} \\p_{I3} &= 1 \times (-24) = -24 \text{ w}\end{aligned}\tag{2.35}$$

From (2.35) we observe that the 12 A and 24 A current sources absorb  $-12 \text{ w}$  and  $-24 \text{ w}$  respectively. This means that these sources deliver (supply) a total of  $36 \text{ w}$  to the rest of the circuit. The 18 A source absorbs power.

Table 2.2 shows that the conservation of energy principle is satisfied since the absorbed power is equal to the power delivered.

TABLE 2.2 Power delivered or absorbed by each device of Figure 2.23

Device	Power Delivered (watts)	Power Absorbed (watts)
12 A Source	12	
18 A Source		18
24 A Source	24	
$4 \Omega^{-1}$ Conductance		4
$6 \Omega^{-1}$ Conductance		6
$8 \Omega^{-1}$ Conductance		8
Total	36	36

---

## 2.10 Voltage and Current Source Combinations

### Definition 2.8

Two or more voltage sources connected in series are said to be *series aiding* when the plus (+) terminal of any one voltage source is connected to the minus (-) terminal of another, or when the minus (-) terminal of any one voltage source is connected to the plus (+) terminal of another.

Two or more series aiding voltage sources may be replaced by an equivalent voltage source whose value is the algebraic sum of the individual voltage sources as shown in Figure 2.25.

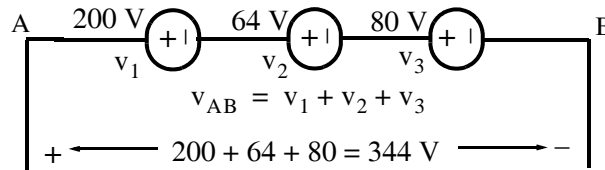


Figure 2.25. Addition of voltage sources in series when all have same polarity

A good example of combining voltage sources as series aiding is when we connect several AA size batteries each rated at 1.5 v to power up a hand calculator, or a small flashlight.

### Definition 2.9

Two or more voltage sources connected in series are said to be *series opposing* when the plus (+) terminal of one voltage source is connected to the plus (+) terminal of the other voltage source or when the minus (-) of one voltage source is connected to the minus (-) terminal of the other voltage source. Two series opposing voltage sources may be replaced by an equivalent voltage source whose value is the algebraic difference of the individual voltage sources as shown in Figure 2.26.

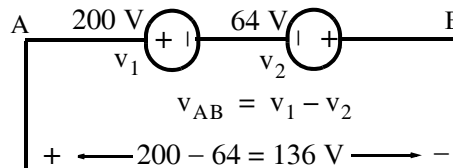


Figure 2.26. Addition of voltage sources in series when they have different polarity

### Definition 2.10

Two or more current sources connected in parallel are said to be *parallel aiding* when the arrows indicating the direction of the current flow have the same direction. They can be combined into a single current source as shown in Figure 2.27.

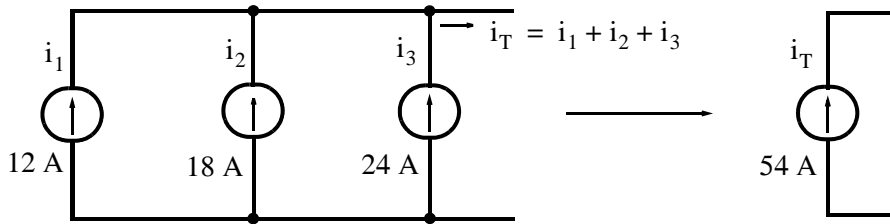


Figure 2.27. Addition of current sources in parallel when all have same direction

### Definition 2.11

Two or more current sources connected in parallel are said to be *parallel opposing* when the arrows indicating the direction of the current flow have opposite direction. They can be replaced by an equivalent current source whose value is the algebraic difference of the individual current sources as shown in Figure 2.28.

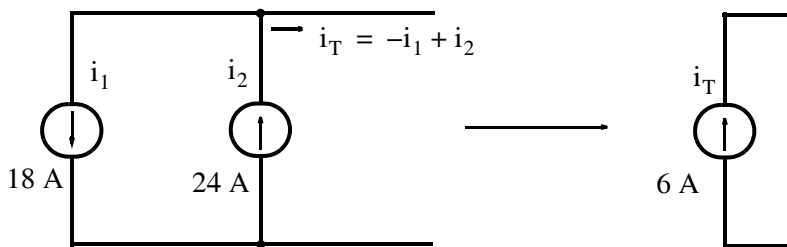


Figure 2.28. Addition of current sources in parallel when they have opposite direction

## 2.11 Resistance and Conductance Combinations

Often, resistors are connected in series or in parallel. With either of these connections, series or parallel, it is possible to replace these resistors by a single resistor to simplify the computations of the voltages and currents. Figure 2.29 shows  $n$  resistors connected in series.

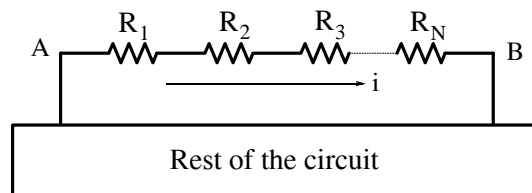


Figure 2.29. Addition of resistances in series

The combined or equivalent resistance  $R_{eq}$  is

$$R_{eq} = \frac{V_{AB}}{i} = \frac{V_{R1}}{i} + \frac{V_{R2}}{i} + \frac{V_{R3}}{i} + \dots + \frac{V_{RN}}{i}$$

or



$$R_{eq} = R_1 + R_2 + R_3 + \dots + R_n = \sum_{k=1}^n R_k \quad (2.36)$$

For Resistors in Series

### Example 2.4

For the circuit of Figure 2.30, find the value of the current  $i$  after combining the voltage sources to a single voltage source and the resistances to a single resistor.

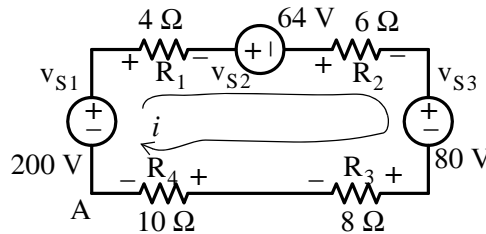


Figure 2.30. Circuit for Example 2.4

#### Solution:

We add the values of the voltage sources as indicated in Definitions 8 and 9, we add the resistances in accordance with (2.36), and we apply Ohm's law. Then,

$$i = \frac{\sum v}{\sum R} = \frac{200 - (64 + 80)}{28} = \frac{56}{28} = 2 \text{ A} \quad (2.37)$$

Next, we consider the case where  $n$  resistors are connected in parallel as shown in Figure 2.31.

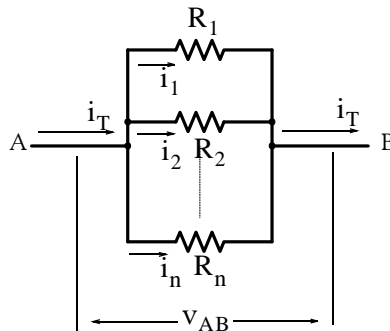


Figure 2.31. Addition of resistances in parallel

By KCL,

$$i_T = i_1 + i_2 + \dots + i_n \quad (2.38)$$

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## Chapter 2 Analysis of Simple Circuits

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The same voltage exists across each resistor; therefore, dividing each term of (2.38) by  $V_{AB}$ , we obtain

$$\frac{i_T}{V_{AB}} = \frac{i_1}{V_{AB}} + \frac{i_2}{V_{AB}} + \dots + \frac{i_n}{V_{AB}} \quad (2.39)$$

and since  $v/i = R$ , then  $i/v = 1/R$  and thus (2.39) can be written as

$$\frac{1}{R_{AB}} = \frac{1}{R_1} + \frac{1}{R_2} + \dots + \frac{1}{R_n}$$

or

$$\boxed{\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \dots + \frac{1}{R_n}} \quad (2.40)$$

For Resistors in Parallel

For the special case of two parallel resistors, (2.40) reduces to

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}$$

or

$$\boxed{R_{eq} = R_1 \parallel R_2 = \frac{R_1 \cdot R_2}{R_1 + R_2}} \quad (2.41)$$

where the designation  $R_1 \parallel R_2$  indicates that  $R_1$  and  $R_2$  are in parallel.

Also, since  $G = 1/R$ , from (2.38),

$$\boxed{G_{eq} = G_1 + G_2 + \dots + G_n = \sum_{k=1}^n G_k} \quad (2.42)$$

that is, *parallel conductances combine as series resistors do.*

---

### Example 2.5

In the circuit of Figure 2.32,

- a. Replace all resistors with a single equivalent resistance  $R_{eq}$
- b. Compute the voltage  $v_{AB}$  across the current source.

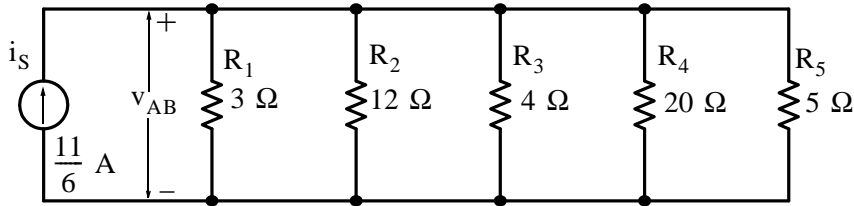


Figure 2.32. Circuit for Example 2.5

**Solution:**

- a. We could use (2.40) to find the equivalent resistance  $R_{eq}$ . However, it is easier to form groups of two parallel resistors as shown in Figure 2.33 and use (2.41) instead.

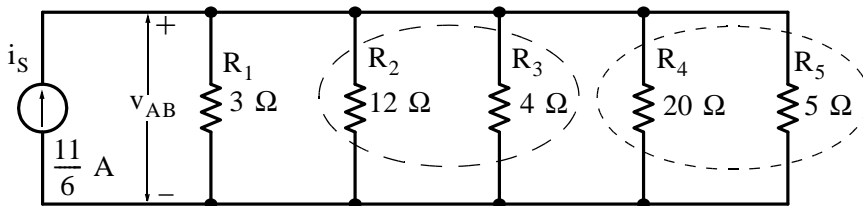


Figure 2.33. Groups of parallel combinations for the circuit of Example 2.5.

Then,

$$R_2 \parallel R_3 = \frac{12 \times 4}{12 + 4} = 3 \Omega$$

Also,

$$R_4 \parallel R_5 = \frac{20 \times 5}{20 + 5} = 4 \Omega$$

and the circuit reduces to that shown in Figure 2.34.

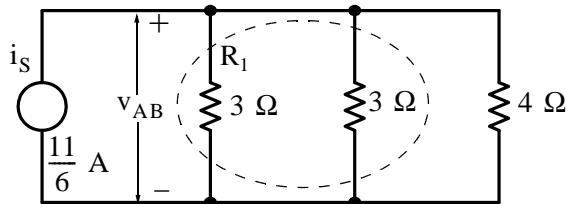


Figure 2.34. Partial reduction for the circuit of Example 2.5

Next,

$$3 \parallel 3 = \frac{3 \times 3}{3 + 3} = 1.5 \Omega$$

Finally,

$$R_{eq} = 1.5 \parallel 4 = \frac{1.5 \times 4}{1.5 + 4} = \frac{12}{11} \Omega$$

and the circuit reduces to that shown in Figure 2.35

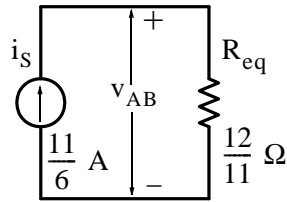


Figure 2.35. Reduction of the circuit of Example 2.5 to its simplest form

b. The voltage  $v_{AB}$  across the current source is

$$v_{AB} = IR_{eq} = \frac{11}{6} \cdot \frac{12}{11} = 2 \text{ V} \quad (2.43)$$

### 2.12 Voltage Division Expressions

In the circuit of Figure 2.36,  $v_S$ ,  $R_1$ , and  $R_2$  are known.

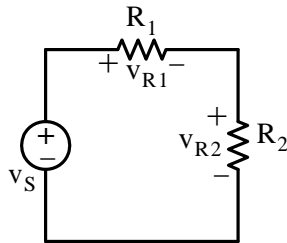


Figure 2.36. Circuit for the derivation of the voltage division expressions

For the circuit of Figure 2.36, we will derive the voltage division expressions which state that:

$$v_{R1} = \frac{R_1}{R_1 + R_2} v_S \quad \text{and} \quad v_{R2} = \frac{R_2}{R_1 + R_2} v_S$$

These expressions enable us to obtain the voltage drops across the resistors in a series circuit simply by observation.

#### Derivation:

By Ohm's law in the circuit of Figure 2.36 where  $i$  is the current flowing through  $i$ , we obtain

$$v_{R1} = R_1 i \quad \text{and} \quad v_{R2} = R_2 i \quad (2.44)$$

Also,

$$(R_1 + R_2)i = v_S$$

or

$$i = \frac{v_S}{R_1 + R_2} \quad (2.45)$$

and by substitution of (2.45) into (2.44) we obtain the voltage division expressions below.

$$v_{R1} = \frac{R_1}{R_1 + R_2} v_S \quad \text{and} \quad v_{R2} = \frac{R_2}{R_1 + R_2} v_S$$

**VOLTAGE DIVISION EXPRESSIONS**

(2.46)

### Example 2.6

In the network of Figure 2.37, the arrows indicate that resistors  $R_1$  and  $R_2$  are variable and that the power supply is set for 12 V.

- a. Compute  $v_{R1}$  and  $v_{R2}$  if  $R_1$  and  $R_2$  are adjusted for  $7 \Omega$  and  $5 \Omega$  respectively.
- b. To what values should  $R_1$  and  $R_2$  be adjusted so that  $v_{R1} = 3 \text{ V}$ ,  $v_{R2} = 9 \text{ V}$ , and  $R_1 + R_2 = 12 \Omega$ ?
- c. Using Simulink and SimPowerSystems\*, create a model to simulate the voltage  $v_{R2}$

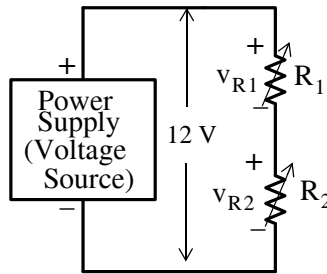


Figure 2.37. Network for Example 2.6

### Solution:

- a. Using the voltage division expressions of (2.46), we obtain

$$v_{R1} = \frac{R_1}{R_1 + R_2} v_S = \frac{7}{7 + 5} \times 12 = 7 \text{ V}$$

and

$$v_{R2} = \frac{R_2}{R_1 + R_2} v_S = \frac{5}{7 + 5} \times 12 = 5 \text{ V}$$

---

\* For an introduction to Simulink and SimPowerSystems, please refer to Appendices B and C respectively.

## Chapter 2 Analysis of Simple Circuits

b. Since  $v_{R_1} + v_{R_2} = 3 + 9 = 12 \text{ V}$ ,  $R_1 + R_2 = 12 \Omega$ , and the voltage drops are proportional to the resistances, it follows that if we let  $R_1 = 3 \Omega$  and  $R_2 = 9 \Omega$ , the voltage drops  $v_{R_1}$  and  $v_{R_2}$  will be 3 V and 9 V respectively.

c.

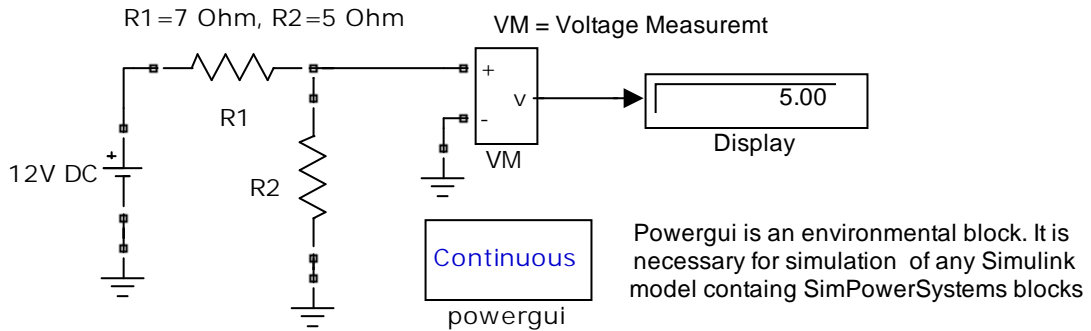


Figure 2.38. Simulink / SimPower Systems model for Example 2.6

### 2.13 Current Division Expressions

In the circuit shown in Figure 2.39,  $i_s$ ,  $G_1$ , and  $G_2$  are known.

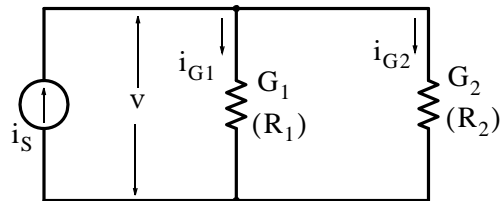


Figure 2.39. Circuit for the derivation of the current division expressions

For the circuit of Figure 2.39, we will derive the current division expressions which state that

$$i_{G1} = \frac{G_1}{G_1 + G_2} i_s \quad \text{and} \quad i_{G2} = \frac{G_2}{G_1 + G_2} i_s$$

and these expressions enable us to obtain the currents through the conductances (or resistances) in a parallel circuit simply by observation.

#### Derivation:

By Ohm's law for conductances, we obtain

$$i_{G1} = G_1 v \quad \text{and} \quad i_{G2} = G_2 v \quad (2.47)$$

Also,

$$(G_1 + G_2)v = i_s$$

or

$$v = \frac{i_s}{G_1 + G_2} \quad (2.48)$$

and by substitution of (2.48) into (2.47)

$$i_{G1} = \frac{G_1}{G_1 + G_2} i_s \quad \text{and} \quad i_{G2} = \frac{G_2}{G_1 + G_2} i_s \quad (2.49)$$

Also, since

$$R_1 = \frac{1}{G_1} \quad R_2 = \frac{1}{G_2}$$

by substitution into (2.49) we obtain

$$i_{R1} = \frac{R_2}{R_1 + R_2} i_s \quad \text{and} \quad i_{R2} = \frac{R_1}{R_1 + R_2} i_s$$

CURRENT DIVISION EXPRESSIONS

(2.50)

### Example 2.7

For the circuit in Figure 2.40, compute the voltage drop  $v$ . Verify your answer with a Simulink / SimPowerSystems model.

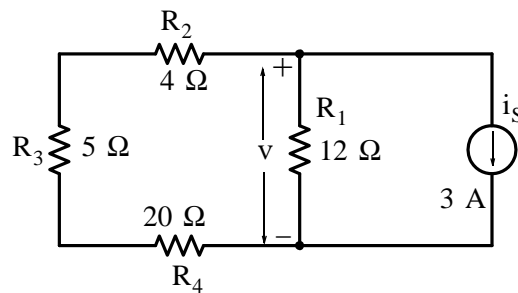


Figure 2.40. Circuit for Example 2.7

### Solution:

The current source  $i_s$  divides into currents  $i_1$  and  $i_2$  as shown in Figure 2.40. We observe that the voltage  $v$  is the voltage across the resistor  $R_1$ . Therefore, we are only interested in current  $i_1$ . This is found by the current division expression as

$$i_1 = \frac{R_2 + R_3 + R_4}{R_1 + R_2 + R_3 + R_4} \cdot i_s = \frac{4 + 5 + 20}{12 + 4 + 5 + 20} \cdot 3 = \frac{87}{41} \text{ A}$$

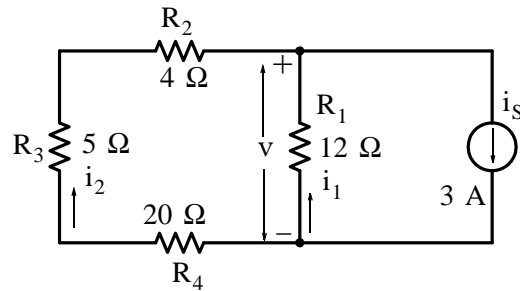


Figure 2.41. Application of current division expressions for the circuit of Example 2.7

and observing the passive sign convention, the voltage  $v$  is

$$v = -i_1 R_1 = -\frac{87}{41} \cdot 12 = -\frac{1044}{41} \text{ V}$$

or

$$v = -25.46 \text{ V}$$

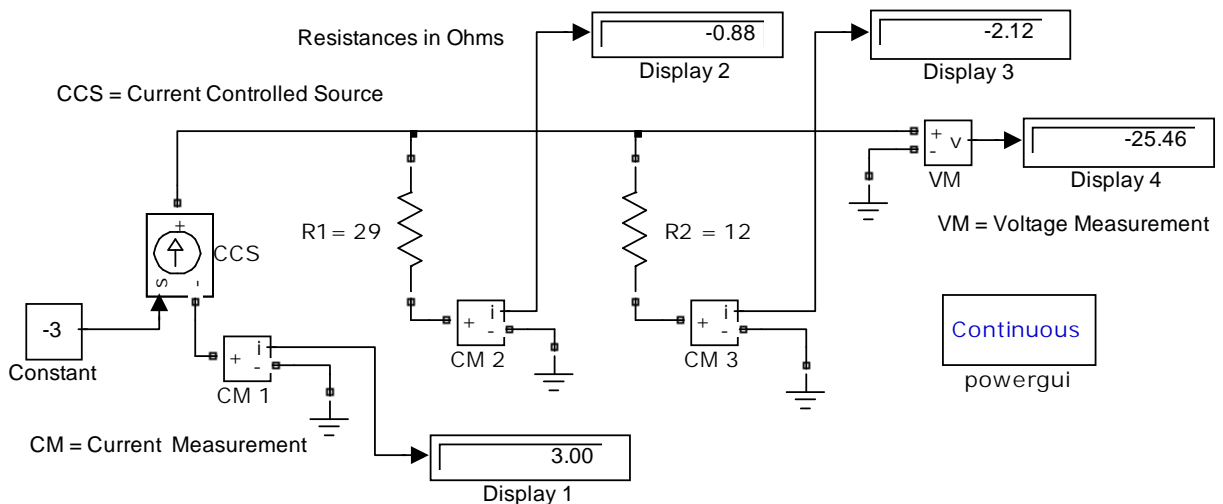


Figure 2.42. Simulink / SimPower Systems model for Example 2.7

### 2.14 Standards for Electrical and Electronic Devices

Standardization of electronic components such as resistors, capacitors and diodes is carried out by various technical committees. In the United States, the Electronics Industries Association (EIA) and the American National Standards Institute (ANSI) have established and published several standards for electrical and electronic devices to provide interchangeability among similar products made by different manufacturers. Also, the U.S. Department of Defense or its agencies issue



standards known as *Military Standards*, or simply MIL–stds. All of the aforementioned standards are updated periodically. The interested reader may find the latest revisions in the Internet or the local library.

## 2.15 Resistor Color Code

The Resistor Color Code is used for marking and identifying pertinent data for standard resistors. Figures 2.43 and 2.44 show the color coding scheme per EIA Standard RS–279 and MIL–STD–1285A respectively.

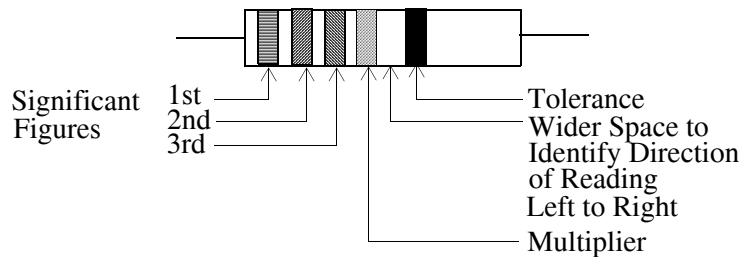


Figure 2.43. Resistor Color Code per EIA Standard RS–279

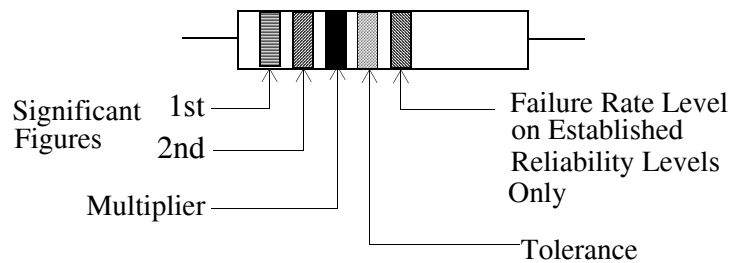


Figure 2.44. Resistor Color Code per MIL–STD–1285A

In a color coded scheme, each color represents a single digit number, or conversely, a single digit number can be represented by a particular color band as shown in Table 2.3 that is based on MIL–STD–1285A color code.

As shown in Figure 2.44, the first and second bands designate the first and second significant digits respectively, the third represents the multiplier, that is, the number by which the first two digits are multiplied, and the fourth and fifth bands, if they exist, indicate the tolerance and failure rate respectively. The *tolerance* is the maximum deviation from the specified nominal value and it is given as a percentage. The *failure rate* is the percent probability of failure in a 1000–hour time interval.

TABLE 2.3 Resistor values per MIL-STD-1285A

Color Code	1st & 2nd Digits	Multiplier (3rd Digit)	Tolerance (Percent)	Fail Rate (Percent)
Black	0	1		
Brown	1	10	1	1
Red	2	100	2	0.1
Orange	3	1000		0.01
Yellow	4	10000		0.001
Green	5	100000	0.5	
Blue	6	1000000	0.25	
Violet	7		0.1	
Gray	8			
White	9			
Gold		0.1	5	
Silver		0.01	10	
No Color			20	

Let A and B represent the first and second significant digits and C represent the multiplier. Then the resistance value is found from the expression

$$R = (10 \times A + B) \times 10^C \quad (2.51)$$

---

### Example 2.8

The value of a resistor is coded with the following colored band code, left to right: Brown, Green, Blue, Gold, Red. What is the value, tolerance, and probability of failure for that resistor?

#### Solution:

Table 2.3 yields the following data: Brown (1st significant digit) = 1, Green (2nd significant digit) = 5, and Blue (multiplier) = 1,000,000. Therefore, the nominal value of this resistor is 15,000,000 Ohms or 15 M $\Omega$ . The 4th band is Gold indicating a  $\pm 5\%$  tolerance meaning that the maximum deviation from the nominal value is  $15,000,000 \pm 5\% = 15,000,000 \times \pm 0.05 = \pm 750,000$  Ohms or  $\pm 0.75$  M $\Omega$ . That is, this resistor can have a value anywhere between 14.25 M $\Omega$  and 15.75 M $\Omega$ . Since the 5th band is Red, there is a 0.1% probability that this resistor will fail after 1000 hours of operation.

---

## 2.16 Power Rating of Resistors

As it was mentioned in Section 2.2, a resistor, besides its resistance rating (ohms) has a power rating (watts) commonly referred to as the *wattage* of the resistor, and common resistor wattage values are  $\frac{1}{4}$  watt,  $\frac{1}{2}$  watt, 1 watt, 2 watts, 5 watts and so on. To appreciate the importance of the wattage of a resistor, let us refer to the voltage divider circuit of Example 2.6, Figure 2.37 where the current is  $12 \text{ V}/12 \text{ } \Omega = 1 \text{ A}$ . Using the power relation  $p_R = i^2 R$ , we find that the wattage of the  $7 \text{ } \Omega$  and  $5 \text{ } \Omega$  resistors would be 7 watts and 5 watts respectively. We could also divide the 12 volt source into two voltages of 7 V and 5 V using a  $7 \text{ k}\Omega$  and a  $5 \text{ k}\Omega$  resistor. Then, with this arrangement the current would be  $12 \text{ V}/12 \text{ k}\Omega = 1 \text{ mA}$ . The wattage of the  $7 \text{ k}\Omega$  and  $5 \text{ k}\Omega$  resistors would then be  $(10^{-3})^2 \times 7 \times 10^3 = 7 \times 10^{-3} \text{ W} = 7 \text{ mW}$  and  $(10^{-3})^2 \times 5 \times 10^3 = 5 \text{ mW}$  respectively.

## 2.17 Temperature Coefficient of Resistance

The resistance of any pure metal, such as copper, changes with temperature. For each degree that the temperature of a copper wire rises above  $20^\circ\text{C}$  Celsius, up to about  $200^\circ\text{C}$ , the resistance increases 0.393 of 1 percent of what it was at 20 degrees Celsius. Similarly, for each degree that the temperature drops below  $20^\circ\text{C}$ , down to about  $-50^\circ\text{C}$ , the resistance decreases 0.393 of 1 percent of what it was at  $20^\circ\text{C}$ . This percentage of change in resistance is called the *Temperature Coefficient of Resistance*. In general, the resistance of any pure metal at temperature  $T$  in degrees Celsius is given by

$$R = R_{20}[1 + \alpha_{20}(T - 20)] \quad (2.52)$$

where  $R_{20}$  is the resistance at  $20^\circ\text{C}$  and  $\alpha_{20}$  is the temperature coefficient of resistance at  $20^\circ\text{C}$ .

### Example 2.9

The resistance of a long piece of copper wire is  $48 \text{ } \Omega$  at  $20^\circ\text{C}$ .

- What would the resistance be at  $50^\circ\text{C}$ ?
- Construct a curve showing the relation between resistance and temperature.

**Solution:**

- The temperature rise is  $50 - 20 = 30$  degrees Celsius and the resistance increases 0.393% for every degree rise. Therefore the resistance increases by  $30 \times 0.393 = 11.79\%$ . This represents

## Chapter 2 Analysis of Simple Circuits

an increase of  $0.1179 \times 48 \Omega$  in resistance or  $5.66 \Omega$ . Therefore, the resistance at 50 degrees Celsius is  $48 + 5.66 = 53.66 \Omega$ .

b. The relation of (2.52) is an equation of a straight line with slope =  $R_{20}\alpha_{20}$ . This straight line is easily constructed with the Microsoft Excel spreadsheet shown in Figure 2.45.

From Figure 2.45, we observe that the resistance reaches zero value at approximately  $-235^\circ\text{C}$ .

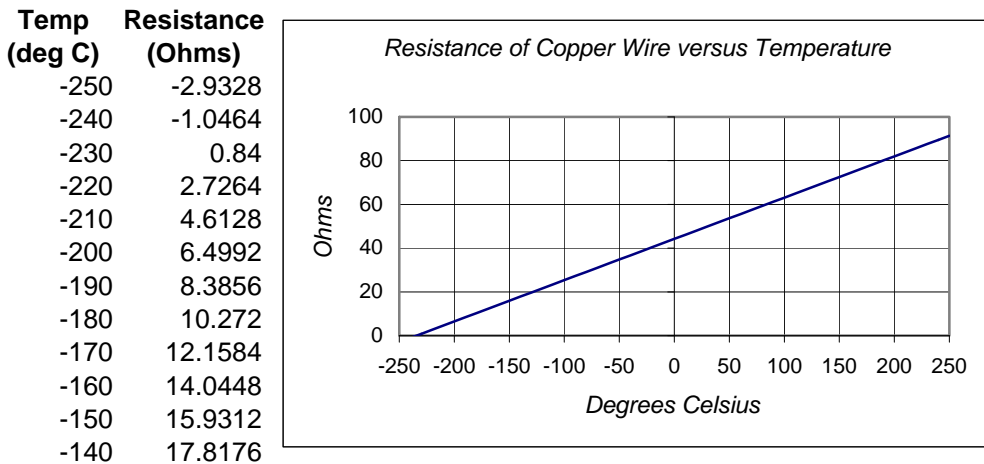


Figure 2.45. Spreadsheet for construction of equation (2.52)

### 2.18 Ampere Capacity of Wires

For public safety, electric power supply (mains) wiring is controlled by local, state and federal boards, primarily on the *National Electric Code* (NEC) and the *National Electric Safety Code*. Moreover, many products such as wire and cable, fuses, circuit breakers, outlet boxes and appliances are governed by *Underwriters Laboratories* (UL) Standards which approves consumer products such as motors, radios, television sets etc.

Table 2.4 shows the NEC allowable current-carrying capacities for copper conductors based on the type of insulation.

The ratings in Table 2.4 are for copper wires. The ratings for aluminum wires are typically 84% of these values. Also, these rating are for not more than three conductors in a cable with temperature  $30^\circ\text{C}$  or  $86^\circ\text{F}$ . The NEC contains tables with correction factors at higher temperatures.

### 2.19 Current Ratings for Electronic Equipment

There are also standards for the internal wiring of electronic equipment and chassis. Table 2.5 provides recommended current ratings for copper wire based on  $45^\circ\text{C}$  ( $40^\circ\text{C}$  for wires smaller

than 22 AWG. Listed also, are the circular mils and these denote the area of the cross section of each wire size. A *circular mil* is the area of a circle whose diameter is 1 mil (one-thousandth of an inch). Since the area of a circle is proportional to the square of its diameter, and the area of a circle one mil in diameter is one circular mil, the area of any circle in circular mils is the square of its diameter in mils.

TABLE 2.4 Current Ratings for Electronic Equipment and Chassis Copper Wires

Wire Size			Maximum Current (Amperes)	
AWG	Circular Mils	Nominal Resistance (Ohms/1000 ft) at 100 °C	Wire in Free Air	Wire Confined in Insulation
32	63.2	188	0.53	0.32
30	100.5	116	0.86	0.52
28	159.8	72	1.4	0.83
26	254.1	45.2	2.2	1.3
24	404	28.4	3.5	2.1
22	642.4	22	7	5
20	1022	13.7	11	7.5
18	1624	6.5	16	10
16	2583	5.15	22	13
14	4107	3.2	32	17
12	6530	2.02	41	23
10	10380	1.31	55	33
8	16510	0.734	73	46
6	26250	0.459	101	60
4	41740	0.29	135	80
2	66370	0.185	181	100
1	83690	0.151	211	125
0	105500	0.117	245	150
00	133100	0.092	283	175
000	167800	0.074	328	200
0000	211600	0.059	380	225

† Dry Locations Only    ‡ Nickel or nickel-coated copper only

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## Chapter 2 Analysis of Simple Circuits

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A *mil-foot wire* is a wire whose length is one foot and has a cross-sectional area of one circular mil.

The resistance of a wire of length  $l$  can be computed with the relation

$$R = \frac{\rho l}{d^2} \quad (2.53)$$

where  $\rho$  = resistance per mil-foot,  $l$  = length of wire in feet,  $d$  = diameter of wire in mils, and  $R$  is the resistance at  $20^\circ\text{C}$ .

---

### Example 2.10

Compute the resistance per mile of a copper conductor  $1/8$  inch in diameter given that the resistance per mil-foot of copper is  $10.4 \Omega$  at  $20^\circ\text{C}$ .

**Solution:**

$$(1/8) \text{ in} = 0.125 \text{ in} = 125 \text{ mils}$$

and from (2.53)

$$R = \frac{\rho l}{d^2} = \frac{10.4 \times 5280}{125^2} = 3.51 \Omega$$

Column 3 of Table 2.5 shows the copper wire resistance at  $100^\circ\text{C}$ . Correction factors must be applied to determine the resistance at other temperatures or for other materials. For copper, the conversion equation is

$$R_T = R_{100}[1 + 0.004(T - 100)] \quad (2.54)$$

where  $R_T$  is the resistance at the desired temperature,  $R_{100}$  is the resistance at  $100^\circ\text{C}$  for copper, and  $T$  is the desired temperature.

---

### Example 2.11

Compute the resistance of 1000 ft of size AWG 12 copper wire at  $30^\circ\text{C}$ .

**Solution:**

From Table 2.5 we find that the resistance of 1000 ft of size AWG 12 copper wire at  $100^\circ\text{C}$  is  $2.02 \Omega$ . Then, by (2.54), the resistance of the same wire at  $30^\circ\text{C}$  is

$$R_{30^\circ\text{C}} = 2.02[1 + 0.004(30 - 100)] = 1.45 \Omega$$

---

## 2.20 Copper Conductor Sizes for Interior Wiring

In the design of an interior electrical installation, the electrical contractor must consider two important factors:

- a. The wiring size in each section must be selected such that the current shall not exceed the current carrying capacities as defined by the NEC tables. Therefore, the electrical contractor must accurately determine the current which each wire must carry and make a tentative selection of the size listed in Table 2.4.
- b. The voltage drop throughout the electrical system must then be computed to ensure that it does not exceed certain specifications. For instance, in the lighting part of the system referred to as the *lighting load*, a variation of more than 5% in the voltage across each lamp causes an unpleasant variation in the illumination. Also, the voltage variation in the heating and air conditioning load must not exceed 10% .

### Important!

The requirements stated here are for instructional purposes only. They change from time to time. It is, therefore, imperative that the designer consults the latest publications of the applicable codes for compliance.

### Example 2.12

Figure 2.46 shows a lighting load distribution diagram for an interior electric installation.

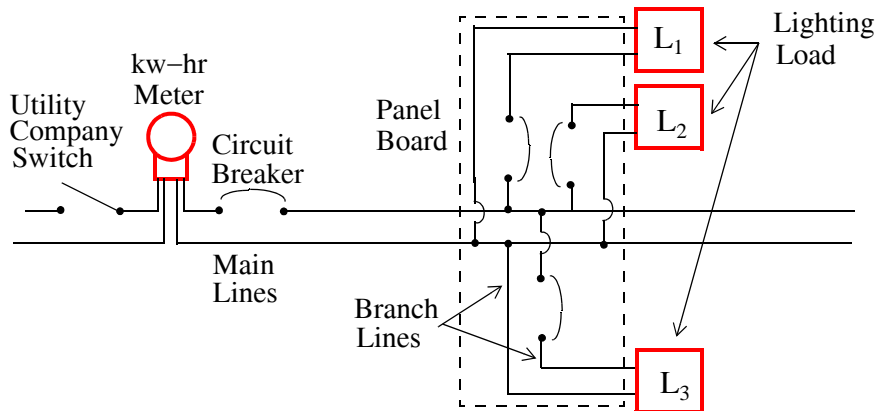


Figure 2.46. Load distribution for an interior electric installation


The panel board is 200 feet from the meter. Each of the three branches has 12 outlets for 75 w, 120 volt lamps. The *load center* is that point on the branch line at which all lighting loads may be considered to be concentrated. For this example, assume that the distance from the panel to the load center is 60 ft. Compute the size of the main lines. Use T (thermoplastic insulation) type copper conductor and base your calculations on 25°C temperature environment.

### Solution:

It is best to use a spreadsheet for the calculations so that we can compute sizes for more and different branches if need be.

The computations for Parts I and II are shown on the spreadsheet in Figures 2.47 and 2.48 where from the last line of Part II we see that the percent line drop is 12.29 and this is more than twice the allowable 5% drop. With the 12.29% voltage variation the brightness of the lamps would vary through wide ranges, depending on how many lamps were in use at one time.

A much higher voltage than the rated 120 V would cause these lamps to glow far above their rated candle power and would either burn them immediately, or shorten their life considerably. It is therefore necessary to install larger than 12 AWG main line. The computations in Parts III through V of the spreadsheet of Figures 2.47 and 2.48 indicate that we should not use a conductor less than size 6 AWG.





Part I	Sizing in Accordance with NEC (Table 2.2)	Calculation	Value	Units
Step	Description			
1	Meter to Panel Board Distance		200 ft	
2	No. of Branches from Panel Board		3	
3	Outlets per branch		12	
4	Outlet Voltage Rating		120 V	
5	Power drawn from each outlet		75 w	
6	Power required by each branch	Step 3 x Step 5	900 w	
7	Current drawn by each branch	Step 6 / Step 4	7.50 A	
8	Required wire size for branches *	Table 2.4	14 AWG	Carries up to 15 A
9	Current required by the main line	Step 2 x Step 7	22.50 A	
10	Required wire size for main line	Table 2.4	12 AWG	Carries up to 20 A
<b>Part II Check for Voltage Drops</b>				
11	Distance from panel to load center		60 ft	
12	Number of wires in each branch		2	
13	Total length of wire in each branch	Step 11 x Step 12	120 ft	
14	Specified Temperature in °C		25 °C	
15	Resistance of 1000 ft 14 AWG wire at 100 °C	Table 2.5	3.20 Ω	
16	Resistance of 1000 ft 14 AWG wire at Spec. Temp	Equation (2.54)	2.24 Ω	
17	Resistance of actual length of wire	(Step 13 x Step 16) / 1000	0.269 Ω	
18	Voltage drop in each branch	Step 7 x Step 17	2.02 V	
19	Number of wires in each main		2	
20	Total length of wire in main line	Step 1 x Step 19	400 ft	
21	Resistance of 1000 ft 12 AWG wire at 100 °C	Table 2.5	2.02 Ω	
22	Resistance of 1000 ft 12 AWG wire at Spec. Temp	Equation (2.54)	1.41 Ω	
23	Resistance of actual length of wire	(Step 20 x Step 22) / 1000	0.57 Ω	
24	Voltage drop in main line	Step 9 x Step 23	12.73 V	
25	Voltage drop from meter to load center	Step 18 + Step 24	14.74 V	
26	Percent line drop	(Step 25 / Step 4) x 100	12.29 %	

\* No size smaller can be installed; smaller sizes have insufficient mechanical strength

Figure 2.47. Spreadsheet for Example 2.12, Parts I and II

<b>Part III Recalculation Using 10 AWG Conductor Size</b>				
27	Next larger conductor size for main line	Table 2.4	10 AWG	Carries up to 30 A
28	Resistance of 1000 ft 10 AWG wire at 100 °C	Table 2.5	1.310 Ω	
29	Resistance of 1000 ft 10 AWG wire at Spec. Temp	Equation (2.54)	0.917 Ω	
30	Resistance of actual length of wire	(Step 20 x Step 29)/1000	0.367 Ω	
31	Voltage drop in main line	Step 9 x Step 30	8.25 V	
32	Voltage drop from meter to load center	Step 18 + Step 24	10.27 V	
33	Percent line drop	(Step 32 / Step 4) x 100	8.56 %	
<b>Part IV Recalculation Using 8 AWG Conductor Size</b>				
34	Next larger conductor size for main line	Table 2.4	8 AWG	Carries up to 40 A
35	Resistance of 1000 ft 8 AWG wire at 100 °C	Table 2.5	0.734 Ω	
36	Resistance of 1000 ft 8 AWG wire at Spec. Temp	Equation (2.54)	0.514 Ω	
37	Resistance of actual length of wire	(Step 20 x Step 36)/1000	0.206 Ω	
38	Voltage drop in main line	Step 9 x Step 37	4.62 V	
39	Voltage drop from meter to load center	Step 18 + Step 38	6.64 V	
40	Percent line drop	(Step 39 / Step 4) x 100	5.53 %	
<b>Part V Recalculation Using 6 AWG Conductor Size</b>				
41	Next larger conductor size for main line	Table 2.4	6 AWG	Carries up to 55 A
42	Resistance of 1000 ft 6 AWG wire at 100 °C	Table 2.5	0.459 Ω	
43	Resistance of 1000 ft 6 AWG wire at Spec. Temp	Equation (2.54)	0.321 Ω	
44	Resistance of actual length of wire	(Step 20 x Step 43)/1000	0.129 Ω	
45	Voltage drop in main line	Step 9 x Step 44	2.89 V	
46	Voltage drop from meter to load center	Step 18 + Step 45	4.91 V	
47	Percent line drop	(Step 46 / Step 4) x 100	4.09 %	

Figure 2.48. Spreadsheet for Example 2.12, Parts III, IV, and V

## 2.21 Summary

- Ohm's Law states that the voltage across a device is proportional to the current through that device and the resistance is the constant of proportionality.
- Open circuit refers to an open branch (defined below) in a network. It can be thought of as a resistor with infinite resistance (or zero conductance). The voltage across the terminals of an open may have a finite value or may be zero whereas the current is always zero.
- Short circuit refers to a branch (defined below) in a network that contains no device between its terminals, that is, a piece of wire with zero resistance. The voltage across the terminals of a short is always zero whereas the current may have a finite value or may be zero.
- A resistor absorbs power.
- A resistor does not store energy. The energy is dissipated in the form of heat.
- A node is a common point where one end of two or more devices are connected.
- A branch is part of a network that contains a device and its nodes.
- A mesh is a closed path that does not contain other closed paths
- A loop contains two or more closed paths.
- Kirchoff's Current Law (KCL) states that the algebraic sum of the currents entering (or leaving) a node is zero.
- Kirchoff's Voltage Law (KVL) states that the algebraic sum of the voltage drops (or voltage rises) around a closed mesh or loop is zero.
- Two or more devices are said to be connected in series if and only if the same current flows through them.
- Two or more devices are said to be connected in parallel if and only if the same voltage exists across their terminals.
- A series circuit with a single mesh can be easily analyzed by KVL.
- A parallel circuit with a single node pair can be easily analyzed by KCL.
- If two or more voltage sources are in series, they can be replaced by a single voltage source with the proper polarity.
- If two or more current sources are in parallel, they can be replaced by a single current source with the proper current direction.
- If two or more resistors are connected in series, they can be replaced by an equivalent resistance whose value is

## Chapter 2 Analysis of Simple Circuits

$$R_{\text{eq}} = R_1 + R_2 + R_3 + \dots + R_n = \sum_{k=1}^n R_k$$

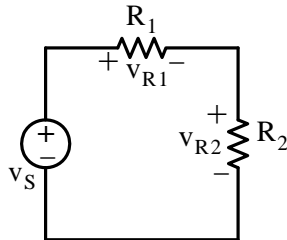
- If two or more resistors are connected in parallel, they can be replaced by an equivalent resistance whose value is

$$\frac{1}{R_{\text{eq}}} = \frac{1}{R_1} + \frac{1}{R_2} + \dots + \frac{1}{R_n}$$

- For the special case of two parallel resistors, the equivalent resistance is found from the relation

$$R_{\text{eq}} = R_1 \parallel R_2 = \frac{R_1 \cdot R_2}{R_1 + R_2}$$

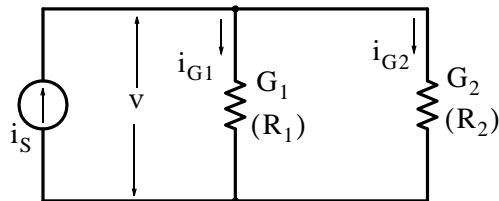
- Conductances connected in series combine as resistors in parallel do.
- Conductances connected in parallel combine as resistors in series do.
- For the simple series circuit below



the voltage division expressions state that:

$$v_{R1} = \frac{R_1}{R_1 + R_2} v_S \quad \text{and} \quad v_{R2} = \frac{R_2}{R_1 + R_2} v_S$$

- For the simple parallel circuit below



the current division expressions state that:

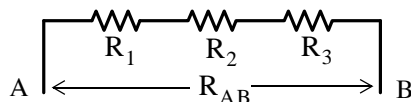
$$i_{R1} = \frac{R_2}{R_1 + R_2} i_S \quad \text{and} \quad i_{R2} = \frac{R_1}{R_1 + R_2} i_S$$

- In the United States, the Electronics Industries Association (EIA) and the American National Standards Institute (ANSI) have established and published several standards for electrical and electronic devices to provide interchangeability among similar products made by different manufacturers.
- The resistor color code is used for marking and identifying pertinent data for standard resistors. Two standards are the EIA Standard RS-279 and MIL-STD-1285A.
- Besides their resistance value, resistors have a power rating.
- The resistance of a wire increases with increased temperature and decreases with decreased temperature.
- The current ratings for wires and electronic equipment are established by national standards and codes.

### 2.22 Exercises

#### Multiple Choice

- Ohm's Law states that
  - the conductance is the reciprocal of resistance
  - the resistance is the slope of the straight line in a voltage versus current plot
  - the resistance is the sum of the voltages across all the devices in a closed path divided by the sum of the currents through all the devices in the closed path
  - the sum of the resistances around a closed loop is zero
  - none of the above
- Kirchoff's Current Law (KCL) states that
  - the sum of the currents in a closed path is zero
  - the current that flows through a device is inversely proportional to the voltage across that device
  - the sum of the currents through all the devices in a closed path is equal to the sum of the voltages across all the devices
  - the sum of the currents entering a node is equal to the sum of the currents leaving that node
  - none of the above
- Kirchoff's Voltage Law (KVL) states that
  - the voltage across a device is directly proportional to the current through that device
  - the voltage across a device is inversely proportional to the current through that device
  - the sum of the voltages across all the devices in a closed path is equal to the sum of the currents through all the devices
  - the sum of the voltages in a node is equal to the sum of the currents at that node
  - none of the above
- For the three resistors connected as shown below, the equivalent resistance  $R_{AB}$  is computed with the formula



A.  $R_{AB} = \sqrt{R_1 + R_2 + R_3}$

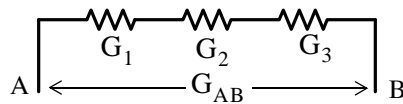
B.  $R_{AB} = \sqrt{R_1^2 + R_2^2 + R_3^2}$

C.  $R_{AB} = \frac{R_1 R_2 R_3}{R_1 + R_2 + R_3}$

D.  $R_{AB} = \sqrt{\frac{R_1 R_2 R_3}{R_1 + R_2 + R_3}}$

E. none of the above

5. For the three conductances connected as shown below, the equivalent conductance  $G_{AB}$  is computed with the formula



A.  $G_{AB} = \sqrt{G_1 + G_2 + G_3}$

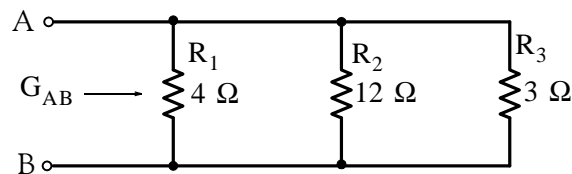
B.  $G_{AB} = \sqrt{G_1^2 + G_2^2 + G_3^2}$

C.  $G_{AB} = \frac{G_1 G_2 G_3}{G_1 + G_2 + G_3}$

D.  $\frac{1}{G_{AB}} = \frac{1}{G_1} + \frac{1}{G_2} + \frac{1}{G_3}$

E. none of the above

6. For the three resistances connected as shown below, the equivalent conductance  $G_{AB}$  is



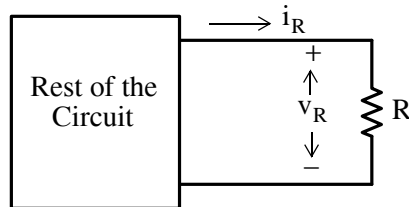
A.  $21 \Omega^{-1}$

B.  $1.5 \Omega^{-1}$

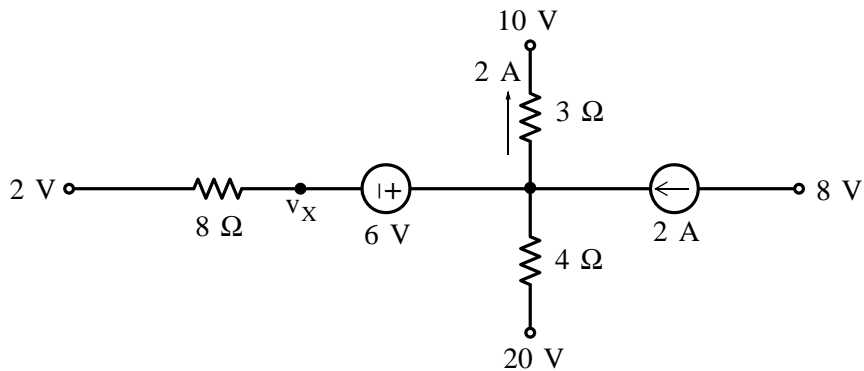
C.  $2/3 \Omega^{-1}$

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- D.  $144/19 \Omega^{-1}$   
E. none of the above
7. In the network shown below, when  $R = 4 \Omega$ , the voltage  $v_R = 6 \text{ V}$ . When  $R = 0 \Omega$ ,  $i_R = 2 \text{ A}$ . When  $R = \infty$ ,  $v_R$  is



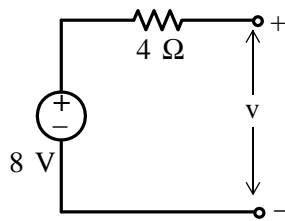
- A. 6 V  
B. 24 V  
C. 8 V  
D. 16 V  
E. none of the above
8. The node voltages shown in the partial network below are relative to some reference node not shown. The value of the voltage  $v_X$  is



- A. -6 V  
B. 16 V  
C. 0 V  
D. 10 V  
E. none of the above

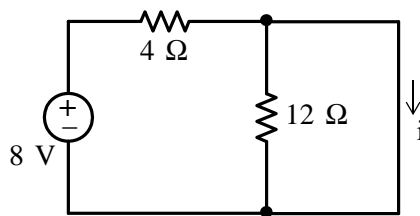


9. For the network below the value of the voltage  $v$  is



- A. 8 V
- B. 2 V
- C. -2 V
- D. -8 V
- E. none of the above

10. For the circuit below the value of the current  $i$  is

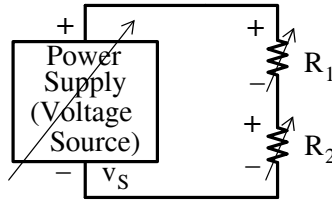


- A. 2 A
- B. 0 A
- C.  $\infty$  A
- D. 1 A
- E. none of the above

## Chapter 2 Analysis of Simple Circuits

### Problems

1. In the circuit below, the voltage source and both resistors are variable.



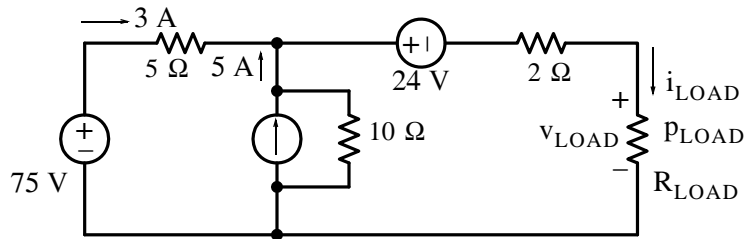
a. With  $v_S = 120 \text{ V}$ ,  $R_1 = 70 \Omega$ , and  $R_2 = 50 \Omega$ , compute the power absorbed by  $R_2$ .

Answer: 50 w

b. With  $v_S = 120 \text{ V}$  and  $R_1 = 0 \Omega$ , to what value should  $R_2$  be adjusted so that the power absorbed by it will be 200 w? Answer: 72  $\Omega$

c. With  $R_1 = 0 \Omega$  and  $R_2 = 100 \Omega$ , to what value should  $v_S$  be adjusted to so that the power absorbed by  $R_2$  will be 100 w? Answer: 100 V

2. In the circuit below,  $R_{\text{LOAD}}$  represents the load of that circuit.



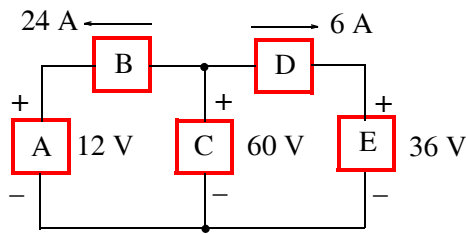
Compute:

a.  $i_{\text{LOAD}}$  Answer: 8 A

b.  $v_{\text{LOAD}}$  Answer: 20 V

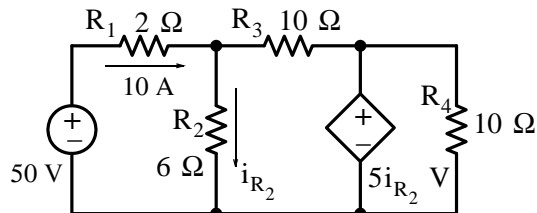
c.  $P_{\text{LOAD}}$  Answer: 160 w

3. For the circuit below, compute the power supplied or absorbed by each device.



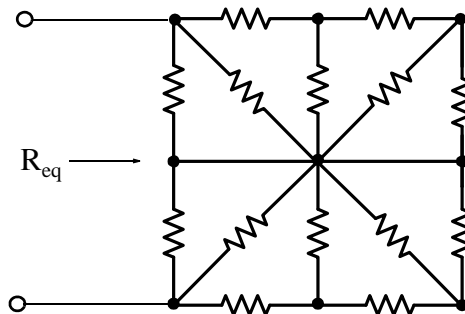
Answers:  $p_A = 288 \text{ w}$ ,  $p_B = 1152 \text{ w}$ ,  $p_C = -1800 \text{ w}$ ,  $p_D = 144 \text{ w}$ ,  $p_E = 216 \text{ w}$

4. In the circuit below, compute the power delivered or absorbed by the dependent voltage source.



Answer: 62.5 w

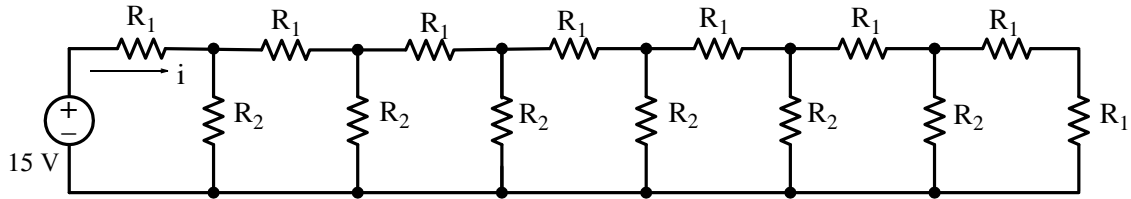
5. In the network below, each resistor is 10 Ω. Compute the equivalent resistance  $R_{eq}$ .



Answer:  $360/21 \text{ } \Omega$

6. In the network below,  $R_1 = 10 \text{ } \Omega$  and  $R_2 = 20 \text{ } \Omega$ . Compute the current  $i$  supplied by the 15 V source.

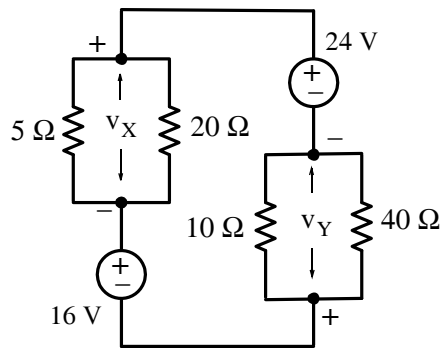
## Chapter 2 Analysis of Simple Circuits



*Hint:* Begin at the right end and by series and parallel combinations of the resistors, reduce the circuit to a simple series circuit. This method is known as *analysis by network reduction*.

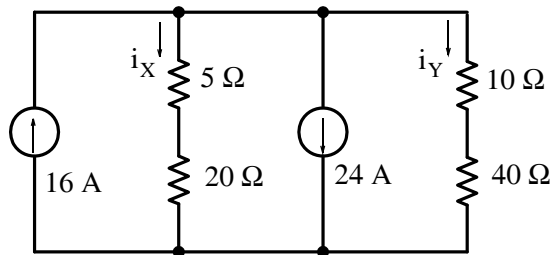
Answer: 0.75 A

7. For the circuit below, use the voltage division expression to compute  $v_X$  and  $v_Y$ .



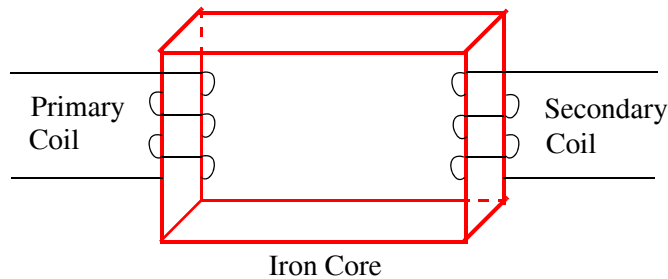
Answers:  $v_X = 8/3$  V,  $v_Y = 16/3$  V

8. For the circuit below, use the current division expression to compute  $i_X$  and  $i_Y$ .



Answers:  $i_X = -16/3$  A,  $i_Y = -8/3$  A

9. A transformer consists of two separate coils (inductors) wound around an *iron core* as shown in below. There are many turns in both the primary and secondary coils but, for simplicity, only few are shown. It is known that the primary coil has a resistance of  $5.48 \Omega$  at 20 degrees Celsius. After two hours of operation, it is found that the primary coil resistance has risen to  $6.32 \Omega$ . Compute the temperature rise of this coil.



Answer:  $36^{\circ}\text{C}$

10. A new facility is to be constructed at a site which is 1.5 miles away from the nearest electric utility company substation. The electrical contractor and the utility company have made load calculations, and decided that the main lines from the substation to the facility will require several copper conductors in parallel. Each of these conductors must have insulation type THHN and must carry a maximum current of 220 A in a  $20^{\circ}\text{C}$  temperature environment.

a. Compute the voltage drop on each of these conductors from the substation to the facility when they carry the maximum required current of 220 A in a  $20^{\circ}\text{C}$  temperature environment.

Answer: 70 V

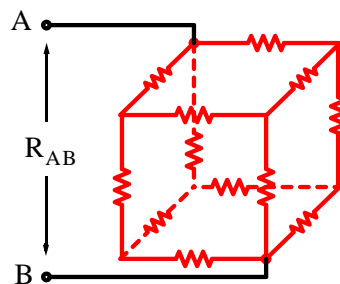
b. The power absorbed by each conductor under the conditions stated above.

Answer: 15.4 Kw

c. The power absorbed per square cm of the surface area of each conductor under the conditions stated above.

Answer:  $0.02 \text{ w/cm}^2$

11. For the network below, each of the 12 resistors along the edges of the cube are  $1 \Omega$  each. Compute the equivalent resistance  $R_{AB}$ . *Hint:* Use any tricks that may occur to you.



Answer:  $5/6 \Omega$

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## Chapter 2 Analysis of Simple Circuits

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12. A heating unit is rated at 5 Kw , and to maintain this rating, it is necessary that a voltage of 220 V is applied to establish an initial temperature of 15 °C . After the heating unit has reached a steady state, it is required that the voltage must be raised to 240 V to maintain the 5 Kw rating. Find the final temperature of the heating element in °C if the temperature coefficient  $\alpha$  is 0.0006  $\Omega$  per 1 °C .

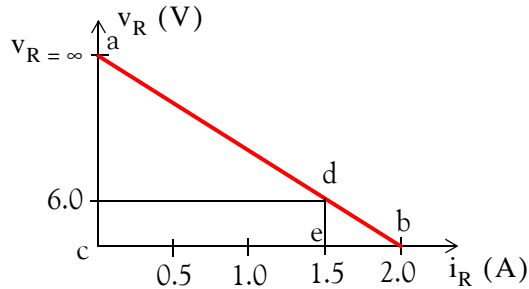
Answer: 332 °C .

2.23 Answers / Solutions to End-of-Chapter Exercises

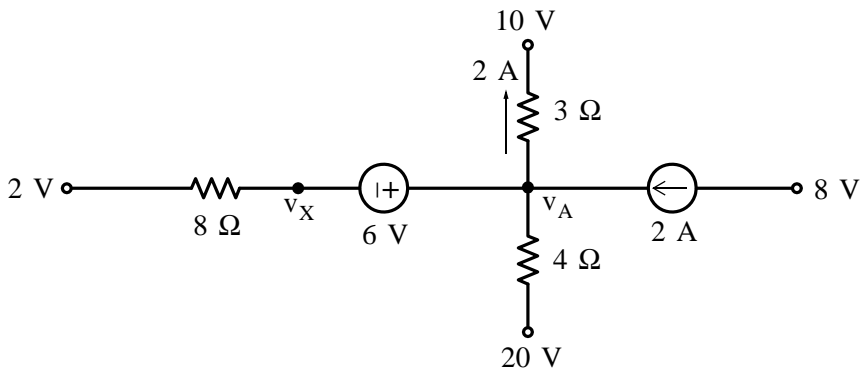
Multiple Choice

1. B
2. D
3. E
4. E
5. D
6. C

7. B When  $R = 4 \Omega$ , the voltage  $v_R = 6 \text{ V}$ . Therefore,  $i_R = 6/4 = 1.5 \text{ A}$ . Also, when  $R = 0 \Omega$ ,  $i_R = 2 \text{ A}$ , and thus  $v_R = 0$  (short circuit). When  $R = \infty$ ,  $i_R = 0$  but  $v_R$  has a finite value and it is denoted as  $v_{R=\infty}$  in the figure below. Now, we observe that the triangles  $abc$  and  $dbe$  are similar. Then  $\frac{be}{bc} = \frac{de}{ac}$  or  $\frac{2.0 - 1.5}{2.0} = \frac{6}{v_{R=\infty}}$  and thus  $v_{R=\infty} = 24 \text{ V}$



8. D We denote the voltage at the common node as  $v_A$  shown on the figure below.



## Chapter 2 Analysis of Simple Circuits

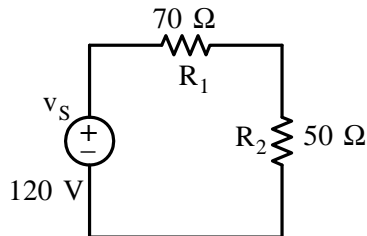
Then, from the branch that contains the  $3\ \Omega$  resistor, we observe that  $\frac{v_A - 10}{3} = 2$  or

$$v_A = 16 \text{ and thus } v_X = -6 + 16 = 10 \text{ V}$$

9. A This is an open circuit and therefore no current flows through the resistor. Accordingly, there is no voltage drop across the resistor and thus  $v = 8 \text{ V}$ .
10. A The  $12\ \Omega$  resistor is shorted out by the short on the right side of the circuit and thus the only resistance in the circuit is the  $4\ \Omega$  resistor.

### Problems

1.  
a. With  $v_S = 120 \text{ V}$ ,  $R_1 = 70\ \Omega$ , and  $R_2 = 50\ \Omega$ , the circuit is as shown below.



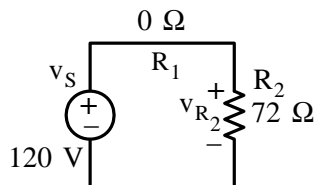
Using the voltage division expression, we obtain

$$v_{R_2} = \frac{50}{70 + 50} \times 120 = 50 \text{ V}$$

Then,

$$p_{R_2} = \frac{v_{R_2}^2}{R_2} = \frac{50^2}{50} = 50 \text{ w}$$

- b. With  $v_S = 120 \text{ V}$  and  $R_1 = 0\ \Omega$ , the circuit is as shown below.



We observe that

$$v_{R_2} = v_S = 120 \text{ V}$$



and

$$\frac{v_{R_2}^2}{R_2} = 200 \text{ w}$$

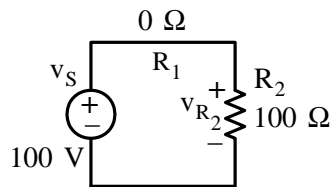
or

$$\frac{120^2}{R_2} = 200 \text{ w}$$

or

$$R_2 = \frac{120^2}{200} = 72 \Omega$$

c. With  $R_1 = 0 \Omega$  and  $R_2 = 100 \Omega$ , the circuit is as shown below.



Then,

$$\frac{v_S^2}{R_2} = 100 \text{ w}$$

or

$$\frac{v_S^2}{100} = 100 \text{ w}$$

or

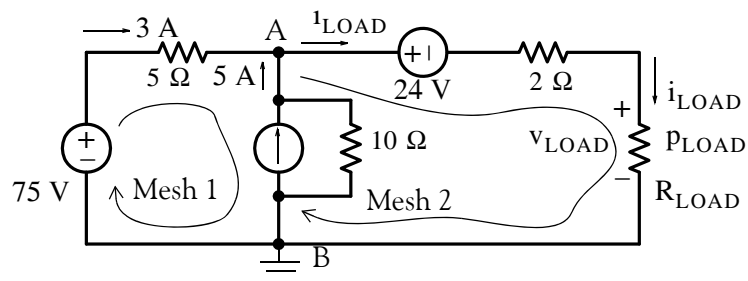
$$v_S^2 = 100 \times 100 = 10,000$$

or

$$v_S = \sqrt{10,000} = 100 \text{ V}$$

2.

a. Application of KCL at node A of the circuit below yields



$$i_{\text{LOAD}} = 3 + 5 = 8 \text{ A}$$

## Chapter 2 Analysis of Simple Circuits

b. Application of KVL around Mesh 1 yields

$$-75 + 3(5) + v_{AB} = 0$$

or

$$v_{AB} = 60 \text{ V}$$

Application of KVL around Mesh 2 yields

$$-v_{AB} + 24 + 2i_{\text{LOAD}} + v_{\text{LOAD}} = 0$$

or

$$-60 + 24 + 2 \times 8 + v_{\text{LOAD}} = 0$$

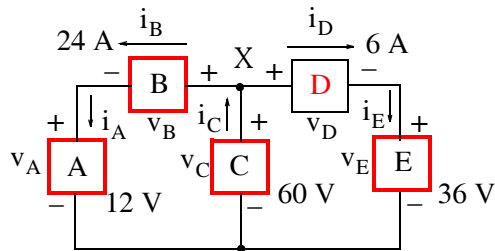
or

$$v_{\text{LOAD}} = 20 \text{ V}$$

c.

$$P_{\text{LOAD}} = v_{\text{LOAD}} \times i_{\text{LOAD}} = 20 \times 8 = 160 \text{ w (absorbed power)}$$

3. Where not shown, we assign plus (+) and minus (-) polarities and current directions in accordance with the passive sign convention as shown below.



We observe that  $i_A = i_B$  and  $i_E = i_D$ . Also, by KCL at Node X

$$i_C = i_B + i_D = 24 + 6 = 30 \text{ A}$$

Then,

$$P_A = v_A i_A = 12 \times 24 = 288 \text{ w (absorbed)}$$

$$P_E = v_E i_E = 36 \times 6 = 216 \text{ w (absorbed)}$$

$$P_C = v_C (-i_C) = 60 \times (-30) = -1800 \text{ w (supplied)}$$

By KVL

$$v_A + v_B = v_C$$

or

$$v_B = v_C - v_A = 60 - 12 = 48 \text{ V}$$

and thus

$$P_B = v_B i_B = 48 \times 24 = 1152 \text{ w (absorbed)}$$

Also by KVL

$$v_D + v_E = v_C$$

or

$$v_D = v_C - v_E = 60 - 36 = 24 \text{ V}$$

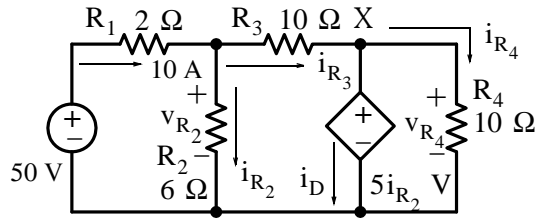
and thus

$$p_D = v_D i_D = 24 \times 6 = 144 \text{ w (absorbed)}$$

Check: We must show that Power supplied = Power absorbed

$$p_C = p_A + p_B + p_C + p_D = 288 + 216 + 1152 + 144 = 1800 \text{ w}$$

4. We assign voltages and currents  $v_{R_2}$ ,  $v_{R_4}$ ,  $i_{R_3}$ ,  $i_{R_4}$ , and  $i_D$  as shown in the circuit below.



By KVL,

$$v_{R_2} = 50 - 2 \times 10 = 30 \text{ V}$$

and by Ohm's law,

$$i_{R_2} = \frac{v_{R_2}}{R_2} = \frac{30}{6} = 5 \text{ A}$$

Therefore, the value of the dependent voltage source is

$$5i_{R_2} = 5 \times 5 = 25 \text{ V}$$

and

$$v_{R_4} = 5i_{R_2} = 25 \text{ V}$$

Then,

$$i_{R_4} = \frac{v_{R_4}}{R_4} = \frac{25}{10} = 2.5 \text{ A}$$

By KCL at Node X

$$i_D = i_{R_3} - i_{R_4}$$

where

$$i_{R_3} = 10 - i_{R_2} = 10 - 5 = 5 \text{ A}$$

and thus

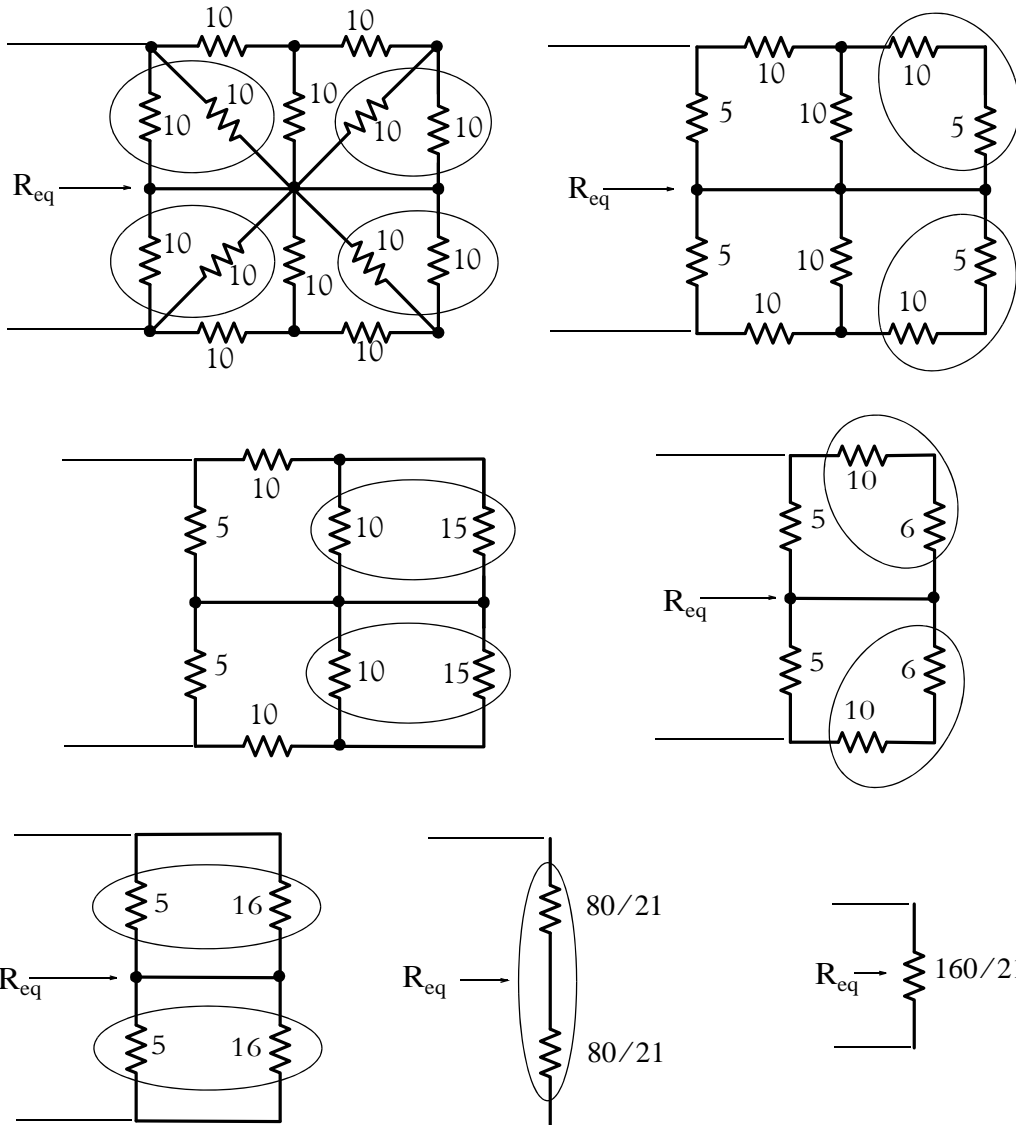
$$i_D = i_{R_3} - i_{R_4} = 5 - 2.5 = 2.5 \text{ A}$$

with the indicated direction through the dependent source. Therefore,

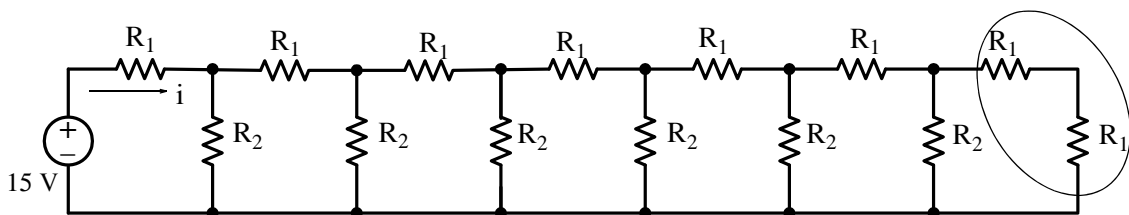
$$p_D = 5i_{R_2} i_D = 25 \times 2.5 = 62.5 \text{ w (absorbed)}$$

## Chapter 2 Analysis of Simple Circuits

5. The simplification procedure begins with the resistors in parallel as indicated below.



6. We begin with the right side of the circuit where the last two resistors are in series as shown below.



Then,

$$R_1 + R_1 = 10 + 10 = 20 \Omega$$

Next,

$$20 \parallel 20 = 10 \Omega$$

$$10 + 10 = 20 \Omega$$

and so on. Finally, addition of the left most resistor with its series equivalent yields

$$10 + 10 = 20 \Omega$$

and thus

$$i = 15/20 = 0.75 \text{ A}$$

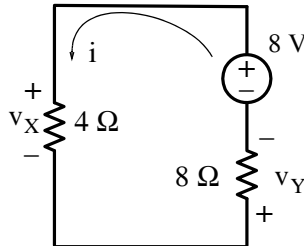
7. We first simplify the given circuit by replacing the parallel resistors by their equivalents. Thus,

$$5 \parallel 20 = \frac{5 \times 20}{5 + 20} = 4 \Omega$$

and

$$10 \parallel 40 = \frac{10 \times 40}{10 + 40} = 8 \Omega$$

The voltage sources are in series opposing connection and they can be replaced by a single voltage source with value  $24 - 16 = 8 \text{ V}$ . The simplified circuit is shown below.



Now, by the voltage division expression,

$$v_X = \frac{4}{4 + 8} \times 8 = \frac{8}{3} \text{ V}$$

and

$$v_Y = \frac{8}{4 + 8} \times 8 = \frac{16}{3} \text{ V}$$

8. We first simplify the given circuit by replacing the series resistors by their equivalents. Thus,

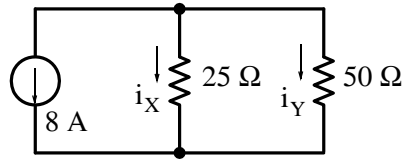
$$5 + 20 = 25 \Omega$$

and

$$10 + 40 = 50 \Omega$$

The current sources are in parallel opposing connection and they can be replaced by a single current source with value  $24 - 16 = 8 \text{ A}$ . The simplified circuit is shown below.

## Chapter 2 Analysis of Simple Circuits



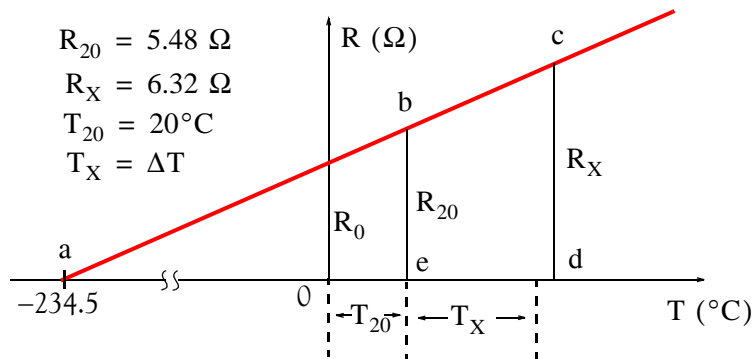
By the current division expression,

$$i_X = \frac{50}{25 + 50} \times (-8) = -\frac{16}{3} \text{ A}$$

and

$$i_Y = \frac{25}{25 + 50} \times (-8) = -\frac{8}{3} \text{ A}$$

9. We construct the resistance versus temperature plot shown below.



From the similar triangles  $acd$  and  $abe$ , we obtain

$$\frac{R_X}{R_{20}} = \frac{234.5 + T_{20} + T_X}{234.5 + T_{20}} = \frac{234.5 + 20 + T_X}{234.5 + 20} = \frac{254.5 + T_X}{254.5}$$

or

$$\Delta T = T_X = \frac{R_X}{R_{20}} \times 254.5 - 254.5 = \frac{6.32}{5.48} \times 254.5 - 254.5 = 36^\circ\text{C}$$

10.

- a. From Table 2.4 we find that the cable size must be 0000 AWG and this can carry up to 235 A. Also, from Table 2.5 we find that the resistance of this conductor is  $0.059 \ \Omega/1000 \text{ ft}$  at  $100^\circ\text{C}$ . Then, the resistance of this conductor that is 1.5 miles long is

$$0.059 \frac{\Omega}{1000 \text{ ft}} \times \frac{5280}{1 \text{ mile}} \times 1.5 \text{ miles} = 0.4673 \ \Omega \text{ at } 100^\circ\text{C}$$

To find the resistance of this cable at  $20^\circ\text{C}$ , we use the relation of (2.54). Thus,

$$R_{20} = R_{100} [1 + 0.004(20 - 100)] = 0.4673(1 - 0.32) = 0.3178 \ \Omega$$

and the voltage drop on each of these conductors is

$$v = iR = 220 \times 0.3178 = 70 \text{ V}$$

b. The power absorbed by each conductor is

$$p = vi = 70 \times 220 = 15,400 \text{ w} = 15.4 \text{ Kw}$$

c. Table 2.5 gives wire sizes in circular mils. We recall that a circular mil is the area of a circle whose diameter is 0.001 in. To find the diameter in cm, we perform the following conversion:

$$\begin{aligned} 1 \text{ circular mil} &= \frac{\pi}{4}d^2 = \frac{\pi}{4}(0.001)^2 = 7.854 \times 10^{-7} \text{ in}^2 \\ &= 7.854 \times 10^{-7} \text{ in}^2 \times \frac{(2.54 \text{ cm})^2}{\text{in}^2} = 5.067 \times 10^{-6} \text{ cm}^2 \end{aligned}$$

From Table 2.5 we find that the cross section of a 0000 AWG cable is 211,600 circular mils. Then, the cross-section of this cable in  $\text{cm}^2$  is

$$211,600 \text{ circular mils} \times \frac{5.067 \times 10^{-6} \text{ cm}^2}{\text{circular mil}} = 1.072 \text{ cm}^2$$

Therefore, the cable diameter in cm is

$$d = \sqrt{1.072} = 1.035 \text{ cm}$$

The cross-section circumference of the cable is

$$\pi d = \pi \times 1.035 = 3.253 \text{ cm}$$

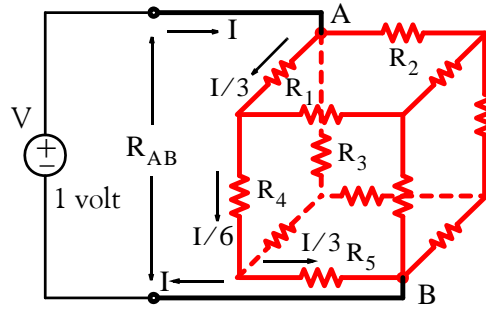
and the surface area of the cable is

$$\text{Surface area} = \pi dl = 3.253 \text{ cm} \times 1.5 \text{ miles} \times \frac{1.609 \text{ Km}}{1 \text{ mile}} \times \frac{10^5 \text{ cm}}{1 \text{ Km}} = 7.851 \times 10^5 \text{ cm}^2$$

Then, the power absorbed per  $\text{cm}^2$  is

$$p_{\text{cm}^2} = \frac{\text{Total power}}{\text{cm}^2} = \frac{15,400 \text{ w}}{7.851 \times 10^5 \text{ cm}^2} = 0.02 \text{ w/cm}^2$$

11. Let us connect a voltage source of 1 volt across the corners A and B of the cube as shown below, and let the current produced by this voltage source be I.



Since all resistors are equal ( $1 \Omega$  each), the current  $I$  entering node  $A$  will be split into 3 equal currents  $I/3$  each. The voltage drop  $V_{AB}$  will be 1 volt regardless of the path from node  $A$  to node  $B$ . Arbitrarily, we choose the path through resistors  $R_1$ ,  $R_4$ , and  $R_5$ , and the currents through these resistors are  $I/3$ ,  $I/6$ , and  $I/3$  respectively. Then, by KVL,

$$\frac{I}{3}R_1 + \frac{I}{6}R_4 + \frac{I}{3}R_5 = IR_{AB} = V = 1 \text{ volt}$$

and since  $R_1 = R_4 = R_5 = 1 \Omega$ ,

$$\frac{I}{3} + \frac{I}{6} + \frac{I}{3} = \frac{5}{6}I = IR_{AB}$$

from which

$$R_{AB} = 5/6 \Omega$$

12. The power  $P_1$  absorbed by the heating unit when the applied voltage is 220 V is 5 Kw and the resistance  $R_1$  is found from the relation  $P_1 = V_1^2/R_1$  or

$$R_1 = \frac{V_1^2}{P_1} = \frac{(220)^2}{5 \text{ Kw}} = \frac{48400}{5000} = 9.68 \Omega$$

The power  $P_2$  absorbed by the heating unit when the applied voltage is 240 V is still 5 Kw and the resistance  $R_2$  is

$$R_2 = \frac{V_1^2}{P_1} = \frac{(240)^2}{5 \text{ Kw}} = \frac{57600}{5000} = 11.52 \Omega$$

From relation (2.52),

$$R_2 = R_1 + R_1\alpha(T_2 - T_1)$$

or

$$R_2 - R_1 = R_1\alpha(T_2 - T_1)$$



or

$$T_2 - T_1 = \frac{R_2 - R_1}{R_1 \alpha} = \frac{11.52 - 9.68}{9.68 \times 0.0006} = 316.8$$

Therefore, the final temperature  $T_2$  of the heating element is

$$T_2 = 316.8 + T_1 = 316.8 + 15 \approx 332 \text{ }^\circ\text{C}$$

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# Chapter 3

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## Nodal and Mesh Equations – Circuit Theorems

This chapter begins with nodal, loop and mesh equations and how they are applied to the solution of circuits containing two or more node-pairs and two or more loops or meshes. Other topics included in this chapter are the voltage-to-current source transformations and vice versa, Thevenin's and Norton's theorems, the maximum power transfer theorem, linearity, superposition, efficiency, and regulation.

### 3.1 Nodal, Mesh, and Loop Equations

*Network Topology* is a branch of network theory concerned with the equations required to completely describe an electric circuit. In this text, we will only be concerned with the following two theorems.

#### Theorem 3.1

Let  $N$  = number of nodes in a circuit; then  $N - 1$  independent nodal equations are required to completely describe that circuit. These equations are obtained by setting the algebraic sum of the currents leaving each of the  $N - 1$  nodes equal to zero.

#### Theorem 3.2

Let  $L = M$  = number of loops or meshes,  $B$  = number of branches,  $N$  = number of nodes in a circuit; then  $L = M = B - N + 1$  independent loop or mesh equations are required to completely describe that circuit. These equations are obtained by setting the algebraic sum of the voltage drops around each of the  $L = M = B - N + 1$  loops or meshes equal to zero.

### 3.2 Analysis with Nodal Equations

In writing nodal equations, we perform the following steps:

1. For a circuit containing  $N$  nodes, we choose one of these as a reference node assumed to be zero volts or ground.
2. At each non-reference node we assign node voltages  $v_1, v_2, \dots, v_{n-1}$  where each of these voltages is measured with respect to the chosen reference node, i.e., ground.
3. If the circuit does not contain any voltage sources between nodes, we apply KCL and write a nodal equation for each of the node voltages  $v_1, v_2, \dots, v_{n-1}$ .

4. If the circuit contains a voltage source between two nodes, say nodes  $j$  and  $k$  denoted as node variables  $v_j$  and  $v_k$ , we replace the voltage source with a short circuit thus forming a combined node and we write a nodal equation for this common node in terms of both  $v_j$  and  $v_k$ ; then we relate the voltage source to the node variables  $v_j$  and  $v_k$ .

### Example 3.1

Write nodal equations for the circuit shown in Figure 3.1, and solve for the unknowns of these equations using matrix theory, Cramer's rule, or the substitution method. Verify your answers with Excel® or MATLAB®. Please refer to Appendix A for discussion and examples.

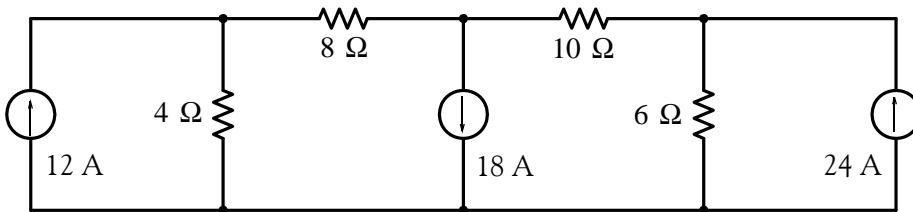


Figure 3.1. Circuit for Example 3.1

### Solution:

We observe that there are 4 nodes and we denote these as ①, ②, ③, and G (for ground) as shown in Figure 3.2.

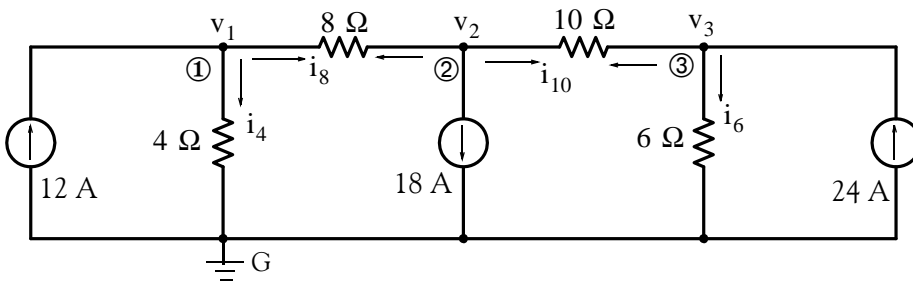


Figure 3.2. Circuit for Example 3.1

For convenience, we have denoted the currents with a subscript that corresponds to the resistor value through which it flows through; thus, the current that flows through the  $4\ \Omega$  resistor is denoted as  $i_4$ , the current through the  $8\ \Omega$  resistor is denoted as  $i_8$ , and so on. We will follow this practice in the subsequent examples.

For the circuit of Figure 3.2, we need  $N - 1 = 4 - 1 = 3$  nodal equations. Let us choose node G (ground) as our reference node, and we assign voltages  $v_1$ ,  $v_2$ , and  $v_3$  at nodes ①, ②, and ③ respectively; these are to be measured with respect to the ground node G. Now, application of KCL at node ① yields

$$i_4 + i_8 - 12 = 0$$

or

$$i_4 + i_8 = 12 \tag{3.1}$$

where  $i_8$  is the current flowing from left to right. Expressing (3.1) in terms of the node voltages, we obtain

$$\frac{v_1}{4} + \frac{v_1 - v_2}{8} = 12$$

or

$$\left(\frac{1}{4} + \frac{1}{8}\right)v_1 - \frac{1}{8}v_2 = 12$$

or

$$\boxed{3v_1 - v_2 = 96} \tag{3.2}$$

Next, application of KCL at node ② yields

$$i_8 + i_{10} + 18 = 0$$

or

$$i_8 + i_{10} = -18 \tag{3.3}$$

where  $i_8$  is the current flowing from right to left\* and  $i_{10}$  is the current that flows from left to right.

Expressing (3.3) in terms of node voltages, we obtain

$$\frac{v_2 - v_1}{8} + \frac{v_2 - v_3}{10} = -18$$

or

$$-\frac{1}{8}v_1 + \left(\frac{1}{8} + \frac{1}{10}\right)v_2 - \frac{1}{10}v_3 = -18$$

or

---

\* The direction of the current through the 8 W resistor from left to right in writing the nodal equation at Node 1, and from right to left in writing the nodal equation at Node 2, should not be confusing. Remember that we wrote independent node equations at independent nodes and, therefore, any assumptions made in writing the first equation need not be held in writing the second since the latter is independent of the first. Of course, we could have assumed that the current through the 8 W resistor flows in the same direction in both nodal equations. It is advantageous, however, to assign a (+) sign to all currents leaving the node in which we apply KCL. The advantage is that we can check, or even write the node equations by inspection. With reference to the above circuit and equation (3.1) for example, since  $G = 1/R$ , we denote the coefficients of  $v_1$  (1/4 and 1/8 siemens) as **self conductances** and the coefficient of  $v_2$  (-1/8) as **mutual conductance**. Likewise, in equation (3.3) the coefficients of  $v_2$  (1/8 and 1/10 siemens) are the self conductances and the coefficients of  $v_1$  (-1/8) and  $v_3$  (-1/10) are the mutual conductances. Therefore, we can write a nodal equation at a particular node by inspection, that is, we assign plus (+) values to self conductances and minus (-) to mutual conductances.

---

$$\boxed{5v_1 - 9v_2 + 4v_3 = 720} \quad (3.4)$$

Similarly, application of KCL at node ③ yields

$$i_{10} + i_6 - 24 = 0$$

or

$$i_{10} + i_6 = 24$$

where  $i_{10}$  is the current flowing from right to left. Then, in terms of node voltages,

$$\frac{v_3 - v_2}{10} + \frac{v_3}{6} = 24 \quad (3.5)$$

or

$$-\frac{1}{10}v_2 + \left(\frac{1}{10} + \frac{1}{6}\right)v_3 = 24$$

or

$$\boxed{-3v_2 + 8v_3 = 720} \quad (3.6)$$

Equations (3.2), (3.4), and (3.6) constitute a set of three simultaneous equations with three unknowns. We write them in matrix form as follows:

$$\underbrace{\begin{bmatrix} 3 & -1 & 0 \\ 5 & -9 & 4 \\ 0 & -3 & 8 \end{bmatrix}}_{\mathbf{G}} \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}}_{\mathbf{V}} = \underbrace{\begin{bmatrix} 96 \\ 720 \\ 720 \end{bmatrix}}_{\mathbf{I}} \quad (3.7)$$

We can use Cramer's rule or Gauss's elimination method as discussed in Appendix A, to solve (3.7) for the unknowns. Simultaneous solution yields  $v_1 = 20.57 \text{ V}$ ,  $v_2 = -34.29 \text{ V}$ , and  $v_3 = 77.14 \text{ V}$ . With these values we can determine the current in each resistor, and the power absorbed or delivered by each device.

Check with MATLAB® :

```
G=[3 -1 0; 5 -9 4; 0 -3 8]; I=[96 720 720]'; V=G\I;...
fprintf('\n'); fprintf('v1 = %5.2f volts \t', V(1)); ...
fprintf('v2 = %5.2f volts \t', V(2)); fprintf('v3 = %5.2f volts', V(3)); fprintf('\n')
v1 = 20.57 volts v2 = -34.29 volts v3 = 77.14 volts
```

Check with Excel® :

The spreadsheet of Figure 3.3 shows the solution of the equations of (3.7). The procedure is discussed in Appendix A.

	A	B	C	D	E	F	G
1	Spreadsheet for Matrix Inversion and Matrix Multiplication						
2							
3		3	-1	0			96
4	G=	5	-9	4		I=	720
5		0	-3	8			720
6							
7		0.429	-0.057	0.029			20.57143
8	G <sup>-1</sup> =	0.286	-0.171	0.086		V=	-34.2857
9		0.107	-0.064	0.157			77.14286

Figure 3.3. Spreadsheet for the solution of (3.7)

### Example 3.2

For the circuit of Figure 3.4, write nodal equations in matrix form and solve for the unknowns using matrix theory, Cramer’s rule, or Gauss’s elimination method. Verify your answers with Excel or MATLAB. Please refer to Appendix A for procedures and examples. Then construct a table showing the voltages across, the currents through and the power absorbed or delivered by each device. Verify your answer with a Simulink / SimPowerSystems model.

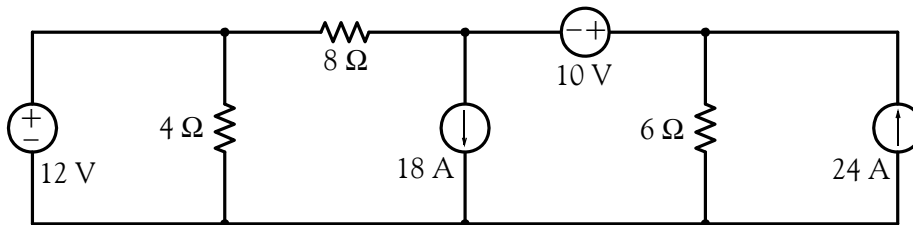


Figure 3.4. Circuit for Example 3.2

#### Solution:

We observe that there are 4 nodes and we denote these as ①, ②, ③, and G (for ground) as shown in Figure 3.5. We assign voltages  $v_1$ ,  $v_2$ , and  $v_3$  at nodes ①, ②, and ③ respectively; these are to be measured with respect to the ground node G. We observe that  $v_1$  is a known voltage, that is,  $v_1 = 12\text{ V}$  and thus our first equation is

$$v_1 = 12 \tag{3.8}$$

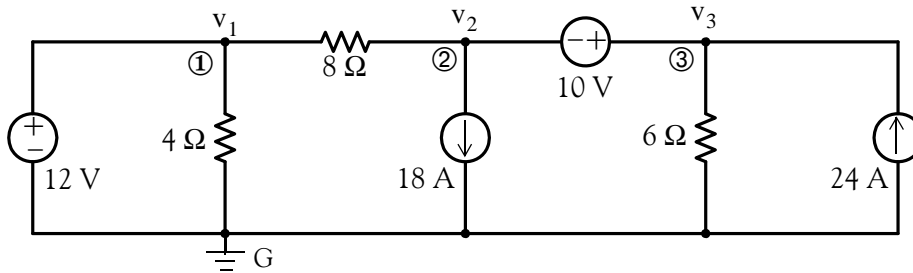


Figure 3.5. Circuit for Example 3.2 with assigned nodes and voltages

Next, we move to node ② where we observe that there are three currents flowing out of this node, one to the left, one to the right, and one down. Therefore, our next nodal equation will contain three terms. We have no difficulty writing the term for the current flowing from node ② to node ①, and for the 18 A source; however, we encounter a problem with the third term because we cannot express it as term representing the current flowing from node ② to node ③. To work around this problem, we temporarily remove the 10 V voltage source and we replace it with a “short” thereby creating a *combined node* (or *generalized node* or *supernode* as some textbooks call it), and the circuit now appears as shown in Figure 3.6.

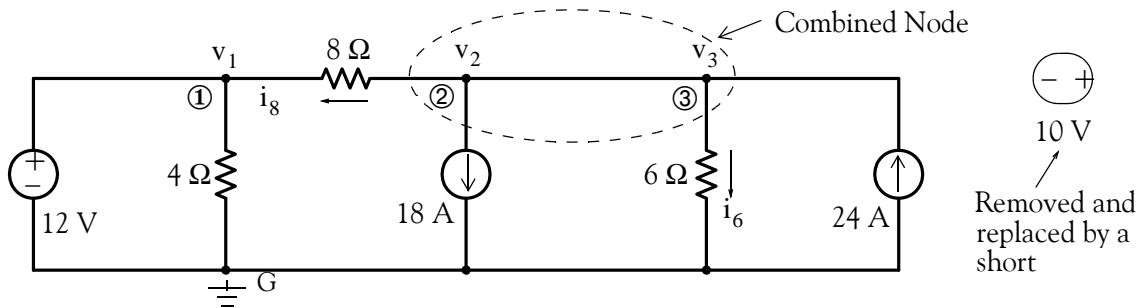


Figure 3.6. Circuit for Example 3.2 with a combined node

Now, application of KCL at this combined node yields the equation

$$i_8 + 18 + i_6 - 24 = 0$$

or

$$i_8 + i_6 = 6$$

or

$$\frac{v_2 - v_1}{8} + \frac{v_3}{6} = 6^* \tag{3.9}$$

\* The combined node technique allows us to combine two nodal equations into one but requires that we use the proper node designations. In this example, to retain the designation of node 2, we express the current  $i_8$  as  $\frac{v_2 - v_1}{8}$ . Likewise, at node 3, we express the current  $i_6$  as  $\frac{v_3}{6}$ .

or

$$-\frac{1}{8}v_1 + \frac{1}{8}v_2 + \frac{1}{6}v_3 = 6$$

or

$$\boxed{-3v_1 + 3v_2 + 4v_3 = 144} \tag{3.10}$$

To obtain the third equation, we reinsert the 10 V source between nodes ② and ③. Then,

$$\boxed{v_3 - v_2 = 10} \tag{3.11}$$

In matrix form, equations (3.8), (3.10), and (3.11) are

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 3 & 4 \\ 0 & -1 & 1 \end{bmatrix}}_{\mathbf{G}} \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}}_{\mathbf{V}} = \underbrace{\begin{bmatrix} 12 \\ 144 \\ 10 \end{bmatrix}}_{\mathbf{I}} \tag{3.12}$$

Simultaneous solution yields  $v_1 = 12 \text{ V}$ ,  $v_2 = 20 \text{ V}$ , and  $v_3 = 30 \text{ V}$ . From these we can find the current through each device and the power absorbed or delivered by each device.

Check with MATLAB:

```
G=[1 0 0; -3 3 4; 0 -1 1]; I=[12 144 10]'; V=G\I;...
fprintf(' \n'); fprintf('v1 = %5.2f volts \t', V(1)); ...
fprintf('v2 = %5.2f volts \t', V(2)); fprintf('v3 = %5.2f volts', V(3)); fprintf(' \n')
```

v1 = 12.00 volts    v2 = 20.00 volts    v3 = 30.00 volts

Check with Excel:

	A	B	C	D	E	F	G	H
1	Spreadsheet for Matrix Inversion and Matrix Multiplication							
2								
3		1	0	0			12	
4	G=	-3	3	4		I=	144	
5		0	-1	1			10	
6								
7		1.000	0.000	0.000			12	
8	G <sup>-1</sup> =	0.429	0.143	-0.571		V=	20	
9		0.429	0.143	0.429			30	

Figure 3.7. Spreadsheet for the solution of (3.12)

Table 3.1 shows that the power delivered is equal to the power absorbed.



TABLE 3.1 Table for Example 3.2

Device	Voltage (volts)	Current (amps)	Power (watts)	
			Delivered	Absorbed
12 V Source	12	2	24	
10 V Source	10	19		190
18 A Source	20	18		360
24 A Source	30	24	720	
4 W Resistor	12	3		36
6 W Resistor	30	5		150
8 W Resistor	8	1		8
Total			744	744

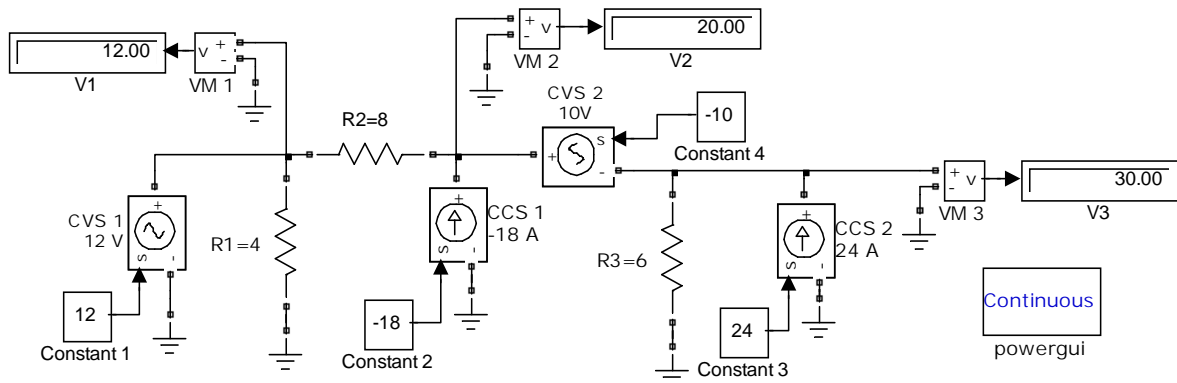


Figure 3.8. Simulink / SimPower Systems model for Example 3.2

### 3.3 Analysis with Mesh or Loop Equations

In writing mesh or loop equations, we follow these steps:

1. For a circuit containing  $M = L = B - N + 1$  meshes (or loops), we assign a mesh or loop current  $i_1, i_2, \dots, i_{n-1}$  for each mesh or loop.
2. If the circuit does not contain any current sources, we apply KVL around each mesh or loop.
3. If the circuit contains a current source between two meshes or loops, say meshes or loops  $j$  and  $k$  denoted as mesh variables  $i_j$  and  $i_k$ , we replace the current source with an open circuit thus forming a common mesh or loop, and we write a mesh or loop equation for this common mesh

or loop in terms of both  $i_j$  and  $i_k$ . Then, we relate the current source to the mesh or loop variables  $i_j$  and  $i_k$ .

**Example 3.3**

For the circuit of Figure 3.9, write mesh equations in matrix form and solve for the unknowns using matrix theory, Cramer's rule, or Gauss's elimination method. Verify your answers with Excel or MATLAB. Please refer to Appendix A for procedures and examples. Then construct a table showing the voltages across, the currents through, and the power absorbed or delivered by each device.

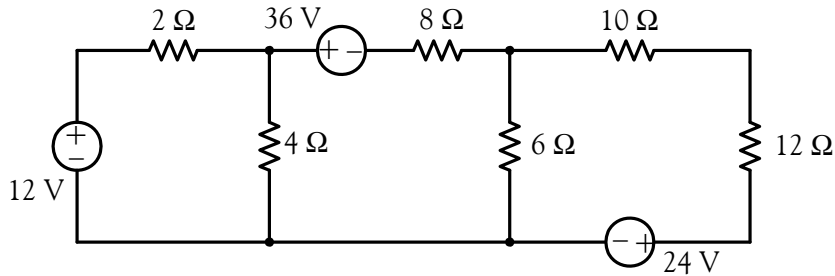


Figure 3.9. Circuit for Example 3.3

**Solution:**

For this circuit we need  $M = L = B - N + 1 = 9 - 7 + 1 = 3$  mesh or loop equations and we arbitrarily assign currents  $i_1$ ,  $i_2$ , and  $i_3$  all in a clockwise direction as shown in Figure 3.10.

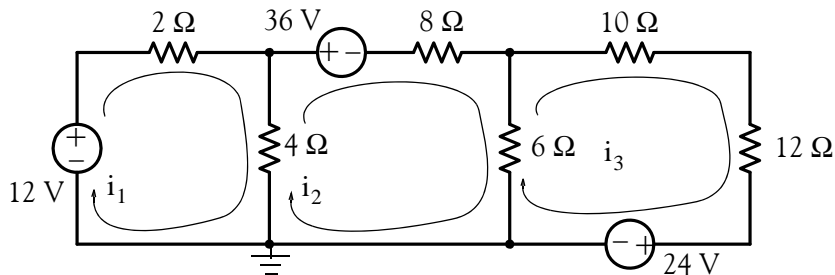


Figure 3.10. Circuit for Example 3.3

Applying KVL around each mesh we obtain:

**Mesh #1:** Starting with the left side of the  $2\ \Omega$  resistor, going clockwise, and observing the passive sign convention, we obtain the equation for this mesh as

$$2i_1 + 4(i_1 - i_2) - 12 = 0$$

or

$$\boxed{6i_1 - 4i_2 = 12} \quad (3.13)$$

**Mesh #2:** Starting with the lower end of the 4  $\Omega$  resistor, going clockwise, and observing the passive sign convention, we obtain the equation

$$4(i_2 - i_1) + 36 + 8i_2 + 6(i_2 - i_3) = 0$$

or

$$\boxed{-4i_1 + 18i_2 - 6i_3 = -36} \quad (3.14)$$

**Mesh #3:** Starting with the lower end of the 6  $\Omega$  resistor, going clockwise, and observing the passive sign convention, we obtain:

$$6(i_3 - i_2) + 10i_3 + 12i_3 + 24 = 0$$

or

$$\boxed{-6i_2 + 28i_3 = -24} \quad (3.15)$$

**Note:** For this example, we assigned all three currents with the same direction, i.e., clockwise. This, of course, was not mandatory; we could have assigned any direction in any mesh. It is advantageous, however, to assign the same direction to all currents. The advantage here is that we can check, or even write the mesh equations by inspection. This is best explained with the following observations:

1. With reference to the circuit of Figure 3.10 and equation (3.13), we see that current  $i_1$  flows through the 2  $\Omega$  and 4  $\Omega$  resistors. We call these the *self resistances* of the first mesh. Their sum, i.e.,  $2 + 4 = 6$  is the coefficient of current  $i_1$  in that equation. We observe that current  $i_2$  also flows through the 4  $\Omega$  resistor. We call this resistance the *mutual resistance* between the first and the second mesh. Since  $i_2$  enters the lower end of the 4  $\Omega$  resistor, and in writing equation (3.13) we have assumed that the upper end of this resistor has the plus (+) polarity, then in accordance with the passive sign convention, the voltage drop due to current  $i_2$  is  $-4i_2$  and this is the second term on the left side of (3.13).
2. In Mesh 2, the self resistances are the 4  $\Omega$ , 8  $\Omega$ , and 6  $\Omega$  resistors whose sum, 18, is the coefficient of  $i_2$  in equation (3.14). The 4  $\Omega$  and 6  $\Omega$  resistors are also the mutual resistances between the first and second, and the second and the third meshes respectively. Accordingly, the voltage drops due to the mutual resistances in the second equation have a minus (-) sign, i.e.,  $-4i_1$  and  $-6i_3$ .
3. The signs of the coefficients of  $i_2$  and  $i_3$  in (3.15) are similarly related to the self and mutual resistances in the third mesh.

Simplifying and rearranging (3.13), (3.14) and (3.15) we obtain:

$$\boxed{3i_1 - 2i_2 = 6} \tag{3.16}$$

$$\boxed{2i_1 - 9i_2 + 3i_3 = 18} \tag{3.17}$$

$$\boxed{3i_2 - 14i_3 = 12} \tag{3.18}$$

and in matrix form

$$\underbrace{\begin{bmatrix} 3 & -2 & 0 \\ 2 & -9 & 3 \\ 0 & 3 & -14 \end{bmatrix}}_{\mathbf{R}} \underbrace{\begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix}}_{\mathbf{I}} = \underbrace{\begin{bmatrix} 6 \\ 18 \\ 12 \end{bmatrix}}_{\mathbf{V}} \tag{3.19}$$

Simultaneous solution yields  $i_1 = 0.4271$ ,  $i_2 = -2.3593$ , and  $i_3 = -1.3627$  where the negative values for  $i_2$  and  $i_3$  indicate that the actual direction for these currents is counterclockwise.

Check with MATLAB:

```
R=[3 -2 0; 2 -9 3; 0 3 -14]; V=[6 18 12]'; I=R\V;...
fprintf(' \n'); fprintf('i1 = %5.2f amps \t', I(1)); ...
fprintf('i2 = %5.2f amps \t', I(2)); fprintf('i3 = %5.2f amps', I(3)); fprintf(' \n')
```

$i_1 = 0.43$  amps     $i_2 = -2.36$  amps     $i_3 = -1.36$  amps

Excel produces the same answers as shown in Figure 3.11.

	A	B	C	D	E	F	G	H
1	Spreadsheet for Matrix Inversion and Matrix Multiplication							
2								
3		3	-2	0			6	
4	R=	2	-9	3		V=	18	
5		0	3	-14			12	
6								
7		0.397	-0.095	-0.020			0.4271	
8	R <sup>-1</sup> =	0.095	-0.142	-0.031		I=	-2.3593	
9		0.020	-0.031	-0.078			-1.3627	

Figure 3.11. Spreadsheet for the solution of (3.19)

Table 3.2 shows that the power delivered by the voltage sources is equal to the power absorbed by the resistors.

TABLE 3.2 Table for Example 3.3

Device	Voltage (volts)	Current (amps)	Power (watts)	
			Delivered	Absorbed
12 V Source	12.000	0.427	5.124	
36 V Source	36.000	2.359	84.924	
24 V Source	24.000	1.363	32.712	
2 W Resistor	0.854	0.427		0.365
4 W Resistor	11.144	2.786		30.964
8 W Resistor	18.874	2.359		44.530
6 W Resistor	5.976	0.996		5.952
10 W Resistor	13.627	1.363		18.570
12 W Resistor	16.352	1.363		22.288
Total			122.760	122.669

**Example 3.4**

For the circuit of Figure 3.12, write loop equations in matrix form, and solve for the unknowns using matrix theory, Cramer’s rule, or Gauss’s elimination method. Verify your answers with Excel or MATLAB. Please refer to Appendix A for procedures and examples. Then, construct a table showing the voltages across, the currents through and the power absorbed or delivered by each device.

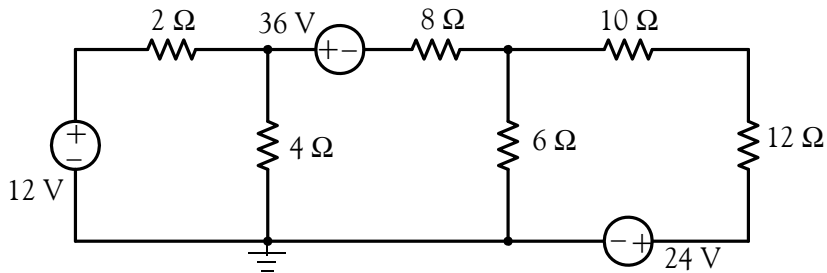


Figure 3.12. Circuit for Example 3.4

**Solution:**

This is the same circuit as that of the previous example where we found that we need 3 mesh or loop equations. We choose our loops as shown in Figure 3.13, and we assign currents  $i_1$ ,  $i_2$ , and  $i_3$ , all in a clockwise direction.

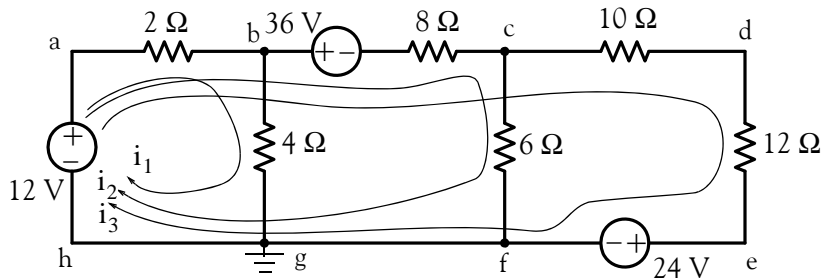


Figure 3.13. Circuit for Example 3.4 with assigned loops

Applying of KVL around each loop, we obtain:

**Loop 1 (abgh):** Starting with the left side of the  $2\ \Omega$  resistor and complying with the passive sign convention, we obtain:

$$2(i_1 + i_2 + i_3) + 4i_1 - 12 = 0$$

or

$$6i_1 + 2i_2 + 2i_3 = 12$$

or

$$\boxed{3i_1 + i_2 + i_3 = 6} \quad (3.20)$$

**Loop 2 (abcfgh):** As before, starting with the left side of the  $2\ \Omega$  resistor and complying with the passive sign convention, we obtain:

$$2(i_1 + i_2 + i_3) + 36 + 8(i_2 + i_3) + 6i_2 - 12 = 0$$

or

$$2i_1 + 16i_2 + 10i_3 = -24$$

or

$$\boxed{i_1 + 8i_2 + 5i_3 = -12} \quad (3.21)$$

**Loop 3 (abcdefgh):** Likewise, starting with the left side of the  $2\ \Omega$  resistor and complying with the passive sign convention, we obtain:

$$2(i_1 + i_2 + i_3) + 36 + 8(i_2 + i_3) + 10i_3 + 12i_3 + 24 - 12 = 0$$

or

$$2i_1 + 10i_2 + 32i_3 = -48$$

or

$$\boxed{i_1 + 5i_2 + 16i_3 = -24} \quad (3.22)$$

and in matrix form

$$\underbrace{\begin{bmatrix} 3 & 1 & 1 \\ 1 & 8 & 5 \\ 1 & 5 & 16 \end{bmatrix}}_R \underbrace{\begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix}}_I = \underbrace{\begin{bmatrix} 6 \\ -12 \\ -24 \end{bmatrix}}_V \quad (3.23)$$

Solving with MATLAB we obtain:

```
R=[3 1 1; 1 8 5; 1 5 16]; V=[6 -12 -24]'; I=R\V;...
fprintf(' \n'); fprintf('i1 = %5.2f amps \t', I(1)); ...
fprintf('i2 = %5.2f amps \t', I(2)); fprintf('i3 = %5.2f amps', I(3)); fprintf(' \n')
```

i1 = 2.79 amps    i2 = -1.00 amps    i3 = -1.36 amps

Excel produces the same answers.

Table 3.3 shows that the power delivered by the voltage sources is equal to the power absorbed by the resistors and the values are approximately the same as those of the previous example.

TABLE 3.3 Table for Example 3.4

			Power (watts)	
Device	Voltage (volts)	Current (amps)	Delivered	Absorbed
12 V Source	12.000	0.427	5.124	
36 V Source	36.000	2.359	84.924	
24 V Source	24.000	1.363	32.712	
2 W Resistor	0.854	0.427		0.365
4 W Resistor	11.146	2.786		31.053
8 W Resistor	18.872	2.359		44.519
6 W Resistor	5.982	0.997		5.964
10 W Resistor	13.627	1.363		18.574
12 W Resistor	16.352	1.363		22.283
Total			122.760	122.758

### Example 3.5

For the circuit of figure 3.14, write mesh equations in matrix form and solve for the unknowns using matrix theory, Cramer’s rule, or the substitution method. Verify your answers with Excel or MATLAB. Please refer to Appendix A for procedures and examples.

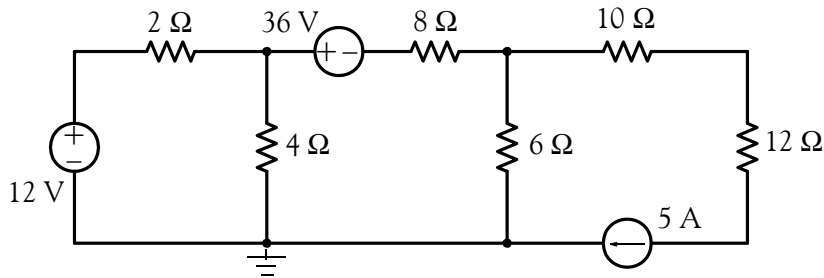


Figure 3.14. Circuit for Example 3.5

**Solution:**

This is the same circuit as those of the two previous examples except that the 24 V voltage source has been replaced by a 5 A current source. As in Examples 3.3 and 3.4, we need  $M = L = B - N + 1 = 9 - 7 + 1 = 3$  mesh or loop equations, and we assign currents  $i_1$ ,  $i_2$ , and  $i_3$  all in a clockwise direction as shown in Figure 3.15.

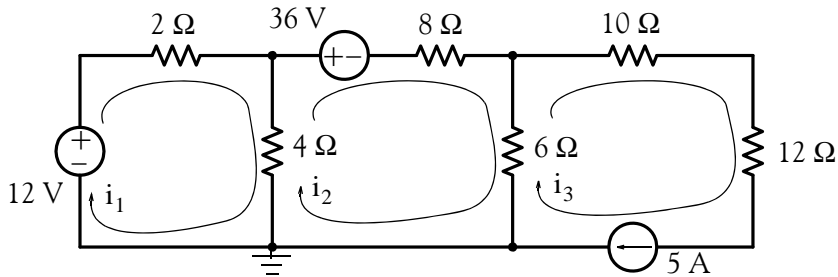


Figure 3.15. Circuit for Example 3.5 with assigned currents

For Meshes 1 and 2, the equations are the same as in Example 3.3 where we found them to be

$$6i_1 - 4i_2 = 12$$

or

$$\boxed{3i_1 - 2i_2 = 6} \quad (3.24)$$

and

$$-4i_1 + 18i_2 - 6i_3 = -36$$

or

$$\boxed{2i_1 - 9i_2 + 3i_3 = 18} \quad (3.25)$$

For Mesh 3, we observe that the current  $i_3$  is just the current of the 5 A current source and thus our third equation is simply

$$\boxed{i_3 = 5} \quad (3.26)$$

and in matrix form,



$$\underbrace{\begin{bmatrix} 3 & -2 & 0 \\ 2 & -9 & 3 \\ 0 & 0 & 1 \end{bmatrix}}_R \underbrace{\begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix}}_I = \underbrace{\begin{bmatrix} 6 \\ 18 \\ 5 \end{bmatrix}}_V \quad (3.27)$$

Solving with MATLAB we obtain:

```
R=[3 -2 0; 2 -9 3; 0 0 1]; V=[6 18 5]'; I=R\V;...
fprintf(' \n'); fprintf('i1 = %5.2f amps \t', I(1)); ...
fprintf('i2 = %5.2f amps \t', I(2)); fprintf('i3 = %5.2f amps', I(3)); fprintf(' \n')
```

i1 = 2.09 amps    i2 = 0.13 amps    i3 = 5.00 amps

**Example 3.6**

Write mesh equations for the circuit of Figure 3.16 and solve for the unknowns using MATLAB or Excel. Then, compute the voltage drop across the 5 A source. Verify your answer with a Simulink / SimPowerSystems model.

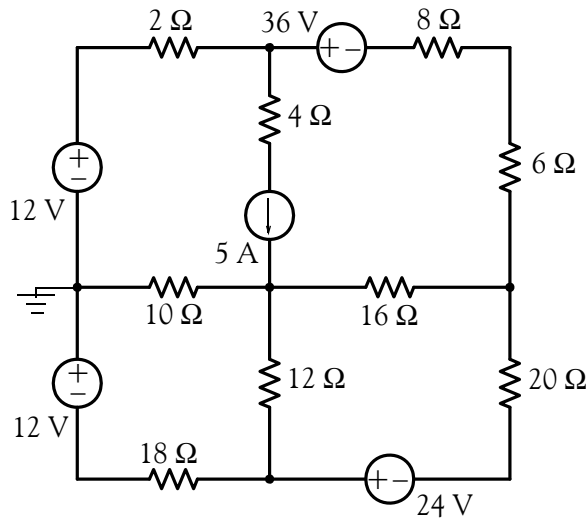


Figure 3.16. Circuit for Example 3.6

**Solution:**

Here, we would be tempted to assign mesh currents as shown in Figure 3.17. However, we will encounter a problem as explained below.

The currents  $i_3$  and  $i_4$  for Meshes 3 and 4 respectively present no problem; but for Meshes 1 and 2 we cannot write mesh equations for the currents  $i_1$  and  $i_2$  as shown because we cannot write a

term which represents the voltage across the 5 A current source. To work around this problem we temporarily remove (open) the 5 A current source and we form a “combined mesh” (or *generalized mesh* or *supermesh* as some textbooks call it) and the current that flows around this combined mesh is as shown in Figure 3.18.

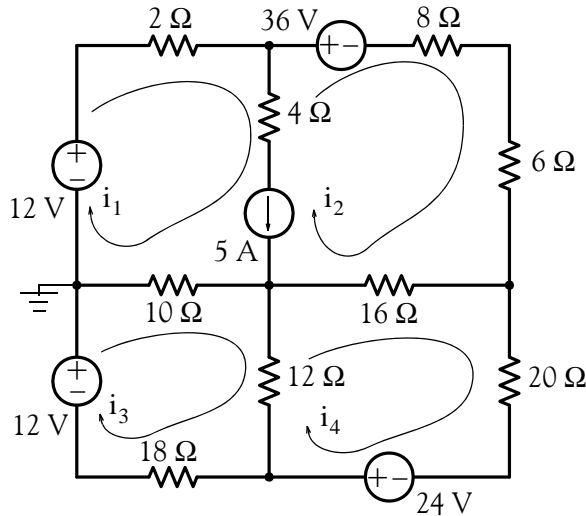


Figure 3.17. Circuit for Example 3.6 with erroneous current assignments

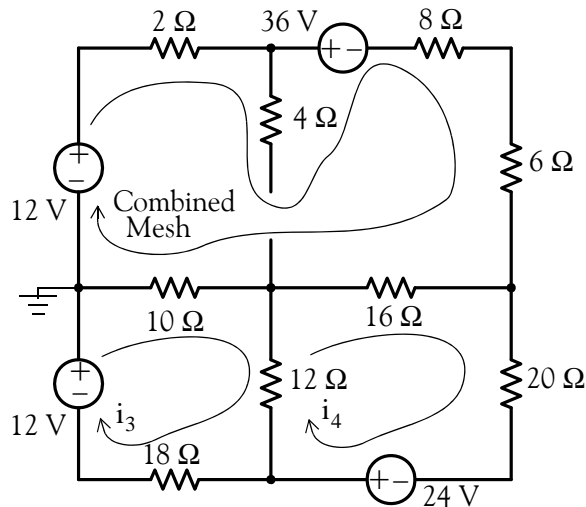


Figure 3.18. Circuit for Example 3.6 with correct current assignments

Now, we apply KVL around this combined mesh. We begin at the left end of the 2 Ω resistor, and we express the voltage drop across this resistor as  $2i_1$  since in Mesh 1 the current is essentially  $i_1$ .

---

## Chapter 3 Nodal and Mesh Equations – Circuit Theorems

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Continuing, we observe that there is no voltage drop across the  $4\ \Omega$  resistor since no current flows through it. The current now enters Mesh 2 where we encounter the  $36\text{ V}$  drop due to the voltage source there, and the voltage drops across the  $8\ \Omega$  and  $6\ \Omega$  resistors are  $8i_2$  and  $6i_2$  respectively since in Mesh 2 the current now is really  $i_2$ . The voltage drops across the  $16\ \Omega$  and  $10\ \Omega$  resistors are expressed as in the previous examples and thus our first mesh equation is

$$2i_1 + 36 + 8i_2 + 6i_2 + 16(i_2 - i_4) + 10(i_1 - i_3) - 12 = 0$$

or

$$12i_1 + 30i_2 - 10i_3 - 16i_4 = -24$$

or

$$\boxed{6i_1 + 15i_2 - 5i_3 - 8i_4 = -12} \quad (3.28)$$

Now, we reinsert the  $5\text{ A}$  current source between Meshes 1 and 2 and we obtain our second equation as

$$\boxed{i_1 - i_2 = 5} \quad (3.29)$$

For meshes 3 and 4, the equations are

$$10(i_3 - i_1) + 12(i_3 - i_4) + 18i_3 - 12 = 0$$

or

$$\boxed{5i_1 - 20i_3 + 6i_4 = -6} \quad (3.30)$$

and

$$16(i_4 - i_2) + 20i_4 - 24 + 12(i_4 - i_3) = 0$$

or

$$\boxed{4i_2 + 3i_3 - 12i_4 = -6} \quad (3.31)$$

and in matrix form

$$\underbrace{\begin{bmatrix} 6 & 15 & -5 & -8 \\ 1 & -1 & 0 & 0 \\ 5 & 0 & -20 & 6 \\ 0 & 4 & 3 & -12 \end{bmatrix}}_{\mathbf{R}} \underbrace{\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix}}_{\mathbf{I}} = \underbrace{\begin{bmatrix} -12 \\ 5 \\ -6 \\ -6 \end{bmatrix}}_{\mathbf{V}} \quad (3.32)$$

We find the solution of (3.32) with the following MATLAB script:

```
R=[6 15 -5 -8; 1 -1 0 0; 5 0 -20 6; 0 4 3 -12]; V=[-12 5 -6 -6]'; I=R\V;...
fprintf(' \n');...
fprintf('i1 = %5.4f amps \t',I(1)); fprintf('i2 = %5.4f amps \t',I(2));...
fprintf('i3 = %5.4f amps \t',I(3)); fprintf('i4 = %5.4f amps',I(4)); fprintf(' \n')
i1=3.3975 amps i2=-1.6025 amps i3=1.2315 amps i4=0.2737 amps
```

Now, we can find the voltage drop across the 5 A current source by application of KVL around Mesh 1 using the following relation:

$$2 \times 3.3975 + 4 \times (3.3975 + 1.6025) + v_{5A} + 10 \times (3.3975 - 1.2315) - 12 = 0$$

This yields

$$v_{5A} = -36.455$$

We can verify this value by application of KVL around Mesh 2 where beginning with the lower end of the 6 w resistor and going counterclockwise we obtain

$$(6 + 8) \times 1.6025 - 36 + 4 \times (3.3975 + 1.6025) - 36.455 + 16 \times (1.6025 + 0.2737) = 0$$

With these values, we can also compute the power delivered or absorbed by each of the voltage sources and the current source.

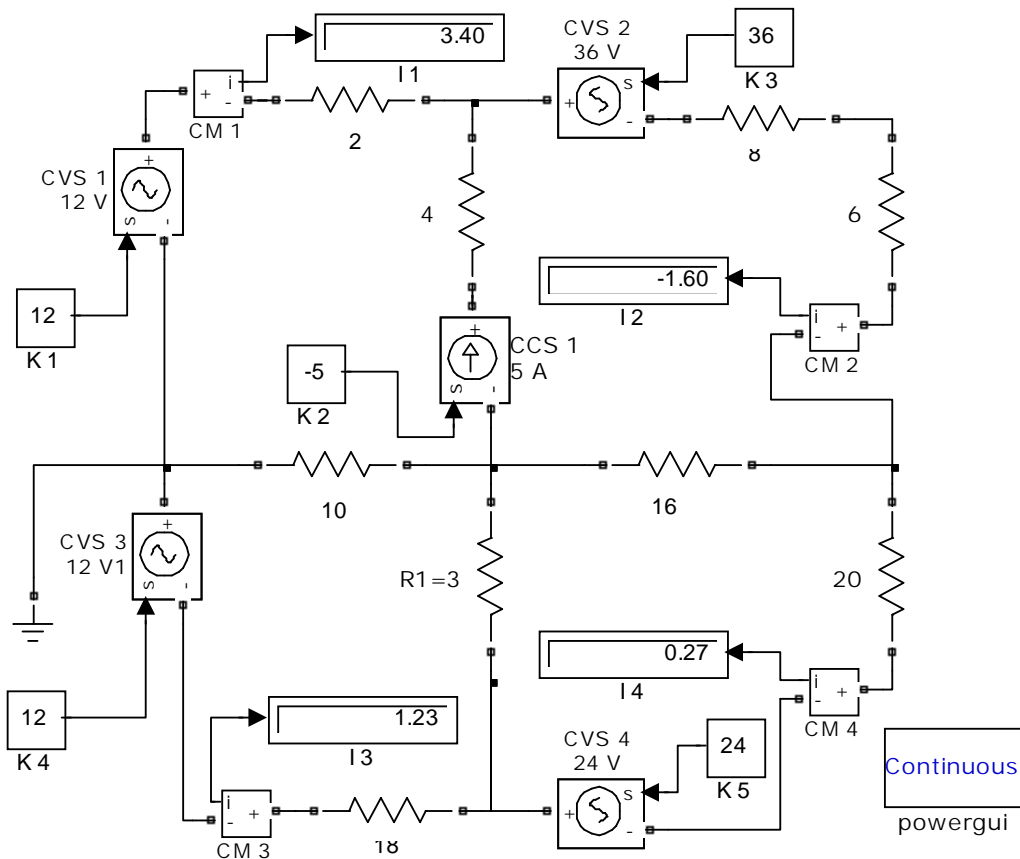


Figure 3.19. Simulink / SimPower Systems model for Example 3.6

### 3.4 Transformation between Voltage and Current Sources

In the previous chapter we stated that a voltage source maintains a constant voltage between its terminals regardless of the current that flows through it. This statement applies to an ideal voltage source which, of course, does not exist; for instance, no voltage source can supply infinite current to a short circuit. We also stated that a current source maintains a constant current regardless of the terminal voltage. This statement applies to an ideal current source which also does not exist; for instance, no current source can supply infinite voltage when its terminals are open-circuited.

A *practical voltage source* has an internal resistance which, to be accounted for, it is represented with an external resistance  $R_S$  in series with the voltage source  $v_S$  as shown in Figure 3.20 (a). Likewise a *practical current source* has an internal conductance which is represented as a resistance  $R_P$  (or conductance  $G_P$ ) in parallel with the current source  $i_S$  as shown in Figure 3.20 (b).



Figure 3.20. Practical voltage and current sources

In Figure 3.20 (a), the voltage of the source will always be  $v_S$  but the terminal voltage  $v_{ab}$  will be  $v_{ab} = v_S - v_{R_S}$  if a load is connected at points a and b. Likewise, in Figure 3.20 (b) the current of the source will always be  $i_S$  but the terminal current  $i_{ab}$  will be  $i_{ab} = i_S - i_{R_P}$  if a load is connected at points a and b.

Now, we will show that the networks of Figures 3.20 (a) and 3.20 (b) can be made equivalent to each other.

In the networks of Figures 3.21 (a) and 3.21 (b), the load resistor  $R_L$  is the same in both.

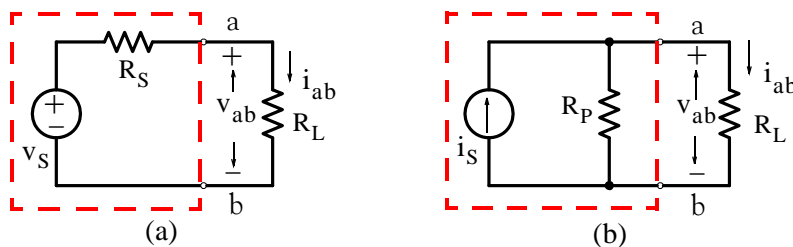


Figure 3.21. Equivalent sources

From the circuit of Figure 3.21 (a),

$$v_{ab} = \frac{R_L}{R_S + R_L} v_S \quad (3.33)$$

and

$$i_{ab} = \frac{v_S}{R_S + R_L} \quad (3.34)$$

From the circuit of Figure 3.21 (b),

$$v_{ab} = \frac{R_p R_L}{R_p + R_L} i_S \quad (3.35)$$

and

$$i_{ab} = \frac{R_p}{R_p + R_L} i_S \quad (3.36)$$

Since we want  $v_{ab}$  to be the same in both circuits 3.21 (a) and 3.21 (b), from (3.33) and (3.35) we must have:

$$v_{ab} = \frac{R_L}{R_S + R_L} v_S = \frac{R_p R_L}{R_p + R_L} i_S \quad (3.37)$$

Likewise, we want  $i_{ab}$  to be the same in both circuits 3.21 (a) and 3.21 (b). Then, from (3.34) and (3.36) we obtain:

$$i_{ab} = \frac{v_S}{R_S + R_L} = \frac{R_p}{R_p + R_L} i_S \quad (3.38)$$

and for any  $R_L$ , from (3.37) and (3.38)

$$\boxed{v_S = R_p i_S} \quad (3.39)$$

and

$$\boxed{R_p = R_S} \quad (3.40)$$

Therefore, a voltage source  $v_S$  in series with a resistance  $R_S$  can be transformed to a current source  $i_S$  whose value is equal to  $v_S/R_S$ , in parallel with a resistance  $R_p$  whose value is the same as  $R_S$ .

Likewise, a current source  $i_S$  in parallel with a resistance  $R_p$  can be transformed to a voltage source  $v_S$  whose value is equal to  $i_S \times R_S$ , in series with a resistance whose value is the same as  $R_p$ .

The voltage-to-current source or current-to-voltage source transformation is not limited to a single resistance load; it applies to any load no matter how complex.

Example 3.7

Find the current  $i_{10}$  through the  $10\ \Omega$  resistor in the circuit of Figure 3.22.

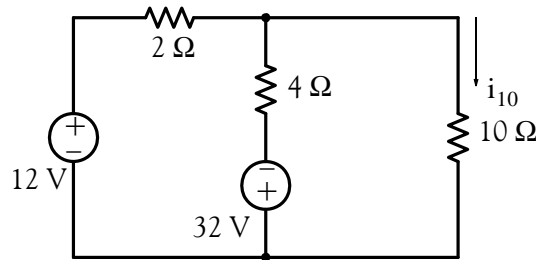


Figure 3.22. Circuit for Example 3.7

Solution:

This problem can be solved either by nodal or by mesh analysis; however, we will transform the voltage sources to current sources and we will replace the resistances with conductances except the  $10\ \Omega$  resistor. We will treat the  $10\ \Omega$  resistor as the load of this circuit so that we can compute the current  $i_{10}$  through it. Then, the circuit becomes as shown in Figure 3.23.

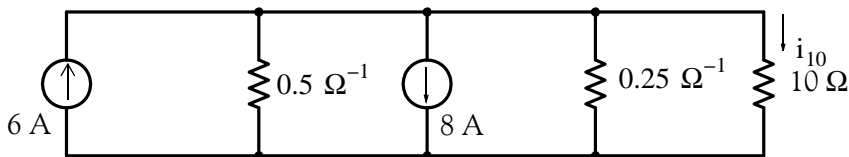


Figure 3.23. Circuit for Example 3.7 with voltage sources transformed to current sources

Combination of the two current sources and their conductances yields the circuit shown in Figure 3.24.

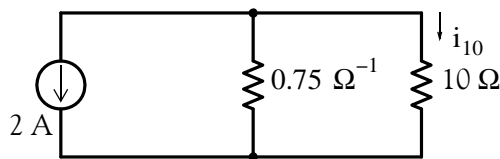


Figure 3.24. Circuit for Example 3.7 after combinations of current sources and conductances

Converting the  $0.75\ \Omega^{-1}$  conductance to a resistance and performing current-to-voltage source transformation, we obtain the circuit of Figure 3.25.

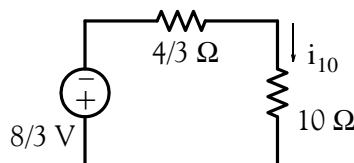


Figure 3.25. Circuit for Example 3.7 in its simplest form

Thus, the current through the  $10\ \Omega$  resistor is

$$i_{10} = \frac{-8/3}{10 + 4/3} = -4/17\ \text{A}$$

### 3.5 Thevenin's Theorem

This theorem is perhaps the greatest time saver in circuit analysis, especially in electronic \* circuits. It states that we can replace a two terminal network by a voltage source  $v_{TH}$  in series with a resistance  $R_{TH}$  as shown in Figure 3.26.

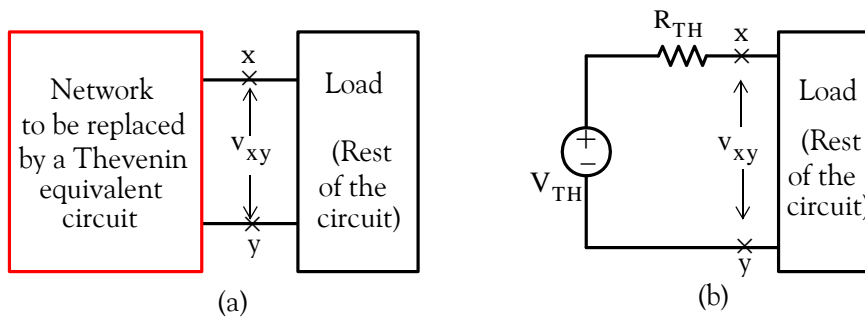


Figure 3.26. Replacement of a network by its Thevenin's equivalent

The network of Figure 3.26 (b) will be equivalent to the network of Figure 3.26 (a) if the load is removed in which case both networks will have the same open circuit voltages  $v_{xy}$  and consequently,

$$v_{TH} = v_{xy}$$

Therefore,

$$v_{TH} = v_{xy\ \text{open}} \tag{3.41}$$

The Thevenin resistance  $R_{TH}$  represents the equivalent resistance of the network being replaced by the Thevenin equivalent, and it is found from the relation

$$R_{TH} = \frac{v_{xy\ \text{open}}}{i_{xy\ \text{short}}} = \frac{v_{TH}}{i_{SC}} \tag{3.42}$$

where  $i_{SC}$  stands for short-circuit current.

\* For an introduction to electronic circuits, please refer to *Electronic Devices and Amplifier Circuits with MATLAB Applications*, ISBN 978-1-934404-13-3.



## Chapter 3 Nodal and Mesh Equations – Circuit Theorems

If the network to be replaced by a Thevenin equivalent contains independent sources only, we can find the Thevenin resistance  $R_{TH}$  by first shorting all (independent) voltage sources, opening all (independent) current sources, and calculating the *resistance looking into the direction that is opposite to the load when it has been disconnected from the rest of the circuit at terminals x and y*.

### Example 3.8

Use Thevenin's theorem to find  $i_{LOAD}$  and  $v_{LOAD}$  for the circuit of Figure 3.27.

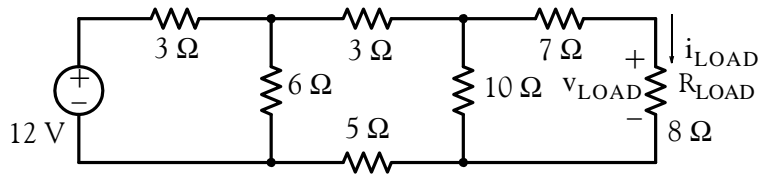


Figure 3.27. Circuit for Example 3.8

#### Solution:

We will apply Thevenin's theorem twice; first at terminals x and y and then at x' and y' as shown in Figure 3.28.

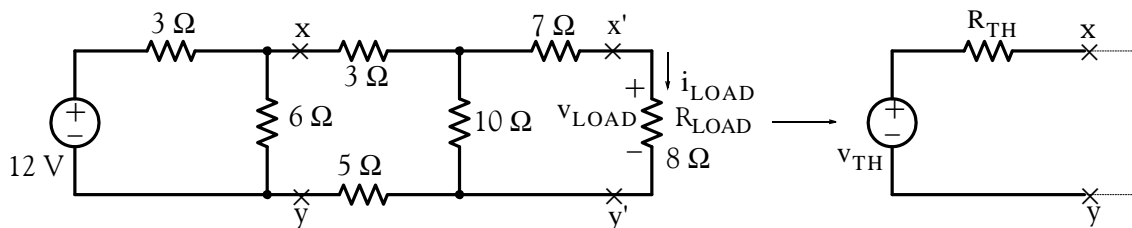


Figure 3.28. First step in finding the Thevenin equivalent of the circuit of Example 3.8

Breaking the circuit at x – y, we are left with the circuit shown in Figure 3.29.

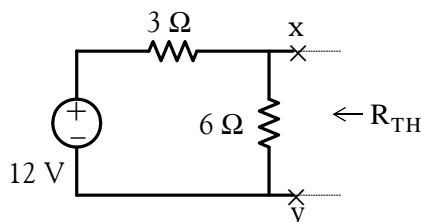


Figure 3.29. Second step in finding the Thevenin equivalent of the circuit of Example 3.8

Applying Thevenin's theorem at x and y and using the voltage division expression, we obtain

$$v_{TH} = v_{xy} = \frac{6}{3+6} \times 12 = 8 \text{ V}$$

$$R_{TH}|_{V_s=0} = \frac{3 \times 6}{3+6} = 2 \text{ } \Omega$$
(3.43)

and thus the equivalent circuit to the left of points x and x is as shown in Figure 3.30.

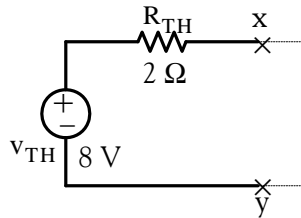


Figure 3.30. First Thevenin equivalent for the circuit of Example 3.8

Next, we attach the remaining part of the given circuit to the Thevenin equivalent of Figure 3.30, and the new circuit now is as shown in Figure 3.31.

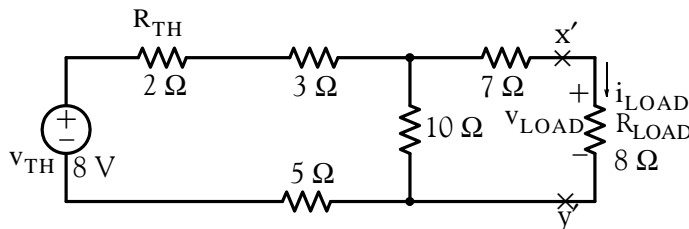


Figure 3.31. Circuit for Example 3.8 with first Thevenin equivalent

Now, we apply Thevenin's theorem at points x' and y' and we obtain the circuit of Figure 3.32.

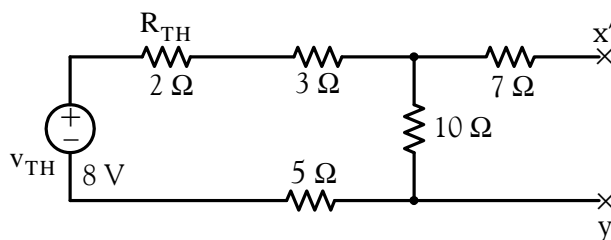


Figure 3.32. Applying Thevenin's theorem at points x' and y' for the circuit for Example 3.8

Using the voltage division expression, we obtain

$$v'_{TH} = v_{x'y'} = \frac{10}{2+3+10+5} \times 8 = 4 \text{ V}$$

$$R'_{TH}|_{v_{TH}=0} = [(2+3+5)||10] + 7 = 12 \text{ } \Omega$$

This Thevenin equivalent with the load resistor attached to it, is shown in Figure 3.33.

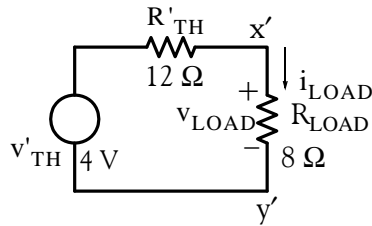


Figure 3.33. Entire circuit of Example 3.8 simplified by Thevenin's theorem

The voltage  $v_{\text{LOAD}}$  is found by application of the voltage division expression, and the current  $i_{\text{LOAD}}$  by Ohm's law as shown below.

$$v_{\text{LOAD}} = \frac{8}{12 + 8} \times 4 = 1.6 \text{ V}$$

$$i_{\text{LOAD}} = \frac{4}{12 + 8} = 0.2 \text{ A}$$

It is imperative to remember that when we compute the Thevenin equivalent resistance, we must always look towards the network portion which remains after disconnecting the load at the  $x$  and  $y$  terminals. This is illustrated with the two examples that follow.

Let us consider the network of Figure 3.34 (a).

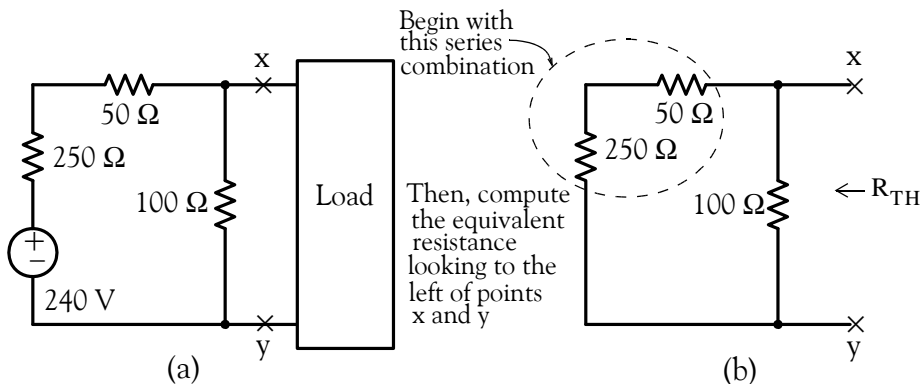


Figure 3.34. Computation of the Thevenin equivalent resistance when the load is to the right

This network contains no dependent sources; therefore, we can find the Thevenin equivalent by shorting the 240 V voltage source, and computing the equivalent resistance looking to the left of points  $x$  and  $y$  as indicated in Figure 3.34 (b). Thus,

$$R_{\text{TH}} = (250 + 50) \parallel 100 = \frac{300 \times 100}{300 + 100} = 75 \text{ } \Omega$$

Now, let us consider the network of Figure 3.35 (a).

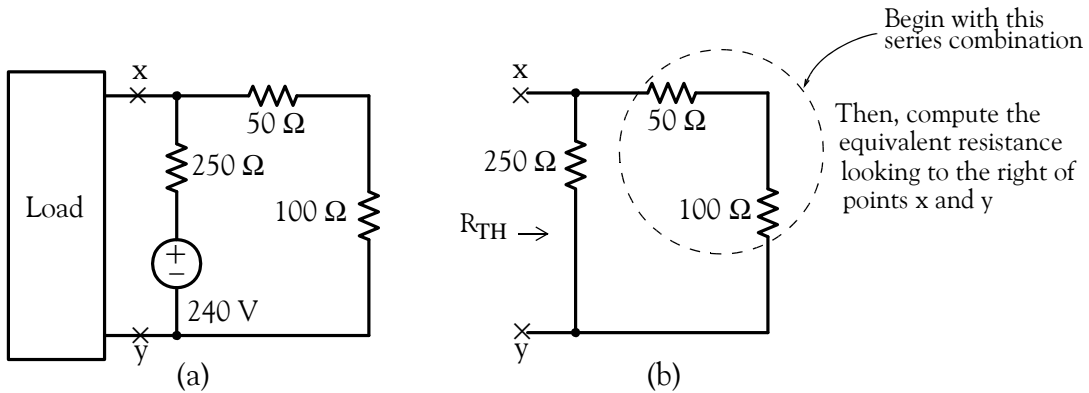


Figure 3.35. Computation of the Thevenin equivalent resistance when the load is to the left

This network contains no dependent sources; therefore, we can find the Thevenin equivalent by shorting the 240 V voltage source, and computing the equivalent resistance looking to the right of points  $x$  and  $y$  as indicated in Figure 3.35 (b). Thus,

$$R_{TH} = (50 + 100) || 250 = \frac{150 \times 250}{150 + 250} = 93.75 \Omega$$

We observe that, although the resistors in the networks of Figures 3.34 (b) and 3.35 (b) have the same values, the Thevenin resistance is different since it depends on the direction in which we look into (left or right).

### Example 3.9

Use Thevenin's theorem to find  $i_{LOAD}$  and  $v_{LOAD}$  for the circuit of Figure 3.36.

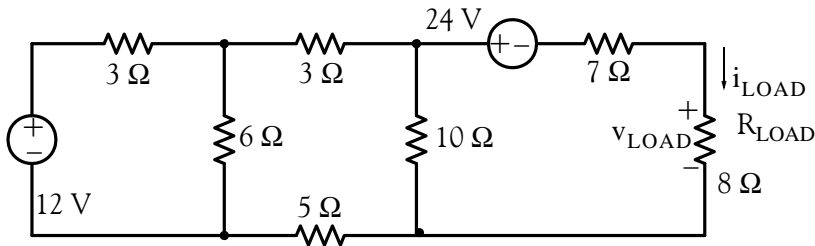


Figure 3.36. Circuit for Example 3.9

#### Solution:

This is the same circuit as the previous example except that a voltage source of 24 V has been placed in series with the 7 Ω resistor. By application of Thevenin's theorem at points  $x$  and  $y$  as before, and connecting the rest of the circuit, we obtain the circuit of Figure 3.37.

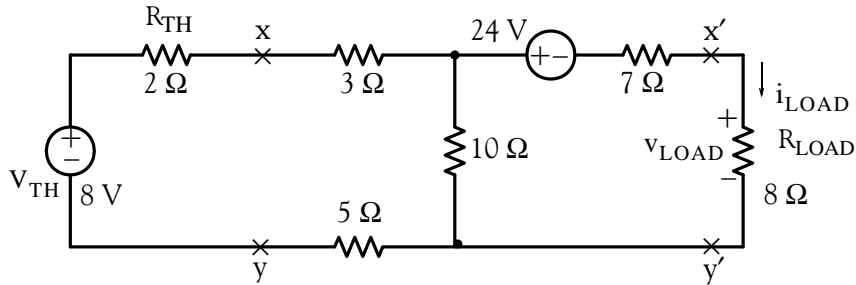


Figure 3.37. Circuit for Example 3.9 with first Thevenin equivalent

Next, disconnecting the load resistor and applying Thevenin's theorem at points  $x'$  and  $y'$  we obtain the circuit of Figure 3.38.

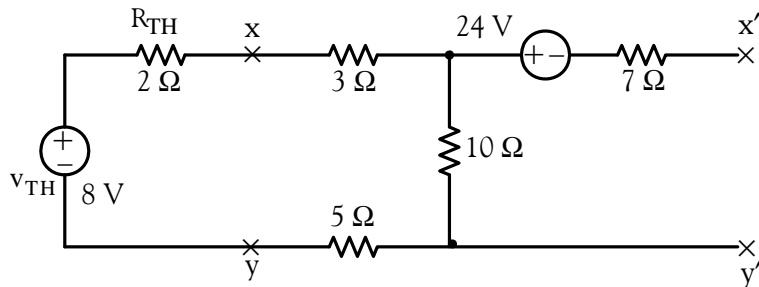


Figure 3.38. Application of Thevenin's theorem at points  $x'$  and  $y'$  for the circuit for Example 3.9

There is no current flow in the  $7\ \Omega$  resistor; thus, the Thevenin voltage across the  $x'$  and  $y'$  points is the algebraic sum of the voltage drop across the  $10\ \Omega$  resistor and the  $24\ \text{V}$  source, i.e.,

$$v'_{\text{TH}} = v_{x'y'} = \frac{10}{2 + 3 + 10 + 5} \times 8 - 24 = -20\ \text{V}$$

and the Thevenin resistance is the same as in the previous example, that is,

$$R'_{\text{TH}}|_{V_{\text{TH}}=0} = [(2 + 3 + 5) || 10] + 7 = 12\ \Omega$$

Finally, connecting the load  $R_{\text{LOAD}}$  as shown in Figure 3.39, we compute  $v_{\text{LOAD}}$  and  $i_{\text{LOAD}}$  as follows:

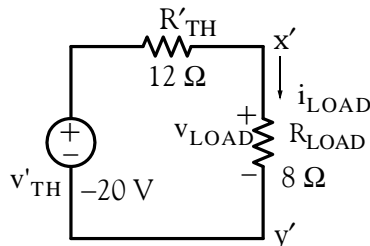


Figure 3.39. Final form of Thevenin equivalent with load connected for circuit of Example 3.9

$$v_{\text{LOAD}} = \frac{8}{12+8} \times (-20) = -8 \text{ V}$$

$$i_{\text{LOAD}} = \frac{-20}{12+8} = -1 \text{ A}$$

### Example 3.10

For the circuit of Figure 3.40, use Thevenin's theorem to find  $i_{\text{LOAD}}$  and  $v_{\text{LOAD}}$ .

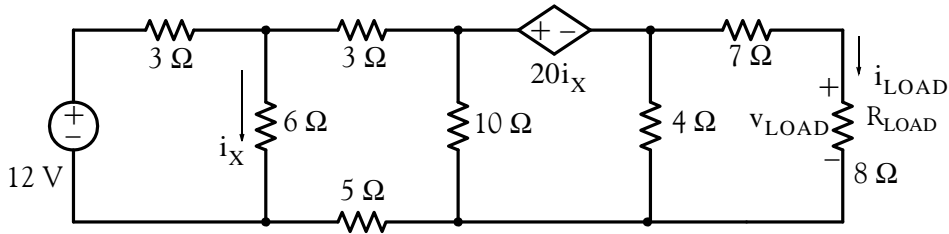


Figure 3.40. Circuit for Example 3.10

#### Solution:

This circuit contains a dependent voltage source whose value is twenty times the current through the  $6 \Omega$  resistor. We will apply Thevenin's theorem at points  $a$  and  $b$  as shown in Figure 3.39.

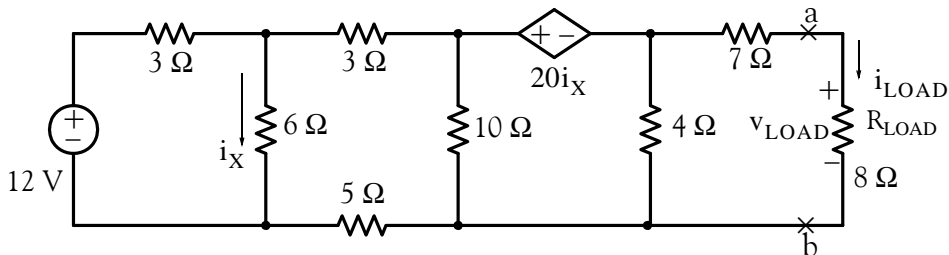


Figure 3.41. Application of Thevenin's theorem for Example 3.10

In the circuit of Figure 3.41, we cannot short the dependent source; therefore, we will find the Thevenin resistance from the relation

$$R_{\text{TH}} = \frac{v_{\text{OC}}}{i_{\text{SC}}} = \frac{v_{\text{LOAD}}|_{R_L \rightarrow \infty}}{i_{\text{LOAD}}|_{R_L \rightarrow 0}} \quad (3.44)$$

To find the open circuit voltage  $v_{\text{OC}} = v_{ab}$ , we disconnect the load resistor and our circuit now is as shown in Figure 3.42.

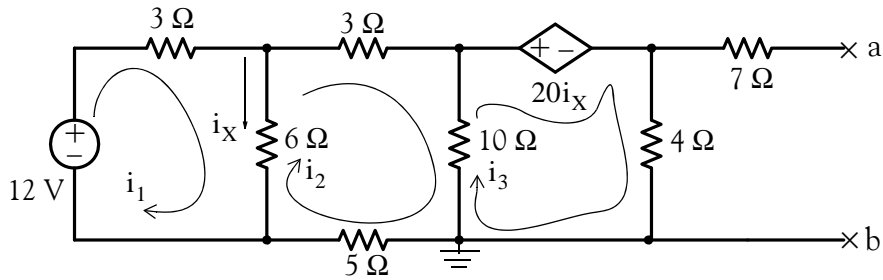


Figure 3.42. Circuit for finding  $v_{OC} = v_{ab}$  of Example 3.10

We will use mesh analysis to find  $v_{OC}$  which is the voltage across the  $4\ \Omega$  resistor. We chose mesh analysis since we only need three mesh equations whereas we would need five equations had we chosen nodal analysis. Please refer to Exercise 16 at the end of this chapter for a solution requiring nodal analysis.

Observing that  $i_x = i_1 - i_2$ , we write the three mesh equations for this network as

$$\begin{aligned} 9i_1 - 6i_2 &= 12 \\ -6i_1 + 24i_2 - 10i_3 &= 0 \\ 20(i_1 - i_2) + 4i_3 + 10(i_3 - i_2) &= 0 \end{aligned} \tag{3.45}$$

and after simplification and combination of like terms, we write them in matrix form as

$$\underbrace{\begin{bmatrix} 3 & -2 & 0 \\ 3 & -12 & 5 \\ 10 & -15 & 7 \end{bmatrix}}_R \underbrace{\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}}_I = \underbrace{\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}}_V \tag{3.46}$$

Using the spreadsheet of Figure 3.43, we find that  $i_3 = -3.53\ \text{A}$

	A	B	C	D	E	F	G	H
1	Spreadsheet for Matrix Inversion and Matrix Multiplication							
3		3	-2	0				4
4	R=	3	-12	5		V=		0
5		10	-15	7				0
7		0.106	-0.165	0.118				0.42
8	R <sup>-1</sup> =	-0.341	-0.247	0.176		I=		-1.36
9		-0.882	-0.294	0.353				-3.53

Figure 3.43. Spreadsheet for Example 3.10

Thus, the Thevenin voltage at points  $a$  and  $b$  is

$$v_{TH} = (-3.53) \times 4 = -14.12 \text{ V}$$

Next, to find the Thevenin resistance  $R_{TH}$ , we must first compute the short circuit current  $I_{SC}$ . Accordingly, we place a short across points  $a$  and  $b$  and the circuit now is as shown in Figure 3.44 and we can find the short circuit current  $i_{SC}$  from the circuit of Figure 3.45 where  $i_{SC} = i_4$

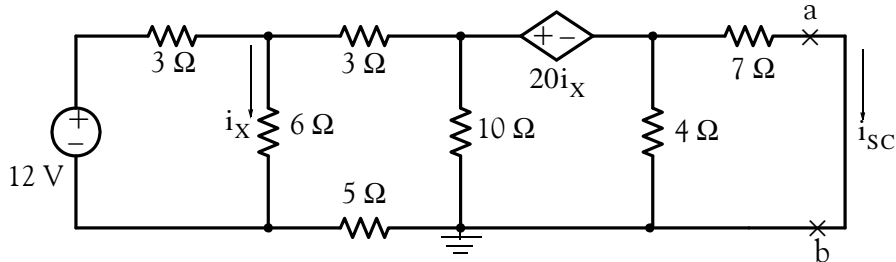


Figure 3.44. Circuit for finding  $i_{SC} = i_{ab}$  in Example 3.10

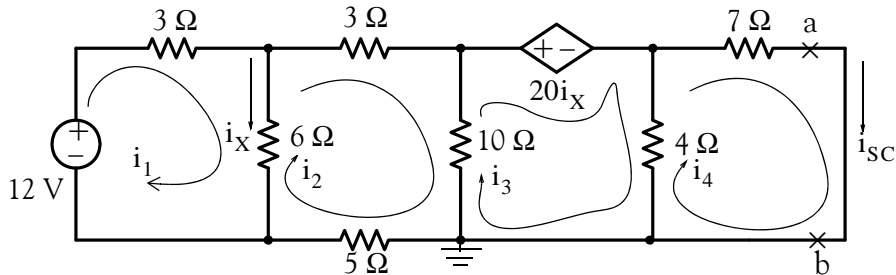


Figure 3.45. Mesh equations for finding  $i_{SC} = i_{ab}$  in Example 3.10

The mesh equations for the circuit of Figure 3.45 are

$$\begin{aligned} 9i_1 - 6i_2 &= 12 \\ -6i_1 + 24i_2 - 10i_3 &= 0 \\ 20(i_1 - i_2) + 4(i_3 - i_4) + 10(i_3 - i_2) &= 0 \\ -4i_3 + 11i_4 &= 0 \end{aligned} \tag{3.47}$$

and after simplification and combination of like terms, we write them in matrix form as

$$\underbrace{\begin{bmatrix} 3 & -2 & 0 & 0 \\ 3 & -12 & 5 & 0 \\ 10 & -15 & 7 & -2 \\ 0 & 0 & -4 & 11 \end{bmatrix}}_{\mathbf{R}} \underbrace{\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix}}_{\mathbf{I}} = \underbrace{\begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{V}} \tag{3.48}$$



## Chapter 3 Nodal and Mesh Equations – Circuit Theorems

We will solve these using MATLAB as follows:

```
R=[3 -2 0 0; 3 -12 5 0; 10 -15 7 -2; 0 0 -4 11]; V=[4 0 0 0]'; I=R\V;
fprintf(' \n');...
fprintf('i1 = %3.4f A \t',I(1,1)); fprintf('i2 = %3.4f A \t',I(2,1));...
fprintf('i3 = %3.4f A \t',I(3,1)); fprintf('i4 = %3.4f A \t',I(4,1));...
fprintf(' \n');...fprintf(' \n')
```

$i_1 = 0.0173 \text{ A}$      $i_2 = -1.9741 \text{ A}$      $i_3 = -4.7482 \text{ A}$      $i_4 = -1.7266 \text{ A}$

Therefore,

$$i_{SC} = i_4 = -1.727$$

and

$$R_{TH} = \frac{v_{OC}}{i_{SC}} = \frac{-14.12}{-1.727} = 8.2 \Omega$$

The Thevenin equivalent is as shown in Figure 3.46.

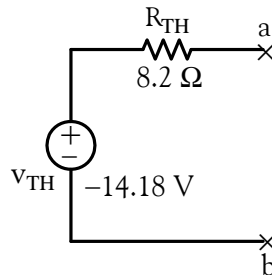


Figure 3.46. Final form of Thevenin's equivalent for the circuit of Example 3.10

Finally, with the load  $R_{LOAD}$  attached to points  $a$  and  $b$ , the circuit is as shown in Figure 3.47.

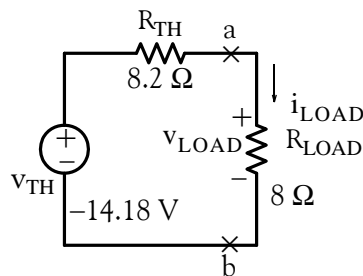


Figure 3.47. Circuit for finding  $v_{LOAD}$  and  $i_{LOAD}$  in Example 3.10

Therefore, using the voltage division expression and Ohm's law we obtain

$$v_{LOAD} = \frac{8}{8.2 + 8} \times (-14.18) = -7.00 \text{ V}$$

$$i_{LOAD} = \frac{-14.18}{8.2 + 8} = -0.875 \text{ A}$$

### 3.6 Norton's Theorem

This theorem is analogous to Thevenin's theorem and states that we can replace everything, except the load, in a circuit by an equivalent circuit containing only an independent current source which we will denote as  $i_N$  in parallel with a resistance which we will denote as  $R_N$ , as shown in Figure 3.48.

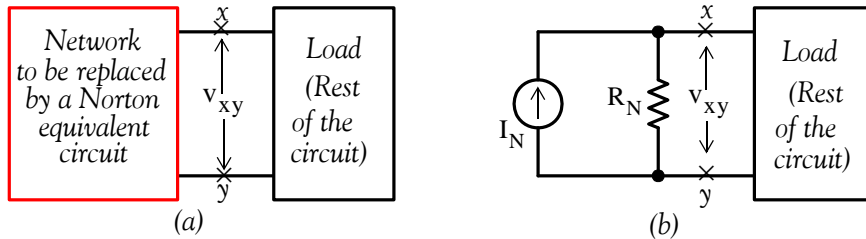


Figure 3.48. Replacement of a network by its Norton equivalent

The current source  $I_N$  has the value of the short circuit current which would flow if a short were connected between the terminals  $x$  and  $y$ , where the Norton equivalent is inserted, and the resistance  $R_N$  is found from the relation

$$R_N = \frac{v_{OC}}{i_{SC}} \quad (3.49)$$

where  $v_{OC}$  is the open circuit voltage which appears across the open terminals  $x$  and  $y$ .

Like Thevenin's, Norton's theorem is most useful when a series of computations involves changing the load of a network while the rest of the circuit remains unchanged.

Comparing the Thevenin's and Norton's equivalent circuits, we see that one can be derived from the other by replacing the Thevenin voltage and its series resistance with the Norton current source and its parallel resistance. Therefore, there is no need to perform separate computations for each of these equivalents; once we know Thevenin's equivalent we can easily draw the Norton equivalent and vice versa.

#### Example 3.11

Replace the network shown in Figure 3.49 by its Thevenin and Norton equivalents.

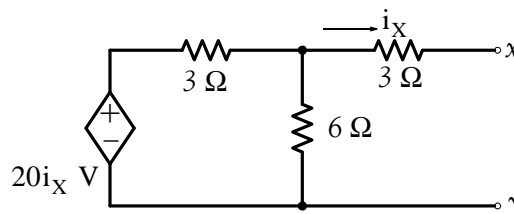


Figure 3.49. Network for Example 3.11

### Solution:

We observe that no current flows through the  $3\ \Omega$  resistor; Therefore,  $i_x = 0$  and the dependent current source is zero, i.e., a short circuit. Thus,

$$v_{TH} = v_{OC} = v_{xy} = 0$$

and also

$$i_{SC} = 0$$

This means that the given network is some mathematical model representing a resistance, but we cannot find this resistance from the expression

$$R_{TH} = R_N = \frac{v_{OC}}{i_{SC}}$$

since this results in the indeterminate form  $0/0$ . In this type of situations, we connect an external source (voltage or current) across the terminals  $x$  and  $y$ . For this example, we arbitrarily choose to connect a 1 volt source as shown in Figure 3.50.

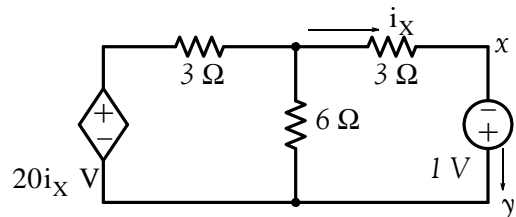


Figure 3.50. Network for Example 3.11 with an external voltage source connected to it.

In the circuit of Figure 3.50, the 1 V source represents the open circuit voltage  $v_{OC}$  and the current  $i$  represents the short circuit current  $i_{SC}$ . Therefore, the Thevenin (or Norton) resistance will be found from the expression

$$R_{TH} = R_N = \frac{v_{OC}}{i_{SC}} = \frac{1\ \text{V}}{i} = \frac{1\ \text{V}}{i_x} \quad (3.50)$$

Now, we can find  $i$  from the circuit of Figure 3.51 by application of KCL at Node ①.

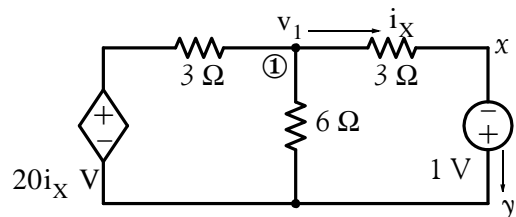


Figure 3.51. Circuit for finding  $i_x$  in Example 3.11

$$\frac{v_1 - 20i_X}{3} + \frac{v_1}{6} + i_X = 0 \quad (3.51)$$

where

$$i_X = \frac{v_1 - 1}{3} \quad (3.52)$$

Simultaneous solution of (3.51) and (3.52) yields  $v_1 = 34/25$  and  $i_X = 3/25$ . Then, from (3.50),

$$R_{TH} = R_N = \frac{1}{3/25} = \frac{25}{3}$$

and the Thevenin and Norton equivalents are shown in Figure 3.52.

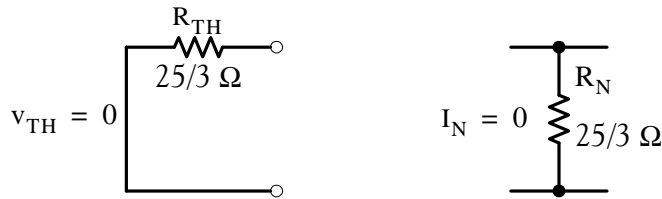


Figure 3.52. Thevenin's and Norton's equivalents for Example 3.11

### 3.7 Maximum Power Transfer Theorem

Consider the circuit shown in Figure 3.53. We want to find the value of  $R_{LOAD}$  that will absorb maximum power from the voltage source  $v_S$  whose internal resistance is  $R_S$ .

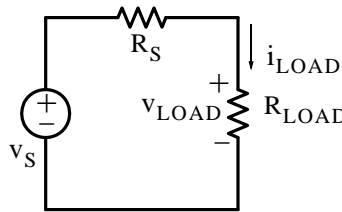


Figure 3.53. Circuit for computation of maximum power delivered to the load  $R_{LOAD}$

The power  $p_{LOAD}$  delivered to the load is found from

$$P_{LOAD} = v_{LOAD} \times i_{LOAD} = \left( \frac{R_{LOAD}}{R_S + R_{LOAD}} v_S \right) \left( \frac{v_S}{R_S + R_{LOAD}} \right)$$

or

$$P_{LOAD} = \frac{R_{LOAD}}{(R_S + R_{LOAD})^2} v_S^2 \quad (3.53)$$

## Chapter 3 Nodal and Mesh Equations – Circuit Theorems

To find the value of  $R_{LOAD}$  which will make  $p_{LOAD}$  maximum, we differentiate (3.53) with respect to  $R_{LOAD}$ . Recalling that

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \frac{d}{dx}(u) - u \frac{d}{dx}(v)}{v^2}$$

and differentiating (3.53), we obtain

$$\frac{dp_{LOAD}}{dR_{LOAD}} = \frac{(R_S + R_{LOAD})^2 v_S^2 - v_S^2 R_{LOAD}(2)(R_S + R_{LOAD})}{(R_S + R_{LOAD})^4} \quad (3.54)$$

and (3.54) will be zero if the numerator is set equal to zero, that is, if

$$(R_S + R_{LOAD})^2 v_S^2 - v_S^2 R_{LOAD}(2)(R_S + R_{LOAD}) = 0$$

or

$$R_S + R_{LOAD} = 2R_{LOAD}$$

or

$$\boxed{R_{LOAD} = R_S} \quad (3.55)$$

Therefore, we conclude that a voltage source with internal series resistance  $R_S$  or a current source with internal parallel resistance  $R_P$  delivers maximum power to a load  $R_{LOAD}$  when  $R_{LOAD} = R_S$  or  $R_{LOAD} = R_P$ . For example, in the circuits of Figure 3.54, the voltage source  $v_S$  and current source  $i_N$  deliver maximum power to the adjustable\* load when  $R_{LOAD} = R_S = R_P = 5 \Omega$

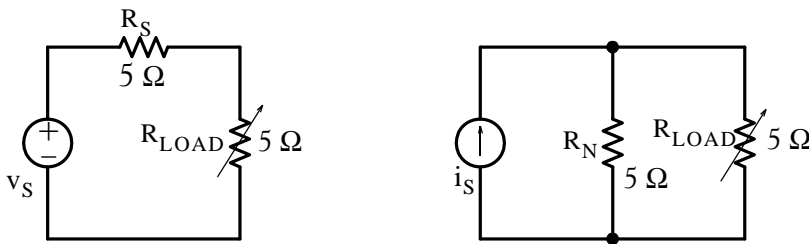


Figure 3.54. Circuits where  $R_{LOAD}$  is set to receive maximum power

We can use Excel or MATLAB to see that the load receives maximum power when it is set to the same value as that of the resistance of the source. Figure 3.55 shows a spreadsheet with various values of an adjustable resistive load. We observe that the power is maximum when  $R_{LOAD} = 5 \Omega$ .

\* An adjustable resistor is usually denoted with an arrow as shown in Figure 3.54.

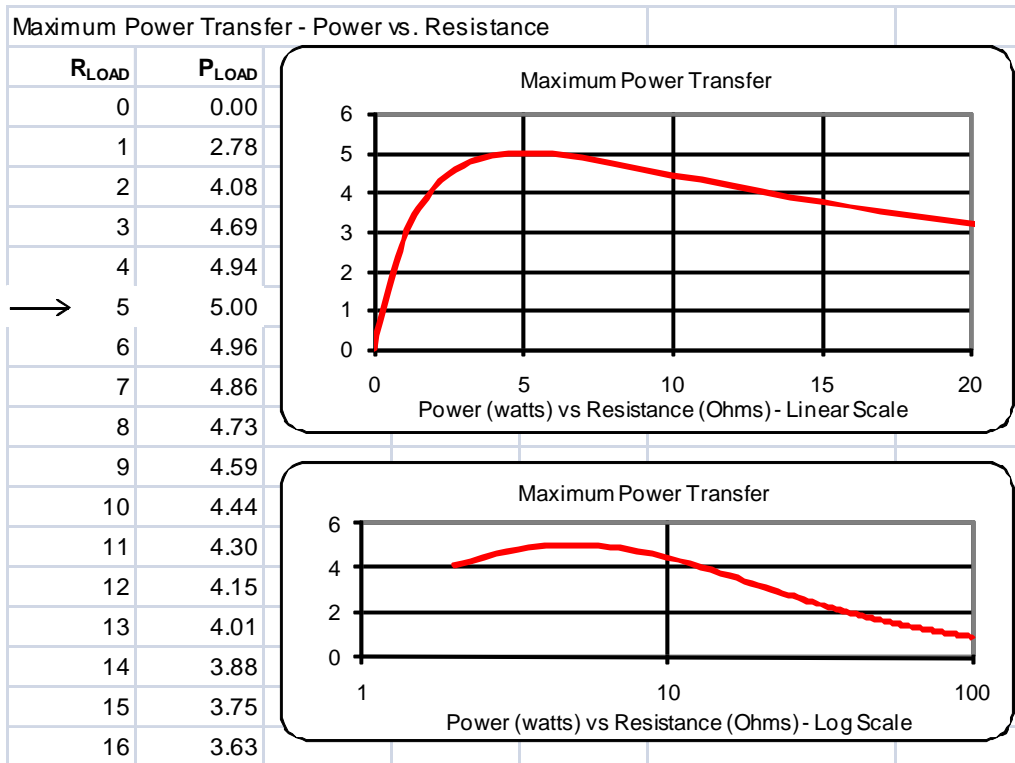


Figure 3.55. Spreadsheet to illustrate maximum power transfer to a resistive load

The condition of maximum power transfer is also referred to as *resistance matching* or *impedance matching*. We will define the term “impedance” in Chapter 6.

The maximum power transfer theorem is of great importance in electronics and communications applications where it is desirable to receive maximum power from a given circuit and efficiency is not an important consideration. On the other hand, in power systems, this application is of no use since the intent is to supply a large amount of power to a given load by making the internal resistance  $R_S$  as small as possible.

### 3.8 Linearity

A *linear passive element* is one in which there is a linear voltage–current relationship such as

$$v_R = R i_R \quad v_L = L \frac{d}{dt} i_L \quad i_C = C \frac{d}{dt} v_C \quad (3.56)$$

#### Definition 3.1

A *linear dependent source* is a dependent voltage or current source whose output voltage or current is proportional only to the first power of some voltage or current variable in the circuit or a

linear combination (the sum or difference of such variables). For example,  $v_{xy} = 2v_1 - 3i_2$  is a linear relationship but  $p = vi = Ri^2 = v^2/R$  and  $i = I_S e^{v/nV_T}$  are non-linear.

### Definition 3.2

A *linear circuit* is a circuit which is composed entirely of independent sources, linear dependent sources and linear passive elements or a combination of these.

## 3.9 Superposition Principle

The *principle of superposition* states that the response (a desired voltage or current) in any branch of a linear circuit having more than one independent source can be obtained as the sum of the responses caused by each independent source acting alone with all other independent voltage sources replaced by short circuits and all other independent current sources replaced by open circuits.

**Note:** *Dependent sources (voltage or current) must not be superimposed* since their values depend on the voltage across or the current through some other branch of the circuit. Therefore, all dependent sources must always be left intact in the circuit while superposition is applied.

---

### Example 3.12

In the circuit of Figure 3.56, compute  $i_6$  by application of the superposition principle.

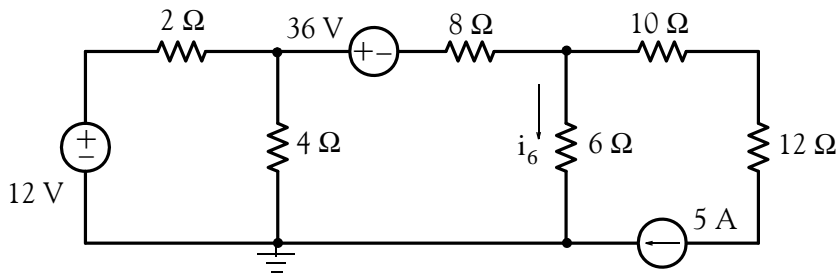


Figure 3.56. Circuit for Example 3.12

### Solution:

Let  $i'_6$  represent the current due to the 12 V source acting alone,  $i''_6$  the current due to the 36 V source acting alone, and  $i'''_6$  the current due to the 5 A source acting alone. Then, by the principle of superposition,

$$i_6 = i'_6 + i''_6 + i'''_6$$

First, to find  $i'_6$  we short the 36 V voltage source and open the 5 A current source. The circuit then reduces to that shown in Figure 3.57.

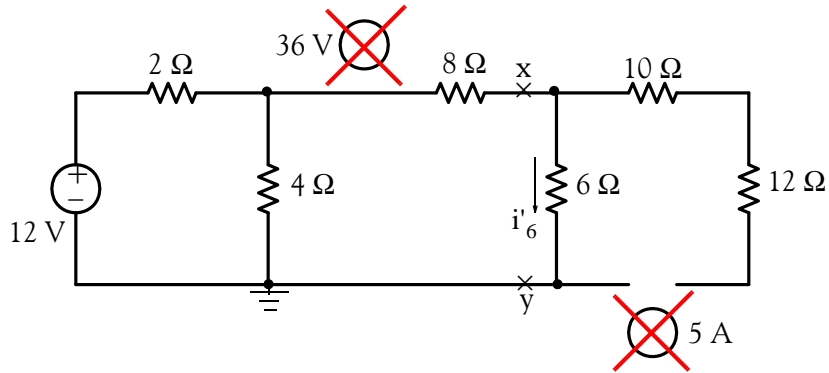


Figure 3.57. Circuit for finding  $i'_6$  in Example 3.12

Applying Thevenin's theorem at points  $x$  and  $y$  of Figure 3.57, we obtain the circuit of Figure 3.58 and from it we obtain

$$v_{xy} = v_{TH} = \frac{4 \times 12}{2 + 4} = 8 \text{ V}$$

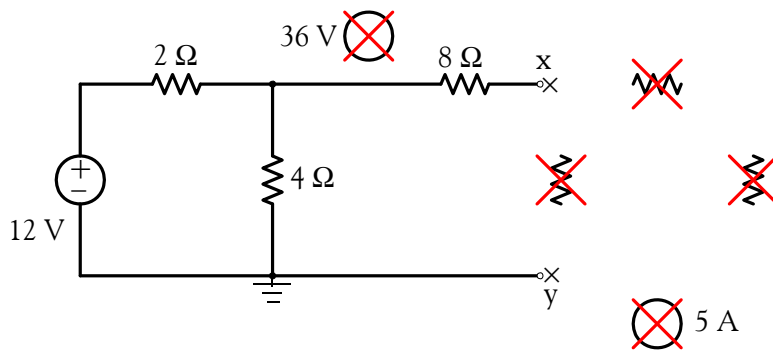


Figure 3.58. Circuit for computing the Thevenin voltage to find  $i'_6$  in Example 3.12

Next, we will use the circuit of Figure 3.59 to find the Thevenin resistance.

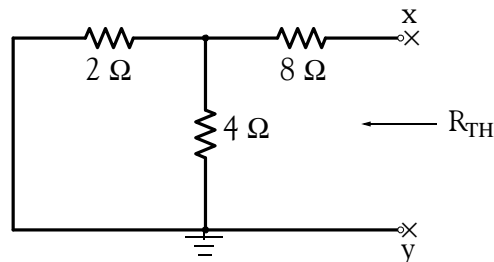


Figure 3.59. Circuit for computing the Thevenin resistance to find  $i'_6$  in Example 3.12

$$R_{TH} = 8 + \frac{4 \times 2}{4 + 2} = \frac{28}{3} \Omega$$



The current  $i'_6$  is found from the circuit of Figure 3.60 below.

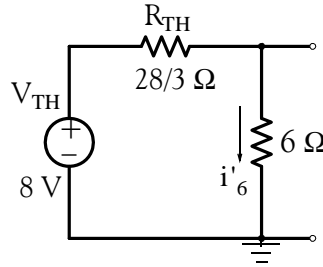


Figure 3.60. Circuit for computing  $i'_6$  in Example 3.12

$$i'_6 = \frac{8}{28/3 + 6} = \frac{12}{23} \text{ A} \quad (3.57)$$

Next, the current  $i''_6$  due to the 36 V source acting alone is found from the circuit of Figure 3.61.

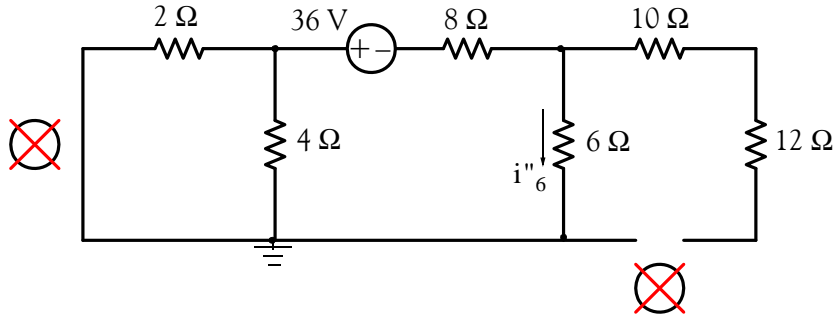


Figure 3.61. Circuit for finding  $i''_6$  in Example 3.12

and after combination of the 2 Ω and 4 Ω parallel resistors to a single resistor, the circuit simplifies to that shown in Figure 3.62.

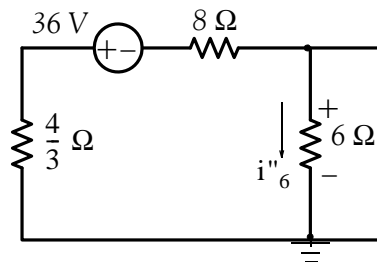


Figure 3.62. Simplification of the circuit of Figure 3.61 to compute  $i''_6$  for Example 3.12

From the circuit of Figure 3.62, we obtain

$$i''_6 = -\frac{36}{4/3 + 8 + 6} = -\frac{54}{23} \text{ A} \quad (3.58)$$

Finally, to find  $i''_6$ , we short the voltage sources, and with the 5 A current source acting alone the circuit reduces to that shown in Figure 3.63.

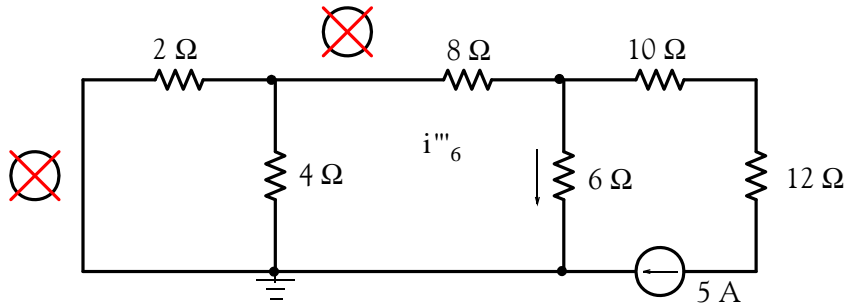


Figure 3.63. Circuit for finding  $i''_6$  in Example 3.12

Replacing the 2  $\Omega$ , 4  $\Omega$ , and 8  $\Omega$  resistors, and 10  $\Omega$  and 12  $\Omega$  by single resistors, we obtain

$$\frac{2 \times 4}{2 + 4} + 8 = \frac{28}{3} \Omega \quad 10 + 12 = 22 \Omega$$

and the circuit of Figure 3.63 reduces to that shown in Figure 3.64.

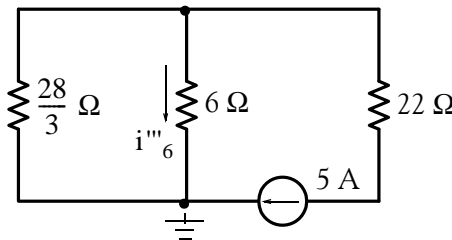


Figure 3.64. Simplification of the circuit of Figure 3.63 to compute  $i''_6$  for Example 3.12

We will use the current division expression in the circuit of Figure 3.64 to find  $i''_6$ . Thus,

$$i''_6 = \frac{28/3}{28/3 + 6} \times (-5) = -\frac{70}{23} \quad (3.59)$$

Therefore, from (3.57), (3.58), and (3.59) we obtain

$$i_6 = i'_6 + i''_6 + i'''_6 = \frac{12}{23} - \frac{54}{23} - \frac{70}{23} = -\frac{112}{23}$$

or

$$i_6 = -4.87 \text{ A} \quad (3.60)$$

and this is the same value as that of Example 3.5.

### 3.10 Circuits with Non-Linear Devices

Most electronic circuits contain non-linear devices such as diodes and transistors whose  $i - v$  (current-voltage) relationships are non-linear. However, for small signals (voltages or currents) these circuits can be represented by linear equivalent circuit models. A detailed discussion of these is beyond the scope of this text; however we will see how operational amplifiers can be represented by equivalent linear circuits in the next chapter.

If a circuit contains only one non-linear device, such as a diode, and all the other devices are linear, we can apply Thevenin's theorem to reduce the circuit to a Thevenin equivalent in series with the non-linear element. Then, we can analyze the circuit using a graphical solution. The procedure is illustrated with the following example.

---

#### Example 3.13

For the circuit of Figure 3.65, the  $i - v$  characteristics of the diode  $D$  are shown in figure 3.66. We wish to find the voltage  $v_D$  across the diode and the current  $i_D$  through this diode using a graphical solution.

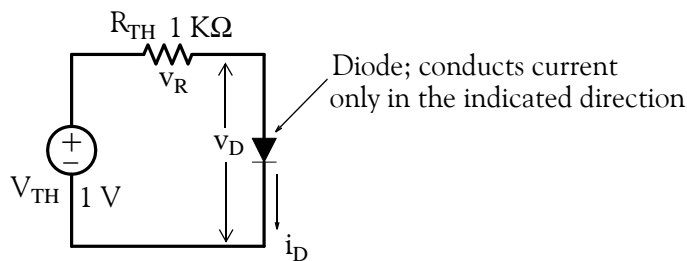


Figure 3.65. Circuit for Example 3.13

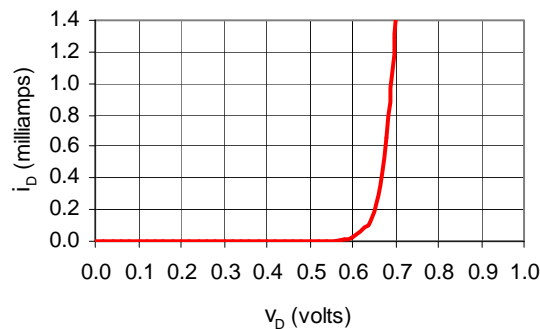


Figure 3.66. Diode  $i - v$  characteristics

**Solution:**

$$v_R + v_D = 1 \text{ V}$$

or

$$Ri_D = -v_D + 1$$

or

$$i_D = -\frac{1}{R}v_D + \frac{1}{R} \tag{3.61}$$

We observe that (3.61) is an equation of a straight line and the two points are obtained from it by first letting  $v_D = 0$ , then,  $i_D = 0$ . We obtain the straight line shown in Figure 3.67 that is plotted on the same graph as the given diode  $i - v$  characteristics.

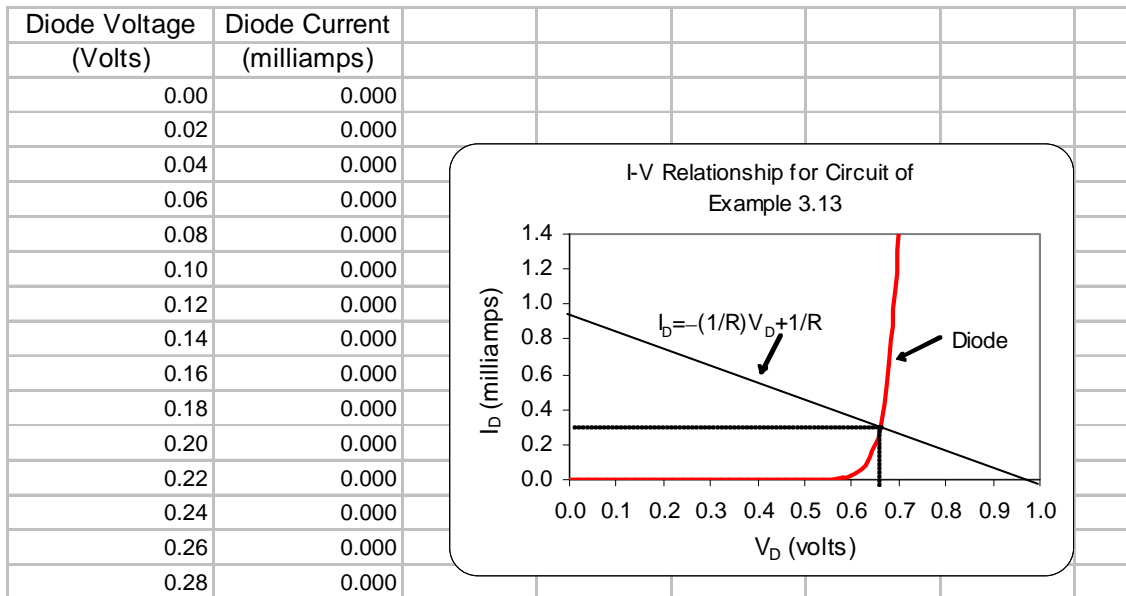


Figure 3.67. Curves for determining voltage and current in a diode

The intersection of the non-linear curve and the straight line yields the voltage and the current of the diode where we find that  $v_D = 0.665\text{V}$  and  $i_D = 0.335 \text{ mA}$ .

**Check:**

Since this is a series circuit,  $i_R = 0.335 \text{ mA}$  also. Therefore, the voltage drop  $v_R$  across the resistor is  $v_R = 1 \text{ k}\Omega \times 0.335 \text{ mA} = 0.335 \text{ V}$ . Then, by KVL

$$v_R + v_D = 0.335 + 0.665 = 1 \text{ V}$$

### 3.11 Efficiency

We have learned that the power absorbed by a resistor can be found from  $p_R = i^2R$  and this power is transformed into heat. In a long length of a conductive material, such as copper, this lost power is known as  $i^2R$  loss and thus the energy received by the load is equal to the energy transmitted minus the  $i^2R$  loss. Accordingly, we define *efficiency*  $\eta$  as

$$\text{Efficiency} = \eta = \frac{\text{Output}}{\text{Input}} = \frac{\text{Output}}{\text{Output} + \text{Loss}}$$

The efficiency  $\eta$  is normally expressed as a percentage. Thus,

$$\% \text{ Efficiency} = \% \eta = \frac{\text{Output}}{\text{Input}} \times 100 = \frac{\text{Output}}{\text{Output} + \text{Loss}} \times 100 \quad (3.62)$$

Obviously, a good efficiency should be close to 100%

#### Example 3.14

In a two-story industrial building, the total load on the first floor draws an average of 60 amperes during peak activity, while the total load of the second floor draws 40 amperes at the same time. The building receives its electric power from a 480 V source. Assuming that the total resistance of the cables (copper conductors) on the first floor is  $1 \Omega$  and on the second floor is  $1.6 \Omega$ , compute the efficiency of transmission.

#### Solution:

First, we draw a circuit that represents the electrical system of this building. This is shown in Figure 3.68.

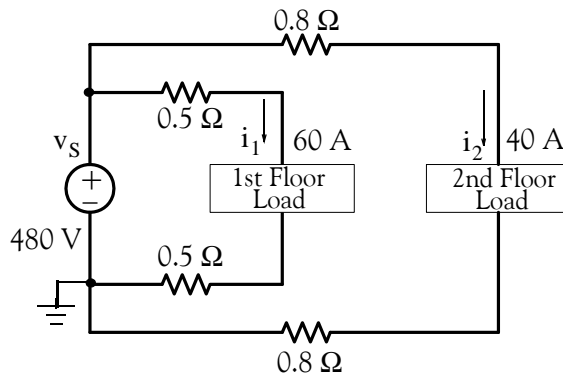


Figure 3.68. Circuit for Example 3.14

Power  $p_S$  supplied by the source:

$$p_S = v_S(i_1 + i_2) = 480 \times (60 + 40) = 48 \text{ kilowatts} \quad (3.63)$$

Power loss between source and 1st floor load:

$$p_{\text{loss1}} = i_1^2(0.5 \Omega + 0.5 \Omega) = 60^2 \times 1 = 3.6 \text{ kilowatts} \quad (3.64)$$

Power loss between source and 2nd floor load:

$$p_{\text{loss2}} = i_2^2(0.8 \Omega + 0.8 \Omega) = 40^2 \times 1.6 = 2.56 \text{ kilowatts} \quad (3.65)$$

Total power loss:

$$p_{\text{loss}} = p_{\text{loss1}} + p_{\text{loss2}} = 3.60 + 2.56 = 6.16 \text{ kilowatts} \quad (3.66)$$

Total power  $p_L$  received by 1st and 2nd floor loads:

$$p_L = p_S - p_{\text{loss}} = 48.00 - 6.16 = 41.84 \text{ kilowatts} \quad (3.67)$$

$$\% \text{ Efficiency} = \% \eta = \frac{\text{Output}}{\text{Input}} \times 100 = \frac{41.84}{48.00} \times 100 = 87.17 \% \quad (3.68)$$

### 3.12 Regulation

The *regulation* is defined as the ratio of the change in load voltage when the load changes from no load (NL) to full load (FL) divided by the full load. Thus, denoting the no-load voltage as  $v_{NL}$  and the full-load voltage as  $v_{FL}$ , the regulation is defined as In other words,

$$\text{Regulation} = \frac{v_{NL} - v_{FL}}{v_{FL}}$$

The regulation is also expressed as a percentage. Thus,

$$\% \text{ Regulation} = \frac{v_{NL} - v_{FL}}{v_{FL}} \times 100 \quad (3.69)$$

#### Example 3.15

Compute the regulation for the 1st floor load of the previous example.

### Solution:

The current drawn by 1st floor load is given as 60 A and the total resistance from the source to the load as  $1 \Omega$ . Then, the total voltage drop in the conductors is  $60 \times 1 = 60 \text{ V}$ . Therefore, the full-load voltage of the load is  $v_{FL} = (480 - 60 = 420 \text{ V})$  and the percent regulation is

$$\% \text{ Regulation} = \frac{v_{NL} - v_{FL}}{v_{FL}} \times 100 = \frac{480 - 420}{420} \times 100 = 14.3 \%$$

### 3.13 Summary

- When using nodal analysis, for a circuit that contains  $N$  nodes, we must write  $N - 1$  independent nodal equations in order to completely describe that circuit. When the presence of voltage sources in a circuit seem to complicate the nodal analysis because we do not know the current through those voltage sources, we create combined nodes as illustrated in Example 3.2.
- When using nodal analysis, for a circuit that contains  $M$  meshes or  $L$  loops,  $B$  branches, and  $N$  nodes, we must write  $L = M = B - N + 1$  independent loop or mesh equations in order to completely describe that circuit. When the presence of current sources in a circuit seem to complicate the mesh or loop analysis because we do not know the voltage across those current sources, we create combined meshes as illustrated in Example 3.6.
- A practical voltage source has an internal resistance and it is represented by a voltage source whose value is the value of the ideal voltage source in series with a resistance whose value is the value of the internal resistance.
- A practical current source has an internal conductance and it is represented by a current source whose value is the value of the ideal current source in parallel with a conductance whose value is the value of the internal conductance.
- A practical voltage source  $v_S$  in series with a resistance  $R_S$  can be replaced by a current source  $i_S$  whose value is  $v_S/i_S$  in parallel with a resistance  $R_P$  whose value is the same as  $R_S$
- A practical current source  $i_S$  in parallel with a resistance  $R_P$  can be replaced by a voltage source  $v_S$  whose value is equal to  $i_S \times R_S$  in series with a resistance  $R_S$  whose value is the same as  $R_P$
- Thevenin's theorem states that in a two terminal network we can be replace everything except the load, by a voltage source denoted as  $v_{TH}$  in series with a resistance denoted as  $R_{TH}$ . The value of  $v_{TH}$  represents the open circuit voltage where the circuit is isolated from the load and  $R_{TH}$  is the equivalent resistance of that part of the isolated circuit. If a given circuit contains independent voltage and independent current sources only, the value of  $R_{TH}$  can be found by first shorting all independent voltage sources, opening all independent current sources, and calculating the resistance looking into the direction which is opposite to the disconnected load. If the circuit contains dependent sources, the value of  $R_{TH}$  must be computed from the relation  $R_{TH} = v_{OC}/i_{SC}$
- Norton's theorem states that in a two terminal network we can be replace everything except the load, by a current source denoted as  $i_N$  in parallel with a resistance denoted as  $R_N$ . The value of  $i_N$  represents the short circuit current where the circuit is isolated from the load and



$R_N$  is the equivalent resistance of that part of the isolated circuit. If the circuit contains independent voltage and independent current sources only, the value of  $R_N$  can be found by first shorting all independent voltage sources, opening all independent current sources, and calculating the resistance looking into the direction which is opposite to the disconnected load. If the circuit contains dependent sources, the value of  $R_N$  must be computed from the relation  $R_N = v_{OC}/i_{SC}$

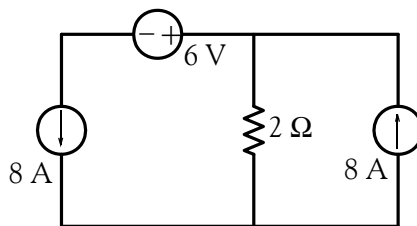
- The maximum power transfer theorem states that a voltage source with a series resistance  $R_S$  or a current source with parallel resistance  $R_S$  delivers maximum power to a load  $R_{LOAD}$  when  $R_{LOAD} = R_S$  or  $R_{LOAD} = R_N$
- Linearity implies that there is a linear voltage–current relationship.
- A linear circuit is composed entirely of independent voltage sources, independent current sources, linear dependent sources, and linear passive devices such as resistors, inductors, and capacitors.
- The principle of superposition states that the response (a desired voltage or current) in any branch of a linear circuit having more than one independent source can be obtained as the sum of the responses caused by each independent source acting alone with all other independent voltage sources replaced by short circuits and all other independent current sources replaced by open circuits.
- Efficiency is defined as the ratio of output to input and thus it is never greater than unity. It is normally expressed as a percentage.
- Regulation is defined as the ratio of  $v_{NL} - v_{FL}$  to  $v_{FL}$  and ideally should be close to zero. It is normally expressed as a percentage.

### 3.14 Exercises

#### Multiple Choice

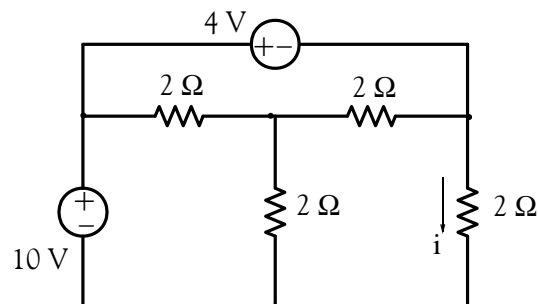
1. The voltage across the  $2\ \Omega$  resistor in the circuit below is

- A. 6 V
- B. 16 V
- C. -8 V
- D. 32 V
- E. none of the above



2. The current  $i$  in the circuit below is

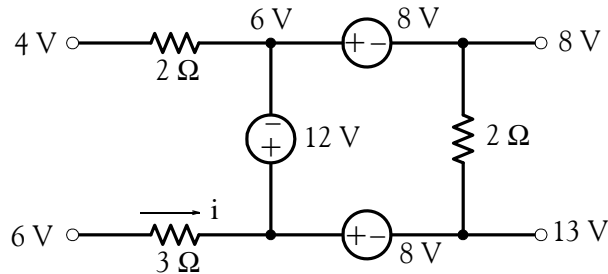
- A. -2 A
- B. 5 A
- C. 3 A
- D. 4 A
- E. none of the above



## Chapter 3 Nodal and Mesh Equations – Circuit Theorems

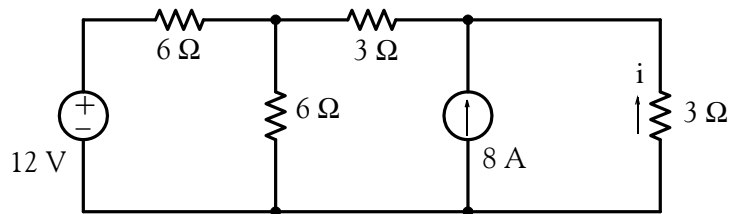
3. The node voltages shown in the partial network below are relative to some reference node which is not shown. The current  $i$  is

- A.  $-4$  A
- B.  $8/3$  A
- C.  $-5$  A
- D.  $-6$  A
- E. none of the above



4. The value of the current  $i$  for the circuit below is

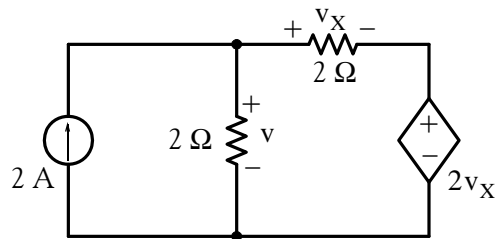
- A.  $-3$  A
- B.  $-8$  A
- C.  $-9$  A
- D.  $6$  A
- E. none of the above



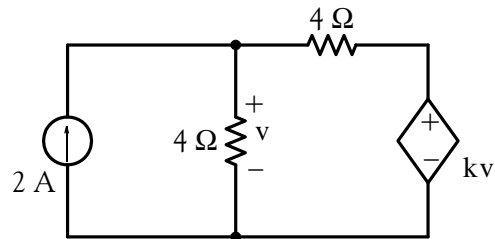
5. The value of the voltage  $v$  for the circuit below is

- A.  $4$  V
- B.  $6$  V
- C.  $8$  V

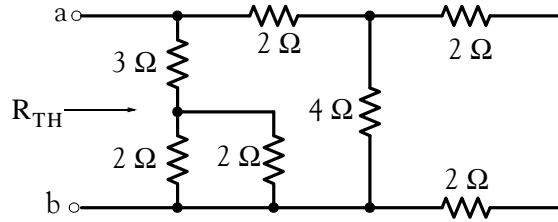
- D. 12 V
- E. none of the above



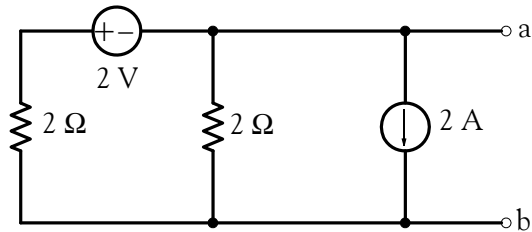
6. For the circuit below, the value of  $k$  is dimensionless. For that circuit, no solution is possible if the value of  $k$  is
- A. 2
  - B. 1
  - C.  $\infty$
  - D. 0
  - E. none of the above



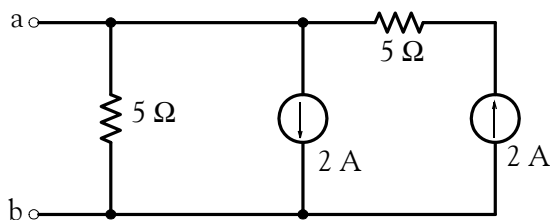
7. For the network below, the Thevenin equivalent resistance  $R_{TH}$  to the right of terminals a and b is
- A. 1
  - B. 2
  - C. 5
  - D. 10
  - E. none of the above



8. For the network below, the Thevenin equivalent voltage  $V_{TH}$  across terminals a and b is
- A. -3 V
  - B. -2 V
  - C. 1 V
  - D. 5 V
  - E. none of the above

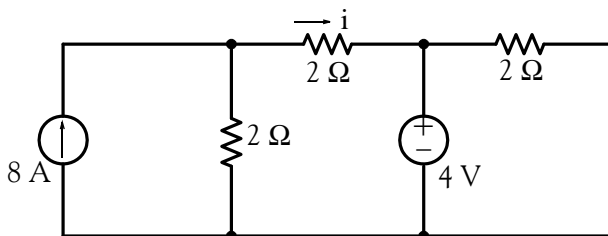


9. For the network below, the Norton equivalent current source  $I_N$  and equivalent parallel resistance  $R_N$  across terminals a and b are
- A. 1 A, 2  $\Omega$
  - B. 1.5 A, 25  $\Omega$
  - C. 4 A, 2.5  $\Omega$
  - D. 0 A, 5  $\Omega$
  - E. none of the above



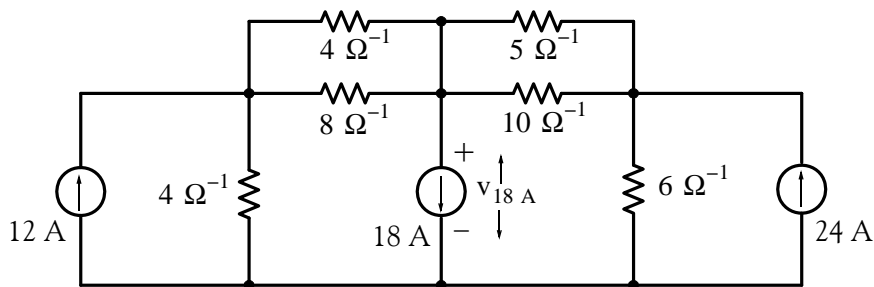
10. In applying the superposition principle to the circuit below, the current  $i$  due to the 4 V source acting alone is

- A. 8 A
- B. -1 A
- C. 4 A
- D. -2 A
- E. none of the above



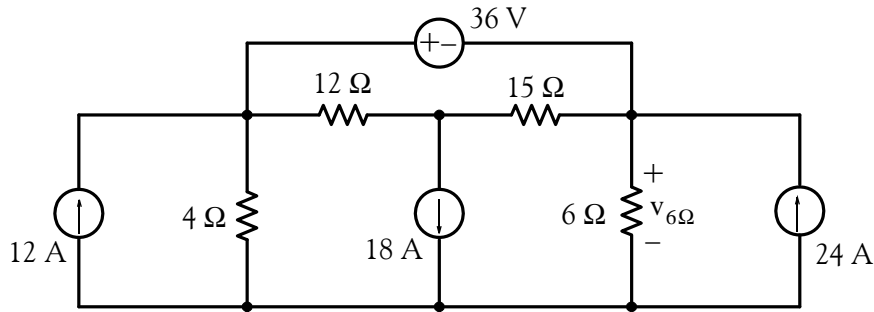
### Problems

1. Use nodal analysis to compute the voltage across the 18 A current source in the circuit below.  
Answer: 1.12 V

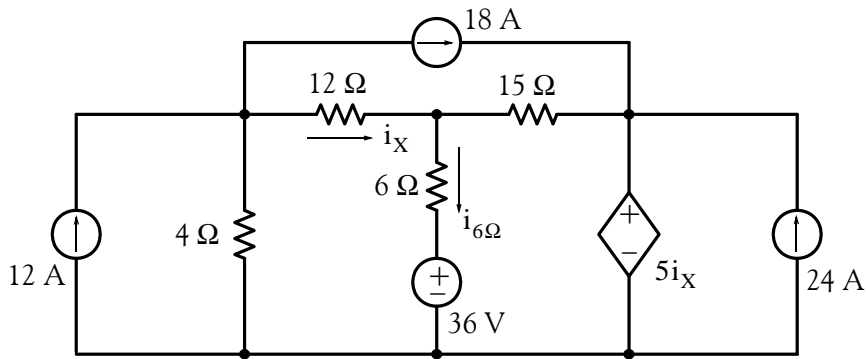


## Chapter 3 Nodal and Mesh Equations – Circuit Theorems

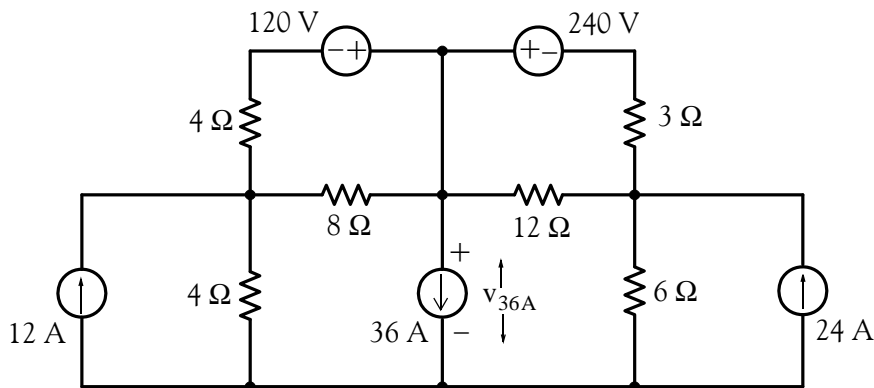
2. Use nodal analysis to compute the voltage  $v_{6\Omega}$  in the circuit below. Answer: 21.6 V



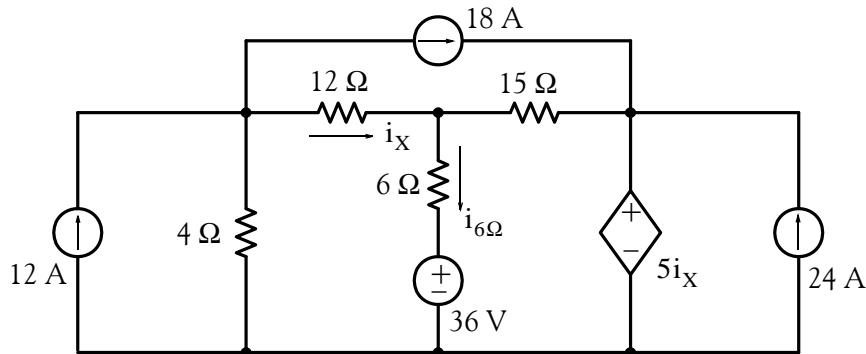
3. Use nodal analysis to compute the current through the 6 Ω resistor and the power supplied (or absorbed) by the dependent source shown below. Answers:  $-3.9$  A,  $-499.17$  w



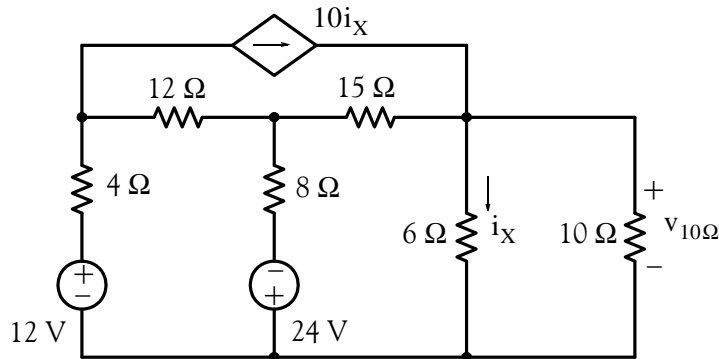
4. Use mesh analysis to compute the voltage  $v_{36A}$  below. Answer: 86.34 V



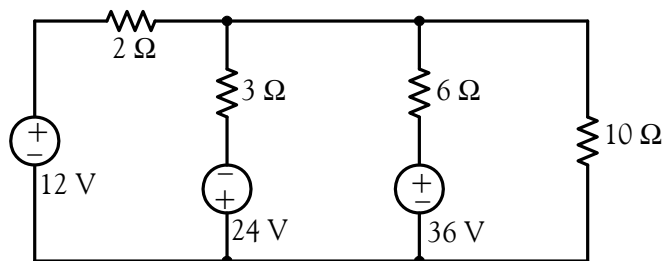
5. Use mesh analysis to compute the current through the  $i_{6\Omega}$  resistor, and the power supplied (or absorbed) by the dependent source shown below. Answers:  $-3.9$  A,  $-499.33$  w



6. Use mesh analysis to compute the voltage  $v_{10\Omega}$  below. Answer:  $0.5$  V

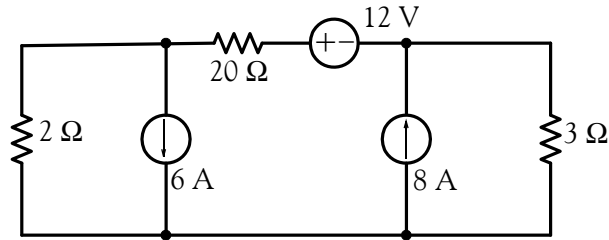


7. Compute the power absorbed by the  $10\ \Omega$  resistor in the circuit below using any method. Answer:  $1.32$  w



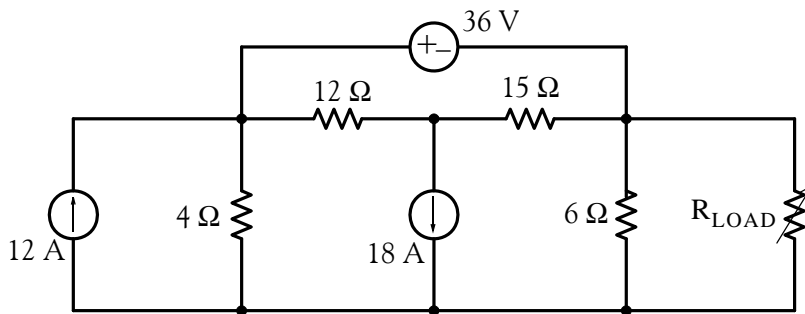
8. Compute the power absorbed by the  $20\ \Omega$  resistor in the circuit below using any method. Answer:  $73.73$  w





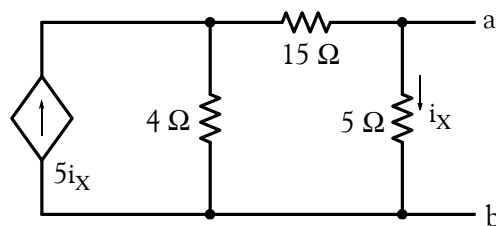
9. For the circuit below:

- To what value should the load resistor  $R_{LOAD}$  should be adjusted to so that it will absorb maximum power? Answer:  $2.4 \Omega$
- What would then the power absorbed by  $R_{LOAD}$  be? Answer:  $135 \text{ w}$



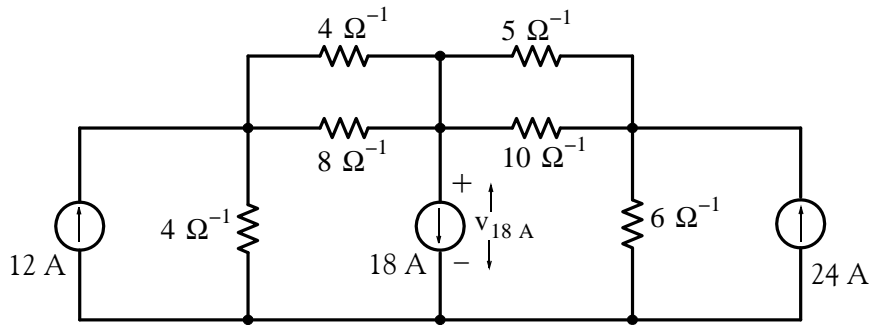
10. Replace the network shown below by its Norton equivalent.

Answers:  $i_N = 0$ ,  $R_N = 23.75 \Omega$



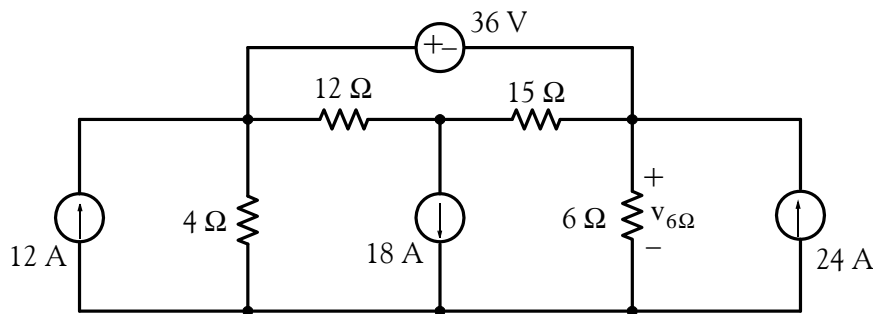
11. Use the superposition principle to compute the voltage  $v_{18A}$  in the circuit below.

Answer:  $1.12 \text{ V}$

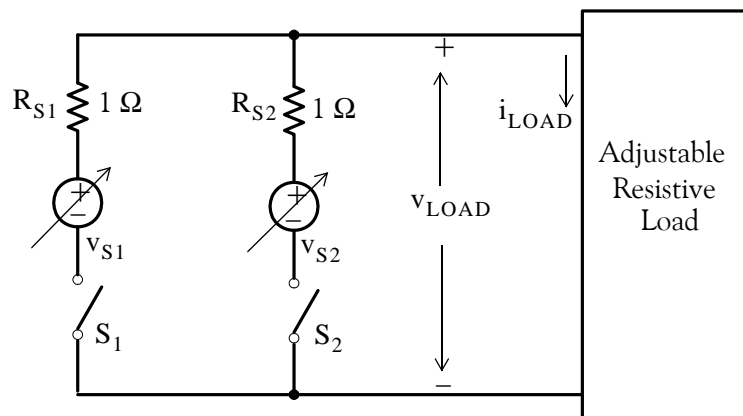


12. Use the superposition principle to compute voltage  $v_{6\ \Omega}$  in the circuit below.

Answer: 21.6 V



13. In the circuit below,  $v_{S1}$  and  $v_{S2}$  are adjustable voltage sources in the range  $-50 \leq V \leq 50$  V, and  $R_{S1}$  and  $R_{S2}$  represent their internal resistances.



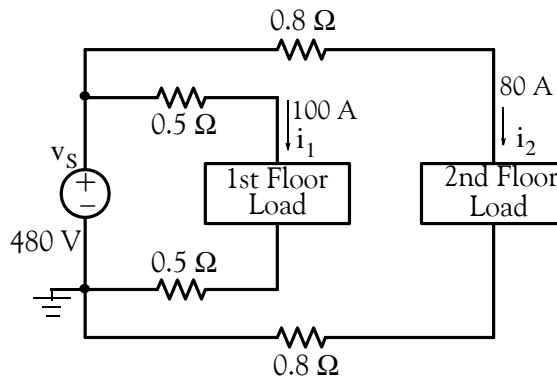
The table below shows the results of several measurements. In Measurement 3 the load resistance is adjusted to the same value as Measurement 1, and in Measurement 4 the load resistance is adjusted to the same value as Measurement 2. For Measurements 5 and 6 the load resistance is adjusted to 1 Ω. Make the necessary computations to fill-in the blank cells of this table.

## Chapter 3 Nodal and Mesh Equations – Circuit Theorems

Measurement	Switch $S_1$	Switch $S_2$	$v_{S1}$ (V)	$v_{S2}$ (V)	$i_{LOAD}$ (A)
1	Closed	Open	48	0	16
2	Open	Closed	0	36	6
3	Closed	Open		0	-5
4	Open	Closed	0	-42	
5	Closed	Closed	15	18	
6	Closed	Closed		24	0

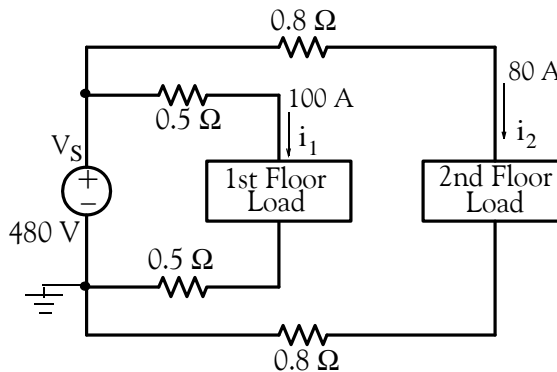
Answers: -15 V, -7 A, 11 A, -24 V

14. Compute the efficiency of the electrical system below. Answer: 76.6%



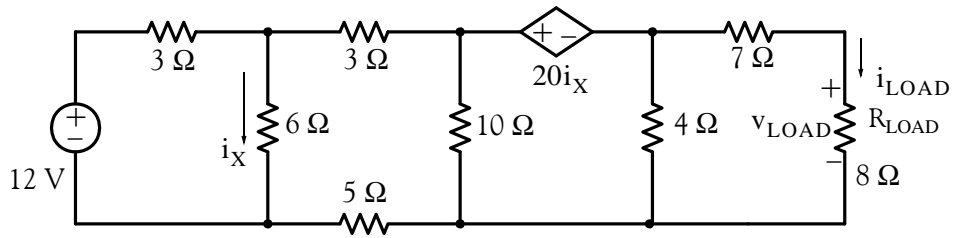
15. Compute the regulation for the 2st floor load of the electrical system below.

Answer: 36.4%



16. Write a set of nodal equations and then use MATLAB to compute  $i_{LOAD}$  and  $v_{LOAD}$  for the circuit of Example 3.10, Page 3-29, which is repeated below for convenience.

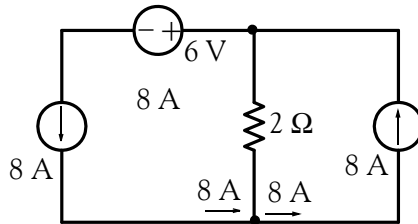
Answers: -0.96 A, -7.68 V



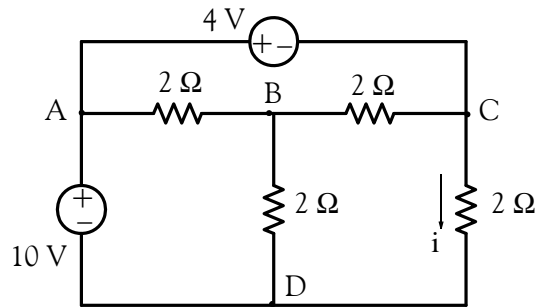
3.15 Answers / Solutions to End-of-Chapter Exercises

Multiple Choice

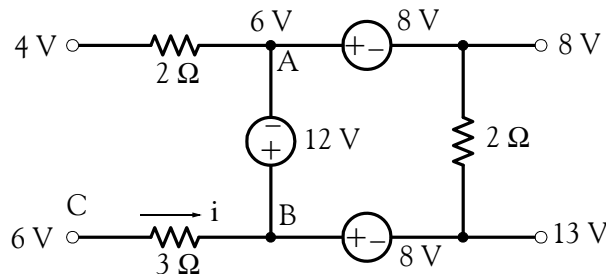
1. E The current entering Node A is equal to the current leaving that node. Therefore, there is no current through the  $2\ \Omega$  resistor and the voltage across it is zero.



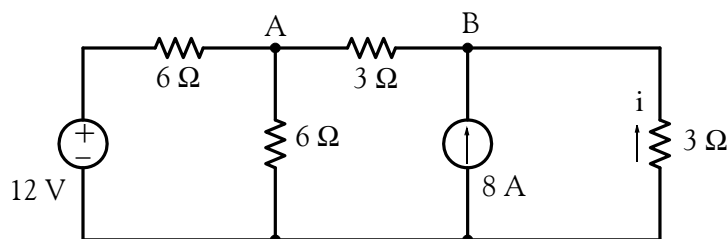
2. C From the figure below,  $V_{AC} = 4\text{ V}$ . Also,  $V_{AB} = V_{BC} = 2\text{ V}$  and  $V_{AD} = 10\text{ V}$ . Then,  $V_{BD} = V_{AD} - V_{AB} = 10 - 2 = 8\text{ V}$  and  $V_{CD} = V_{BD} - V_{BC} = 8 - 2 = 6\text{ V}$ . Therefore,  $i = 6/2 = 3\text{ A}$ .



3. A From the figure below we observe that the node voltage at A is  $6\text{ V}$  relative to the reference node which is not shown. Therefore, the node voltage at B is  $6 + 12 = 18\text{ V}$  relative to the same reference node. The voltage across the resistor is  $V_{BC} = 18 - 6 = 12\text{ V}$  and the direction of current through the  $3\ \Omega$  resistor is opposite to that shown since Node B is at a higher potential than Node C. Thus  $i = -12/3 = -4\text{ A}$



4. E We assign node voltages at Nodes A and B as shown below.



At Node A

$$\frac{V_A - 12}{6} + \frac{V_A}{6} + \frac{V_A - V_B}{3} = 0$$

and at Node B

$$\frac{V_B - V_A}{3} + \frac{V_B}{3} = 8$$

These simplify to

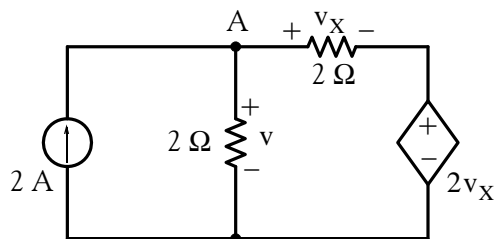
$$\frac{2}{3}V_A - \frac{1}{3}V_B = 2$$

and

$$-\frac{1}{3}V_A + \frac{2}{3}V_B = 8$$

Multiplication of the last equation by 2 and addition with the first yields  $V_B = 18$  and thus  $i = -18/3 = -6$  A.

5. E Application of KCL at Node A of the circuit below yields



$$\frac{v}{2} + \frac{v - 2v_X}{2} = 2$$

or

$$v - v_X = 2$$

Also by KVL

$$v = v_X + 2v_X$$

and by substitution

$$v_X + 2v_X - v_X = 2$$

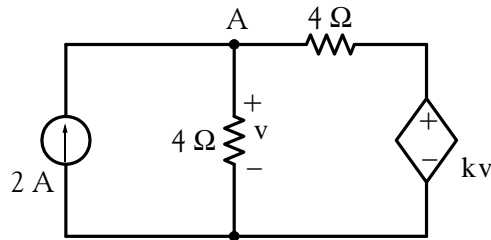
or

$$v_X = 1$$

and thus

$$v = v_X + 2v_X = 1 + 2 \times 1 = 3 \text{ V}$$

6. A Application of KCL at Node A of the circuit below yields



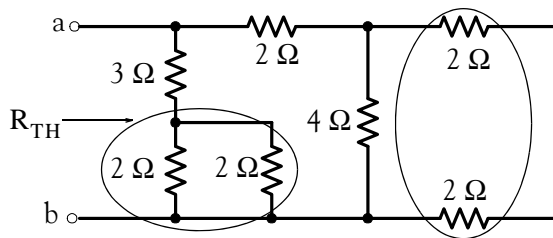
$$\frac{v}{4} + \frac{v - kv}{4} = 2$$

or

$$\frac{1}{4}(2v - kv) = 2$$

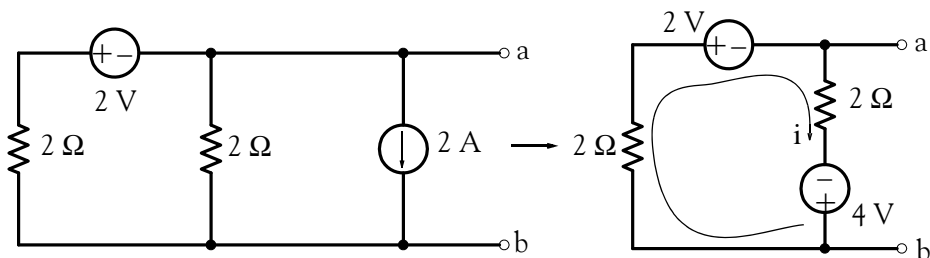
and this relation is meaningless if  $k = 2$ . Thus, this circuit has solutions only if  $k \neq 2$ .

7. B The two  $2 \Omega$  resistors on the right are in series and the two  $2 \Omega$  resistors on the left shown in the figure below are in parallel.



Beginning on the right side and proceeding to the left we obtain  $2 + 2 = 4$ ,  $4 \parallel 4 = 2$ ,  $2 + 2 = 4$ ,  $4 \parallel (3 + 2 \parallel 2) = 4 \parallel (3 + 1) = 4 \parallel 4 = 2 \Omega$ .

8. A Replacing the current source and its  $2 \Omega$  parallel resistance with an equivalent voltage source in series with a  $2 \Omega$  resistance we obtain the network shown below.



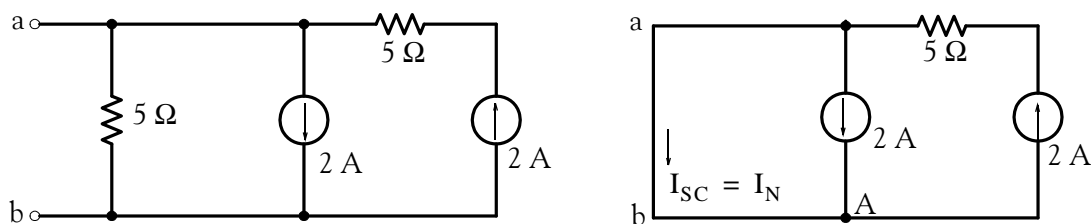
By Ohm's law,

$$i = \frac{4-2}{2+2} = 0.5 \text{ A}$$

and thus

$$v_{\text{TH}} = v_{\text{ab}} = 2 \times 0.5 + (-4) = -3 \text{ V}$$

9. D The Norton equivalent current source  $I_N$  is found by placing a short across the terminals a and b. This short shorts out the  $5 \Omega$  resistor and thus the circuit reduces to the one shown below.

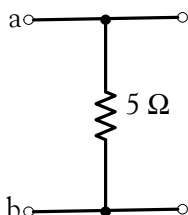


By KCL at Node A,

$$I_N + 2 = 2$$

and thus  $I_N = 0$

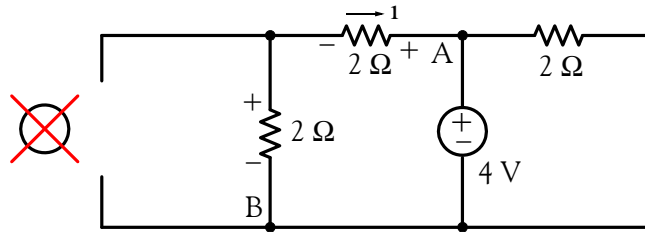
The Norton equivalent resistance  $R_N$  is found by opening the current sources and looking to the right of terminals a and b. When this is done, the circuit reduces to the one shown below.



Therefore,  $R_N = 5 \Omega$  and the Norton equivalent circuit consists of just a  $5 \Omega$  resistor.



10. B With the 4 V source acting alone, the circuit is as shown below.

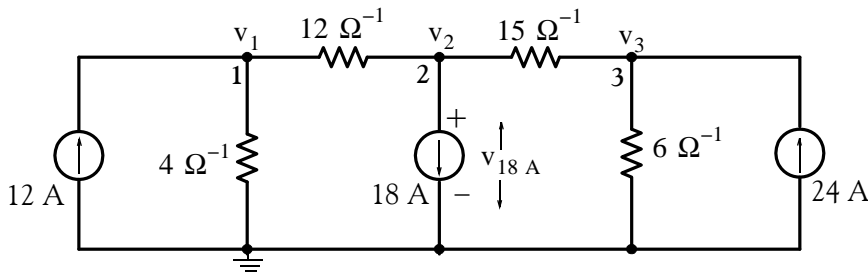


We observe that  $v_{AB} = 4\text{ V}$  and thus the voltage drop across each of the  $2\ \Omega$  resistors to the left of the 4 V source is 2 V with the indicated polarities. Therefore,

$$i = -2/2 = -1\text{ A}$$

### Problems

1. We first replace the parallel conductances with their equivalents and the circuit simplifies to that shown below.



Applying nodal analysis at Nodes 1, 2, and 3 we obtain:

Node 1:

$$16v_1 - 12v_2 = 12$$

Node 2:

$$-12v_1 + 27v_2 - 15v_3 = -18$$

Node 3:

$$-15v_2 + 21v_3 = 24$$

Simplifying the above equations, we obtain:

$$\begin{aligned} 4v_1 - 3v_2 &= 3 \\ -4v_1 + 9v_2 - 5v_3 &= -6 \\ -5v_2 + 7v_3 &= 8 \end{aligned}$$

Addition of the first two equations above and grouping with the third yields

$$6v_2 - 5v_3 = -3$$

$$-5v_2 + 7v_3 = 8$$

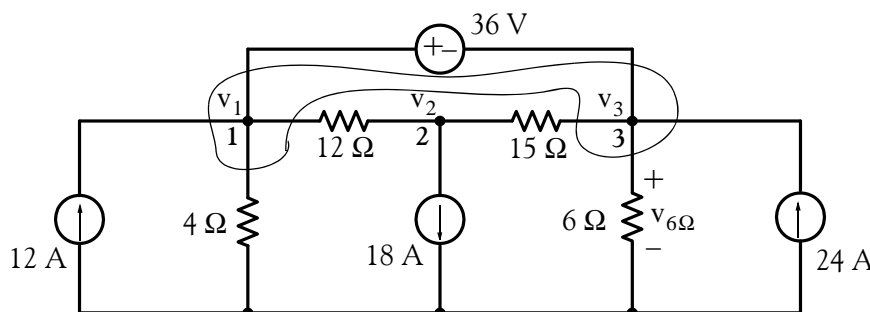
For this problem we are only interested in  $v_2 = v_{18\text{ A}}$ . Therefore, we will use Cramer's rule to solve for  $v_2$ . Thus,

$$v_2 = \frac{D_2}{\Delta} \quad D_2 = \begin{bmatrix} -3 & -5 \\ 8 & 7 \end{bmatrix} = -21 + 40 = 19 \quad \Delta = \begin{bmatrix} 6 & -5 \\ -5 & 7 \end{bmatrix} = 42 - 25 = 17$$

and

$$v_2 = v_{18\text{ A}} = 19/17 = 1.12\text{ V}$$

2. Since we cannot write an expression for the current through the 36 V source, we form a combined node as shown on the circuit below.



At Node 1 (combined node):

$$\frac{v_1}{4} + \frac{v_1 - v_2}{12} + \frac{v_3 - v_2}{15} + \frac{v_3}{6} - 12 - 24 = 0$$

and at Node 2,

$$\frac{v_2 - v_1}{12} + \frac{v_2 - v_3}{15} = -18$$

Also,

$$v_1 - v_3 = 36$$

Simplifying the above equations, we obtain:

$$\begin{aligned} \frac{1}{3}v_1 - \frac{3}{20}v_2 + \frac{7}{30}v_3 &= 36 \\ -\frac{1}{12}v_1 + \frac{3}{20}v_2 - \frac{1}{15}v_3 &= -18 \\ v_1 &\quad -v_3 = 36 \end{aligned}$$

Addition of the first two equations above and multiplication of the third by  $-1/4$  yields

$$\frac{1}{4}v_1 + \frac{1}{6}v_3 = 18$$

$$-\frac{1}{4}v_1 + \frac{1}{4}v_3 = -9$$

and by adding the last two equations we obtain

$$\frac{5}{12}v_3 = 9$$

or

$$v_3 = v_{6\Omega} = \frac{108}{5} = 21.6\text{V}$$

Check with MATLAB:

```
format rat
```

```
R=[1/3 -3/20 7/30; -1/12 3/20 -1/15; 1 0 -1];
```

```
I=[36 -18 36]';
```

```
V=R\I;
```

```
fprintf('\n'); disp('v1='); disp(V(1)); disp('v2='); disp(V(2)); disp('v3='); disp(V(3))
```

```
v1=
```

```
288/5
```

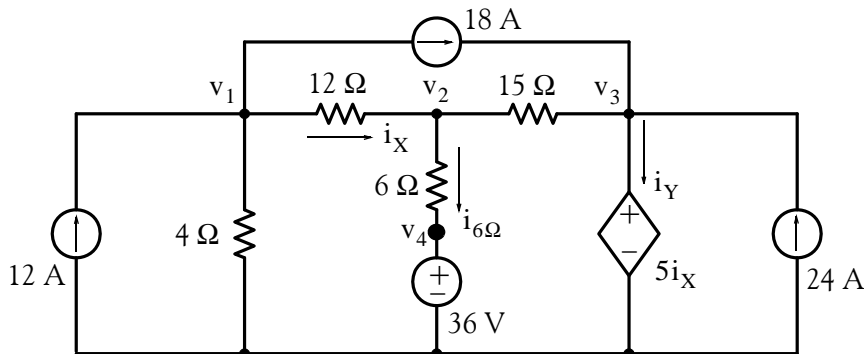
```
v2=
```

```
-392/5
```

```
v3=
```

```
108/5
```

3. We assign node voltages  $v_1, v_2, v_3, v_4$  and current  $i_Y$  as shown in the circuit below.



Then,

$$\frac{v_1}{4} + \frac{v_1 - v_2}{12} + 18 - 12 = 0$$

and

$$\frac{v_2 - v_1}{12} + \frac{v_2 - v_3}{12} + \frac{v_2 - v_4}{6} = 0$$

Simplifying the last two equations above, we obtain

$$\frac{1}{3}v_1 - \frac{1}{12}v_2 = -6$$

and

$$-\frac{1}{12}v_1 + \frac{19}{60}v_2 - \frac{1}{15}v_3 - \frac{1}{6}v_4 = 0$$

Next, we observe that  $i_X = \frac{v_1 - v_2}{12}$ ,  $v_3 = 5i_X$  and  $v_4 = 36$  V. Then  $v_3 = \frac{5}{12}(v_1 - v_2)$  and by substitution into the last equation above, we obtain

$$-\frac{1}{12}v_1 + \frac{19}{60}v_2 - \frac{1}{15} \times \frac{5}{12}(v_1 - v_2) - \frac{1}{6}36 = 0$$

or

$$-\frac{1}{9}v_1 + \frac{31}{90}v_2 = 6$$

Thus, we have two equations with two unknowns, that is,

$$\begin{aligned} \frac{1}{3}v_1 - \frac{1}{12}v_2 &= -6 \\ -\frac{1}{9}v_1 + \frac{31}{90}v_2 &= 6 \end{aligned}$$

Multiplication of the first equation above by  $1/3$  and addition with the second yields

$$\frac{19}{60}v_2 = 4$$

or

$$v_2 = 240/19$$

We find  $v_1$  from

$$\frac{1}{3}v_1 - \frac{1}{12}v_2 = -6$$

Thus,

$$\frac{1}{3}v_1 - \frac{1}{12} \times \frac{240}{19} = -6$$

or

$$v_1 = -282/19$$

Now, we find  $v_3$  from

$$v_3 = \frac{5}{12}(v_1 - v_2) = \frac{5}{12}\left(\frac{-282}{19} - \frac{240}{19}\right) = -\frac{435}{38}$$

Therefore, the node voltages of interest are:

$$v_1 = -282/19 \text{ V}$$

$$v_2 = 240/19 \text{ V}$$

$$v_3 = -435/38 \text{ V}$$

$$v_4 = 36 \text{ V}$$

The current through the 6  $\Omega$  resistor is

$$i_{6\Omega} = \frac{v_2 - v_4}{6} = \frac{240/19 - 36}{6} = -\frac{74}{19} = -3.9 \text{ A}$$

To compute the power supplied (or absorbed) by the dependent source, we must first find the current  $i_Y$ . It is found by application of KCL at node voltage  $v_3$ . Thus,

$$i_Y - 24 - 18 + \frac{v_3 - v_2}{15} = 0$$

or

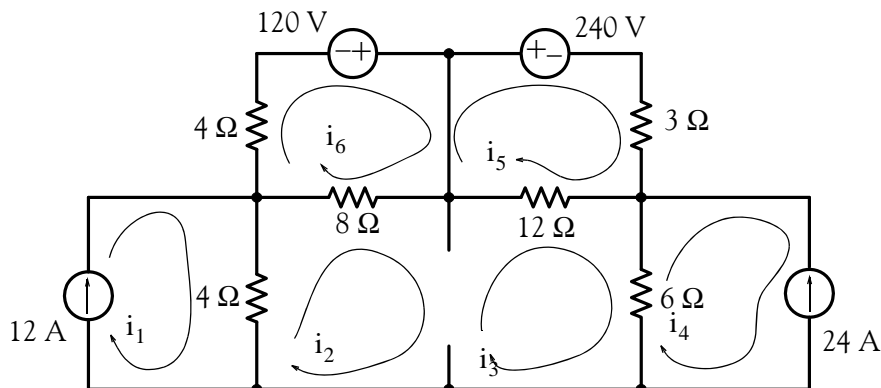
$$\begin{aligned} i_Y &= 42 - \frac{-435/38 - 240/19}{15} \\ &= 42 + \frac{915/38}{15} = \frac{1657}{38} \end{aligned}$$

and

$$p = v_3 i_Y = -\frac{435}{38} \times \frac{1657}{38} = -\frac{72379}{145} = -499.17 \text{ w}$$

that is, the dependent source supplies power to the circuit.

4. Since we cannot write an expression for the 36 A current source, we temporarily remove it and we form a combined mesh for Meshes 2 and 3 as shown below.



Mesh 1:

$$i_1 = 12$$

Combined mesh (2 and 3):

$$-4i_1 + 12i_2 + 18i_3 - 6i_4 - 8i_5 - 12i_6 = 0$$

or

$$-2i_1 + 6i_2 + 9i_3 - 3i_4 - 4i_5 - 6i_6 = 0$$

We now re-insert the 36 A current source and we write the third equation as

$$i_2 - i_3 = 36$$

Mesh 4:

$$i_4 = -24$$

Mesh 5:

$$-8i_2 + 12i_5 = 120$$

or

$$-2i_2 + 3i_5 = 30$$

Mesh 6:

$$-12i_3 + 15i_6 = -240$$

or

$$-4i_3 + 5i_6 = -80$$

Thus, we have the following system of equations:

$$\begin{array}{rcl} i_1 & & = 12 \\ -2i_1 + 6i_2 + 9i_3 - 3i_4 - 4i_5 - 6i_6 & & = 0 \\ i_2 - i_3 & & = 36 \\ & i_4 & = -24 \\ -2i_2 & + 3i_5 & = 30 \\ & -4i_3 & + 5i_6 = -80 \end{array}$$

and in matrix form

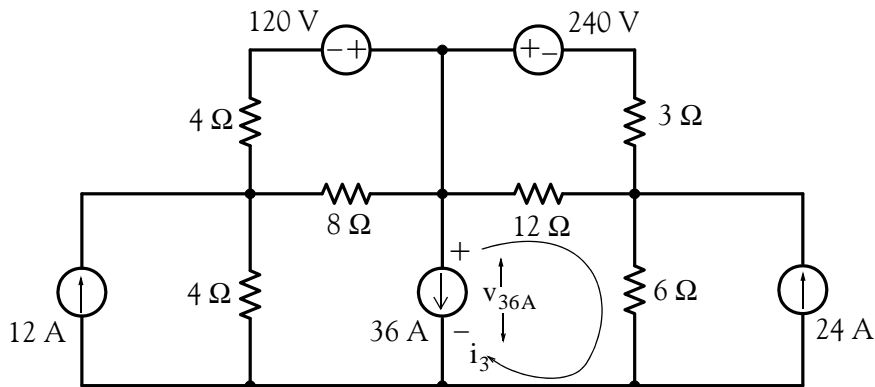
$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & 6 & 9 & -3 & -4 & -6 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 3 & 0 \\ 0 & 0 & -4 & 0 & 0 & 5 \end{bmatrix}}_{\mathbf{R}} \cdot \underbrace{\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix}}_{\mathbf{I}} = \underbrace{\begin{bmatrix} 12 \\ 0 \\ 36 \\ -24 \\ 30 \\ -80 \end{bmatrix}}_{\mathbf{V}}$$

We find the currents  $i_1$  through  $i_6$  with the following MATLAB script:

```
R=[1 0 0 0 0 0; -2 6 9 -3 -4 -6;...
    0 1 -1 0 0 0; 0 0 0 1 0 0;...
    0 -2 0 0 3 0; 0 0 -4 0 0 5];
V=[12 0 36 -24 30 -80]';
I=R\V;
fprintf('\n');...
fprintf('i1=%7.2f A \t', I(1));...
fprintf('i2=%7.2f A \t', I(2));...
fprintf('i3=%7.2f A \t', I(3));...
fprintf('\n');...
fprintf('i4=%7.2f A \t', I(4));...
fprintf('i5=%7.2f A \t', I(5));...
fprintf('i6=%7.2f A \t', I(6));...
fprintf('\n')
```

$i_1 = 12.00 \text{ A}$      $i_2 = 6.27 \text{ A}$      $i_3 = -29.73 \text{ A}$   
 $i_4 = -24.00 \text{ A}$      $i_5 = 14.18 \text{ A}$      $i_6 = -39.79 \text{ A}$

Now, we can find the voltage  $v_{36 \text{ A}}$  by application of KVL around Mesh 3.



Thus,

$$v_{36\text{ A}} = v_{12\ \Omega} + v_{6\ \Omega} = 12 \times [(-29.73) - (-39.79)] + 6 \times [(-29.73) - (24.00)]$$

or

$$v_{36\text{ A}} = 86.34\text{ V}$$

To verify that this value is correct, we apply KVL around Mesh 2. Thus, we must show that

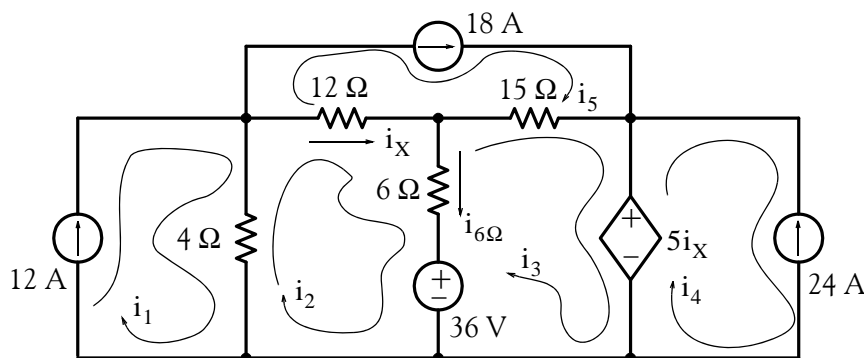
$$v_{4\ \Omega} + v_{8\ \Omega} + v_{36\text{ A}} = 0$$

By substitution of numerical values, we find that

$$4 \times [6.27 - 12] + 8 \times [6.27 - 14.18] + 86.34 = 0.14$$

5. This is the same circuit as that of Problem 3. We will show that we obtain the same answers using mesh analysis.

We assign mesh currents as shown below.



Mesh 1:

$$i_1 = 12$$

Mesh 2:

$$-4i_1 + 22i_2 - 6i_3 - 12i_5 = -36$$

or

$$-2i_1 + 11i_2 - 3i_3 - 6i_5 = -18$$

Mesh 3:

$$-6i_2 + 21i_3 - 15i_5 + 5i_X = 36$$

and since  $i_X = i_2 - i_5$ , the above reduces to

$$-6i_2 + 21i_3 - 15i_5 + 5i_2 - 5i_5 = 36$$

or

$$-i_2 + 21i_3 - 20i_5 = 36$$

Mesh 4:

$$i_4 = -24$$



Mesh 5:

$$i_5 = 18$$

Grouping these five independent equations we obtain:

$$\begin{aligned} i_1 &= 12 \\ -2i_1 + 11i_2 - 3i_3 - 6i_5 &= -18 \\ -i_2 + 21i_3 - 20i_5 &= 36 \\ i_4 &= -24 \\ i_5 &= 18 \end{aligned}$$

and in matrix form,

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 11 & -3 & 0 & -6 \\ 0 & -1 & 21 & 0 & -20 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathbf{R}} \cdot \underbrace{\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \end{bmatrix}}_{\mathbf{I}} = \underbrace{\begin{bmatrix} 12 \\ -18 \\ 36 \\ -24 \\ 18 \end{bmatrix}}_{\mathbf{V}}$$

We find the currents  $i_1$  through  $i_5$  with the following MATLAB script:

```
R=[1 0 0 0 0; -2 11 -3 0 -6; 0 -1 21 0 -20; ...
  0 0 0 1 0; 0 0 0 0 1];
V=[12 -18 36 -24 18]';
I=R\V;
fprintf('\n');...
fprintf('i1=%7.2f A \t', I(1));...
fprintf('i2=%7.2f A \t', I(2));...
fprintf('i3=%7.2f A \t', I(3));...
fprintf('\n');...
fprintf('i4=%7.2f A \t', I(4));...
fprintf('i5=%7.2f A \t', I(5));...
fprintf('\n')

i1= 12.00 A   i2= 15.71 A   i3= 19.61 A
i4= -24.00 A  i5= 18.00 A
```

By inspection,

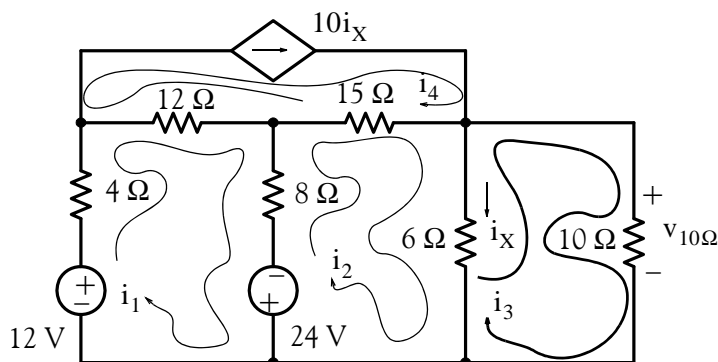
$$i_{6\Omega} = i_2 - i_3 = 15.71 - 19.61 = -3.9 \text{ A}$$

Next,

$$\begin{aligned}
 p_{5i_X} &= 5i_X(i_3 - i_4) = 5(i_2 - i_5)(i_3 - i_4) \\
 &= 5(15.71 - 18.00)(19.61 + 24.00) = -499.33 \text{ w}
 \end{aligned}$$

These are the same answers as those we found in Problem 3.

6. We assign mesh currents as shown below and we write mesh equations.



Mesh 1:

$$24i_1 - 8i_2 - 12i_4 - 24 - 12 = 0$$

or

$$6i_1 - 2i_2 - 3i_4 = 9$$

Mesh 2:

$$-8i_1 + 29i_2 - 6i_3 - 15i_4 = -24$$

Mesh 3:

$$-6i_2 + 16i_3 = 0$$

or

$$-3i_2 + 8i_3 = 0$$

Mesh 4:

$$i_4 = 10i_X = 10(i_2 - i_3)$$

or

$$10i_2 - 10i_3 - i_4 = 0$$

Grouping these four independent equations we obtain:

$$\begin{aligned}
 6i_1 - 2i_2 \quad \quad \quad - 3i_4 &= 9 \\
 -8i_1 + 29i_2 - 6i_3 - 15i_4 &= -24 \\
 -3i_2 + 8i_3 &= 0 \\
 10i_2 - 10i_3 - i_4 &= 0
 \end{aligned}$$

and in matrix form,

$$\underbrace{\begin{bmatrix} 6 & -2 & 0 & -3 \\ -8 & 29 & -6 & -15 \\ 0 & -3 & 8 & 0 \\ 0 & 10 & -10 & -1 \end{bmatrix}}_{\mathbf{R}} \cdot \underbrace{\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix}}_{\mathbf{I}} = \underbrace{\begin{bmatrix} 9 \\ -24 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{V}}$$

We find the currents  $i_1$  through  $i_4$  with the following MATLAB script:

```
R=[6 -2 0 -3; -8 29 -6 -15; 0 -3 8 0; 0 10 -10 -1];
V=[9 -24 0 0]'; I=R\V;
fprintf('\n');...
fprintf('i1=%7.2f A \t', I(1));...
fprintf('i2=%7.2f A \t', I(2));...
fprintf('i3=%7.2f A \t', I(3));...
fprintf('i4=%7.2f A \t', I(4));...
fprintf('\n')
```

$$i_1 = 1.94 \text{ A} \quad i_2 = 0.13 \text{ A} \quad i_3 = 0.05 \text{ A} \quad i_4 = 0.79 \text{ A}$$

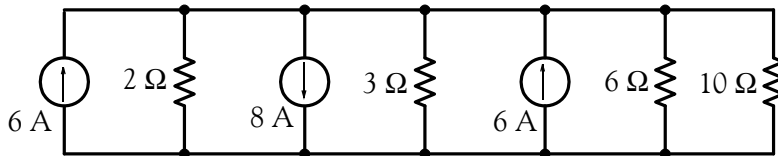
Now, we find  $v_{10\Omega}$  by Ohm's law, that is,

$$v_{10\Omega} = 10i_3 = 10 \times 0.05 = 0.5 \text{ V}$$

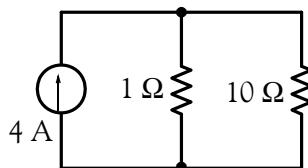
The same value is obtained by computing the voltage across the  $6 \Omega$  resistor, that is,

$$v_{6\Omega} = 6(i_2 - i_3) = 6(0.13 - 0.05) = 0.48 \text{ V}$$

7. Voltage-to-current source transformation yields the circuit below.



By combining all current sources and all parallel resistors except the  $10 \Omega$  resistor, we obtain the simplified circuit below.



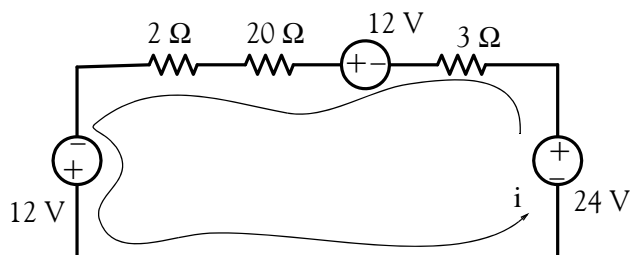
Applying the current division expression, we obtain

$$i_{10\ \Omega} = \frac{1}{1+10} \times 4 = \frac{4}{11}\ \text{A}$$

and thus

$$p_{10\ \Omega} = i_{10\ \Omega}^2(10) = \left(\frac{4}{11}\right)^2 \times 10 = \frac{16}{121} \times 10 = \frac{160}{121} = 1.32\ \text{w}$$

8. Current-to-voltage source transformation yields the circuit below.



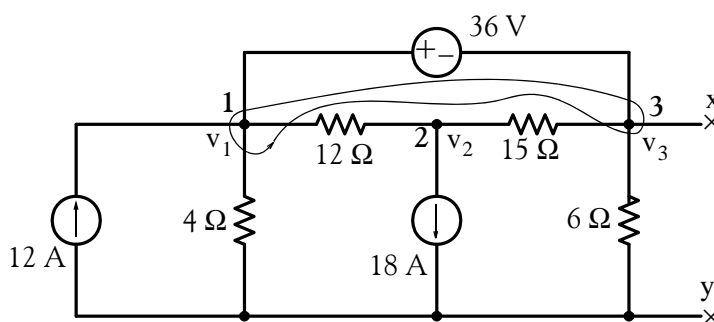
From this series circuit,

$$i = \frac{\sum v}{\sum R} = \frac{48}{25}\ \text{A}$$

and thus

$$p_{20\ \Omega} = i^2(20) = \left(\frac{48}{25}\right)^2 \times 20 = \frac{2304}{625} \times 20 = 73.73\ \text{w}$$

9. We remove  $R_{\text{LOAD}}$  from the rest of the rest of the circuit and we assign node voltages  $v_1$ ,  $v_2$ , and  $v_3$ . We also form the combined node as shown on the circuit below.



Node 1:

$$\frac{v_1}{4} + \frac{v_1 - v_2}{12} - 12 + \frac{v_3 - v_2}{15} + \frac{v_3}{6} = 0$$

or

$$\frac{1}{3}v_1 - \frac{3}{20}v_2 + \frac{7}{30}v_3 = 12$$

Node 2:

$$\frac{v_2 - v_1}{12} + \frac{v_2 - v_3}{15} = -18$$

or

$$-\frac{1}{12}v_1 + \frac{3}{20}v_2 - \frac{1}{15}v_3 = -18$$

Also,

$$v_1 - v_3 = 36$$

For this problem, we are interested only in the value of  $v_3$  which is the Thevenin voltage  $v_{TH}$ , and we could find it by Gauss's elimination method. However, for convenience, we will group these three independent equations, express these in matrix form, and use MATLAB for their solution.

$$\begin{aligned} \frac{1}{3}v_1 - \frac{3}{20}v_2 + \frac{7}{30}v_3 &= 12 \\ -\frac{1}{12}v_1 + \frac{3}{20}v_2 - \frac{1}{15}v_3 &= -18 \\ v_1 - v_3 &= 36 \end{aligned}$$

and in matrix form,

$$\underbrace{\begin{bmatrix} \frac{1}{3} & -\frac{3}{20} & \frac{7}{30} \\ -\frac{1}{12} & \frac{3}{20} & -\frac{1}{15} \\ 1 & 0 & -1 \end{bmatrix}}_G \cdot \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}}_V = \underbrace{\begin{bmatrix} 12 \\ -18 \\ 36 \end{bmatrix}}_I$$

We find the voltages  $v_1$  through  $v_3$  with the following MATLAB script:

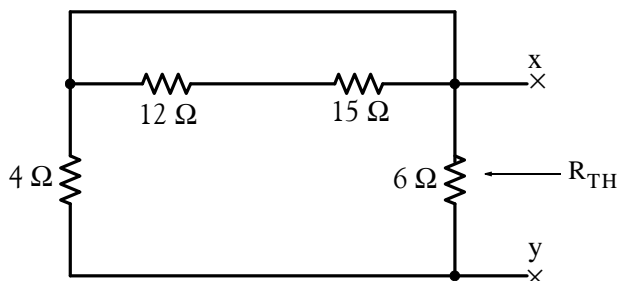
```
G=[1/3 -3/20 7/30; -1/12 3/20 -1/15; 1 0 -1];
I=[12 -18 36]'; V=G\I;
fprintf('\n');...
fprintf('v1=%7.2f V \t', V(1)); fprintf('v2=%7.2f V \t', V(2)); fprintf('v3=%7.2f V \t', V(3));
fprintf('\n')
```

$v_1 = 0.00 \text{ V}$      $v_2 = -136.00 \text{ V}$      $v_3 = -36.00 \text{ V}$

Thus,

$$v_{TH} = v_3 = -36 \text{ V}$$

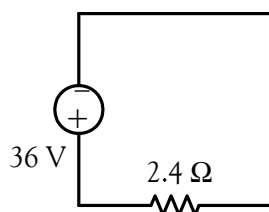
To find  $R_{TH}$  we short circuit the voltage source and we open the current sources. The circuit then reduces to the resistive network below.



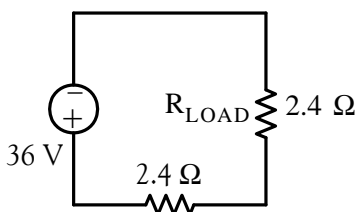
We observe that the resistors in series are shorted out and thus the Thevenin resistance is the parallel combination of the  $4\ \Omega$  and  $6\ \Omega$  resistors, that is,

$$4\ \Omega \parallel 6\ \Omega = 2.4\ \Omega$$

and the Thevenin equivalent circuit is as shown below.



Now, we connect the load resistor  $R_{LOAD}$  at the open terminals and we obtain the simple series circuit shown below.



a. For maximum power transfer,

$$R_{LOAD} = 2.4\ \Omega$$

b. Power under maximum power transfer condition is

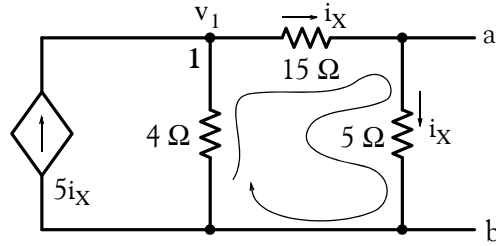
$$P_{MAX} = i^2 R_{LOAD} = \left( \frac{36}{2.4 + 2.4} \right)^2 \times 2.4 = 7.5^2 \times 2.4 = 135\ \text{w}$$

---

## Chapter 3 Nodal and Mesh Equations – Circuit Theorems

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10. We assign a node voltage Node 1 and a mesh current for the mesh on the right as shown below.



At Node 1:

$$\frac{v_1}{4} + i_X = 5i_X$$

Mesh on the right:

$$(15 + 5)i_X = v_1$$

and by substitution into the node equation above,

$$\frac{20i_X}{4} + i_X = 5i_X$$

or

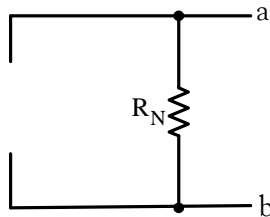
$$6i_X = 5i_X$$

but this can only be true if  $i_X = 0$ .

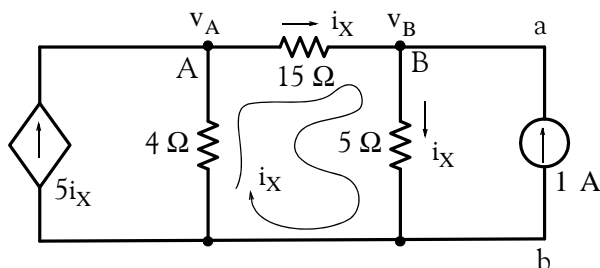
Then,

$$i_N = \frac{v_{OC}}{R_N} = \frac{v_{ab}}{R_N} = \frac{5 \times i_X}{R_N} = \frac{5 \times 0}{R_N} = 0$$

Thus, the Norton current source is open as shown below.



To find the value of  $R_N$  we insert a 1 A current source as shown below.



At Node A:

$$\frac{v_A}{4} + \frac{v_A - v_B}{15} = 5i_X$$

But

$$v_B = (5 \Omega) \times i_X = 5i_X$$

and by substitution into the above relation

$$\frac{v_A}{4} + \frac{v_A - v_B}{15} = v_B$$

or

$$\frac{19}{60}v_A - \frac{16}{15}v_B = 0$$

At Node B:

$$\frac{v_B - v_A}{15} + \frac{v_B}{5} = 1$$

or

$$-\frac{1}{15}v_A + \frac{4}{15}v_B = 1$$

For this problem, we are interested only in the value of  $v_B$  which we could find by Gauss's elimination method. However, for convenience, we will use MATLAB for their solution.

$$\frac{19}{60}v_A - \frac{16}{15}v_B = 0$$

$$-\frac{1}{15}v_A + \frac{4}{15}v_B = 1$$

and in matrix form,

$$\underbrace{\begin{bmatrix} \frac{19}{60} & -\frac{16}{15} \\ -\frac{1}{15} & \frac{4}{15} \end{bmatrix}}_{\mathbf{G}} \cdot \underbrace{\begin{bmatrix} v_A \\ v_B \end{bmatrix}}_{\mathbf{V}} = \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\mathbf{I}}$$

We find the voltages  $v_1$  and  $v_2$  with the following MATLAB script:



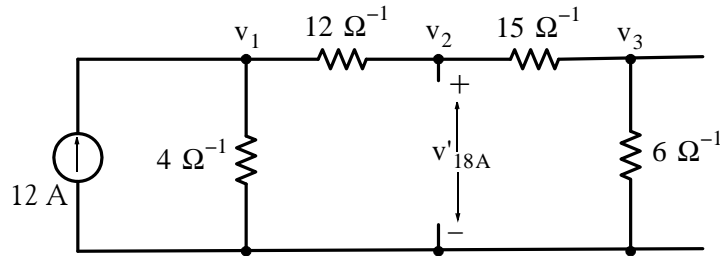
```
G=[19/60 -16/15; -1/15 4/15];
I=[0 1]'; V=G\I;
fprintf('\n');...
fprintf('vA=%7.2f V \t', V(1)); fprintf('vB=%7.2f V \t', V(2));
fprintf('\n')
```

vA= 80.00 V      vB= 23.75 V

Now, we can find the Norton equivalent resistance from the relation

$$R_N = \frac{V_{ab}}{I_{SC}} = \frac{V_B}{1} = 23.75 \Omega$$

11. This is the same circuit as that of Problem 1. Let  $v'_{18A}$  be the voltage due to the 12 A current source acting alone. The simplified circuit with assigned node voltages is shown below where the parallel conductances have been replaced by their equivalents.



The nodal equations at the three nodes are

$$\begin{aligned} 16v_1 - 12v_2 &= 12 \\ -12v_1 + 27v_2 - 15v_3 &= 0 \\ -15v_2 + 21v_3 &= 0 \end{aligned}$$

or

$$\begin{aligned} 4v_1 - 3v_2 &= 3 \\ -4v_1 + 9v_2 - 5v_3 &= 0 \\ -5v_2 + 7v_3 &= 0 \end{aligned}$$

Since  $v_2 = v'_{18A}$ , we only need to solve for  $v_2$ . Adding the first 2 equations above and grouping with the third we obtain

$$\begin{aligned} 6v_2 - 5v_3 &= 3 \\ -5v_2 + 7v_3 &= 0 \end{aligned}$$

Multiplying the first by 7 and the second by 5 we obtain

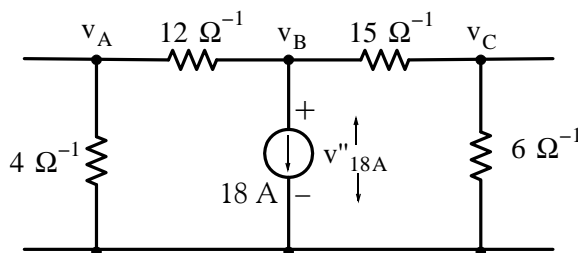
$$42v_2 - 35v_3 = 21$$

$$-25v_2 + 35v_3 = 0$$

and by addition of these we obtain

$$v_2 = v'_{18A} = \frac{21}{17} \text{ V}$$

Next, we let  $v''_{18A}$  be the voltage due to the 18 A current source acting alone. The simplified circuit with assigned node voltages is shown below where the parallel conductances have been replaced by their equivalents.



The nodal equations at the three nodes are

$$16v_A - 12v_B = 0$$

$$-12v_A + 27v_B - 15v_C = -18$$

$$-15v_B + 21v_C = 0$$

or

$$4v_A - 3v_B = 0$$

$$-4v_A + 9v_B - 5v_C = -6$$

$$-5v_B + 7v_C = 0$$

Since  $v_B = v''_{18A}$ , we only need to solve for  $v_B$ . Adding the first 2 equations above and grouping with the third we obtain

$$6v_B - 5v_C = -6$$

$$-5v_B + 7v_C = 0$$

Multiplying the first by 7 and the second by 5 we obtain

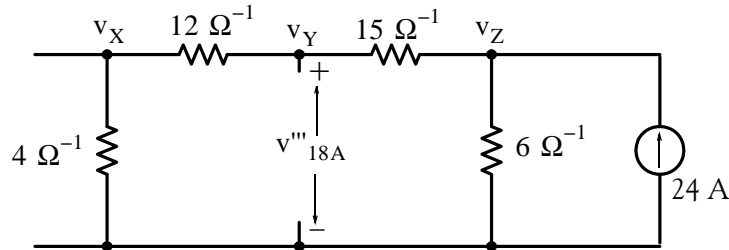
$$42v_B - 35v_C = -42$$

$$-25v_B + 35v_C = 0$$

and by addition of these we obtain

$$v_B = v''_{18A} = \frac{-42}{17} \text{ V}$$

Finally, we let  $v'''_{18A}$  be the voltage due to the 24 A current source acting alone. The simplified circuit with assigned node voltages is shown below where the parallel conductances have been replaced by their equivalents.



The nodal equations at the three nodes are

$$\begin{aligned} 16v_X - 12v_Y &= 0 \\ -12v_X + 27v_Y - 15v_Z &= 0 \\ -15v_Y + 21v_Z &= 24 \end{aligned}$$

or

$$\begin{aligned} 4v_X - 3v_Y &= 0 \\ -4v_X + 9v_Y - 5v_Z &= 0 \\ -5v_Y + 7v_Z &= 8 \end{aligned}$$

Since  $v_Y = v'''_{18A}$ , we only need to solve for  $v_Y$ . Adding the first 2 equations above and grouping with the third we obtain

$$\begin{aligned} 6v_Y - 5v_Z &= 0 \\ -5v_Y + 7v_Z &= 0 \end{aligned}$$

Multiplying the first by 7 and the second by 5 we obtain

$$\begin{aligned} 42v_Y - 35v_Z &= 0 \\ -25v_Y + 35v_Z &= 40 \end{aligned}$$

and by addition of these we obtain

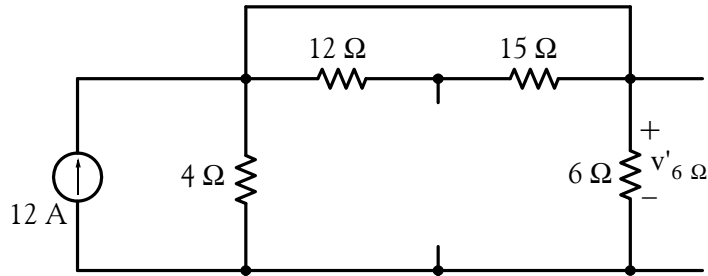
$$v_Y = v'''_{18A} = \frac{40}{17} \text{ V}$$

and thus

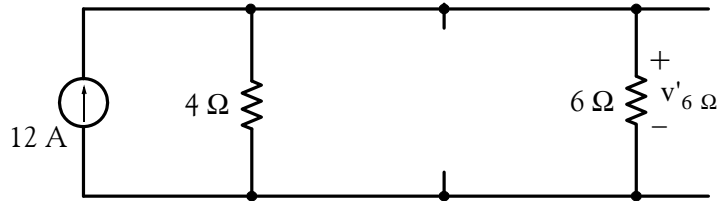
$$v_{18A} = v'_{18A} + v''_{18A} + v'''_{18A} = \frac{21}{17} + \frac{-42}{17} + \frac{40}{17} = \frac{19}{17} = 1.12 \text{ V}$$

This is the same answer as in Problem 1.

12. This is the same circuit as that of Problem 2. Let  $v'_{6\ \Omega}$  be the voltage due to the 12 A current source acting alone. The simplified circuit is shown below.



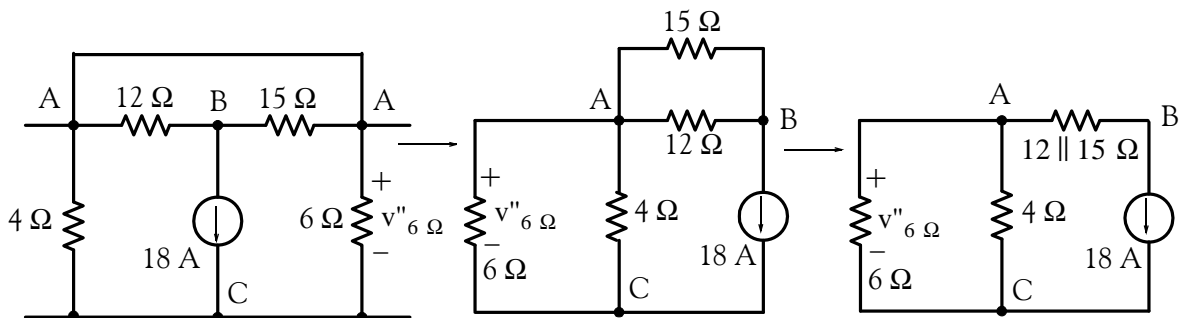
The 12 Ω and 15 Ω resistors are shorted out and the circuit is further simplified to the one shown below.



The voltage  $v'_{6\ \Omega}$  is computed easily by application of the current division expression and multiplication by the 6 Ω resistor. Thus,

$$v'_{6\ \Omega} = \left( \frac{4}{4+6} \times 12 \right) \times 6 = \frac{144}{5} \text{ V}$$

Next, we let  $v''_{6\ \Omega}$  be the voltage due to the 18 A current source acting alone. The simplified circuit is shown below. The letters A, B, and C are shown to visualize the circuit simplification process.

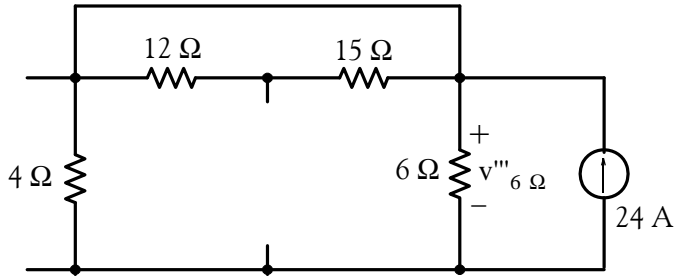


## Chapter 3 Nodal and Mesh Equations – Circuit Theorems

The voltage  $v''_{6\Omega}$  is computed easily by application of the current division expression and multiplication by the  $6\Omega$  resistor. Thus,

$$v''_{6\Omega} = \left[ \frac{4}{4+6} \times (-18) \right] \times 6 = \frac{-216}{5} \text{ V}$$

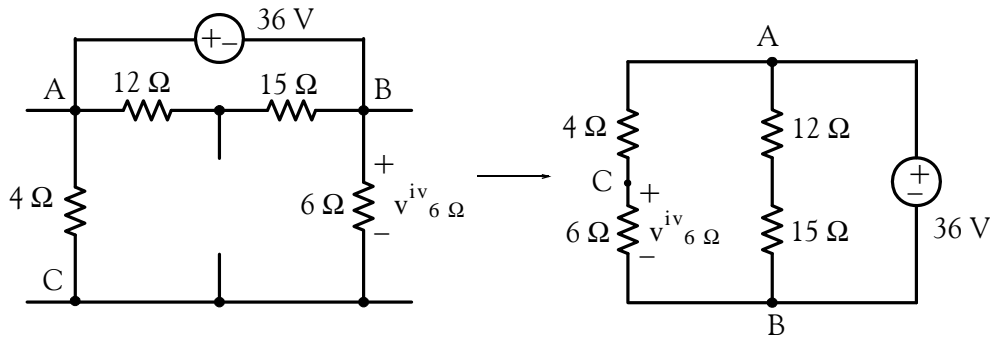
Now, we let  $v'''_{6\Omega}$  be the voltage due to the  $24\text{ A}$  current source acting alone. The simplified circuit is shown below.



The  $12\Omega$  and  $15\Omega$  resistors are shorted out and voltage  $v'''_{6\Omega}$  is computed by application of the current division expression and multiplication by the  $6\Omega$  resistor. Thus,

$$v'''_{6\Omega} = \left( \frac{4}{4+6} \times 24 \right) \times 6 = \frac{288}{5} \text{ V}$$

Finally, we let  $v^{iv}_{6\Omega}$  be the voltage due to the  $36\text{ V}$  voltage source acting alone. The simplified circuit is shown below.



By application of the voltage division expression we find that

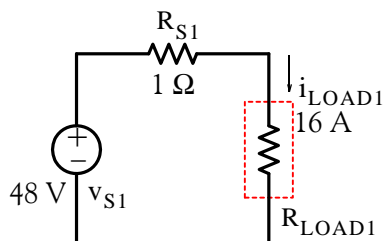
$$v^{iv}_{6\Omega} = \frac{6}{4+6} \times (-36) = \frac{-108}{5}$$

Therefore,

$$v_{6\ \Omega} = v'_{6\ \Omega} + v''_{6\ \Omega} + v'''_{6\ \Omega} + v^{iv}_{6\ \Omega} = \frac{144}{5} - \frac{216}{5} + \frac{288}{5} - \frac{108}{5} = \frac{108}{5} = 21.6\ \text{V}$$

This is the same answer as that of Problem 2.

13. The circuit for Measurement 1 is shown below.



Let  $R_{eq1} = R_{S1} + R_{LOAD1}$ . Then,

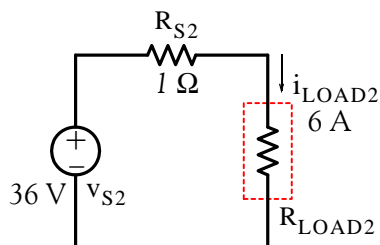
$$R_{eq1} = \frac{v_{S1}}{i_{LOAD1}} = \frac{48}{16} = 3\ \Omega$$

For Measurement 3 the load resistance is the same as for Measurement 1 and the load current is given as  $-5\ \text{A}$ . Therefore, for Measurement 3 we find that

$$v_{S1} = R_{eq1}(-5) = 3 \times (-5) = -15\ \text{V}$$

and we enter this value in the table below.

The circuit for Measurement 2 is shown below.



Let  $R_{eq2} = R_{S1} + R_{LOAD2}$ . Then,

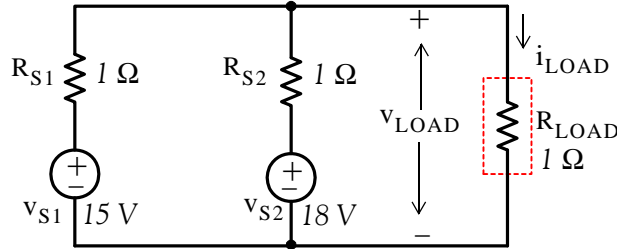
$$R_{eq2} = \frac{v_{S2}}{i_{LOAD2}} = \frac{36}{6} = 6\ \Omega$$

For Measurement 4 the load resistance is the same as for Measurement 2 and  $v_{S2}$  is given as  $-42\ \text{V}$ . Therefore, for Measurement 4 we find that

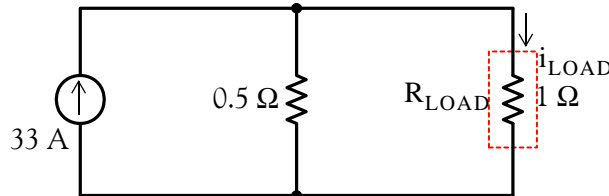
$$i_{\text{LOAD}2} = \frac{v_{\text{S}2}}{R_{\text{eq}2}} = -\frac{42}{6} = -7 \text{ A}$$

and we enter this value in the table below.

The circuit for Measurement 5 is shown below.



Replacing the voltage sources with their series resistances to their equivalent current sources with their parallel resistances and simplifying, we obtain the circuit below.

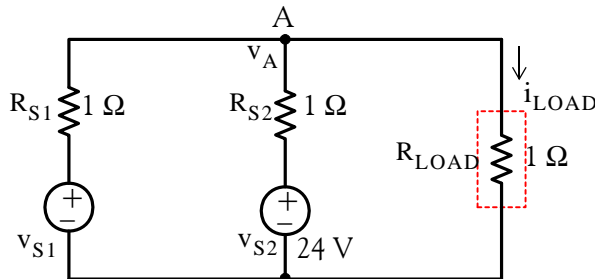


Application of the current division expression yields

$$i_{\text{LOAD}} = \frac{0.5}{0.5 + 1} \times 33 = 11 \text{ A}$$

and we enter this value in the table below.

The circuit for Measurement 6 is shown below.



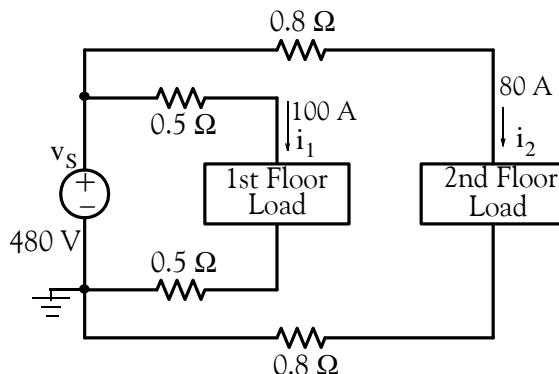
We observe that  $i_{\text{LOAD}}$  will be zero if  $v_A = 0$  and this will occur when  $v_{\text{S}1} = -24$ . This can be shown to be true by writing a nodal equation at Node A. Thus,

$$\frac{v_A - (-24)}{1} + \frac{v_A - 24}{1} + 0 = 0$$

or  $v_A = 0$

Measurement	Switch $S_1$	Switch $S_2$	$v_{S1}$ (V)	$v_{S2}$ (V)	$i_L$ (A)
1	Closed	Open	48	0	16
2	Open	Closed	0	36	6
3	Closed	Open	-15	0	-5
4	Open	Closed	0	-42	-7
5	Closed	Closed	15	18	11
6	Closed	Closed	-24	24	0

14.



The power supplied by the voltage source is

$$p_S = v_S (i_1 + i_2) = 480(100 + 80) = 86,400 \text{ w} = 86.4 \text{ Kw}$$

The power loss on the 1st floor is

$$p_{\text{LOSS1}} = i_1^2 (0.5 + 0.5) = 100^2 \times 1 = 10,000 \text{ w} = 10 \text{ Kw}$$

The power loss on the 2nd floor is

$$p_{\text{LOSS2}} = i_2^2 (0.8 + 0.8) = 80^2 \times 1.6 = 10,240 \text{ w} = 10.24 \text{ Kw}$$

and thus the total loss is

$$\text{Total loss} = 10 + 10.24 = 20.24 \text{ Kw}$$

Then,



## Chapter 3 Nodal and Mesh Equations – Circuit Theorems

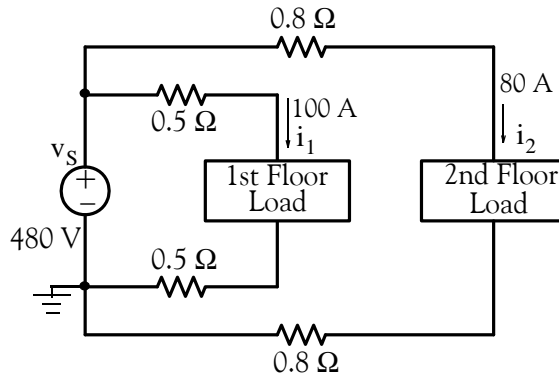
$$\text{Output power} = \text{Input power} - \text{power losses} = 86.4 - 20.24 = 66.16 \text{ Kw}$$

and

$$\% \text{ Efficiency} = \eta = \frac{\text{Output}}{\text{Input}} \times 100 = \frac{66.16}{86.4} \times 100 = 76.6\%$$

This is indeed a low efficiency.

15.



The voltage drop on the second floor conductor is

$$v_{\text{cond}} = R_T i_2 = 1.6 \times 80 = 128 \text{ V}$$

and thus the full-load voltage is

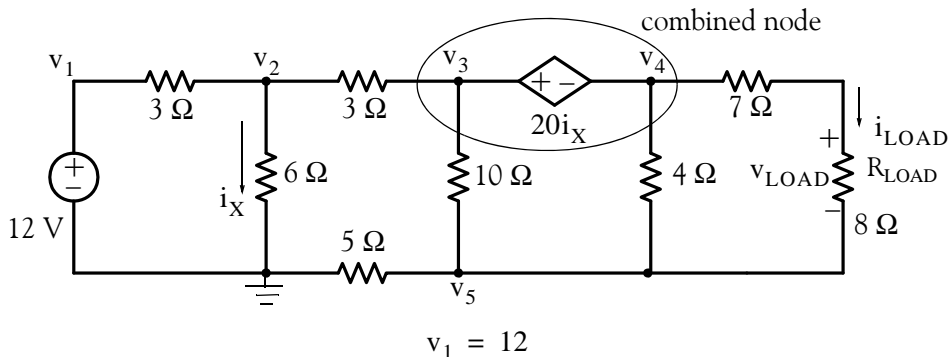
$$v_{\text{FL}} = 480 - 128 = 352 \text{ V}$$

Then,

$$\% \text{ Regulation} = \frac{v_{\text{NL}} - v_{\text{FL}}}{v_{\text{FL}}} \times 100 = \frac{480 - 352}{352} \times 100 = 36.4\%$$

This is a very poor regulation.

16. We assign node voltages and we write nodal equations as shown below.



$$\frac{v_2 - v_1}{3} + \frac{v_2}{6} + \frac{v_2 - v_3}{3} = 0$$

$$\frac{v_3 - v_2}{3} + \frac{v_3 - v_5}{10} + \frac{v_4 - v_5}{4} + \frac{v_4 - v_5}{7+8} = 0$$

$$v_3 - v_4 = 20i_X$$

where  $i_X = v_2/6$  and thus

$$v_5 = \frac{10}{3}v_2$$

$$\frac{v_5}{5} + \frac{v_5 - v_3}{10} + \frac{v_5 - v_4}{4} + \frac{v_5 - v_4}{7+8} = 0$$

Collecting like terms and rearranging we obtain

$$\begin{aligned} v_1 &= 12 \\ -\frac{1}{3}v_1 + \frac{5}{6}v_2 + \frac{-1}{3}v_3 &= 0 \\ -\frac{1}{3}v_2 + \frac{13}{30}v_3 + \frac{19}{60}v_4 - \frac{19}{60}v_5 &= 0 \\ -\frac{10}{3}v_2 + v_3 - v_4 &= 0 \\ -\frac{1}{10}v_3 - \frac{19}{60}v_4 + \frac{37}{60}v_5 &= 0 \end{aligned}$$

and in matrix form

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{5}{6} & -\frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & \frac{13}{30} & \frac{19}{60} & -\frac{19}{60} \\ 0 & -\frac{10}{3} & 1 & -1 & 0 \\ 0 & 0 & -\frac{1}{10} & -\frac{19}{60} & \frac{37}{60} \end{bmatrix}}_{\mathbf{G}} \cdot \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix}}_{\mathbf{V}} = \underbrace{\begin{bmatrix} 12 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{I}}$$

We use MATLAB to solve the above.

```
G=[1 0 0 0 0;...
  -1/3 5/6 -1/3 0 0;...
  0 -1/3 13/30 19/60 -19/60;...
  0 -10/3 1 -1 0;...
  0 0 -1/10 -19/60 37/60];
I=[12 0 0 0 0]'; V=G\I;
fprintf('\n');...
fprintf('v1 = %7.2f V \n',V(1));...
fprintf('v2 = %7.2f V \n',V(2));...
fprintf('v3 = %7.2f V \n',V(3));...
fprintf('v4 = %7.2f V \n',V(4));...
fprintf('v5 = %7.2f V \n',V(5));...
fprintf('\n'); fprintf('\n')

v1 = 12.00 V
v2 = 13.04 V
v3 = 20.60 V
v4 = -22.87 V
v5 = -8.40 V
```

Now,

$$i_{\text{LOAD}} = \frac{v_4 - v_5}{8 + 7} = \frac{-22.87 - (-8.40)}{15} = -0.96 \text{ A}$$

and

$$v_{\text{LOAD}} = 8i_{\text{LOAD}} = 8 \times (-0.96) = -7.68 \text{ V}$$

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# Chapter 4

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## Introduction to Operational Amplifiers

This chapter is an introduction to amplifiers. It discusses amplifier gain in terms of decibels (dB) and provides an overview of operational amplifiers, their characteristics and applications. Numerous formulas for the computation of the gain are derived and several practical examples are provided.

### 4.1 Signals

A *signal* is any waveform that serves as a means of communication. It represents a fluctuating electric quantity, such as voltage, current, electric or magnetic field strength, sound, image, or any message transmitted or received in telegraphy, telephony, radio, television, or radar. A typical signal which varies with time is shown in figure 4.1 where  $f(t)$  can be any physical quantity such as voltage, current, temperature, pressure, and so on.

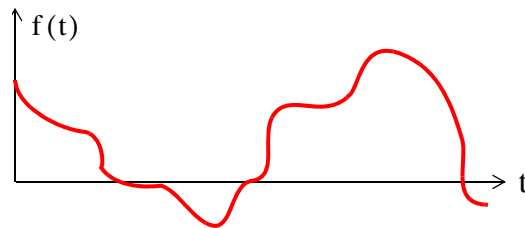


Figure 4.1. A signal that changes with time

### 4.2 Amplifiers

An *amplifier* is an electronic circuit which increases the magnitude of the input signal. The symbol of a typical amplifier is a triangle as shown in Figure 4.2.

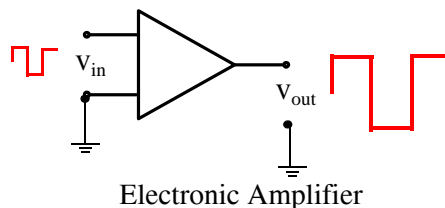


Figure 4.2. Symbol for electronic amplifier

An electronic (or electric) circuit which produces an output that is smaller than the input is called an *attenuator*. A resistive voltage divider is a typical attenuator.

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## Chapter 4 Introduction to Operational Amplifiers

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An amplifier can be classified as a voltage amplifier, current amplifier, or power amplifier. The *gain* of an amplifier is the ratio of the output to the input. Thus for a voltage amplifier,

$$\text{Voltage Gain} = \frac{\text{Output Voltage}}{\text{Input Voltage}}$$

or

$$\boxed{G_v = \frac{V_{\text{out}}}{V_{\text{in}}}} \quad (4.1)$$

The *current gain*  $G_i$  and *power gain*  $G_p$  are defined similarly.

**Note 1:** Throughout this text, the common (base 10) logarithm of a number  $x$  will be denoted as  $\log(x)$  while its natural (base  $e$ ) logarithm will be denoted as  $\ln(x)$ .

### 4.3 Decibels

The ratio of any two values of the same quantity (power, voltage or current) can be expressed in decibels (dB). For instance, we say that an amplifier has 10 dB power gain or a transmission line has a power loss of 7 dB (or gain  $-7$  dB). If the gain (or loss) is 0 dB, the output is equal to the input.

We must remember that a negative voltage or current gain  $G_v$  or  $G_i$  indicates that there is a  $180^\circ$  phase difference between the input and the output waveforms. For instance, if an amplifier has a gain of  $-100$  (dimensionless number), it means that the output is 180 degrees out-of-phase with the input. Therefore, to avoid misinterpretation of gain or loss, we use absolute values of power, voltage and current when these are expressed in dB.

By definition,

$$\text{dB} = 10 \log \left| \frac{P_{\text{out}}}{P_{\text{in}}} \right| \quad (4.2)$$

Therefore,

10 dB represents a power ratio of 10

10n dB represents a power ratio of  $10^n$

It is useful to remember that

20 dB represents a power ratio of 100

30 dB represents a power ratio of 1000

60 dB represents a power ratio of 1000000

Also,

1 dB represents a power ratio of approximately 1.25

3 dB represents a power ratio of approximately 2

7 dB represents a power ratio of approximately 5

From these, we can estimate other values. For instance, 4 dB = 3 dB + 1 dB which is equivalent to a power ratio of approximately  $2 \times 1.25 = 2.5$ . Likewise, 27 dB = 20 dB + 7 dB and this is equivalent to a power ratio of approximately  $100 \times 5 = 500$ .

Since  $y = \log x^2 = 2 \log x$  and  $p = v^2/R = i^2R$ , if we let  $R = 1$ , the dB values for voltage and current ratios become:

$$\text{dB}_v = 10 \log \left| \frac{v_{\text{out}}}{v_{\text{in}}} \right|^2 = 20 \log \left| \frac{v_{\text{out}}}{v_{\text{in}}} \right| \quad (4.3)$$

and

$$\text{dB}_i = 10 \log \left| \frac{i_{\text{out}}}{i_{\text{in}}} \right|^2 = 20 \log \left| \frac{i_{\text{out}}}{i_{\text{in}}} \right| \quad (4.4)$$

### Example 4.1

Compute the gain in  $\text{dB}_w$  for the amplifier shown in Figure 4.3.

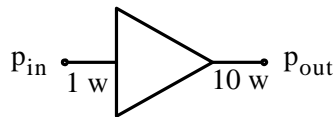


Figure 4.3. Amplifier for Example 4.1

**Solution:**

$$\text{dB}_w = 10 \log \frac{P_{\text{out}}}{P_{\text{in}}} = 10 \log \frac{10}{1} = 10 \log 10 = 10 \times 1 = 10 \text{ dB}_w$$

### Example 4.2

Compute the gain in  $\text{dB}_v$  for the amplifier shown in Figure 4.4, given that  $\log 2 = 0.3$ .

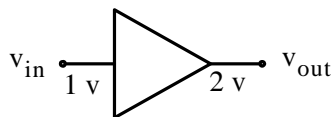


Figure 4.4. Amplifier for Example 4.2

**Solution:**

$$dB_v = 20 \log \frac{v_{out}}{v_{in}} = 20 \log \frac{2}{1} = 20 \log 0.3 = 20 \times 0.3 = 6 \text{ dBv}$$

### 4.4 Bandwidth and Frequency Response

Like electric filters, amplifiers exhibit a band of frequencies over which the output remains nearly constant. Consider, for example, the magnitude of the output voltage  $|v_{out}|$  of an electric or electronic circuit as a function of radian frequency  $\omega$  as shown in Figure 4.5.

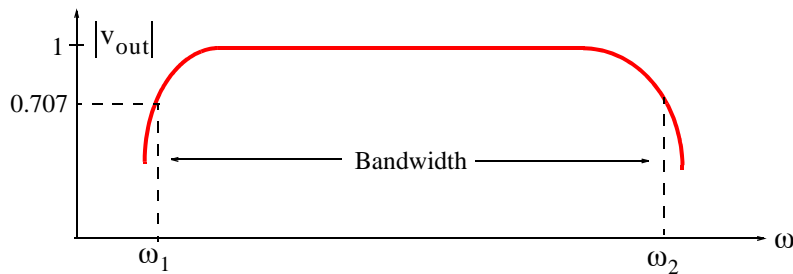


Figure 4.5. Typical bandwidth of an amplifier

As shown above, the *bandwidth* is  $BW = \omega_2 - \omega_1$  where  $\omega_1$  and  $\omega_2$  are the *lower and upper cutoff frequencies* respectively. At these frequencies,  $|v_{out}| = \sqrt{2}/2 = 0.707$  and these two points are known as the *3-dB down* or *half-power points*. They derive their name from the fact that power  $p = v^2/R = i^2R$ , and for  $R = 1$  and  $v = \sqrt{2}/2 = 0.707$  or  $i = \sqrt{2}/2 = 0.707$ , the power is  $1/2$ , that is, the power is “halved”. Alternately, we can define the bandwidth as the frequency band between half-power points.

Most amplifiers are used with a *feedback* path which returns (feeds) some or all its output to the input as shown in Figure 4.6.

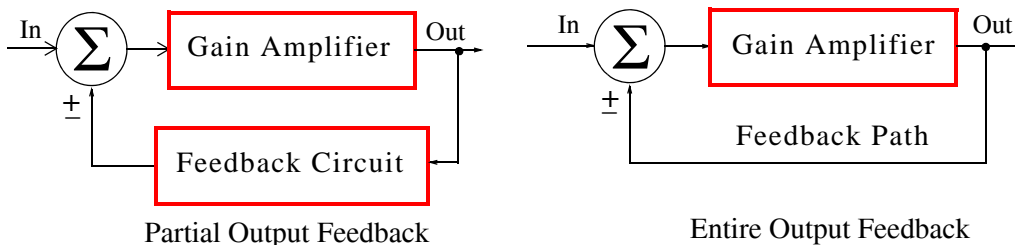


Figure 4.6. Gain amplifiers used with feedback

In Figure 4.6, the symbol  $\Sigma$  (Greek capital letter sigma) inside the circle denotes the summing point where the output signal, or portion of it, is combined with the input signal. This summing point may be also indicated with a large plus (+) symbol inside the circle. The positive (+) sign below the summing point implies *positive feedback* which means that the output, or portion of it, is added to the input. On the other hand, the negative (–) sign implies *negative feedback* which means that the output, or portion of it, is subtracted from the input. Practically, all amplifiers use used with negative feedback since positive feedback causes circuit instability.

### 4.5 The Operational Amplifier

The *operational amplifier* or simply *op amp* is the most versatile electronic amplifier. It derives its name from the fact that it is capable of performing many mathematical operations such as addition, multiplication, differentiation, integration, analog-to-digital conversion or vice versa. It can also be used as a comparator and electronic filter. It is also the basic block in analog computer design. Its symbol is shown in Figure 4.7.

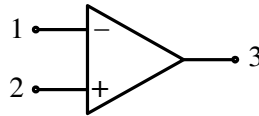


Figure 4.7. Symbol for operational amplifier

As shown above the op amp has two inputs but only one output. For this reason it is referred to as *differential input, single ended output* amplifier. Figure 4.8 shows the internal construction of a typical op amp. This figure also shows terminals  $V_{CC}$  and  $V_{EE}$ . These are the voltage sources required to power up the op amp. Typically,  $V_{CC}$  is +15 volts and  $V_{EE}$  is –15 volts. These terminals are not shown in op amp circuits since they just provide power, and do not reveal any other useful information for the op amp’s circuit analysis.

### 4.6 An Overview of the Op Amp

The op amp has the following important characteristics:

1. Very high input impedance (resistance)
2. Very low output impedance (resistance)
3. Capable of producing a very large gain that can be set to any value by connection of external resistors of appropriate values
4. Frequency response from DC to frequencies in the MHz range
5. Very good stability
6. Operation to be performed, i.e., addition, integration etc. is done externally with proper selection of passive devices such as resistors, capacitors, diodes, and so on.



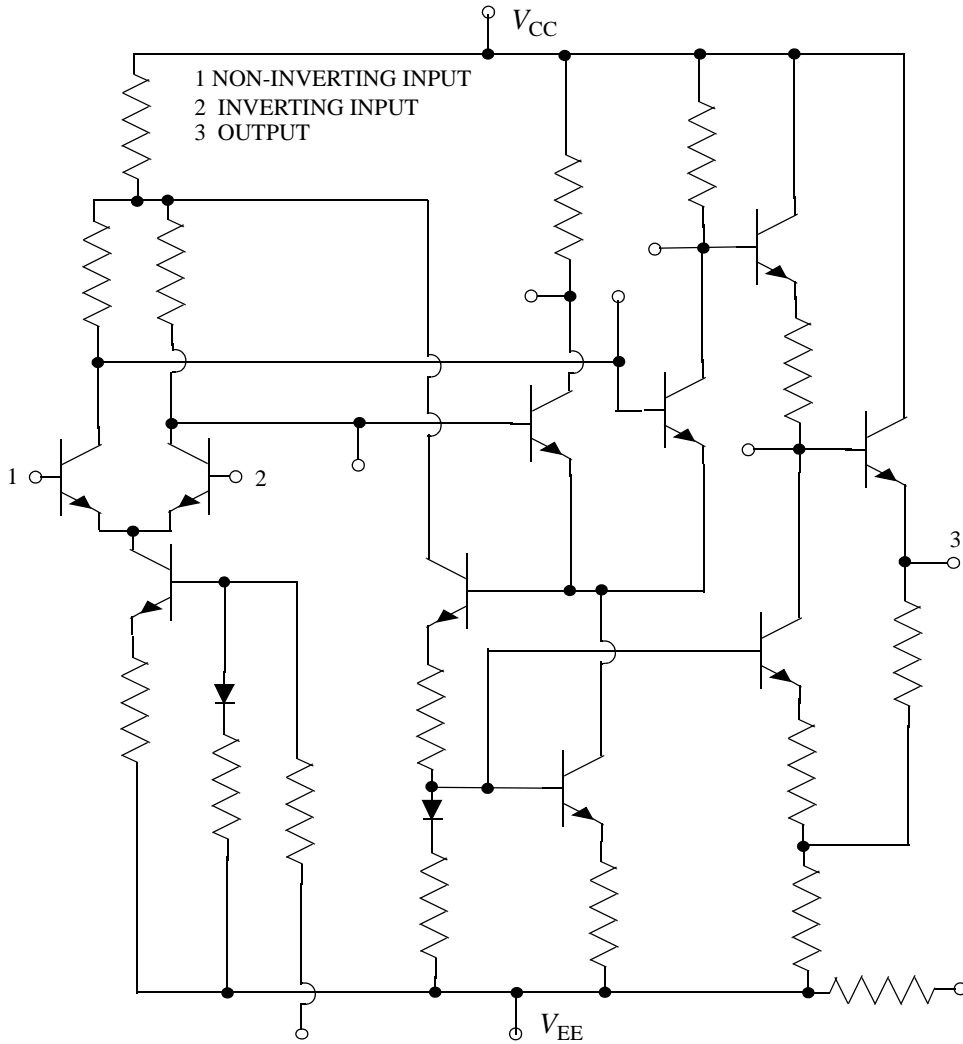


Figure 4.8. Internal Devices of a Typical Op Amp

An op amp is said to be connected in the *inverting mode* when an input signal is connected to the inverting (–) input through an external resistor  $R_{in}$  whose value along with the feedback resistor  $R_f$  determine the op amp’s gain. The non-inverting (+) input is grounded through an external resistor  $R$  as shown in Figure 4.9.

For the circuit of Figure 4.9, the voltage gain  $G_v$  is

$$G_v = \frac{v_{out}}{v_{in}} = -\frac{R_f}{R_{in}} \quad (4.5)$$

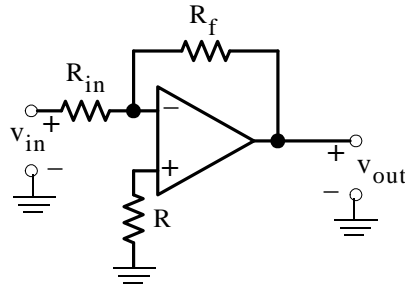


Figure 4.9. Circuit of Inverting op amp

**Note 2:** The resistor  $R$  connected between the non-inverting (+) input and ground serves only as a current limiting device, and thus it does not influence the op amp's gain. It will be omitted in our subsequent discussion.

**Note 3:** The input voltage  $v_{in}$  and the output voltage  $v_{out}$  as indicated in the circuit of Figure 4.9, should not be interpreted as open circuits; these designations imply that an input voltage of any waveform may be applied at the input terminals and the corresponding output voltage appears at the output terminals.

As shown in the formula of (4.5), the gain for this op amp configuration is the ratio  $-R_f/R_{in}$  where  $R_f$  is the feedback resistor which allows portion of the output to be fed back to the input. The minus (-) sign in the gain ratio  $-R_f/R_{in}$  implies that the output signal has opposite polarity from that of the input signal; hence the name inverting amplifier. Therefore, when the input signal is positive (+) the output will be negative (-) and vice versa. For example, if the input is +1 volt DC and the op amp gain is 100, the output will be -100 volts DC. For AC (sinusoidal) signals, the output will be 180 degrees out-of-phase with the input. Thus, if the input is 1 volt AC and the op amp gain is 5, the output will be -5 volts AC or 5 volts AC with 180 degrees out-of-phase with the input.

### Example 4.3

Compute the voltage gain  $G_v$  and then the output voltage  $v_{out}$  for the inverting op amp circuit shown in Figure 4.10, given that  $v_{in} = 1$  mV. Plot  $v_{in}$  and  $v_{out}$  as mV versus time on the same set of axes.

#### Solution:

This is an inverting amplifier and thus the voltage gain  $G_v$  is

$$G_v = -\frac{R_f}{R_{in}} = -\frac{120 \text{ K}\Omega}{20 \text{ K}\Omega}$$

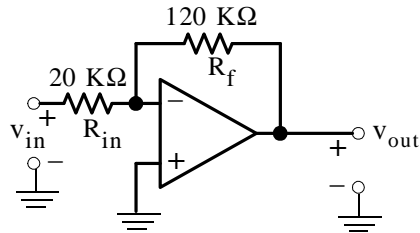


Figure 4.10. Circuit for Example 4.3

or

$$G_v = -6$$

and since

$$G_v = \frac{v_{out}}{v_{in}}$$

the output voltage is

$$v_{out} = G_v v_{in} = -6 \times 1$$

or

$$v_{out} = -6 \text{ mV}$$

The voltages  $v_{in}$  and  $v_{out}$  are plotted as shown in Figure 4.11.

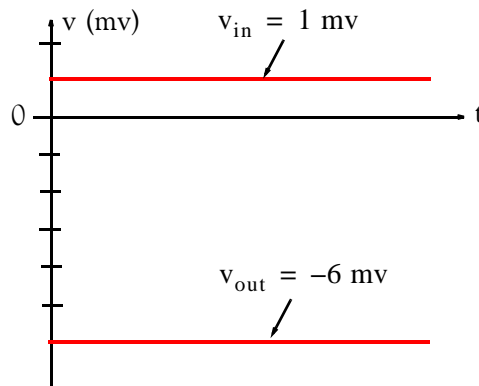


Figure 4.11. Input and output waveforms for the circuit of Example 4.3

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### Example 4.4

Compute the voltage gain  $G_v$  and then the output voltage  $v_{out}$  for the inverting op amp circuit shown in Figure 4.12, given that  $v_{in} = \sin t \text{ mV}$ . Plot  $v_{in}$  and  $v_{out}$  as mV versus time on the same set of axes.

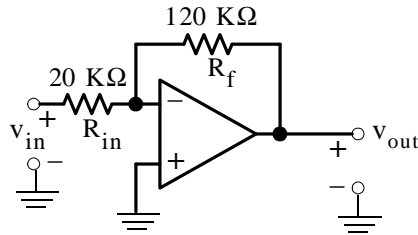


Figure 4.12. Circuit for Example 4.4

**Solution:**

This is the same circuit as that of the previous example except that the input is a sine wave with unity amplitude and the voltage gain  $G_v$  is the same as before, that is,

$$G_v = -\frac{R_f}{R_{in}} = -\frac{120 \text{ K}\Omega}{20 \text{ K}\Omega} = -6$$

and the output voltage is

$$v_{out} = G_v v_{in} = -6 \times \text{sint} = -6\text{sint mV}$$

The voltages  $v_{in}$  and  $v_{out}$  are plotted as shown in Figure 4.13.

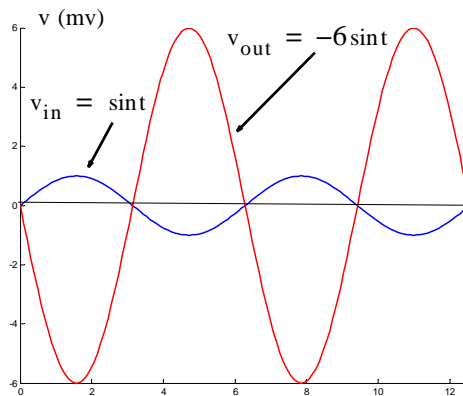


Figure 4.13. Input and output waveforms for the circuit of Example 4.4

An op amp is said to be connected in the *non-inverting mode* when an input signal is connected to the non-inverting (+) input through an external resistor  $R$  which serves as a current limiter, and the inverting (-) input is grounded through an external resistor  $R_{in}$  as shown in Figure 4.14. In our subsequent discussion, the resistor  $R$  will represent the internal resistance of the applied voltage  $v_{in}$ .

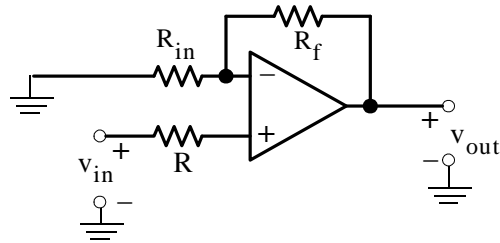


Figure 4.14. Circuit of non-inverting op amp

For the circuit of Figure 4.14, the voltage gain  $G_v$  is

$$G_v = \frac{v_{out}}{v_{in}} = 1 + \frac{R_f}{R_{in}} \quad (4.6)$$

As indicated by the relation of (4.6), the gain for this op amp configuration is  $1 + R_f/R_{in}$  and therefore, in the non-inverting mode the op amp output signal has the same polarity as the input signal; hence, the name non-inverting amplifier. Thus, when the input signal is positive (+) the output will be also positive and if the input is negative, the output will be also negative. For example, if the input is +1 mV DC and the op amp gain is 75, the output will be +75 mV DC. For AC signals the output will be in-phase with the input. For example, if the input is 0.5 V AC and the op amp gain is  $G_v = 1 + 19 \text{ K}\Omega / 1 \text{ K}\Omega = 20$ , the output will be 10 V AC and in-phase with the input.

### Example 4.5

Compute the voltage gain  $G_v$  and then the output voltage  $v_{out}$  for the non-inverting op amp circuit shown in Figure 4.15, given that  $v_{in} = 1 \text{ mV}$ . Plot  $v_{in}$  and  $v_{out}$  as mV versus time on the same set of axes.

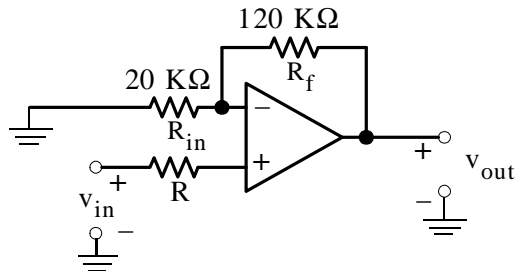


Figure 4.15. Circuit for Example 4.5

**Solution:**

The voltage gain  $G_v$  is

$$G_v = \frac{v_{out}}{v_{in}} = 1 + \frac{R_f}{R_{in}} = 1 + \frac{120 \text{ K}\Omega}{20 \text{ K}\Omega} = 1 + 6 = 7$$

and thus

$$v_{out} = G_v v_{in} = 7 \times 1 \text{ mV} = 7 \text{ mV}$$

The voltages  $v_{in}$  and  $v_{out}$  are plotted as shown in Figure 4.16.

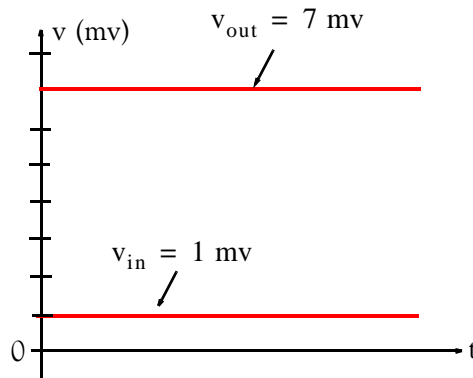


Figure 4.16. Input and output waveforms for the circuit of Example 4.5

**Example 4.6**

Compute the voltage gain  $G_v$  and then the output voltage  $v_{out}$  for the non-inverting op amp circuit shown in Figure 4.17, given that  $v_{in} = \sin t \text{ mV}$ . Plot  $v_{in}$  and  $v_{out}$  as mV versus time on the same set of axes.

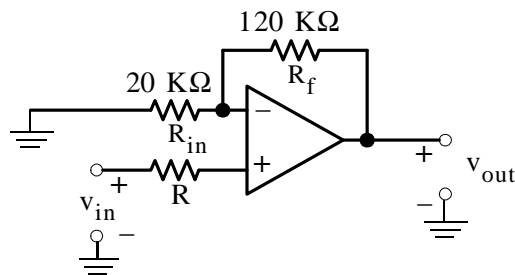


Figure 4.17. Circuit for Example 4.6

**Solution:**

This is the same circuit as in the previous example except that the input is a sinusoid. Therefore, the voltage gain  $G_v$  is the same as before, that is,

$$G_v = \frac{v_{\text{out}}}{v_{\text{in}}} = 1 + \frac{R_f}{R_{\text{in}}} = 1 + \frac{120 \text{ K}\Omega}{20 \text{ K}\Omega} = 1 + 6 = 7$$

and the output voltage is

$$v_{\text{out}} = G_v v_{\text{in}} = 7 \times \text{sint} = 7 \text{ sint mV}$$

The voltages  $v_{\text{in}}$  and  $v_{\text{out}}$  are plotted as shown in Figure 4.18.

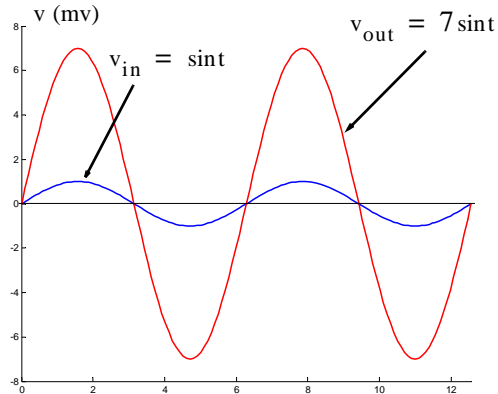


Figure 4.18. Input and output waveforms for the circuit of Example 4.6

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Quite often an op amp is connected as shown in Figure 4.19.

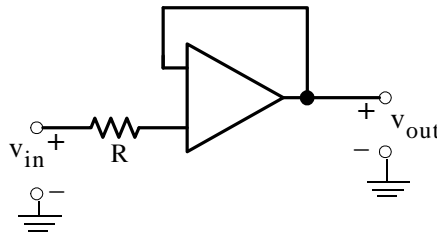


Figure 4.19. Circuit of unity gain op amp

For the circuit of Figure 4.19, the voltage gain  $G_v$  is

$$G_v = \frac{v_{\text{out}}}{v_{\text{in}}} = 1 \quad (4.7)$$

and thus

$$v_{\text{out}} = v_{\text{in}} \quad (4.8)$$

For this reason, the op amp circuit of Figure 4.19 it is called *unity gain amplifier*. For example, if the input voltage is 5 mV DC the output will also be 5 mV DC, and if the input voltage is 2 mV AC, the output will also be 2 mV AC. The unity gain op amp is used to provide a very high resistance between a voltage source and the load connected to it. An example will be given in Section 4.8.

## 4.7 Active Filters

An *active filter* is an electronic circuit consisting of an amplifier and other devices such as resistors and capacitors. In contrast, a *passive filter* is a circuit which consists of passive devices such as resistors, capacitors and inductors. Operational amplifiers are used extensively as active filters.

A *low-pass filter* transmits (passes) all frequencies below a *critical (cutoff)* frequency denoted as  $\omega_c$ , and *attenuates* (blocks) all frequencies above this cutoff frequency. An op amp low-pass filter is shown in Figure 4.20 and its frequency response in Figure 4.21.

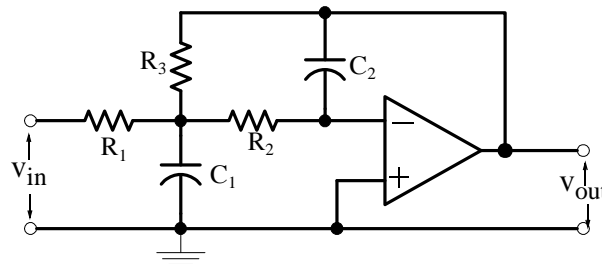


Figure 4.20. A low-pass active filter

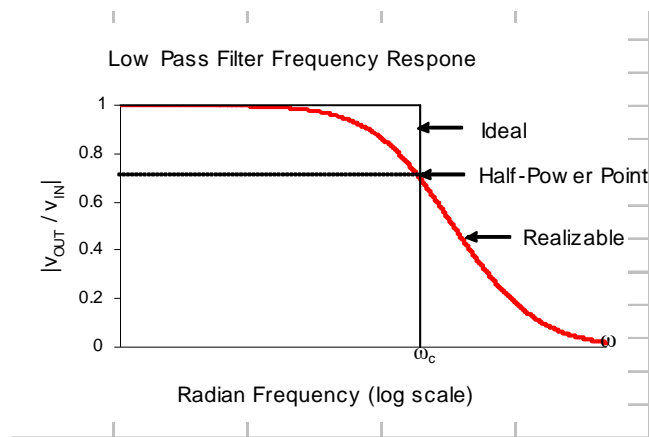


Figure 4.21. Frequency response for amplitude of a low-pass filter

In Figure 4.21, the straight vertical and horizontal lines represent the ideal (unrealizable) and the smooth curve represents the practical (realizable) low-pass filter characteristics. The vertical



scale represents the magnitude of the ratio of output-to-input voltage  $v_{out}/v_{in}$ , that is, the gain  $G_v$ . The cutoff frequency  $\omega_c$  is the frequency at which the maximum value of  $v_{out}/v_{in}$  which is unity, falls to  $0.707 \times G_v$ , and as mentioned before, this is the *half power* or the  $-3$  dB point.

A *high-pass filter* transmits (passes) all frequencies above a critical (cutoff) frequency  $\omega_c$ , and attenuates (blocks) all frequencies below the cutoff frequency. An op amp high-pass filter is shown in Figure 4.22 and its frequency response in Figure 4.23.

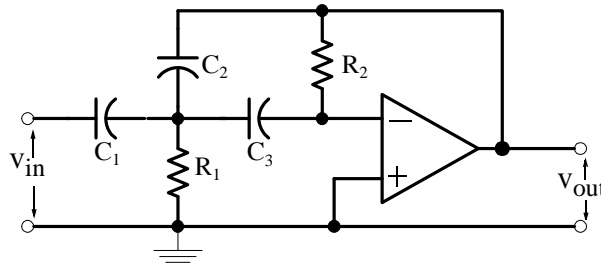


Figure 4.22. A high-pass active filter

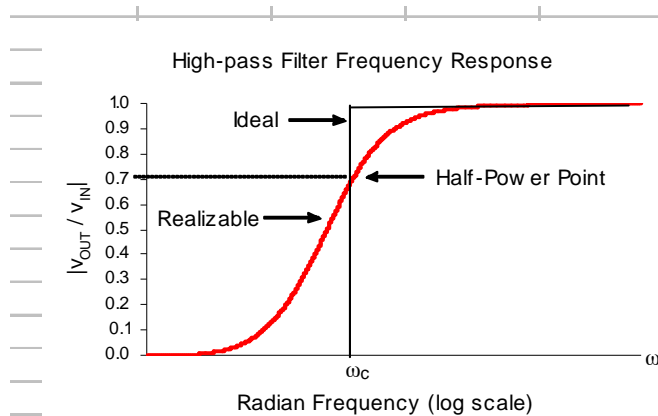


Figure 4.23. Frequency response for amplitude of a high-pass filter

In Figure 4.23, the straight vertical and horizontal lines represent the ideal (unrealizable) and the smooth curve represents the practical (realizable) high-pass filter characteristics. The vertical scale represents the magnitude of the ratio of output-to-input voltage  $v_{out}/v_{in}$ , that is, the gain  $G_v$ . The cutoff frequency  $\omega_c$  is the frequency at which the maximum value of  $v_{out}/v_{in}$  which is unity, falls to  $0.707 \times G_v$ , i.e., the *half power* or the  $-3$  dB point.

A *band-pass filter* transmits (passes) the band (range) of frequencies between the critical (cutoff) frequencies denoted as  $\omega_1$  and  $\omega_2$ , where the maximum value of  $G_v$  which is unity, falls to

$0.707 \times G_v$ , while it attenuates (blocks) all frequencies outside this band. An op amp band-pass filter is shown in Figure 4.24 and its frequency response in Figure 4.25.

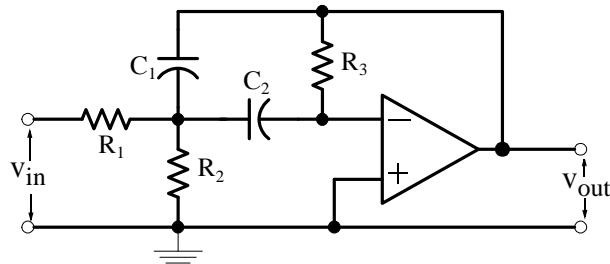


Figure 4.24. An active band-pass filter

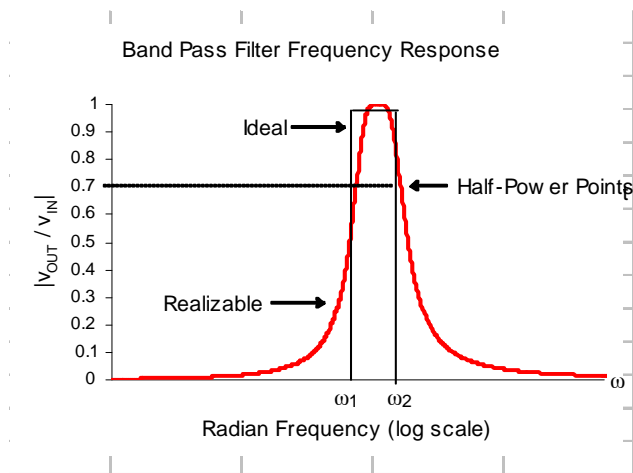


Figure 4.25. Frequency response for amplitude of a band-pass filter

A *band-elimination* or *band-stop* or *band-rejection* filter attenuates (rejects) the band (range) of frequencies between the critical (cutoff) frequencies denoted as  $\omega_1$  and  $\omega_2$ , where the maximum value of  $G_v$  which is unity, falls to  $0.707 \times G_v$ , while it transmits (passes) all frequencies outside this band. An op amp band-stop filter is shown in Figure 4.26 and its frequency response in Figure 4.27.

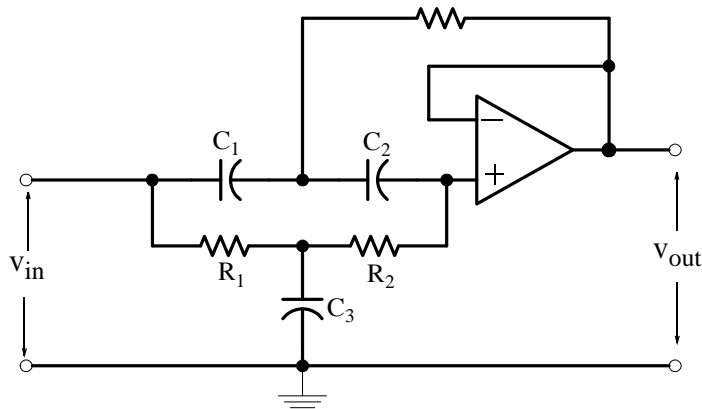


Figure 4.26. An active band-elimination filter

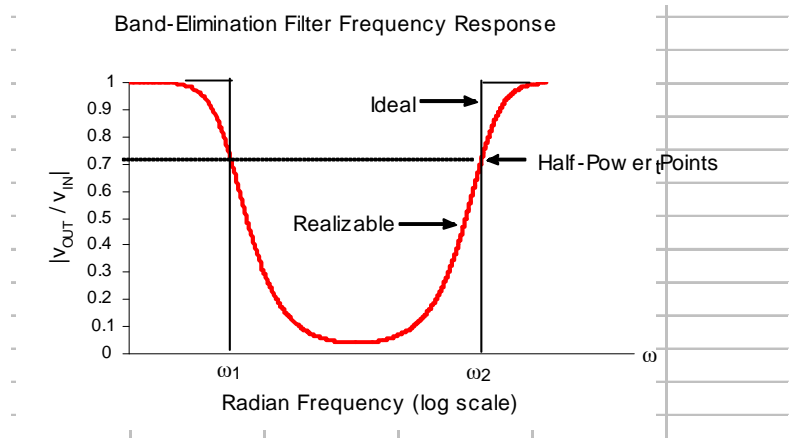


Figure 4.27. Frequency response for amplitude of a band-elimination filter

## 4.8 Analysis of Op Amp Circuits

The procedure for analyzing an op amp circuit (finding voltages, currents and power) is the same as for the other circuits which we have studied thus far. That is, we can apply Ohm's law, KCL and KVL, superposition, Thevenin's and Norton's theorems. When analyzing an op amp circuit, we must remember that in any op amp:

- a. The currents into both input terminals are zero
- b. The voltage difference between the input terminals of an op amp is zero
- c. For circuits containing op amps, we will assume that the reference (ground) is the common terminal of the two power supplies. For simplicity, the power supplies will not be shown.

We will provide several examples to illustrate the analysis of op amp circuits without being concerned about its internal operation; this is discussed in electronic circuit analysis books.

**Example 4.7**

The op amp circuit shown in Figure 4.28 is called *inverting op amp*. Prove that the voltage gain  $G_v$  is as given in (4.9) below, and draw its equivalent circuit showing the output as a dependent source.

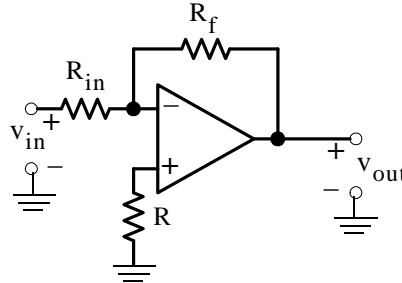


Figure 4.28. Circuit for deriving the gain of an inverting op amp

$$G_v = \frac{v_{out}}{v_{in}} = -\frac{R_f}{R_{in}} \tag{4.9}$$

**Proof:**

No current flows through the (-) input terminal of the op amp; therefore the current  $i$  which flows through resistor  $R_{in}$  flows also through resistor  $R_f$ . Also, since the (+) input terminal is grounded and there is no voltage drop between the (-) and (+) terminals, the (-) input is said to be at *virtual ground*. From the circuit of Figure 4.28,

$$v_{out} = -R_f i$$

where

$$i = \frac{v_{in}}{R_{in}}$$

and thus

$$v_{out} = -\frac{R_f}{R_{in}} v_{in}$$

or

$$G_v = \frac{v_{out}}{v_{in}} = -\frac{R_f}{R_{in}}$$

## Chapter 4 Introduction to Operational Amplifiers

The input and output parts of the circuit are shown in Figure 4.29 with the virtual ground being the same as the circuit ground.

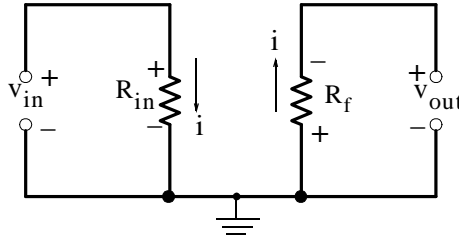


Figure 4.29. Input and output parts of the inverting op amp

These two circuits are normally drawn with the output as a dependent source as shown in Figure 4.30. This is the *equivalent circuit of the inverting op amp* and, as mentioned in Chapter 1, the dependent source is a Voltage Controlled Voltage Source (VCVS).

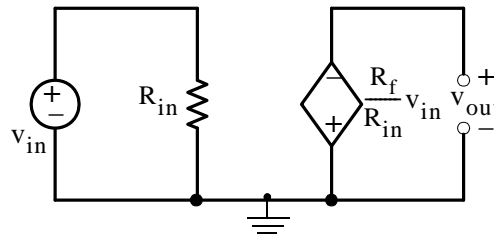


Figure 4.30. Equivalent circuit of the inverting op amp

### Example 4.8

The op amp circuit shown in Figure 4.31 is called *non-inverting op amp*. Prove that the voltage gain  $G_v$  is as given in (4.10) below, and draw its equivalent circuit showing the output as a dependent source.

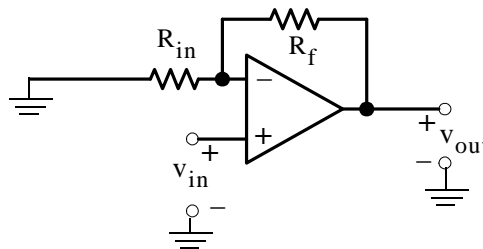


Figure 4.31. Circuit of non-inverting op amp

$$G_v = \frac{v_{\text{out}}}{v_{\text{in}}} = 1 + \frac{R_f}{R_{\text{in}}} \quad (4.10)$$

**Proof:**

Let the voltages at the (-) and (+) terminals be denoted as  $v_1$  and  $v_2$  respectively as shown in Figure 4.32.

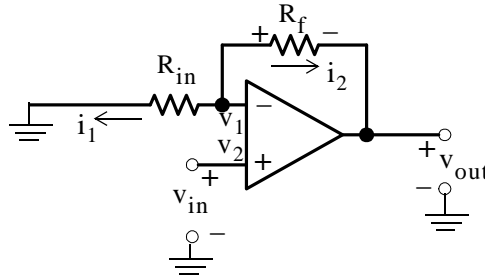


Figure 4.32. Non-inverting op amp circuit for derivation of (4.10)

By application of KCL at  $v_1$

$$i_1 + i_2 = 0$$

or

$$\frac{v_1}{R_{in}} + \frac{v_1 - v_{out}}{R_f} = 0 \tag{4.11}$$

There is no potential difference between the (-) and (+) terminals; therefore,  $v_1 - v_2 = 0$  or  $v_1 = v_2 = v_{in}$ . Relation (4.11) then can be written as

$$\frac{v_{in}}{R_{in}} + \frac{v_{in} - v_{out}}{R_f} = 0$$

or

$$\left( \frac{1}{R_{in}} + \frac{1}{R_f} \right) v_{in} = \frac{v_{out}}{R_f}$$

Rearranging, we obtain

$$G_v = \frac{v_{out}}{v_{in}} = 1 + \frac{R_f}{R_{in}}$$

and its equivalent circuit is as shown in Figure 4.33. The dependent source of this equivalent circuit is also a VCVS.

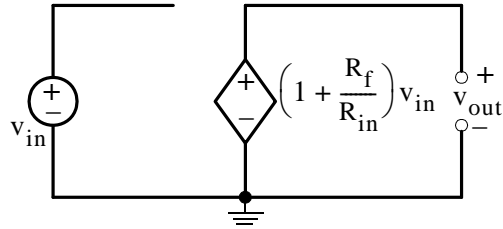


Figure 4.33. Equivalent circuit of the non-inverting op amp

### Example 4.9

If, in the non-inverting op amp circuit of the previous example, we replace  $R_{in}$  with an open circuit ( $R_{in} \rightarrow \infty$ ) and  $R_f$  with a short circuit ( $R_f \rightarrow 0$ ), prove that the voltage gain  $G_v$  is

$$G_v = \frac{v_{out}}{v_{in}} = 1 \quad (4.12)$$

and thus

$$v_{out} = v_{in} \quad (4.13)$$

Proof:

With  $R_{in}$  open and  $R_f$  shorted, the non-inverting amplifier of the previous example reduces to the circuit of Figure 4.34.

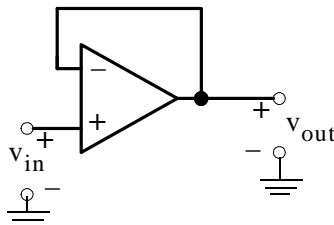


Figure 4.34. Circuit of Figure 4.32 with  $R_{in}$  open and  $R_f$  shorted

The voltage difference between the (+) and (-) terminals is zero; then  $v_{out} = v_{in}$ .

We will obtain the same result if we consider the non-inverting op amp gain  $G_v = 1 + R_f/R_{in}$ . Then, letting  $R_f \rightarrow 0$ , the gain reduces to  $G_v = 1$  and for this reason this circuit is called *unity gain amplifier* or *voltage follower*. It is also called *buffer amplifier* because it can be used to “buffer” (isolate) one circuit from another when one “loads” the other as we will see on the next example.

**Example 4.10**

For the circuit of Figure 4.35

- With the load  $R_{LOAD}$  disconnected, compute the open circuit voltage  $v_{ab}$
- With the load connected, compute the voltage  $v_{LOAD}$  across the load  $R_{LOAD}$
- Insert a buffer amplifier between  $a$  and  $b$  and compute the new voltage  $v_{LOAD}$  across the same load  $R_{LOAD}$

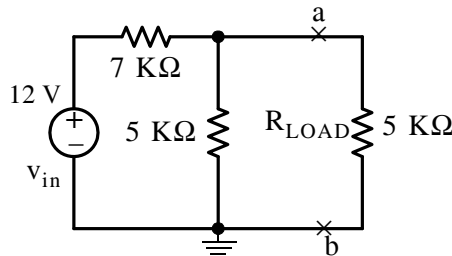


Figure 4.35. Circuit for Example 4.10

**Solution:**

- With the load  $R_{LOAD}$  disconnected the circuit is as shown in Figure 4.36.

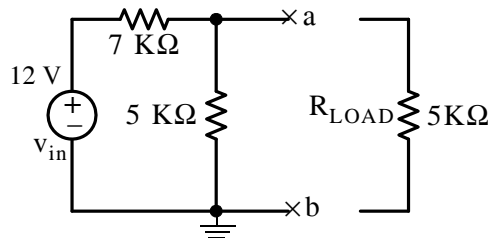


Figure 4.36. Circuit for Example 4.10 with the load disconnected

The voltage across terminals  $a$  and  $b$  is

$$v_{ab} = \frac{5 \text{ K}\Omega}{7 \text{ K}\Omega + 5 \text{ K}\Omega} \times 12 = 5 \text{ V}$$

- With the load  $R_{LOAD}$  reconnected the circuit is as shown in Figure 4.37. Then,

$$v_{LOAD} = \frac{5 \text{ K}\Omega \parallel 5 \text{ K}\Omega}{7 \text{ K}\Omega + 5 \text{ K}\Omega \parallel 5 \text{ K}\Omega} \times 12 = 3.16 \text{ V}$$

Here, we observe that the load  $R_{LOAD}$  “loads down” the load voltage from 5 V to 3.16 V and this voltage may not be sufficient for proper operation of the load.



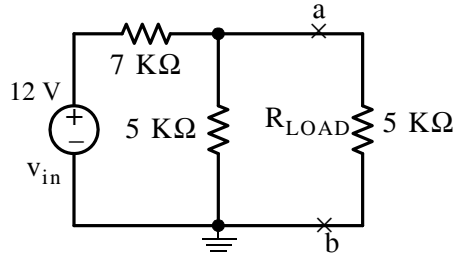


Figure 4.37. Circuit for Example 4.10 with the load reconnected

- c. With the insertion of the buffer amplifier between points *a* and *b* and the load, the circuit now is as shown in Figure 4.38.

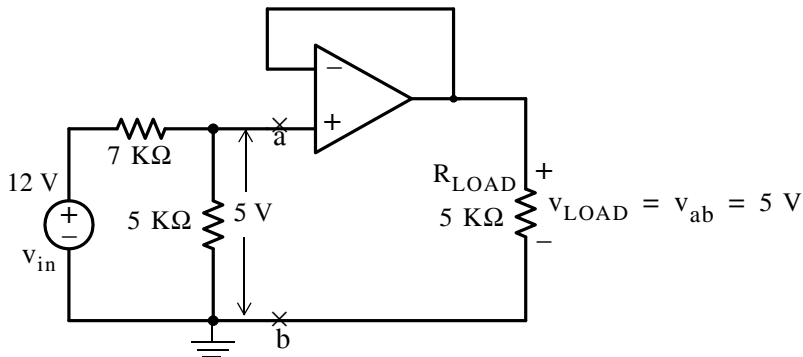


Figure 4.38. Circuit for Example 4.10 with the insertion of a buffer op amp

From the circuit of Figure 4.38, we observe that the voltage across the load is 5 V as desired.

### Example 4.11

The op amp circuit shown in Figure 4.39 is called *summing circuit* or *summer* because the output is the summation of the weighted inputs.

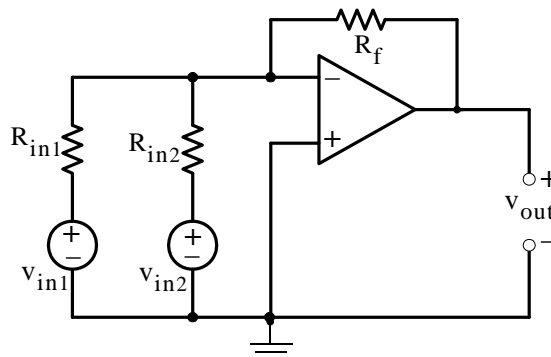


Figure 4.39. Two-input summing op amp circuit

Prove that for this circuit,

$$v_{out} = -R_f \left( \frac{v_{in1}}{R_{in1}} + \frac{v_{in2}}{R_{in2}} \right) \quad (4.14)$$

**Proof:**

We recall that the voltage across the (-) and (+) terminals is zero. We also observe that the (+) input is grounded, and thus the voltage at the (-) terminal is at “virtual ground”. Then, by application of KCL at the (-) terminal, we obtain

$$\frac{v_{in1}}{R_{in1}} + \frac{v_{in2}}{R_{in2}} + \frac{v_{out}}{R_f} = 0$$

and solving for  $v_{out}$  we obtain (4.14). Alternately, we can apply the principle of superposition to derive this relation.

**Example 4.12**

Compute the output voltage  $v_{out}$  for the amplifier circuit shown in Figure 4.40.

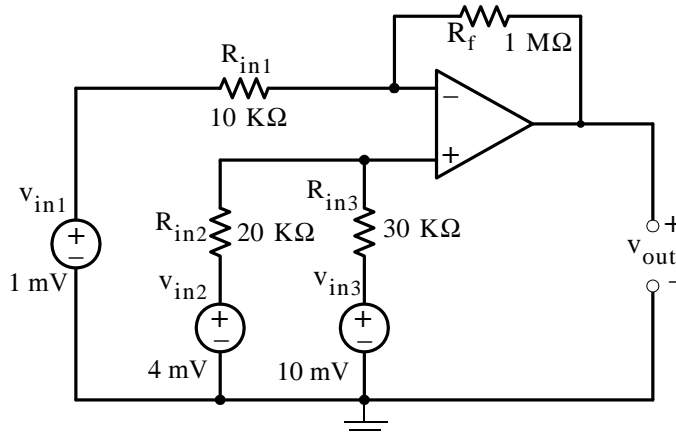


Figure 4.40. Circuit for Example 4.12

**Solution:**

Let  $v_{out1}$  be the output due to  $v_{in1}$  acting alone,  $v_{out2}$  be the output due to  $v_{in2}$  acting alone, and  $v_{out3}$  be the output due to  $v_{in3}$  acting alone. Then by superposition,

$$v_{out} = v_{out1} + v_{out2} + v_{out3}$$

First, with  $v_{in1}$  acting alone and  $v_{in2}$  and  $v_{in3}$  shorted, the circuit becomes as shown in Figure 4.41.

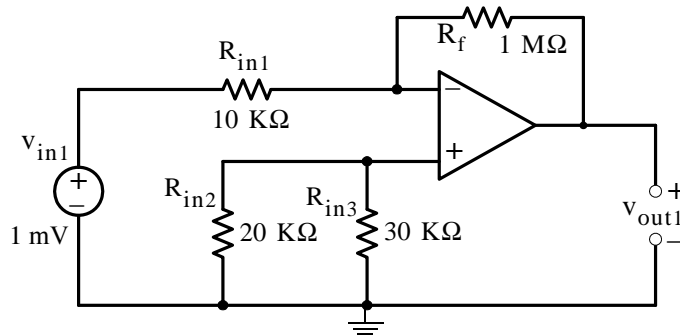


Figure 4.41. Circuit for Example 4.12 with  $v_{in1}$  acting alone

We recognize this as an inverting amplifier whose voltage gain  $G_v$  is

$$G_v = 1 \text{ M}\Omega / 10 \text{ K}\Omega = 100$$

and thus

$$v_{out1} = (100)(-1 \text{ mV}) = -100 \text{ mV} \quad (4.15)$$

Next, with  $v_{in2}$  acting alone and  $v_{in1}$  and  $v_{in3}$  shorted, the circuit becomes as shown in Figure 4.42.

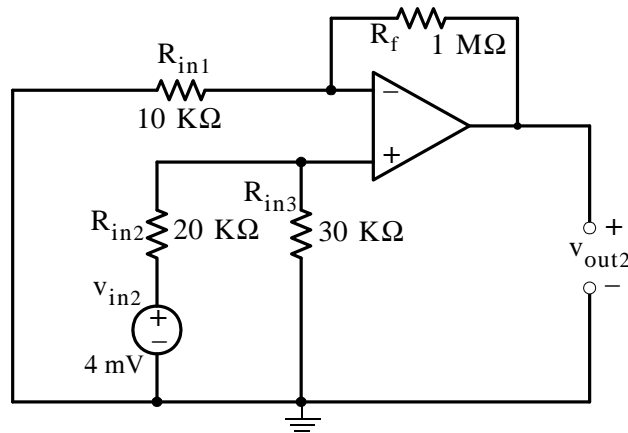


Figure 4.42. Circuit for Example 4.12 with  $v_{in2}$  acting alone

The circuit of Figure 4.42 as a non-inverting op amp whose voltage gain  $G_v$  is

$$G_v = 1 + 1 \text{ M}\Omega / 10 \text{ K}\Omega = 101$$

and the voltage at the plus (+) input is computed from the voltage divider circuit shown in Figure 4.43.

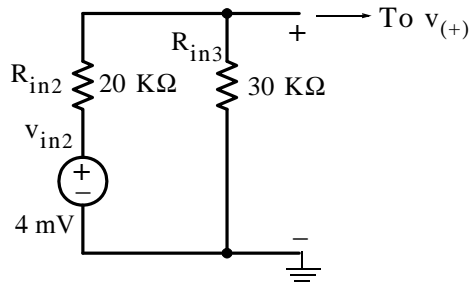


Figure 4.43. Voltage divider circuit for the computation of  $v_{(+)}$  with  $v_{in2}$  acting alone

Then,

$$v_{(+)} = \frac{R_{in3}}{R_{in2} + R_{in3}} \times v_{in2} = \frac{30 \text{ K}\Omega}{50 \text{ K}\Omega} \times 4 \text{ mV} = 2.4 \text{ mV}$$

and thus

$$v_{out2} = 101 \times 2.4 \text{ mV} = 242.4 \text{ mV} \quad (4.16)$$

Finally, with  $v_{in3}$  acting alone and  $v_{in1}$  and  $v_{in2}$  shorted, the circuit becomes as shown in Figure 4.44.

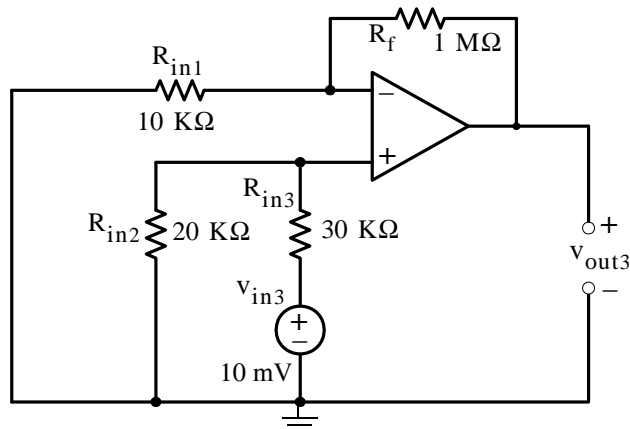


Figure 4.44. Circuit for Example 4.12 with  $v_{in3}$  acting alone

The circuit of Figure 4.44 is also a non-inverting op amp whose voltage gain  $G_v$  is

$$G_v = 1 + 1 \text{ M}\Omega / 10 \text{ K}\Omega = 101$$

and the voltage at the plus (+) input is computed from the voltage divider circuit shown in Figure 4.45.

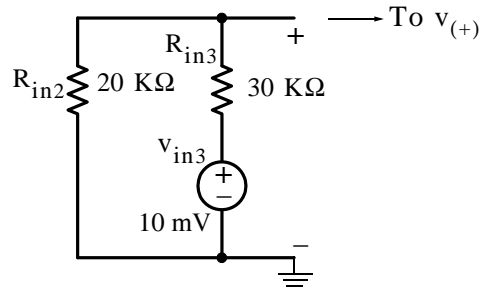


Figure 4.45. Voltage divider circuit for the computation of  $v_{(+)}$  with  $v_{in3}$  acting alone

Then,

$$v_{(+)} = \frac{R_{in2}}{R_{in2} + R_{in3}} \times v_{in2} = \frac{20 \text{ K}\Omega}{50 \text{ K}\Omega} \times 10 \text{ mV} = 4 \text{ mV}$$

and thus

$$v_{out3} = 101 \times 4 \text{ mV} = 404 \text{ mV} \quad (4.17)$$

Therefore, from (4.15), (4.16) and (4.17),

$$v_{out} = v_{out1} + v_{out2} + v_{out3} = -100 + 242.4 + 404 = 546.4 \text{ mV}$$

### Example 4.13

For the circuit shown in Figure 4.46, derive an expression for the voltage gain  $G_v$  in terms of the external resistors  $R_1$ ,  $R_2$ ,  $R_3$ , and  $R_f$ .

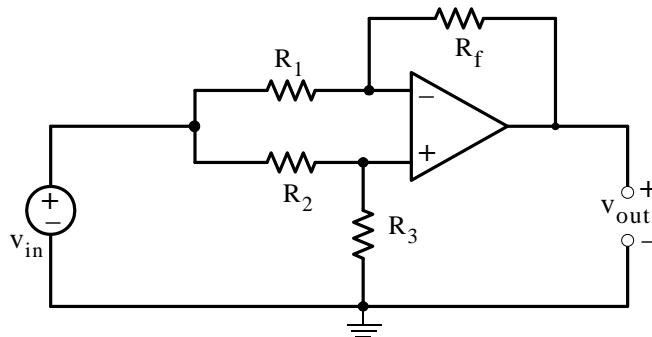


Figure 4.46. Circuit for Example 4.13

**Solution:**

We apply KCL at nodes  $v_1$  and  $v_2$  as shown in Figure 4.47.

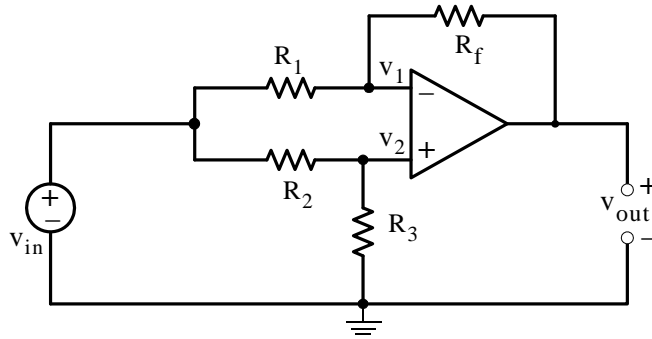


Figure 4.47. Application of KCL for the circuit of Example 4.13

At node  $v_1$ :

$$\frac{v_1 - v_{in}}{R_1} + \frac{v_1 - v_{out}}{R_f} = 0$$

or

$$\left( \frac{1}{R_1} + \frac{1}{R_f} \right) v_1 = \frac{v_{in}}{R_1} + \frac{v_{out}}{R_f}$$

or

$$\left( \frac{R_1 + R_f}{R_1 R_f} \right) v_1 = \frac{R_f v_{in} + R_1 v_{out}}{R_1 R_f}$$

or

$$v_1 = \frac{R_f v_{in} + R_1 v_{out}}{R_1 + R_f} \quad (4.18)$$

At node  $v_2$ :

$$\frac{v_2 - v_{in}}{R_2} + \frac{v_2}{R_3} = 0$$

or

$$v_2 = \frac{R_3 v_{in}}{R_2 + R_3} \quad (4.19)$$

and since  $v_2 = v_1$ , we rewrite (4.19) as

$$v_1 = \frac{R_3 v_{in}}{R_2 + R_3} \quad (4.20)$$

Equating the right sides of (4.18) and (4.20) we obtain

$$\frac{R_f v_{in} + R_1 v_{out}}{R_1 + R_f} = \frac{R_3 v_{in}}{R_2 + R_3}$$

or

$$R_f v_{in} + R_1 v_{out} = \frac{R_3 v_{in}}{R_2 + R_3} (R_1 + R_f)$$

Dividing both sides of the above relation by  $R_1 v_{in}$  and rearranging, we obtain

$$\frac{v_{out}}{v_{in}} = \frac{R_3(R_1 + R_f)}{R_1(R_2 + R_3)} - \frac{R_f}{R_1}$$

and after simplification

$$G_v = \frac{v_{out}}{v_{in}} = \frac{R_1 R_3 - R_2 R_f}{R_1(R_2 + R_3)} \quad (4.21)$$

### 4.9 Input and Output Resistance

The input and output resistances are very important parameters in amplifier circuits.

The *input resistance*  $R_{in}$  of a circuit is defined as the ratio of the applied voltage  $v_s$  to the current  $i_s$  drawn by the circuit, that is,

$$\boxed{R_{in} = \frac{v_s}{i_s}} \quad (4.22)$$

Therefore, in an op amp circuit the input resistance provides a measure of the current  $i_s$  which the amplifier draws from the voltage source  $v_s$ . Of course, we want  $i_s$  to be as small as possible; accordingly, we must make the input resistance  $R_{in}$  as high as possible.

---

#### Example 4.14

Compute the input resistance  $R_{in}$  of the inverting op amp amplifier shown in Figure 4.48 in terms of  $R_1$  and  $R_f$ .

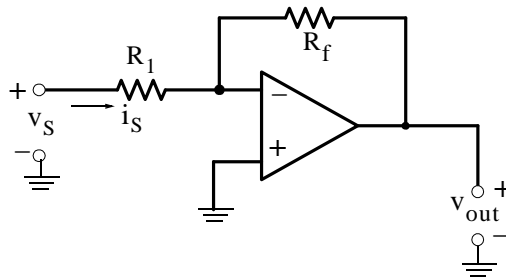


Figure 4.48. Circuit for Example 4.14

**Solution:**

By definition,

$$R_{in} = \frac{v_S}{i_S} \tag{4.23}$$

and since no current flows into the minus (-) terminal of the op amp and this terminal is at virtual ground, it follows that

$$i_S = \frac{v_S}{R_1} \tag{4.24}$$

From (4.23) and (4.24) we observe that

$$\boxed{R_{in} = R_1} \tag{4.25}$$

It is therefore, desirable to make  $R_1$  as high as possible. However, if we make  $R_1$  very high such as  $10\text{ M}\Omega$ , for a large gain, say 100, the value of the feedback resistor  $R_f$  should be  $1\text{ G}\Omega$ . Obviously, this is an impractical value. Fortunately, a large gain can be achieved with the circuit of Problem 8 at the end of this chapter.

**Example 4.15**

Compute the input resistance  $R_{in}$  of the op amp shown in Figure 4.49.

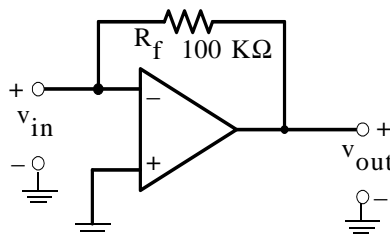


Figure 4.49. Circuit for Example 4.15



**Solution:**

In the circuit of Figure 4.49,  $v_{in}$  is the voltage at the minus (–) terminal; not the source voltage  $v_s$ . Therefore, there is no current  $i_s$  drawn by the op amp. In this case, we apply a test (hypothetical) current  $i_x$  as shown in Figure 4.49, and we treat  $v_{in}$  as the source voltage.

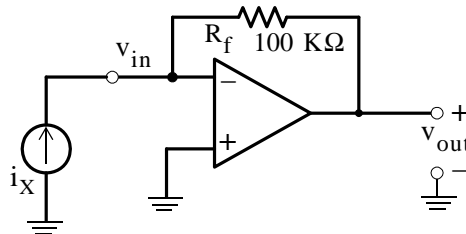


Figure 4.50. Circuit for Example 4.15 with a test current source

We observe that  $v_{in}$  is zero (virtual ground). Therefore,

$$R_{in} = \frac{v_{in}}{i_x} = \frac{0}{i_x} = 0$$

By definition, the output resistance  $R_{out}$  is the ratio of the open circuit voltage to the short circuit current, that is,

$$\boxed{R_{out} = \frac{v_{OC}}{i_{SC}}} \quad (4.26)$$

The output resistance  $R_{out}$  is not the same as the load resistance. The output resistance provides a measure of the change in output voltage when a load which is connected at the output terminals draws current from the circuit. It is desirable to have an op amp with very low output resistance as illustrated by the following example.

---

**Example 4.16**

The output voltage of an op amp decreases by 10% when a  $5 \text{ K}\Omega$  load is connected at the output terminals. Compute the output resistance  $R_{out}$ .

**Solution:**

Consider the output portion of the op amp shown in Figure 4.51.

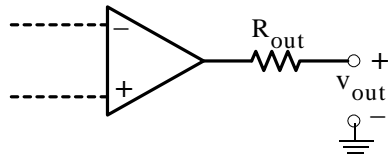


Figure 4.51. Partial circuit for Example 4.16

With no load connected at the output terminals,

$$v_{\text{out}} = v_{\text{OC}} = G_v v_{\text{in}} \quad (4.27)$$

With a load  $R_{\text{LOAD}}$  connected at the output terminals, the load voltage  $v_{\text{LOAD}}$  is

$$v_{\text{LOAD}} = \frac{R_{\text{LOAD}}}{R_{\text{out}} + R_{\text{LOAD}}} \times v_{\text{out}} \quad (4.28)$$

and from (4.27) and (4.28)

$$v_{\text{LOAD}} = \frac{R_{\text{LOAD}}}{R_{\text{out}} + R_{\text{LOAD}}} \times G_v v_{\text{in}} \quad (4.29)$$

Therefore,

$$\frac{v_{\text{LOAD}}}{v_{\text{OC}}} = 0.9 = \frac{5 \text{ K}\Omega}{R_{\text{out}} + 5 \text{ K}\Omega}$$

and solving for  $R_{\text{out}}$  we obtain

$$R_{\text{out}} = 555 \Omega$$

We observe from (4.29) that as  $R_{\text{out}} \rightarrow 0$ , relation (4.29) reduces to  $v_{\text{LOAD}} = G_v v_{\text{in}}$  and by comparison with (4.27), we see that  $v_{\text{LOAD}} = v_{\text{OC}}$

### 4.10 Summary

- A signal is any waveform representing a fluctuating electric quantity, such as voltage, current, electric or magnetic field strength, sound, image, or any message transmitted or received in telegraphy, telephony, radio, television, or radar. al that changes with time.
- An amplifier is an electronic circuit which increases the magnitude of the input signal.
- The gain of an amplifier is the ratio of the output to the input. It is normally expressed in decibel (dB) units where by definition  $\text{dB} = 10\log|p_{\text{out}}/p_{\text{in}}|$
- Frequency response is the band of frequencies over which the output remains fairly constant.
- The lower and upper cutoff frequencies are those where the output is 0.707 of its maximum value. They are also known as half-power points.
- Most amplifiers are used with feedback where the output, or portion of it, is fed back to the input.
- The operational amplifier (op amp) is the most versatile amplifier and its main features are:
  1. Very high input impedance (resistance)
  2. Very low output impedance (resistance)
  3. Capable of producing a very large gain that can be set to any value by connection of external resistors of appropriate values
  4. Frequency response from DC to frequencies in the MHz range
  5. Very good stability
  6. Operation to be performed, i.e., addition, integration etc. is done externally with proper selection of passive devices such as resistors, capacitors, diodes, and so on.
- The gain of an inverting op amp is the ratio  $-R_f/R_{\text{in}}$  where  $R_f$  is the feedback resistor which allows portion of the output to be fed back to the minus (-) input. The minus (-) sign implies that the output signal has opposite polarity from that of the input signal.
- The gain of a non-inverting op amp is  $1 + R_f/R_{\text{in}}$  where  $R_f$  is the feedback resistor which allows portion of the output to be fed back to the minus (-) input which is grounded through the  $R_{\text{in}}$  resistor. The output signal has the same polarity from that of the input signal.
- In a unity gain op amp the output is the same as the input. A unity gain op amp is used to provide a very high resistance between a voltage source and the load connected to it.
- Op amps are also used as active filters.

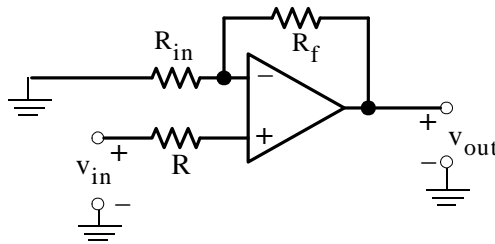
- A low-pass filter transmits (passes) all frequencies below a critical (cutoff) frequency denoted as  $\omega_c$  and attenuates (blocks) all frequencies above this cutoff frequency.
- A high-pass filter transmits (passes) all frequencies above a critical (cutoff) frequency  $\omega_c$ , and attenuates (blocks) all frequencies below the cutoff frequency.
- A band-pass filter transmits (passes) the band (range) of frequencies between the critical (cutoff) frequencies denoted as  $\omega_1$  and  $\omega_2$ , where the maximum value of  $G_v$  which is unity, falls to  $0.707 \times G_v$ , while it attenuates (blocks) all frequencies outside this band.
- A band-elimination or band-stop or band-rejection filter attenuates (rejects) the band (range) of frequencies between the critical (cutoff) frequencies denoted as  $\omega_1$  and  $\omega_2$ , where the maximum value of  $G_v$  which is unity, falls to  $0.707 \times G_v$ , while it transmits (passes) all frequencies outside this band.
- A summing op amp is a circuit with two or more inputs.
- The input resistance is the ratio of the applied voltage  $v_s$  to the current  $i_s$  drawn by the circuit, that is,  $R_{in} = v_s/i_s$
- The output resistance (not to be confused with the load resistance) is the ratio of the open circuit voltage when the load is removed from the circuit, to the short circuit current which is the current that flows through a short circuit connected at the output terminals, that is,  $R_o = v_{OC}/i_{SC}$

### 4.11 Exercises

#### Multiple Choice

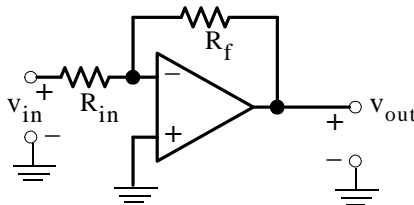
1. In the op amp circuit below  $v_{in} = 2\text{ V}$ ,  $R_{in} = 1\text{ k}\Omega$ , and it is desired to have  $v_{out} = 8\text{ V}$ . This will be obtained if the feedback resistor  $R_f$  has a value of

- A.  $1\text{ k}\Omega$
- B.  $2\text{ k}\Omega$
- C.  $3\text{ k}\Omega$
- D.  $4\text{ k}\Omega$
- E. none of the above



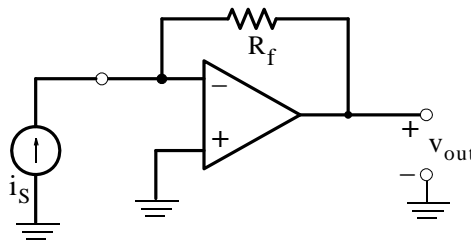
2. In the circuit below  $v_{in} = 6\text{ V}$ ,  $R_{in} = 2\text{ k}\Omega$ , and  $R_f = 3\text{ k}\Omega$ . Then  $v_{out}$  will be

- A.  $-9\text{ V}$
- B.  $9\text{ V}$
- C.  $-4\text{ V}$
- D.  $4\text{ V}$
- E. none of the above



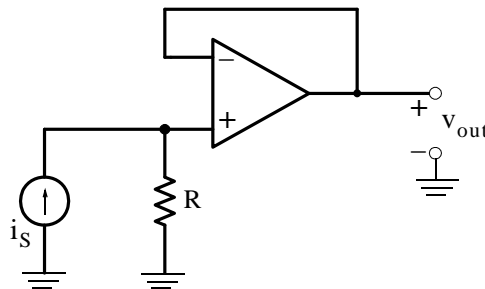
3. In the circuit below  $i_s = 2 \text{ mA}$  and  $R_f = 5 \text{ K}\Omega$ . Then  $v_{\text{out}}$  will be

- A.  $\infty \text{ V}$
- B.  $0 \text{ V}$
- C.  $10 \text{ V}$
- D.  $-10 \text{ V}$
- E. none of the above



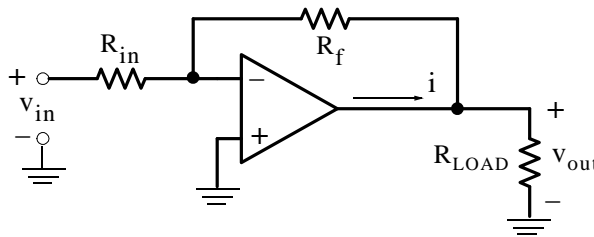
4. In the circuit below  $i_s = 4 \text{ mA}$  and  $R = 3 \text{ K}\Omega$ . Then  $v_{\text{out}}$  will be

- A.  $\infty \text{ V}$
- B.  $0 \text{ V}$
- C. indeterminate
- D.  $-12 \text{ V}$
- E. none of the above

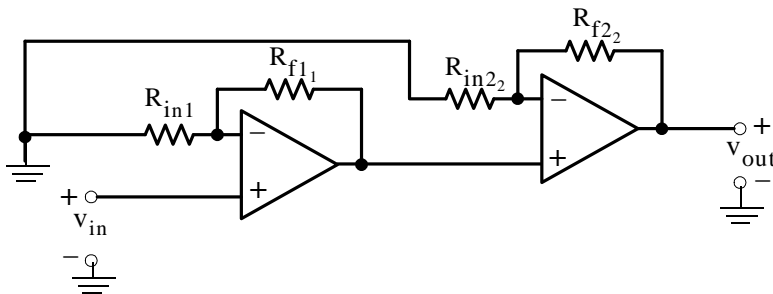


## Chapter 4 Introduction to Operational Amplifiers

5. In the circuit below  $v_{in} = 4 \text{ V}$ ,  $R_{in} = 12 \text{ K}\Omega$ ,  $R_f = 18 \text{ K}\Omega$ , and  $R_{LOAD} = 6 \text{ K}\Omega$ . Then  $i$  will be
- A.  $-1 \text{ mA}$
  - B.  $1 \text{ mA}$
  - C.  $-4/3 \text{ mA}$
  - D.  $4/3 \text{ mA}$
  - E. none of the above

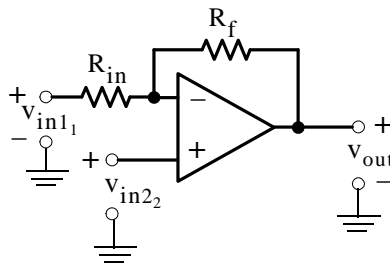


6. In the circuit below  $v_{in} = 1 \text{ V}$  and all resistors have the same value. Then  $v_{out}$  will be
- A.  $-2 \text{ V}$
  - B.  $2 \text{ V}$
  - C.  $-4 \text{ V}$
  - D.  $4 \text{ V}$
  - E. none of the above



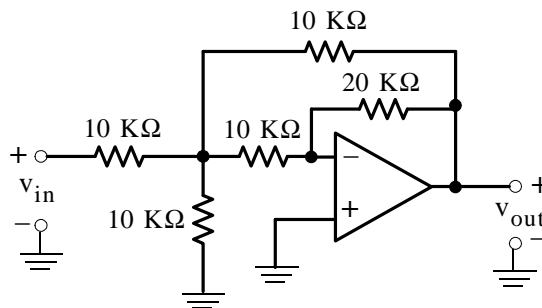
7. In the circuit below  $v_{in1} = 2\text{ V}$ ,  $v_{in2} = 4\text{ V}$ , and  $R_{in} = R_f = 1\text{ K}\Omega$ . Then  $v_{out}$  will be

- A.  $-2\text{ V}$
- B.  $2\text{ V}$
- C.  $-8\text{ V}$
- D.  $8\text{ V}$
- E. none of the above



8. In the circuit below  $v_{in} = 30\text{ mV}$ . Then  $v_{out}$  will be

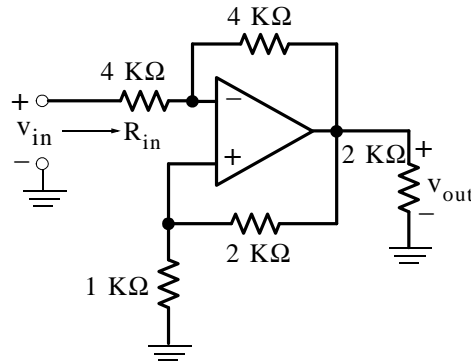
- A.  $-5\text{ mV}$
- B.  $-10\text{ mV}$
- C.  $-15\text{ mV}$
- D.  $-90\text{ mV}$
- E. none of the above





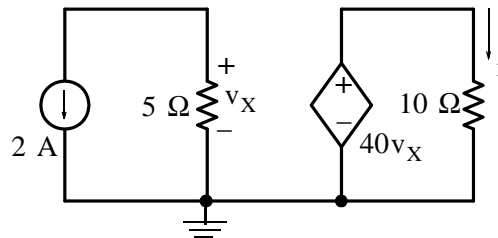
9. For the circuit below the input resistance  $R_{in}$  is

- A. 1 K $\Omega$
- B. 2 K $\Omega$
- C. 4 K $\Omega$
- D. 8 K $\Omega$
- E. none of the above



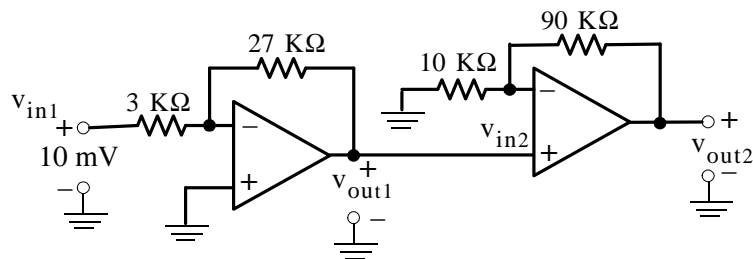
10. For the circuit below the current  $i$  is

- A. -40 A
- B. 40 A
- C. -400 A
- D. 400 A
- E. none of the above

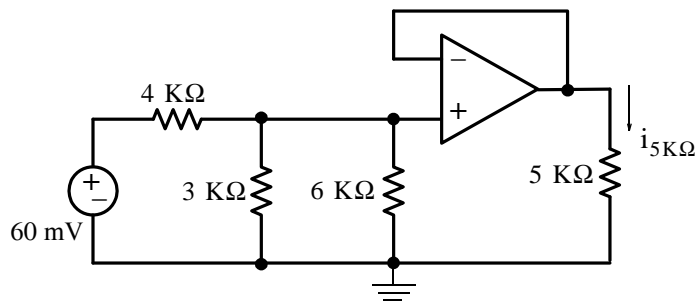


Problems

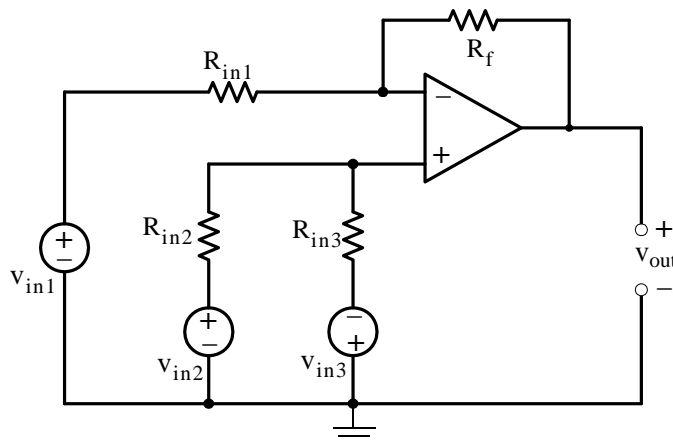
1. For the circuit below compute  $v_{out2}$ . Answer:  $-0.9\text{ V}$



2. For the circuit below compute  $i_{5K\Omega}$ . Answer:  $4\mu\text{A}$



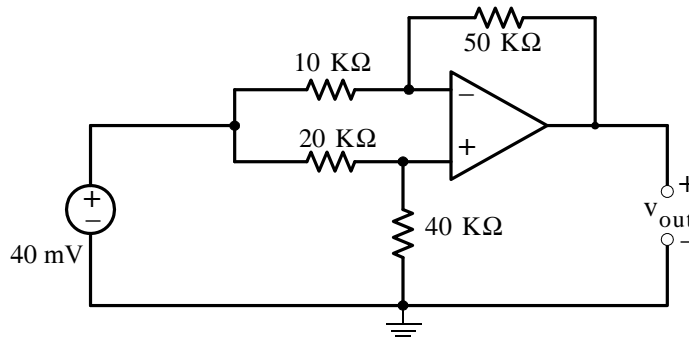
3. For the circuit below  $R_{in1}$ ,  $R_{in2}$ , and  $R_{in3}$  represent the internal resistances of the input voltages  $v_{in1}$ ,  $v_{in2}$ , and  $v_{in3}$  respectively. Derive an expression for  $v_{out}$  in terms of the input voltage sources and their internal resistances, and the feedback resistance  $R_f$ .



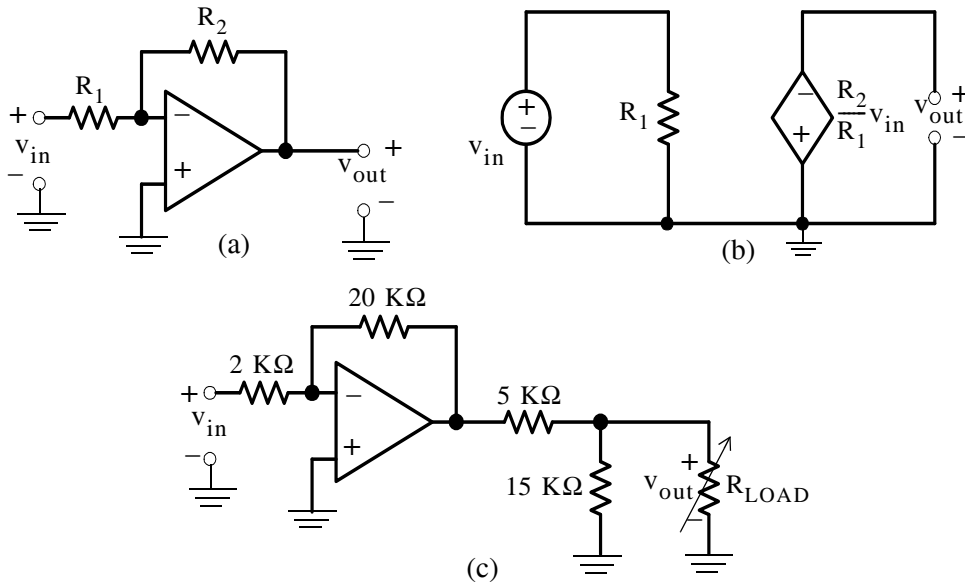
$$\text{Answer: } v_{out} = R_f \left( \frac{v_{in3}}{R_{in3}} - \frac{v_{in2}}{R_{in2}} - \frac{v_{in1}}{R_{in1}} \right)$$

## Chapter 4 Introduction to Operational Amplifiers

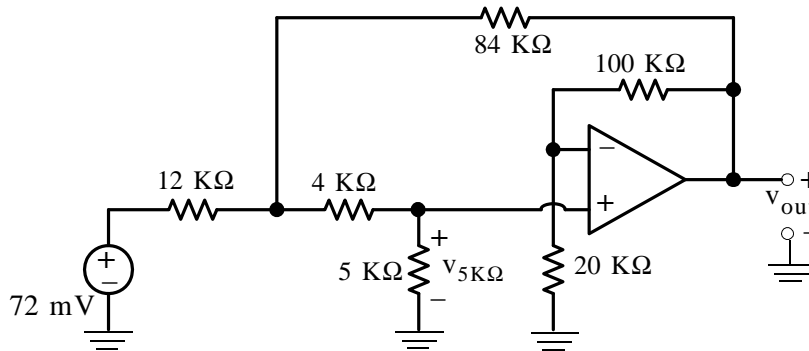
4. For the circuit below compute  $v_{out}$ . Answer:  $-40\text{ mV}$



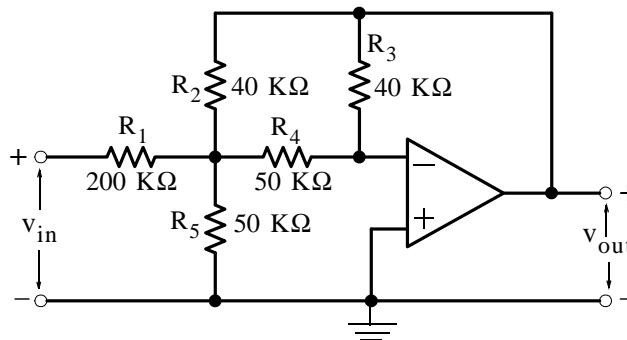
5. The op amp circuit (a) below can be represented by its equivalent circuit (b). For the circuit (c), compute the value of  $R_L$  so that it will receive maximum power. Answer:  $3.75\text{ k}\Omega$



6. For the circuit below compute  $v_{5\text{k}\Omega}$  using Thevenin's theorem. Answer:  $20\text{ mV}$

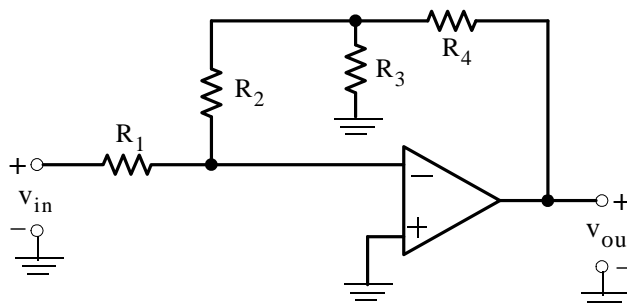


7. For the circuit below compute the gain  $G_v = v_{out}/v_{in}$ . Answer:  $-(2/37)$

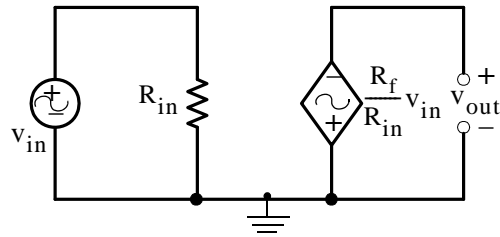


8. For the circuit below, show that the gain is given by

$$G_v = \frac{v_{out}}{v_{in}} = -\frac{1}{R_1} \left[ R_4 + R_2 \left( \frac{R_4}{R_3} + 1 \right) \right]$$



9. Create a Simulink / SimPowerSystems model for the equivalent circuit of the inverting op amp shown below.



$$v_{in} = 1 \text{ v peak} \quad f = 0.2 \text{ Hz} \quad \text{Phase} = 0 \text{ deg}$$

$$R_{in} = 1 \text{ K}\Omega \quad R_f = 10 \text{ K}\Omega$$

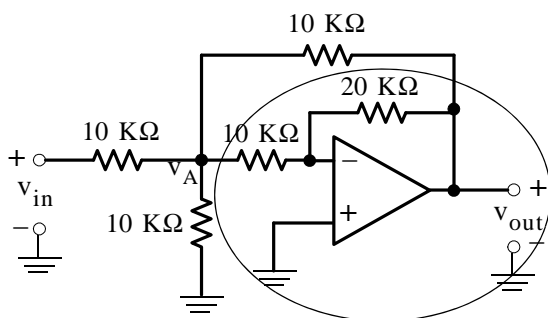
### 4.12 Answers / Solutions to End-of-Chapter Exercises

#### Multiple Choice

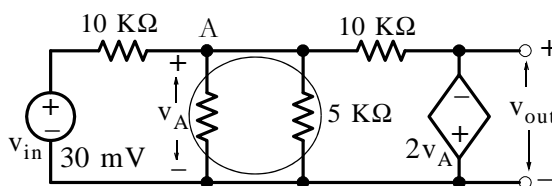
1. C For  $v_{in} = 2$  and  $v_{out} = 8$ , the gain must be  $G_v = 4$  or  $1 + R_f/R_{in} = 4$  and  $R_f = 3 \text{ K}\Omega$
2. A  $v_{out} = -R_f/R_{in} \times v_{in} = -9 \text{ V}$
3. D All current flows through  $R_f$  and the voltage drop across it is  $-(2 \text{ mA} \times 5 \text{ K}\Omega) = -10 \text{ V}$
4. E All current flows through  $R$  and the voltage drop across it is  $4 \text{ mA} \times 3 \text{ K}\Omega = 12 \text{ V}$ . Since this circuit is a unity gain amplifier, it follows that  $v_{out} = 12 \text{ V}$  also.
5. C  $v_{out} = -(18/12) \times 4 = -6 \text{ V}$ . Therefore,  $i_{LOAD} = v_{out}/R_{LOAD} = -6 \text{ V}/6 \text{ K}\Omega = -1 \text{ mA}$ . Applying KCL at the plus (+) terminal of  $v_{out}$  we obtain

$$i = \frac{-6 \text{ V}}{6 \text{ K}\Omega} + \frac{-6 \text{ V} - 4 \text{ V}}{18 \text{ K}\Omega + 12 \text{ K}\Omega} = -1 - \frac{1}{3} = -\frac{4}{3} \text{ mA}$$

6. D The gain of each of the non-inverting op amps is 2. Thus, the output of the first op amp is 2 V and the output of the second is 4 V.
7. E By superposition,  $v_{out1}$  due to  $v_{in1}$  acting alone is -2 V and  $v_{out2}$  due to  $v_{in2}$  acting alone is 8 V. Therefore,  $v_{out} = -2 + 8 = 6 \text{ V}$
8. B We assign node voltage  $v_A$  as shown below and we replace the encircled part by its equivalent.



We now attach the remaining resistors and the entire equivalent circuit is shown below.



Application of KCL at Node A yields

$$\frac{v_A - 30}{10} + \frac{v_A}{5} + \frac{v_A - (-2v_A)}{10} = 0$$

and thus

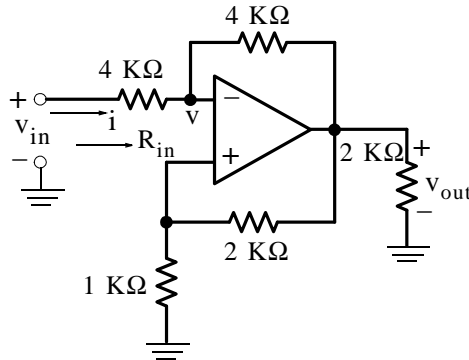
$$v_A = 30/6 = 5 \text{ mV}$$

Therefore,

$$v_{\text{out}} = -2v_A = -10 \text{ mV}$$

NOTE: For this circuit, the magnitude of the voltage is less than the magnitude of the input voltage. Therefore, this circuit is an attenuator, not an amplifier. Op amps are not configured for attenuation. This circuit is presented just for instructional purposes. A better and simpler attenuator is a voltage divider circuit.

9. C The voltage gain for this circuit is  $4 \text{ K}\Omega / 4 \text{ K}\Omega = 1$  and thus  $v_{\text{out}} = -v_{\text{in}}$ . The voltage  $v$  at the minus (-) input of the op amp is zero as proved below.



$$\frac{v - v_{\text{in}}}{4} + \frac{v - (-v_{\text{in}})}{4} = 0$$

or

$$v = 0$$

Then,

$$i = \frac{v_{\text{in}}}{4 \text{ K}\Omega}$$

and

$$R_{\text{in}} = \frac{v_{\text{in}}}{v_{\text{in}}/4 \text{ K}\Omega} = 4 \text{ K}\Omega$$

10. A For this circuit,  $v_X = -10 \text{ V}$  and thus  $40v_X = -400 \text{ V}$ . Then,  $i = -400/10 = -40 \text{ A}$

### Problems

1.

$$v_{\text{out1}} = -(27/3) \times 10 = -90 \text{ mV}$$

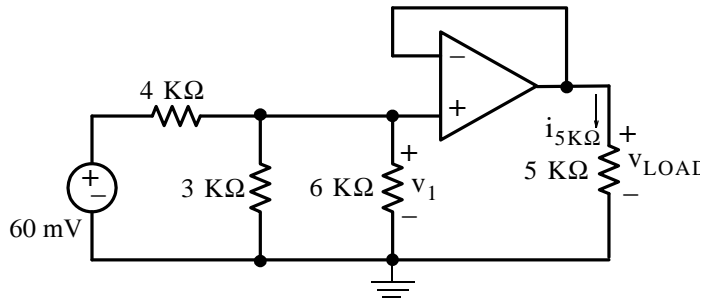
and thus

$$v_{in2} = v_{out1} = -90 \text{ mV}$$

Then,

$$v_{out2} = \left(1 + \frac{90}{10}\right) \times (-90) = -0.9 \text{ V}$$

2. We assign  $R_{LOAD}$ ,  $v_1$ , and  $v_{LOAD}$  as shown below.



$$3 \text{ K}\Omega \parallel 6 \text{ K}\Omega = 2 \text{ K}\Omega$$

and by the voltage division expression

$$v_1 = \frac{2 \text{ K}\Omega}{4 \text{ K}\Omega + 2 \text{ K}\Omega} \times 60 \text{ mV} = 20 \text{ mV}$$

and since this is a unity gain amplifier, we obtain

$$v_{LOAD} = v_1 = 20 \text{ mV}$$

Then,

$$i_{5\text{K}\Omega} = \frac{v_{LOAD}}{R_{LOAD}} = \frac{20 \text{ mV}}{5 \text{ K}\Omega} = \frac{20 \times 10^{-3}}{5 \times 10^3} = 4 \times 10^{-6} \text{ A} = 4 \mu\text{A}$$

3. By superposition

$$v_{out} = v_{out1} + v_{out2} + v_{out3}$$

where

$$v_{out1} \Big|_{\substack{v_{in2} = 0 \\ v_{in3} = 0}} = -\frac{R_f}{R_{in1}} v_{in1}$$

We observe that the minus (-) is a virtual ground and thus there is no current flow in  $R_{in1}$  and  $R_{in2}$ . Also,

$$v_{out2} \Big|_{\substack{v_{in1} = 0 \\ v_{in3} = 0}} = -\frac{R_f}{R_{in2}} v_{in2}$$

and

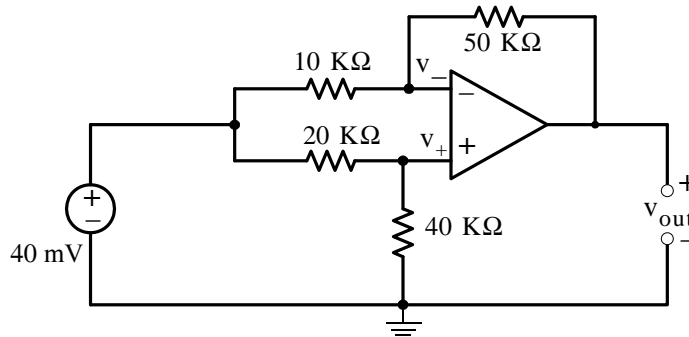


$$v_{out3} \Big|_{\substack{v_{in1} = 0 \\ v_{in2} = 0}} = -\frac{R_f}{R_{in3}}(-v_{in3})$$

Then,

$$v_{out} = R_f \left( \frac{v_{in3}}{R_{in3}} - \frac{v_{in2}}{R_{in2}} - \frac{v_{in1}}{R_{in1}} \right)$$

4. We assign voltages  $v_-$  and  $v_+$  as shown below.



At the minus (-) terminal

$$\frac{v_- - 40 \text{ mV}}{10 \text{ k}\Omega} + \frac{v_- - v_{out}}{50 \text{ k}\Omega} = 0$$

or

$$\frac{6}{50 \times 10^3} v_- - \frac{1}{50 \times 10^3} v_{out} = 4 \times 10^{-6}$$

At the plus (+) terminal

$$\frac{v_+ - 40 \text{ mV}}{20 \text{ k}\Omega} + \frac{v_+}{40 \text{ k}\Omega} = 0$$

or

$$\frac{3}{40 \times 10^3} v_+ = 2 \times 10^{-6}$$

or

$$v_+ = \frac{80 \times 10^{-3}}{3}$$

Since  $v_+ = v_-$  we equate the nodal equations and we obtain

$$\frac{6}{50 \times 10^3} \left( \frac{80 \times 10^{-3}}{3} \right) - \frac{1}{50 \times 10^3} v_{out} = 4 \times 10^{-6}$$

Multiplication by  $50 \times 10^3$  yields

$$\frac{2 \times 80 \times 10^{-3} \times 50 \times 10^3}{50 \times 10^3} - v_{\text{out}} = 4 \times 10^{-6} \times 50 \times 10^3$$

or

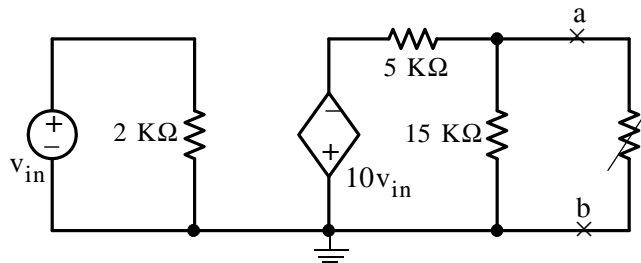
$$v_{\text{out}} = -40 \text{ mV}$$

Check using MATLAB:

R1=10000; R2=20000; R3=40000; Rf=50000; Vin=40\*10<sup>(-3)</sup>;  
 Vout=(R1\*R3-R2\*Rf)\*Vin/(R1\*(R2+R3))

$$V_{\text{out}} = -0.0400$$

5. We attach the 5 K $\Omega$ , 15 K $\Omega$ , and R<sub>LOAD</sub> resistors to the equivalent circuit as shown below.



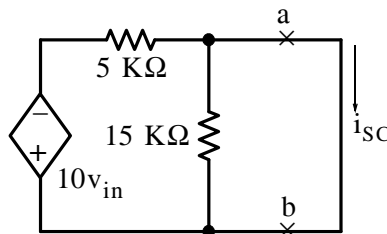
By Thevenin's theorem

$$v_{\text{TH}} = v_{\text{OC}} = v_{\text{ab}} = \frac{15 \text{ K}\Omega}{5 \text{ K}\Omega + 15 \text{ K}\Omega} (-10v_{\text{in}})$$

or

$$v_{\text{TH}} = -7.5v_{\text{in}}$$

Because the circuit contains a dependent source, we must compute the Thevenin resistance using the relation  $R_{\text{TH}} = v_{\text{TH}}/i_{\text{SC}}$  where  $i_{\text{SC}}$  is found from the circuit below.



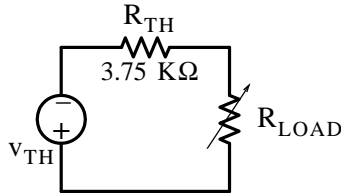
We observe that the short circuit shorts out the 15 K $\Omega$  and thus

$$i_{\text{SC}} = \frac{-10v_{\text{in}}}{5 \text{ K}\Omega} = -2 \times 10^{-3} v_{\text{in}}$$

Then,

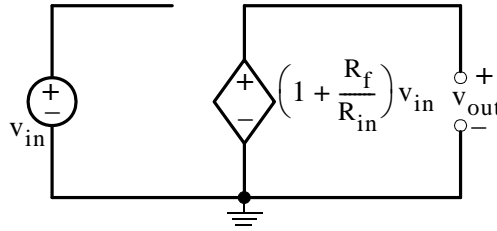
$$R_{TH} = \frac{-7.5v_{in}}{-2 \times 10^{-3}v_{in}} = 3.75 \text{ K}\Omega$$

and the Thevenin equivalent circuit is shown below.



Therefore, for maximum power transfer we must have  $R_{LOAD} = R_{TH} = 3.75 \text{ K}\Omega$

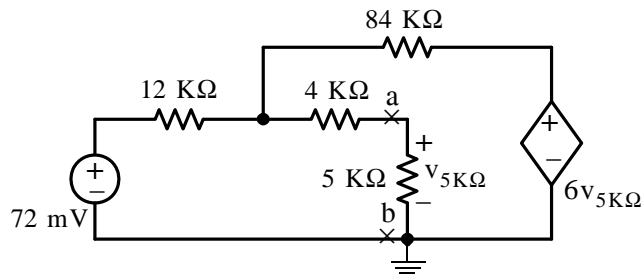
6. This is a non-inverting op amp whose equivalent circuit is shown below.



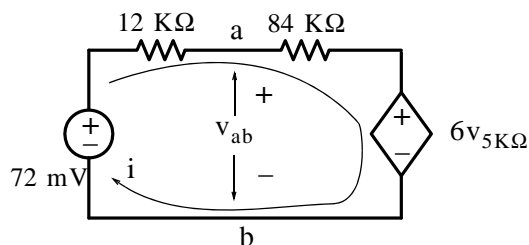
For this circuit  $v_{in} = v_{5K\Omega}$  and the value of the VCVS is

$$\left(1 + \frac{R_f}{R_{in}}\right)v_{5K\Omega} = \left(1 + \frac{100}{20}\right)v_{5K\Omega} = 6v_{5K\Omega}$$

Attaching the external resistors to the equivalent circuit above we obtain the circuit below.



To find the Thevenin equivalent at points  $a$  and  $b$  we disconnect the  $5 \text{ K}\Omega$  resistor. When this is done there is no current in the  $4 \text{ K}\Omega$  and the circuit simplifies to the one shown below.



By KVL

$$(12 \text{ K}\Omega + 84 \text{ K}\Omega)i + 6v_{5\text{K}\Omega} = 72 \text{ mV}$$

or

$$i = \frac{72 \text{ mV} - 6v_{5\text{K}\Omega}}{(12 \text{ K}\Omega + 84 \text{ K}\Omega)}$$

Also,

$$\begin{aligned} v_{\text{TH}} = v_{\text{ab}} = v_{5\text{K}\Omega} &= 72 \text{ mV} - (12 \text{ K}\Omega)i = 72 \text{ mV} - 12 \text{ K}\Omega \left( \frac{72 \text{ mV} - 6v_{5\text{K}\Omega}}{96 \text{ K}\Omega} \right) \\ &= 72 \text{ mV} - 9 \text{ mV} + \frac{3}{4}v_{5\text{K}\Omega} \end{aligned}$$

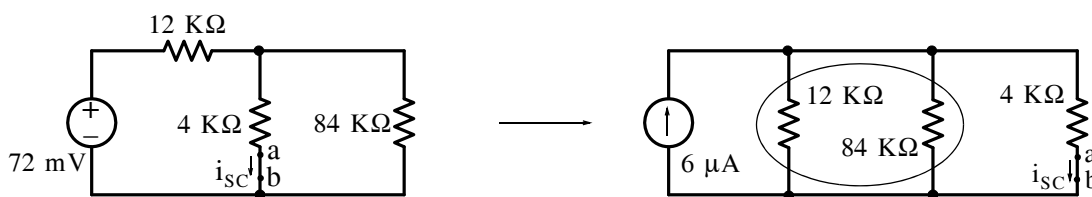
or

$$v_{5\text{K}\Omega} - \frac{3}{4}v_{5\text{K}\Omega} = 63 \text{ mV}$$

and thus

$$v_{\text{TH}} = v_{\text{ab}} = v_{5\text{K}\Omega} = 252 \text{ mV}$$

The Thevenin resistance is found from  $R_{\text{TH}} = v_{\text{OC}}/i_{\text{SC}}$  where  $i_{\text{SC}}$  is computed with the terminals  $a$  and  $b$  shorted making  $v_{5\text{K}\Omega} = 0$  and the circuit is as shown on the left below. We also perform voltage-source to current-source transformation and we obtain the circuit on the right below.



Now,

$$12 \text{ K}\Omega \parallel 84 \text{ K}\Omega = 10.5 \text{ K}\Omega$$

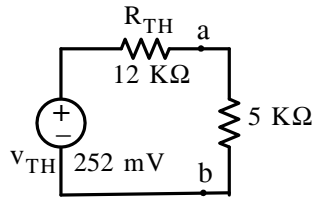
and by the current division expression

$$i_{\text{SC}} = i_{\text{ab}} = \frac{10.5 \text{ K}\Omega}{10.5 \text{ K}\Omega + 4 \text{ K}\Omega} \times 6 \mu\text{A} = \frac{126}{29} \mu\text{A}$$

Therefore,

$$R_{\text{TH}} = \frac{v_{\text{OC}}}{i_{\text{SC}}} = \frac{252}{126/29} = 58 \text{ K}\Omega$$

and the Thevenin equivalent circuit with the 5 KΩ resistor is shown below.



Finally,

$$v_{5\text{K}\Omega} = \frac{5}{58 + 5} \times 252 = 20 \text{ mV}$$

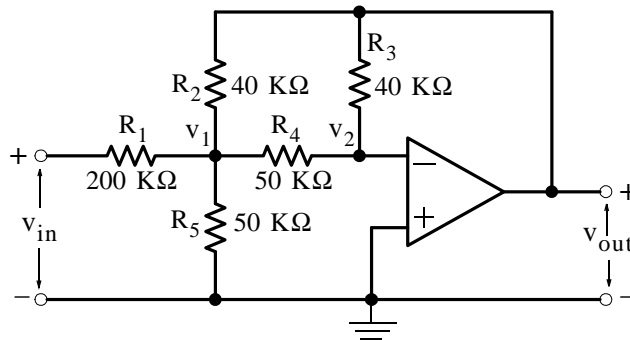
7. We assign node voltages  $v_1$  and  $v_2$  as shown below and we write node equations observing that  $v_2 = 0$  (virtual ground).

Node 1:

$$\frac{v_1 - v_{in}}{200 \text{ K}\Omega} + \frac{v_1 - v_{out}}{40 \text{ K}\Omega} + \frac{v_1 - 0}{50 \text{ K}\Omega} + \frac{v_1}{50 \text{ K}\Omega} = 0$$

or

$$\left( \frac{1}{200 \text{ K}\Omega} + \frac{1}{40 \text{ K}\Omega} + \frac{1}{50 \text{ K}\Omega} + \frac{1}{50 \text{ K}\Omega} \right) v_1 = \frac{v_{in}}{200 \text{ K}\Omega} + \frac{v_{out}}{40 \text{ K}\Omega}$$



Multiplication of each term by 200 KΩ and simplification yields

$$v_1 = \frac{1}{14}(v_{in} + 5v_{out})$$

Node 2:

$$\frac{0 - v_1}{50 \text{ K}\Omega} + \frac{0 - v_{out}}{40 \text{ K}\Omega} = 0$$

or

$$v_1 = -\frac{5}{4}v_{out}$$

Equating the right sides we obtain

$$\frac{1}{14}(v_{in} + 5v_{out}) = -\frac{5}{4}v_{out}$$

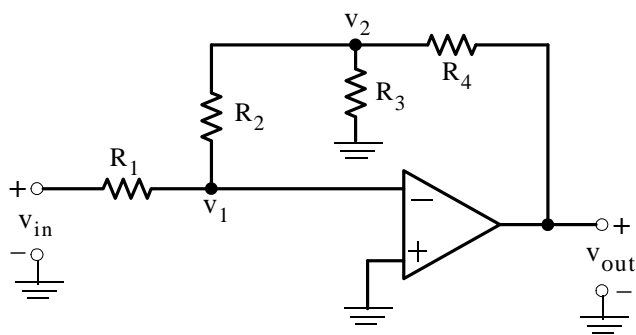
or

$$\frac{37}{28}v_{out} = -\frac{1}{14}v_{in}$$

Simplifying and dividing both sides by  $v_{in}$  we obtain

$$G_v = \frac{v_{out}}{v_{in}} = -\frac{2}{37}$$

8. We assign node voltages  $v_1$  and  $v_2$  as shown below and we write node equations observing that  $v_1 = 0$  (virtual ground).



Node 1:

$$\frac{0 - v_{in}}{R_1} + \frac{0 - v_2}{R_2} = 0$$

or

$$v_2 = -\frac{R_2}{R_1}v_{in}$$

Node 2:

$$\frac{v_2 - 0}{R_2} + \frac{v_2}{R_3} + \frac{v_2 - v_{out}}{R_4} = 0$$

or

$$\left(\frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}\right)v_2 = \frac{v_{out}}{R_4}$$

or

$$v_2 = \frac{1}{R_4/R_2 + R_4/R_3 + 1}v_{out}$$

Equating the right sides we obtain

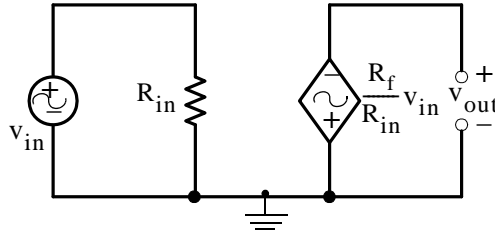
$$\frac{1}{R_4/R_2 + R_4/R_3 + 1}v_{out} = -\frac{R_2}{R_1}v_{in}$$

Simplifying and dividing both sides by  $v_{in}$  we obtain

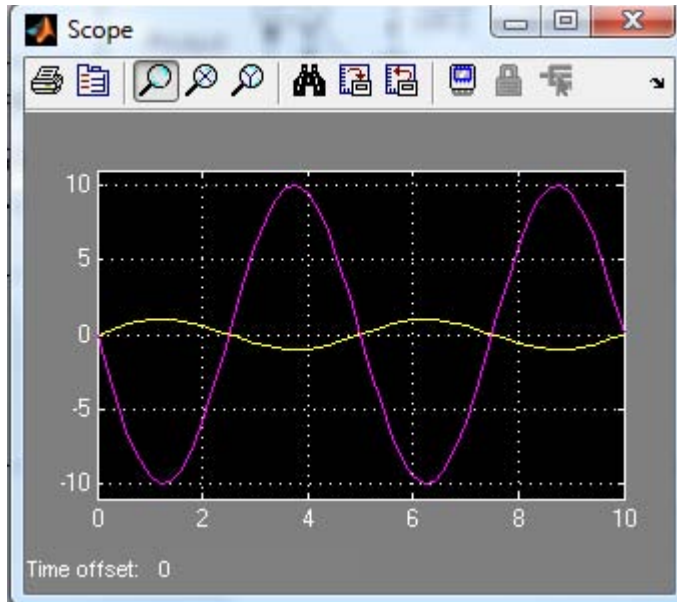
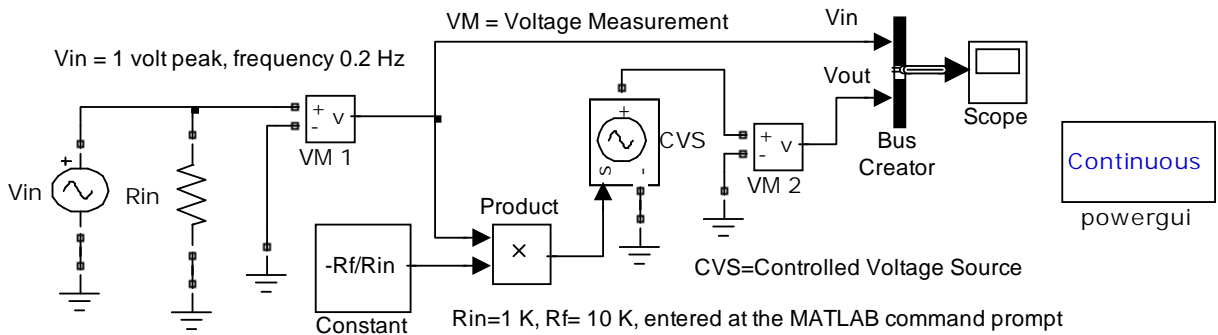
## Chapter 4 Introduction to Operational Amplifiers

$$G_v = \frac{V_{out}}{v_{in}} = -\frac{1}{R_1} \left[ R_4 + R_2 \left( \frac{R_4}{R_3} + 1 \right) \right]$$

9.



$v_{in} = 1 \text{ v peak}$      $f = 0.2 \text{ Hz}$     Phase = 0 deg     $R_{in} = 1 \text{ K}\Omega$      $R_f = 10 \text{ K}\Omega$



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# Chapter 5

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## Inductance and Capacitance

This chapter is an introduction to inductance and capacitance, their voltage–current relationships, power absorbed, and energy stored in inductors and capacitors. Procedures for analyzing circuits with inductors and capacitors are presented along with several examples.

### 5.1 Energy Storage Devices

In the first four chapters we considered resistive circuits only, that is, circuits with resistors and constant voltage and current sources. However, resistance is not the only property that an electric circuit possesses; in every circuit there are two other properties present and these are the inductance and the capacitance. We will see through some examples that will be presented later in this chapter, that inductance and capacitance have an effect on an electric circuit as long as there are changes in the voltages and currents in the circuit.

The effects of the inductance and capacitance properties can best be stated in simple differential equations since they involve the changes in voltage or current with time. We will study inductance first.

### 5.2 Inductance

Inductance is associated with the magnetic field which is always present when there is an electric current. Thus, when current flows in an electric circuit the conductors (wires) connecting the devices in the circuit are surrounded by a magnetic field. Figure 5.1 shows a simple loop of wire and its magnetic field represented by the small loops.

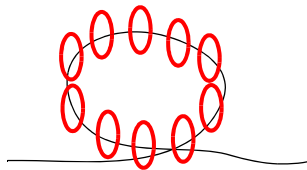


Figure 5.1. Magnetic field around a loop of wire

The direction of the magnetic field (not shown) can be determined by the left–hand rule if conventional current flow is assumed, or by the right–hand rule if electron current flow is assumed. The magnetic field loops are circular in form and are referred to as *lines of magnetic flux*. The unit of magnetic flux is the *weber* (Wb).



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## Chapter 5 Inductance and Capacitance

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In a loosely wound coil of wire such as the one shown in Figure 5.2, the current through the wound coil produces a denser magnetic field and many of the magnetic lines link the coil several times.

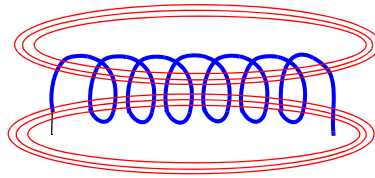


Figure 5.2. Magnetic field around several loops of wire

The magnetic flux is denoted as  $\phi$  and, if there are  $N$  turns and we assume that the flux  $\phi$  passes through each turn, the total flux, denoted as  $\lambda$ , is called *flux linkage*. Then,

$$\lambda = N\phi \quad (5.1)$$

Now, we define a linear inductor one in which the flux linkage is proportional to the current through it, that is,

$$\lambda = Li \quad (5.2)$$

where the constant of proportionality  $L$  is called *inductance* in webers per ampere.

We also recall Faraday's law of electromagnetic induction which states that

$$v = \frac{d\lambda}{dt} \quad (5.3)$$

and from (5.2) and (5.3),

$$v = L \frac{di}{dt} \quad (5.4)$$

Alternately, the inductance  $L$  is defined as the constant which relates the voltage across and the current through a device called *inductor* by the relation of (5.4).

The symbol and the voltage–current\* designations for the inductor are shown in Figure 5.3.

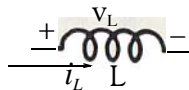


Figure 5.3. Symbol for inductor

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\* In the first four chapters we have used the subscript **LOAD** to denote a voltage across a load, a current through a load, and the resistance of a such load as  $R_{\text{LOAD}}$  to avoid confusion with the subscript **L** which henceforth will denote inductance. We will continue using the subscript **LOAD** for any load connected to a circuit.

For an inductor, the voltage–current relationship is

$$v_L = L \frac{di_L}{dt} \quad (5.5)$$

where  $v_L$  and  $i_L$  have the indicated polarity and direction. Obviously,  $v_L$  has a non-zero value only when  $i_L$  changes with time.

The unit of inductance is the *Henry* abbreviated as H. Since

$$L = \frac{v_L}{\frac{di_L}{dt}} = \frac{\text{volts}}{\frac{\text{amperes}}{\text{seconds}}} \quad (5.6)$$

we can say that *one henry is the inductance in a circuit in which a voltage of one volt is induced by a current changing at the rate of one ampere per second.*

By separation of the variables we rewrite (5.5) as

$$di_L = \frac{1}{L} v_L dt \quad (5.7)$$

and integrating both sides we obtain:

$$\int_{i_L(t_0)}^{i(t)} di_L = \frac{1}{L} \int_{t_0}^t v_L dt$$

or

$$i_L(t) - i_L(t_0) = \frac{1}{L} \int_{t_0}^t v_L dt$$

or

$$i_L(t) = \frac{1}{L} \int_{t_0}^t v_L dt + i_L(t_0) \quad (5.8)$$

where  $i_L(t_0)$ , more often denoted as  $i_L(0)$ , is the current flowing through the inductor at some reference time usually taken as  $t = 0$ , and it is referred to as the *initial condition*.

We can also express (5.8) as

$$i_L(t) = \frac{1}{L} \int_{-\infty}^t v_L dt = \frac{1}{L} \int_{-\infty}^0 v_L dt + \frac{1}{L} \int_0^t v_L dt \quad (5.9)$$

where the first integral on the right side represents the initial condition.

### Example 5.1

The current  $i_L(t)$  passing through a 50 mH inductor is shown in Figure 5.4.

- a. Compute the flux linkage  $\lambda$  at  $t = 2, 5, 9,$  and  $11$  ms
- b. Compute and sketch the voltage  $v_L(t)$  for the time interval  $-\infty < t < 14$  ms

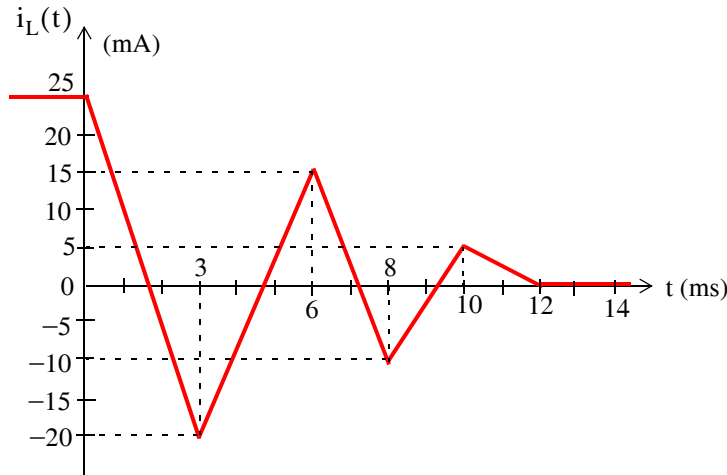


Figure 5.4. Waveform for Example 5.1

### Solution:

- a. The flux linkage  $\lambda$  is directly proportional to the current; then from (5.1) and (5.2)

$$\lambda = N\phi = Li$$

Therefore, we need to compute the current  $i$  at  $t = 2$  ms,  $t = 5$  ms,  $t = 9$  ms, and  $t = 11$  ms

For time interval  $0 < t < 3$  ms,  $i = mt + b$  where  $m$  is the slope of the straight line segment, and  $b$  is the  $i$ -axis intercept which, by inspection, is 25 mA. The slope  $m$  is

$$m = \frac{-20 - 25}{3 - 0} = -15$$

and thus

$$i|_{t=0}^{3 \text{ ms}} = -15t + 25 \quad (5.10)$$

At  $t = 2$  ms, (5.10) yields  $i = -5$  mA. Then, the flux linkage is

$$\lambda = Li = 50 \times 10^{-3} \times (-5) \times 10^{-3}$$

and

$$\lambda|_{t=2 \text{ ms}} = -250 \mu\text{Wb} \quad (5.11)$$

For the time interval  $3 < t < 6$  ms,  $i = mt + b$  where

$$m = \frac{15 - (-20)}{3 - 0} = \frac{35}{3}$$

and thus

$$i = \frac{35}{3}t + b$$

To find  $b$  we use the fact that at  $t = 3$  ms,  $i = -20$  mA as seen in Figure 5.4. Then,

$$-20 = \frac{35}{3} \times 3 + b$$

from which  $b = -55$ .

Thus, the straight line equation for the time interval  $3 < t < 6$  ms is

$$i|_{t=3 \text{ ms}}^{6 \text{ ms}} = \frac{35}{3}t - 55 \quad (5.12)$$

and therefore at  $t = 5$  ms,  $i = 10/3$  mA, and the flux linkage is

$$\lambda = Li = 50 \times 10^{-3} \times \frac{10}{3} \times 10^{-3}$$

or

$$\lambda|_{t=5 \text{ ms}} = \frac{500}{3} \mu\text{Wb} \quad (5.13)$$

Using the same procedure we find that

$$i|_{t=6 \text{ ms}}^{8 \text{ ms}} = -12.5t + 90 \quad (5.14)$$

Also,

$$i|_{t=8 \text{ ms}}^{10 \text{ ms}} = 7.5t - 70 \quad (5.15)$$

and with (5.15),

$$\lambda|_{t=9 \text{ ms}} = Li = -125 \mu\text{Wb} \quad (5.16)$$

Likewise,

$$i|_{t=10 \text{ ms}}^{12 \text{ ms}} = -2.5t + 30 \quad (5.17)$$

and with (5.17),

$$\lambda|_{t=11 \text{ ms}} = Li = 125 \mu\text{Wb} \quad (5.18)$$

b. Since

$$v_L = L \frac{di_L}{dt}$$

to compute and sketch the voltage  $v_L(t)$  for the time interval  $-\infty < t < 14$  ms, we only need to differentiate, that is, compute the slope of the straight line segments for this interval. These were found in part (a) as (5.10), (5.12), (5.14), (5.15), and (5.17). Then,

$$\text{slope}|_{-\infty < t < 0} = 0$$

$$v_L|_{-\infty < t < 0} = L \times \text{slope} = 0 \quad (5.19)$$

$$\text{slope}|_{0 < t < 3 \text{ ms}} = -15 \text{ mA/ms} = -15 \text{ A/s}$$

$$v_L|_{0 < t < 3 \text{ ms}} = L \times \text{slope} = 50 \times 10^{-3} \frac{\text{V}}{\text{A/s}} \times (-15 \text{ A/s}) = -750 \text{ mV} \quad (5.20)$$

$$\text{slope}|_{3 < t < 6 \text{ ms}} = 35/3 \text{ mA/ms} = 35/3 \text{ A/s}$$

$$v_L|_{3 < t < 6 \text{ ms}} = L \times \text{slope} = 50 \times 10^{-3} \times (35/3) = 583.3 \text{ mV} \quad (5.21)$$

$$\text{slope}|_{6 < t < 8 \text{ ms}} = -12.5 \text{ mA/ms} = -12.5 \text{ A/s}$$

$$v_L|_{6 < t < 8 \text{ ms}} = L \times \text{slope} = 50 \times 10^{-3} \times (-12.5) = -625 \text{ mV} \quad (5.22)$$

$$\text{slope}|_{8 < t < 10 \text{ ms}} = 7.5 \text{ mA/ms} = 7.5 \text{ A/s}$$

$$v_L|_{8 < t < 10 \text{ ms}} = L \times \text{slope} = 50 \times 10^{-3} \times 7.5 = 375 \text{ mV} \quad (5.23)$$

$$\text{slope}|_{10 < t < 12 \text{ ms}} = -2.5 \text{ mA/ms} = -2.5 \text{ A/s}$$

$$v_L|_{10 < t < 12 \text{ ms}} = L \times \text{slope} = 50 \times 10^{-3} \times (-2.5) = -125 \text{ mV} \quad (5.24)$$

$$\text{slope}|_{12 < t < 14 \text{ ms}} = 0$$

$$v_L|_{12 < t < 14 \text{ ms}} = L \times \text{slope} = 0 \quad (5.25)$$

We now have all values given by (5.19) through (5.25) to sketch  $v_L$  as a function of time. We can do this easily with a spreadsheet such as Excel as shown in Figure 5.5.

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### Example 5.2

The voltage across a 50 mH inductor is as shown on the waveform of Figure 5.6, and it is given that the initial condition is  $i_L(t_0) = i_L(0) = 25 \text{ mA}$ . Compute and sketch the current which flows through this inductor in the interval  $-5 < t < 5 \text{ ms}$

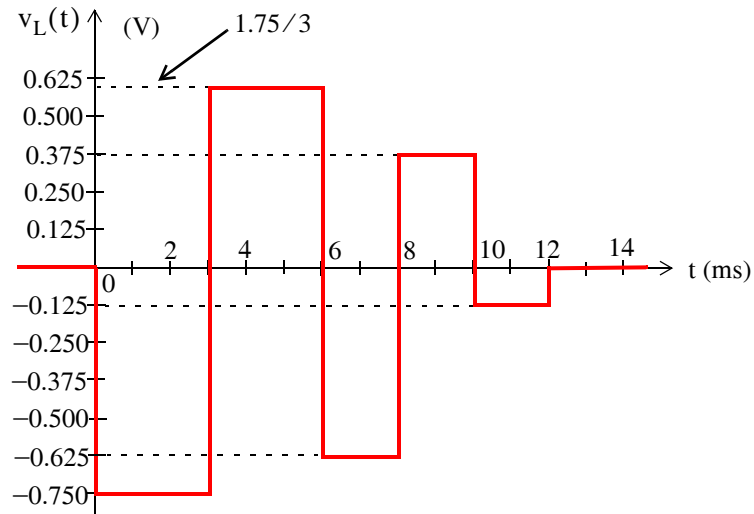


Figure 5.5. Voltage waveform for Example 5.1

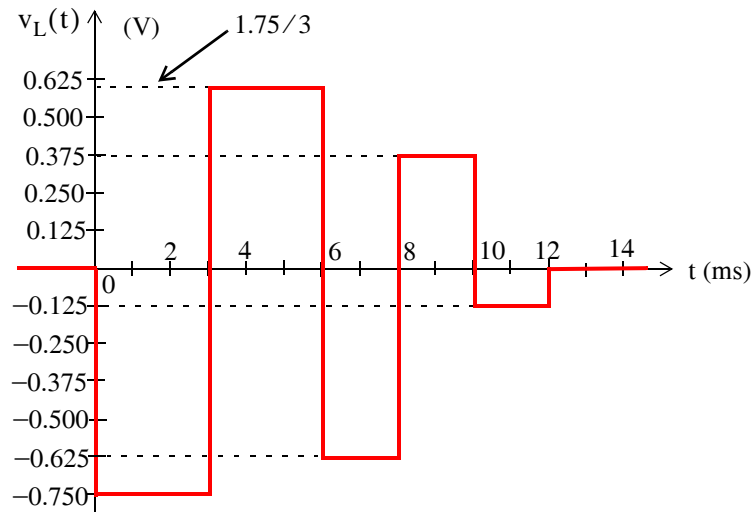


Figure 5.6. Waveform for Example 5.2

**Solution:**

The current  $i_L(t)$  in an inductor is related to the voltage  $v_L(t)$  by (5.8) which is repeated here for convenience.

$$i_L(t) = \frac{1}{L} \int_{t_0}^t v_L dt + i_L(t_0)$$

where  $i_L(t_0) = i_L(0) = 25 \text{ mA}$  is the initial condition, that is,

$$i_L|_{-\infty < t < 0} = 25 \text{ mA}$$

## Chapter 5 Inductance and Capacitance

From the given waveform,

$$v_L|_{0 < t < 3 \text{ ms}} = -0.75 \text{ V}$$

Then,

$$\begin{aligned} i_L|_{0 < t < 3 \text{ ms}} &= \frac{1}{50 \times 10^{-3}} \int_0^{3 \text{ ms}} (-0.75) dt + 25 \times 10^{-3} \\ &= 20 \left( -0.75 t \Big|_0^{3 \times 10^{-3}} \right) + 25 \times 10^{-3} = 20(-2.25 \times 10^{-3}) + 20 \times 0 + 25 \times 10^{-3} \\ &= -45 \times 10^{-3} + 25 \times 10^{-3} = -20 \times 10^{-3} = -20 \text{ mA} \end{aligned}$$

that is, the current has dropped linearly from 25 mA at  $t = 0$  to  $-20$  mA at  $t = 3$  ms as shown in Figure 5.7.

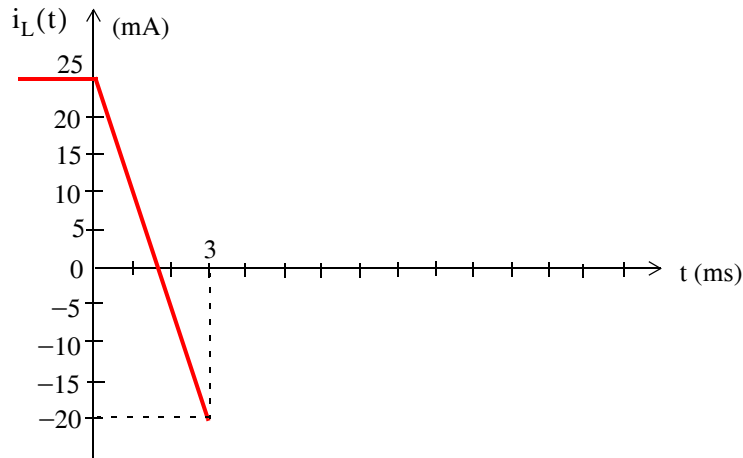


Figure 5.7. Inductor current for  $0 < t < 3$  ms, Example 5.2

The same result can be obtained by graphical integration. Thus,

$$\begin{aligned} i_L|_{t=3 \text{ ms}} &= \frac{1}{L} (\text{Area}|_{t=0}^{3 \text{ ms}}) + \text{initial condition} \\ &= 20(-0.750 \times 3 \times 10^{-3}) + 25 \times 10^{-3} = -20 \text{ mA} \end{aligned}$$

and the value of  $i_L|_{t=3 \text{ ms}} = -20$  mA now becomes our initial condition for the time interval  $3 < t < 6$  ms.

Continuing with graphical integration, we obtain

$$\begin{aligned} i_L|_{t=6 \text{ ms}} &= \frac{1}{L} (\text{Area}|_{t=3}^{6 \text{ ms}}) + \text{initial condition} \\ &= 20 \left( \frac{1.75}{3} \times 3 \times 10^{-3} \right) - 20 \times 10^{-3} = 15 \text{ mA} \end{aligned}$$

and now the current has increased linearly from  $-20$  mA at  $t = 3$  ms to  $15$  mA at  $t = 6$  ms as shown in Figure 5.8.

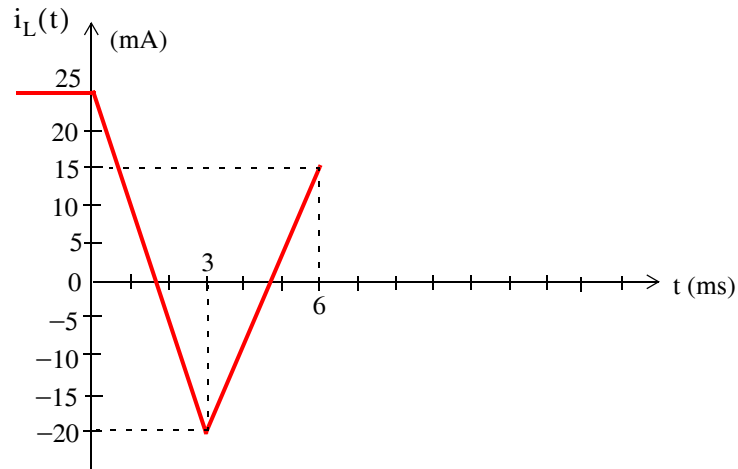


Figure 5.8. Inductor current for  $0 < t < 6$  ms, Example 5.2

For the time interval  $6 < t < 8$  ms, we obtain

$$\begin{aligned} i_L|_{t=8 \text{ ms}} &= \frac{1}{L}(\text{Area}|_{t=6}^{8 \text{ ms}}) + \text{initial condition} \\ &= 20(-0.625 \times 2 \times 10^{-3}) + 15 \times 10^{-3} = -10 \text{ mA} \end{aligned}$$

Therefore, the current has decreased linearly from  $15$  mA at  $t = 6$  ms to  $-10$  mA at  $t = 8$  ms as shown in Figure 5.9.

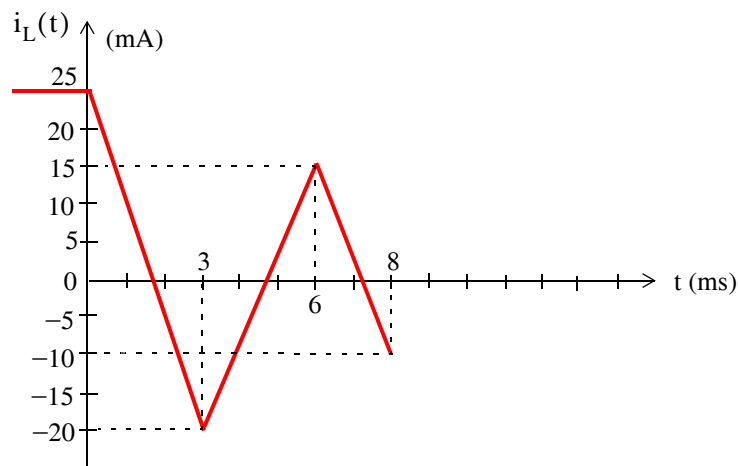


Figure 5.9. Inductor current for  $0 < t < 8$  ms, Example 5.2

For the time interval  $8 \text{ ms} < t < 10 \text{ ms}$  we obtain



$$\begin{aligned}
 i_L|_{t=10 \text{ ms}} &= \frac{1}{L}(\text{Area}|_{t=8}^{10 \text{ ms}}) + \text{initial condition} \\
 &= 20(0.375 \times 2 \times 10^{-3}) - 10 \times 10^{-3} = 5 \text{ mA}
 \end{aligned}$$

that is, the current has increased linearly from  $-10 \text{ mA}$  at  $t = 8 \text{ ms}$  to  $5 \text{ mA}$  at  $t = 10 \text{ ms}$  as shown in Figure 5.10.

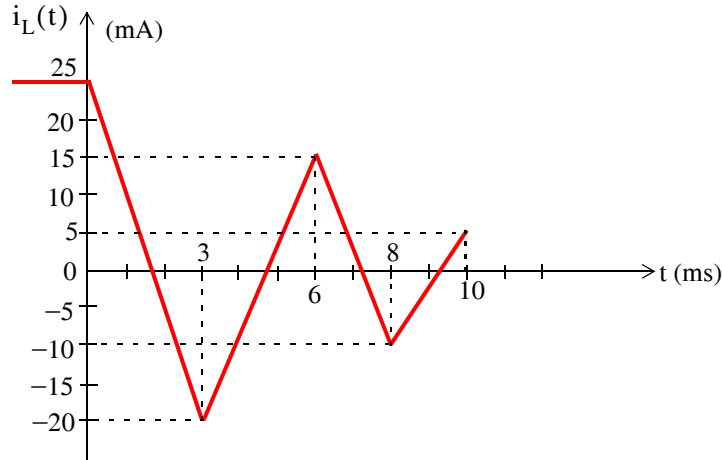


Figure 5.10. Inductor current for  $0 < t < 10 \text{ ms}$ , Example 5.2

Finally, for the time interval  $10 \text{ ms} < t < 12 \text{ ms}$  we obtain

$$\begin{aligned}
 i_L|_{t=12 \text{ ms}} &= \frac{1}{L}(\text{Area}|_{t=10}^{12 \text{ ms}}) + \text{initial condition} \\
 &= 20(-0.125 \times 2 \times 10^{-3}) + 5 \times 10^{-3} = 0
 \end{aligned}$$

that is, the current has decreased linearly from  $5 \text{ mA}$  at  $t = 10 \text{ ms}$  to  $0 \text{ mA}$  at  $t = 12 \text{ ms}$  and remains at zero for  $t > 12 \text{ ms}$  as shown in Figure 5.11.

Example 5.2 confirms the well known fact that *the current through an inductor cannot change instantaneously*. This can be observed from the voltage and current waveforms for this and the previous example. We observe that the voltage across the inductor can change instantaneously as shown by the discontinuities at  $t = 0, 3, 6, 8, 10,$  and  $12 \text{ ms}$ . However, the current through the inductor never changes instantaneously, that is, it displays no discontinuities since its value is explicitly defined at all instances of time.

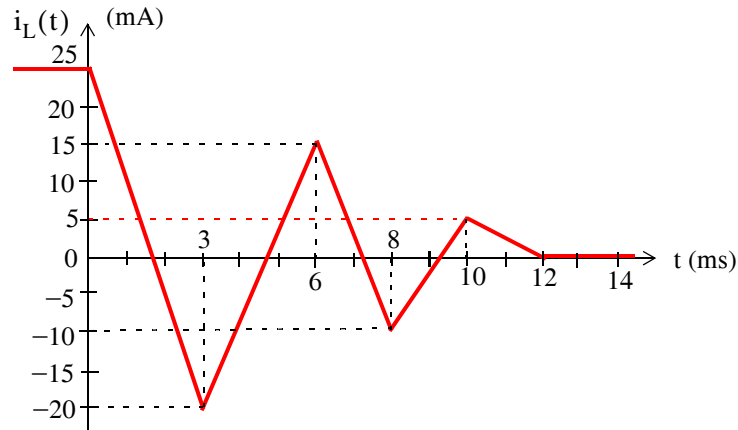


Figure 5.11. Inductor current for  $0 < t < 12$  ms, Example 5.2

### 5.3 Power and Energy in an Inductor

Power in an inductor with inductance  $L$  is found from

$$p_L = v_L i_L = \left( L \frac{di_L}{dt} \right) i_L = L i_L \frac{di_L}{dt} \quad (5.26)$$

and the energy in an inductor, designated as  $W_L$  is the integral of the power, that is,

$$W_L \Big|_{t_0}^t = \int_{t_0}^t p_L dt = L \int_{i(t_0)}^{i(t)} i_L \frac{di_L}{dt} dt = L \int_{i(t_0)}^{i(t)} i_L di_L$$

or

$$W_L \Big|_{t_0}^t = \frac{1}{2} L i_L^2 \Big|_{i(t_0)}^{i(t)} = \frac{1}{2} L [i_L^2(t) - i_L^2(t_0)]$$

or

$$W_L(t) - W_L(t_0) = \frac{1}{2} L [i_L^2(t) - i_L^2(t_0)]$$

and letting  $i_L = 0$  at  $t = 0$ , we obtain the *energy stored in an inductor* as

$$\boxed{W_L(t) = \frac{1}{2} L i_L^2(t)} \quad (5.27)$$

Unlike the resistor which dissipates energy (in the form of heat), the (ideal) inductor is a physical device capable of storing energy in analogy to the potential energy of a stretched spring.

Electric circuits which contain inductors can be simplified if the applied voltage and current sources are constant as shown by the following example.

## Example 5.3

For the circuit shown in Figure 5.12, compute  $v_1$ ,  $v_2$ , and  $v_3$ , after steady-state\* conditions have been reached. Then, compute the power absorbed and the energy consumed by the 5 mH inductor.

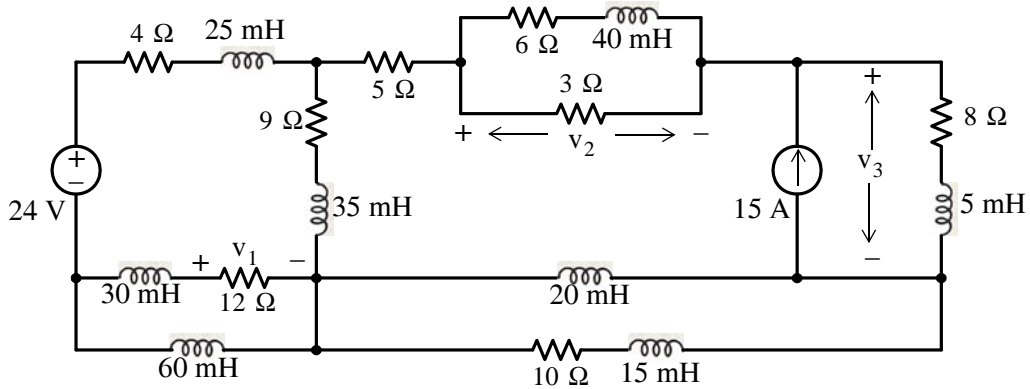


Figure 5.12. Circuit for Example 5.3

### Solution:

Since both the voltage and the current sources are constant, the voltages and the currents in all branches of the circuit will be constant after *steady-state conditions* have been reached.

Since

$$v_L = L \frac{di_L}{dt} = L \frac{d(\text{constant})}{dt} = 0$$

then, all voltages across the inductors will be zero and therefore we can replace all inductors by short circuits. The given circuit then reduces to the one shown in Figure 5.13 where the 3 Ω and 6 Ω parallel resistors have been combined into a single 2 Ω resistor.

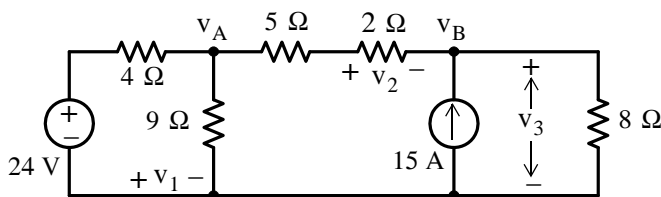


Figure 5.13. Circuit for Example 5.3 after steady-state conditions have been reached

\* By steady state conditions we mean the condition (state) where the voltages and currents, after some transient disturbances, have subsided. Transients will be in Chapter 10.

Now, in Figure 5.13, by inspection,  $v_1 = 0$  since the  $12\ \Omega$  resistor was shorted out by the  $60\ \text{mH}$  inductor. To find  $v_2$  and  $v_3$ , let us first find  $v_A$  and  $v_B$  using nodal analysis.

At Node  $v_A$ ,

$$\frac{v_A - 24}{4} + \frac{v_A}{9} + \frac{v_A - v_B}{5 + 2} = 0$$

or

$$\left(\frac{1}{4} + \frac{1}{9} + \frac{1}{7}\right)v_A - \frac{1}{7}v_B = 6 \quad (5.28)$$

At Node  $v_B$

$$\frac{v_B - v_A}{5 + 2} - 15 + \frac{v_B}{8} = 0$$

or

$$-\frac{1}{7}v_A + \left(\frac{1}{7} + \frac{1}{8}\right)v_B = 15 \quad (5.29)$$

We will use the MATLAB script below to find the solution of (5.28) and (5.29).

```
format rat    % Express answers in rational form
G=[1/4+1/9+1/7  -1/7; -1/7  1/7+1/8]; I=[6  15]'; V=G\I;
disp('vA='); disp(V(1)); disp('vB='); disp(V(2))
```

```
vA=
    360/11
```

```
vB=
    808/11
```

Therefore,

$$v_A = 360/11\ \text{V}$$

$$v_B = 808/11\ \text{V}$$

$$v_2 = v_A - v_2 = -448/11\ \text{V}$$

$$v_3 = v_2 = 808/11\ \text{V}$$

and

$$p_{5\ \text{mH}} = v_{5\ \text{mH}} \times i_{5\ \text{mH}} = 0 \times i_{5\ \text{mH}} = 0$$

that is,

$$p_{5\ \text{mH}} = 0\ \text{watts}$$

Also,

$$W_{5\ \text{mH}} = \frac{1}{2}L i_{5\ \text{mH}}^2 = \frac{1}{2}L \left(\frac{v_3}{8}\right)^2 = 0.5 \times 5 \times 10^{-3} \times \left(\frac{808/11}{8}\right)^2$$

or

$$W_{5\ \text{mH}} = 0.211\ \text{J}$$

5.4 Combinations of Inductors in Series and in Parallel

Consider the circuits of figures 5.14 (a) and 5.14 (b) where the source voltage  $v_S$  is the same for both circuits. We wish to find an expression for the equivalent inductance which we denote as  $L_{Seq}$  in terms of  $L_1, L_2, \dots, L_N$  in Figure 5.14 (a) so that the current  $i$  will be the same for both circuits.

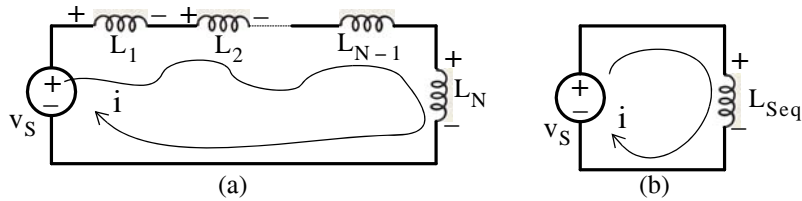


Figure 5.14. Circuits for derivation of equivalent inductance for inductors in series

From the circuit of Figure 5.14 (a),

$$L_1 \frac{di}{dt} + L_2 \frac{di}{dt} + \dots + L_{N-1} \frac{di}{dt} + L_N \frac{di}{dt} = v_S$$

or

$$(L_1 + L_2 + \dots + L_{N-1} + L_N) \frac{di}{dt} = v_S \tag{5.30}$$

From the circuit of Figure 5.14 (b),

$$L_{Seq} \frac{di}{dt} = v_S \tag{5.31}$$

Equating the left sides of (5.30) and (5.31) we obtain:

$$L_{Seq} = L_1 + L_2 + \dots + L_{N-1} + L_N \tag{5.32}$$

Thus, inductors in series combine as resistors in series do.

Next, we will consider the circuits of Figures 5.15 (a) and 5.15 (b) where the source current  $i_S$  is the same for both circuits. We wish to find an expression for the equivalent inductance which we denote as  $L_{Peq}$  in terms of  $L_1, L_2, \dots, L_N$  in Figure 5.15 (a) so that the voltage  $v$  will be the same for both circuits.

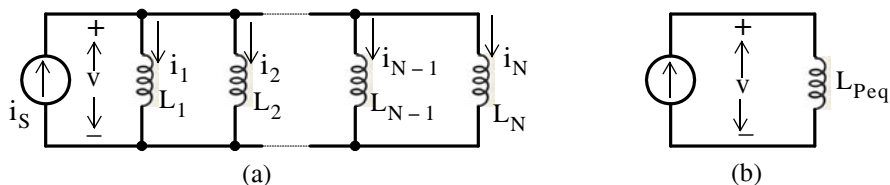


Figure 5.15. Circuits for derivation of equivalent inductance for inductors in parallel

From the circuit of Figure 5.15 (a)

$$i_1 + i_2 + \dots + i_{N-1} + i_N = i_s$$

or

$$\frac{1}{L_1} \int_{-\infty}^t v dt + \frac{1}{L_2} \int_{-\infty}^t v dt + \dots + \frac{1}{L_{N-1}} \int_{-\infty}^t v dt + \frac{1}{L_N} \int_{-\infty}^t v dt = i_s$$

or

$$\left( \frac{1}{L_1} + \frac{1}{L_2} + \dots + \frac{1}{L_{N-1}} + \frac{1}{L_N} \right) \int_{-\infty}^t v dt = i_s \quad (5.33)$$

From the circuit of Figure 5.15 (b)

$$\frac{1}{L_{Peq}} \int_{-\infty}^t v dt = i_s \quad (5.34)$$

Equating the left sides of (5.33) and (5.34) we obtain:

$$\boxed{\frac{1}{L_{Peq}} = \frac{1}{L_1} + \frac{1}{L_2} + \dots + \frac{1}{L_N}} \quad (5.35)$$

and for the special case of two parallel inductors

$$L_{Peq} = \frac{L_1 L_2}{L_1 + L_2} \quad (5.36)$$

Thus, inductors in parallel combine as resistors in parallel do.

### Example 5.4

For the network of Figure 5.16, replace all inductors by a single equivalent inductor.

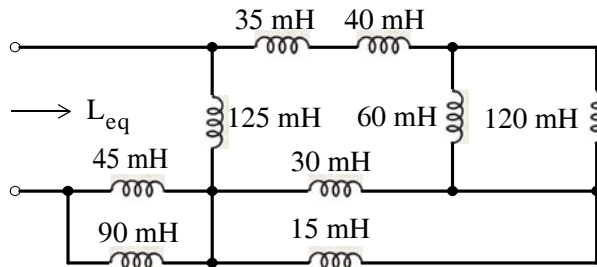


Figure 5.16. Network for Example 5.4

### Solution:

Starting at the right end of the network and moving towards the left end, we find that  $60 \text{ mH} \parallel 120 \text{ mH} = 40 \text{ mH}$ ,  $30 \text{ mH} \parallel 15 \text{ mH} = 10 \text{ mH}$ ,  $40 \text{ mH} + 35 \text{ mH} = 75 \text{ mH}$ , and also  $45 \text{ mH} \parallel 90 \text{ mH} = 30 \text{ mH}$ . The network then reduces to that shown in Figure 5.17.

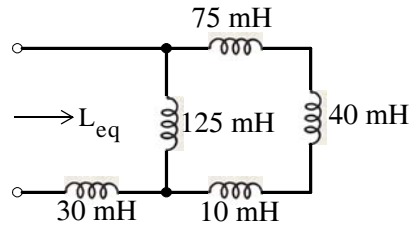


Figure 5.17. First step in combination of inductances

Finally, with reference to Figure 5.17,  $(40 \text{ mH} + 35 \text{ mH} + 10 \text{ mH}) \parallel 125 \text{ mH} = 62.5 \text{ mH}$ , and  $L_{\text{eq}} = 30 \text{ mH} + 62.5 \text{ mH} = 92.5 \text{ mH}$  as shown in Figure 5.18.

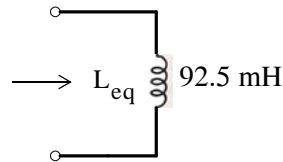


Figure 5.18. Network showing the equivalent inductance of Figure 5.16

### 5.5 Capacitance

In Section 5.2 we learned that inductance is associated with a *magnetic field* which is created whenever there is current flow. Similarly, *capacitance* is associated with an *electric field*. In a simple circuit we can represent the entire capacitance with a device called *capacitor*, just as we considered the entire inductance to be concentrated in a single inductor. A capacitor consists of two parallel metal plates separated by an air space or by a sheet of some type of insulating material called the *dielectric*.

Now, let us consider the simple series circuit of Figure 5.19 where the device denoted as  $C$ , is the standard symbol for a capacitor.

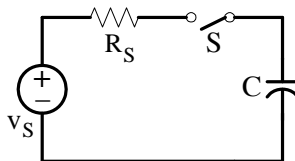


Figure 5.19. Simple circuit to illustrate a charged capacitor

When the switch  $S$  closes in the circuit of Figure 5.19, the voltage source will force electrons from its negative terminal through the conductor to the lower plate of the capacitor and it will accumulate *negative charge*. At the same time, electrons which were present in the upper plate of the capacitor will move towards the positive terminal of the voltage source. This action leaves the

upper plate of the capacitor deficient in electrons and thus it becomes *positively charged*. Therefore, an *electric field* has been established between the plates of the capacitor.

The distribution of the electric field set up in a capacitor is usually represented by lines of force similar to the lines of force in a magnetic field. However, in an electric field the lines of force start at the positive plate and terminate at the negative plate, whereas magnetic lines of force are always complete loops.

Figure 5.20 shows the distribution of the electric field between the two plates of a capacitor.

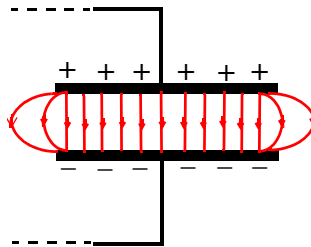


Figure 5.20. Electric field between the plates of a capacitor

We observe that the electric field has an almost uniform density in the area directly between the plates, but it decreases in density beyond the edges of the plates.

The charge  $q$  on the plates is directly proportional to the voltage between the plates and the capacitance  $C$  is the constant of proportionality. Thus,

$$q = Cv \tag{5.37}$$

and recalling that the current  $i$  is the rate of change of the charge  $q$ , we have the relation

$$i = \frac{dq}{dt} = \frac{d}{dt}(Cv)$$

or

$$i_C = C \frac{dv_C}{dt} \tag{5.38}$$

where  $i_C$  and  $v_C$  in (5.38) obey the passive sign convention.

The unit of capacitance is the *Farad* abbreviated as *F* and since

$$C = \frac{i_C}{\frac{dv_C}{dt}} = \frac{\text{amperes}}{\frac{\text{volts}}{\text{seconds}}} \tag{5.39}$$



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## Chapter 5 Inductance and Capacitance

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we can say that *one farad is the capacitance in a circuit in which a current of one ampere flows when the voltage is changing at the rate of a one volt per second.*

By separation of the variables we rewrite (5.38) as

$$dv_C = \frac{1}{C} i_C dt \quad (5.40)$$

and integrating both sides we obtain:

$$\int_{v_C(t_0)}^{v_C(t)} dv_C = \frac{1}{C} \int_{t_0}^t i_C dt$$

or

$$v_C(t) - v_C(t_0) = \frac{1}{C} \int_{t_0}^t i_C dt$$

or

$$\boxed{v_C(t) = \frac{1}{C} \int_{t_0}^t i_C dt + v_C(t_0)} \quad (5.41)$$

where  $v_C(t_0)$  is the initial condition, that is, the voltage across a capacitor at some reference time usually taken as  $t = 0$ , and denoted as  $v_C(0)$ .

We can also write (5.41) as

$$v_C(t) = \frac{1}{C} \int_{-\infty}^t i_C dt = \frac{1}{C} \int_{-\infty}^0 i_C dt + \frac{1}{C} \int_0^t i_C dt$$

where the initial condition is represented by the first integral on the right side.

---

### Example 5.5

The waveform shown in Figure 5.21 represents the current flowing through a  $1 \mu\text{F}$  capacitor. Compute and sketch the voltage across this capacitor for the time interval  $0 < t < 4 \text{ ms}$  given that the initial condition is  $v_C(0) = 0$ .

#### Solution:

The initial condition  $v_C(0) = 0$ , establishes the first point at the coordinates  $(0, 0)$  on the  $v_C(t)$  versus time plot of Figure 5.22.

Next,

$$v_C|_{t=1 \text{ ms}} = \frac{1}{C} \int_0^{1 \times 10^{-3}} i_C dt + \underbrace{\frac{v_C(0)}{0}}$$

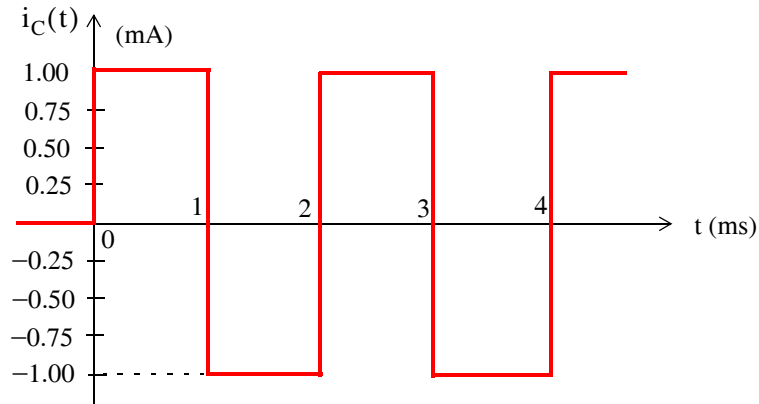


Figure 5.21. Waveform for Example 5.5

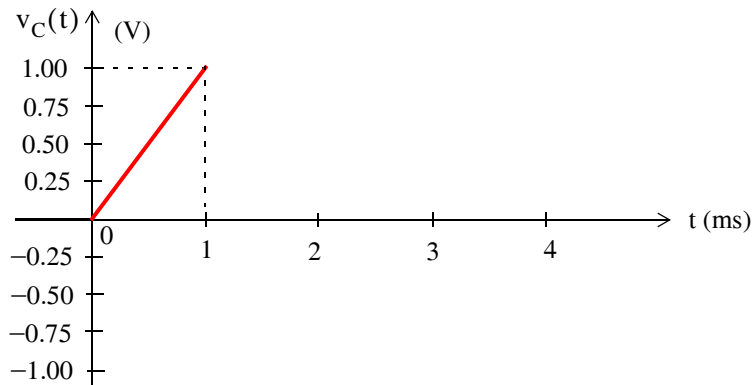


Figure 5.22. Straight line segment for  $0 < t < 1$  ms of the voltage waveform for Example 5.5

or

$$v_C|_{t=1 \text{ ms}} = \frac{1}{C} \left( \text{Area} \Big|_{t=0}^{1 \times 10^{-3}} \right) = \frac{1}{1 \times 10^{-6}} (1 \times 10^{-3} \times 1 \times 10^{-3}) = 1 \text{ volt}$$

and this value establishes the second point of the straight line segment passing through the origin as shown in Figure 5.22.

This value of 1 volt at  $t = 1$  ms becomes our initial condition for the time interval  $1 < t < 2$ . Continuing, we obtain

$$\begin{aligned} v_C|_{t=2 \text{ ms}} &= \frac{1}{C} \left( \text{Area} \Big|_{t=0}^{1 \times 10^{-3}} \right) + 1 \\ &= \frac{1}{1 \times 10^{-6}} (-1 \times 10^{-3} \times (2 - 1) \times 10^{-3}) + 1 = 0 \text{ volts} \end{aligned}$$

Thus, the capacitor voltage then decreases linearly from 1 volt at  $t = 1$  ms to 0 volts at  $t = 2$  ms as shown in Figure 5.23.

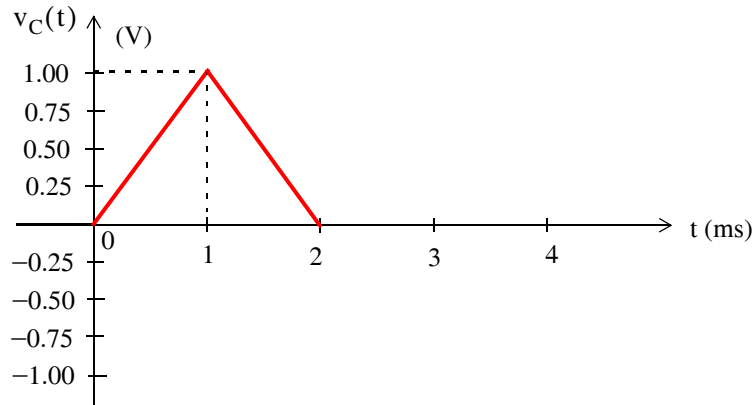


Figure 5.23. Voltage waveform for  $0 < t < 2$  ms of Example 5.5

There is no need to calculate the values of the capacitor voltage  $v_c$  at  $t = 3$  ms and at  $t = 4$  ms because the waveform of the current  $i_c$  starts repeating itself at  $t = 2$  ms, and the initial conditions and the areas are the same as before. Accordingly, the capacitor voltage  $v_c$  waveform of figure (b) starts repeating itself also as shown in Figure 5.24.

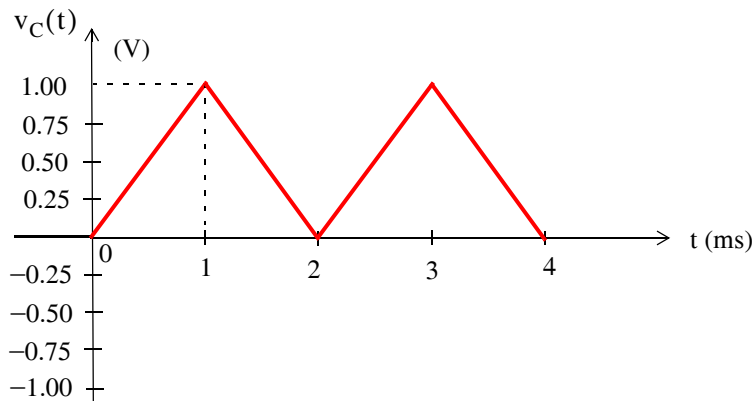


Figure 5.24. Voltage waveform for  $0 < t < 4$  ms of Example 5.5

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Example 5.5 has illustrated the well known fact that *the voltage across* a capacitor cannot change instantaneously. Referring to the current and voltage waveforms for this example, we observe that the current through the capacitor can change instantaneously as shown by the discontinuities at  $t = 1, 2, 3,$  and  $4$  ms in Figure 5.21. However, the voltage across the capacitor never changes instantaneously, that is, it displays no discontinuities since its value is explicitly defined at all instances of time as shown in Figure 5.24.

## 5.6 Power and Energy in a Capacitor

Power in a capacitor with capacitance  $C$  is found from

$$p_C = v_C i_C = v_C \left( C \frac{dv_C}{dt} \right)$$

and the energy in a capacitor, denoted as  $W_C$  is the integral of the power, that is,

$$\begin{aligned} W_C \Big|_{t_0}^t &= \int_{t_0}^t p_C dt = C \int_{v(t_0)}^{v(t)} v_C \frac{dv_C}{dt} dt = C \int_{v(t_0)}^{v(t)} v_C dv_C \\ &= \frac{1}{2} C v_C^2 \Big|_{i(t_0)}^{i(t)} = \frac{1}{2} C [v_C^2(t) - v_C^2(t_0)] \end{aligned}$$

or

$$W_C(t) - W_C(t_0) = \frac{1}{2} C [v_C^2(t) - v_C^2(t_0)]$$

and letting  $v_C = 0$  at  $t = 0$ , we obtain the *energy stored in a capacitor* as

$$\boxed{W_C(t) = \frac{1}{2} C v_C^2(t)} \quad (5.42)$$

Like an inductor, a capacitor is a physical device capable of storing energy.

It was stated earlier that the current through an inductor and the voltage across a capacitor cannot change instantaneously. These facts can also be seen from the expressions of the energy in an inductor and in a capacitor, equations (5.27) and (5.42) where we observe that if the current in an inductor or the voltage across a capacitor could change instantaneously, then the energies  $W_L$  and  $W_C$  would also change instantaneously but this is, of course, a physical impossibility.

### Example 5.6

In the circuit of figure 5.25, the voltage and current sources are constant.

- Compute  $i_{L1}$  and  $v_{C2}$
- Compute the power and energy in the  $2 \mu\text{F}$  capacitor.

**Solution:**

- The voltage and current sources are constant; thus, after steady-state conditions have been reached, the voltages across the inductors will be zero and the currents through the capacitors will be zero. Therefore, we can replace the inductors by short circuits and the capacitors by open circuits and the given circuit reduces to that shown in Figure 5.26.

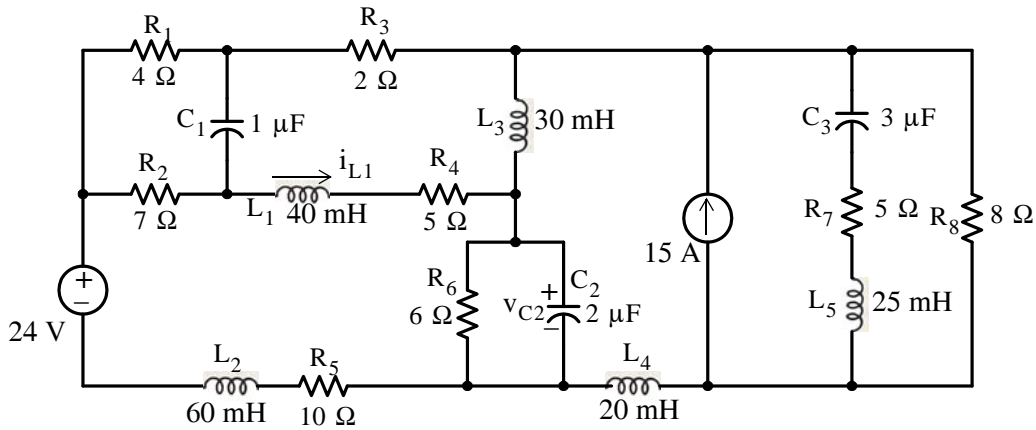


Figure 5.25. Circuit for Example 5.6

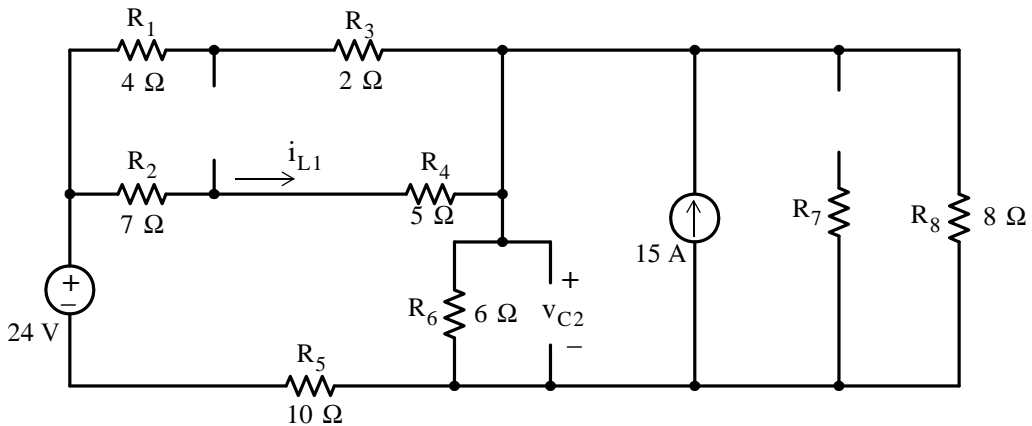


Figure 5.26. First simplification of the circuit of Example 5.6

We can simplify the circuit of figure 5.26 by first exchanging the 15 A current source and resistor  $R_8$  for a voltage source of  $15 \times 8 = 120$  V in series with  $R_8$  as shown in Figure 5.27. We also combine the series-parallel resistors  $R_1$  through  $R_4$ . Thus,  $R_{eq} = (4 + 2) \parallel (7 + 5) = 4 \Omega$ . But now we observe that the branch in which the current  $i_{L1}$  flows has disappeared; however, this presents no problem since we can apply the current division expression once  $i$ , shown in Figure 5.27, is found. The simplified circuit then is as shown in Figure 5.27.

We can apply superposition here. Instead, we will write two mesh equations and we will solve using MATLAB. These in matrix form are

$$\begin{bmatrix} 20 & -6 \\ -6 & 14 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 24 \\ -120 \end{bmatrix}$$

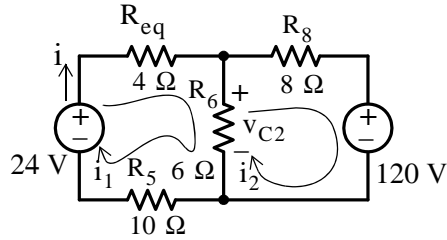


Figure 5.27. Final simplification of the circuit of Example 5.6

Solution using MATLAB:

```
format rat; R=[20 -6; -6 14]; V=[24 -120]; I=R\V; disp('i1='); disp(I(1)); disp('i2='); disp(I(2))
```

```
i1=
-96/61
i2=
-564/61
```

Therefore, with reference to the circuit of Figure 5.28 below, we obtain

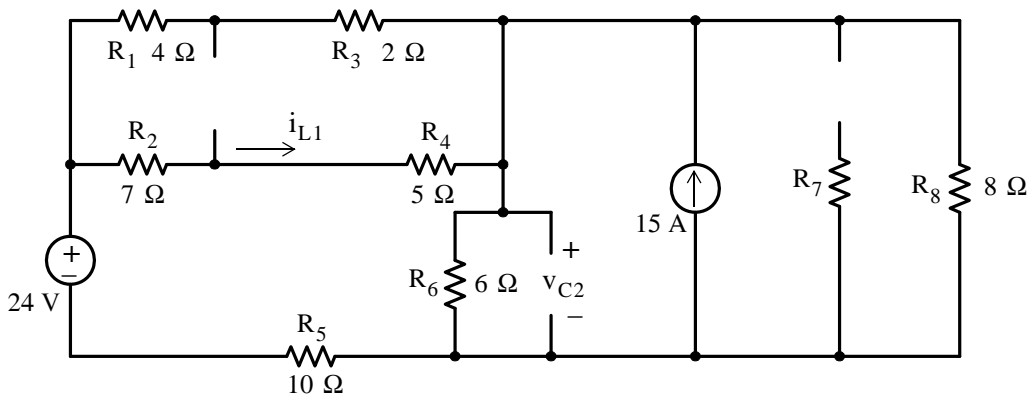


Figure 5.28. Circuit for computation of  $i_{L1}$  and  $v_{C2}$  for Example 5.6

$$i_{L1} = \frac{(4 + 2)}{(4 + 2) + (7 + 5)} \times \left(-\frac{96}{61}\right) = -\frac{32}{61} = -0.525 \text{ A}$$

and

$$v_{C2} = 6 \left(-\frac{96}{61} + \frac{564}{61}\right) = \frac{2808}{61} = 46.03 \text{ V}$$

b.

$$P_{2 \mu\text{F}} = v_{2 \mu\text{F}} \times i_{2 \mu\text{F}} = v_{C2} \times 0 = 0$$

and

$$W_{2 \mu\text{F}} = \frac{1}{2} C v_{2 \mu\text{F}}^2 = 0.5 \times 2 \times 10^{-6} \times \left(\frac{2808}{61}\right)^2 = 2 \text{ mJ}$$

### 5.7 Combinations of Capacitors in Series and in Parallel

Consider the circuits of figures 5.29 (a) and 5.29 (b) in which the source voltage  $v_S$  is the same for both circuits. We want to find an expression for the equivalent capacitance which we denote as  $C_{Seq}$  in terms of  $C_1, C_2, \dots, C_{N-1}, C_N$  in Figure 5.29 (a) so that the current  $i$  will be the same in both circuits.

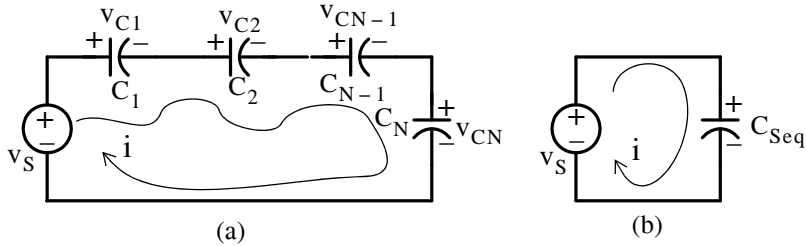


Figure 5.29. Circuits for derivation of equivalent capacitance for capacitors in series

From the circuit of Figure 5.29 (a),

$$v_{C1} + v_{C2} + \dots + v_{CN-1} + v_{CN} = v_S$$

or

$$\frac{1}{C_1} \int_{-\infty}^t i dt + \frac{1}{C_2} \int_{-\infty}^t i dt + \dots + \frac{1}{C_{N-1}} \int_{-\infty}^t i dt + \frac{1}{C_N} \int_{-\infty}^t i dt = v_S$$

or

$$\left( \frac{1}{C_1} + \frac{1}{C_2} + \dots + \frac{1}{C_{N-1}} + \frac{1}{C_N} \right) \int_{-\infty}^t i dt = v_S \quad (5.43)$$

From the circuit of Figure 5.29 (b)

$$\frac{1}{C_{Seq}} \int_{-\infty}^t i dt = v_S \quad (5.44)$$

Equating the left sides of (5.43) and (5.44) we obtain:

$$\boxed{\frac{1}{C_{Seq}} = \frac{1}{C_1} + \frac{1}{C_2} + \dots + \frac{1}{C_{N-1}} + \frac{1}{C_N}} \quad (5.45)$$

and for the special case of two capacitors in series

$$C_{Seq} = \frac{C_1 C_2}{C_1 + C_2} \quad (5.46)$$

Thus capacitors in series combine as resistors in parallel do.

Next, we will consider the circuits of figures 5.30 (a) and 5.30 (b) where the source current  $i_s$  is the same for both circuits. We wish to find an expression for the equivalent capacitance which we denote as  $C_{Peq}$  in terms of  $C_1, C_2, \dots, C_N$  in Figure 5.30 (a) so that the voltage  $v$  will be the same in both circuits.

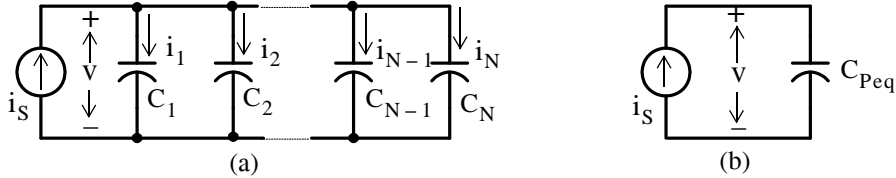


Figure 5.30. Circuits for derivation of equivalent capacitance for capacitors in parallel

From the circuit of Figure 5.30 (a),

$$i_1 + i_2 + \dots + i_{N-1} + i_N = i_s$$

or

$$C_1 \frac{dv}{dt} + C_2 \frac{dv}{dt} + \dots + C_{N-1} \frac{dv}{dt} + C_N \frac{dv}{dt} = i_s$$

or

$$(C_1 + C_2 + \dots + C_{N-1} + C_N) \frac{dv}{dt} = i_s \quad (5.47)$$

From the circuit of Figure 5.30 (b),

$$C_{Peq} \frac{dv}{dt} = i_s \quad (5.48)$$

Equating the left sides of (5.47) and (5.48) we obtain:

$$\boxed{C_{Peq} = C_1 + C_2 + \dots + C_{N-1} + C_N} \quad (5.49)$$

Thus, capacitors in parallel combine as resistors in series do.

### Example 5.7

For the network of Figure 5.31, replace all capacitors by a single equivalent capacitor.

**Solution:**

Beginning at the right of the network and moving towards the left, we find that

$$3 \mu\text{F} \parallel 1 \mu\text{F} = 4 \mu\text{F}$$

$$2 \mu\text{F} \parallel 4 \mu\text{F} = 6 \mu\text{F}$$



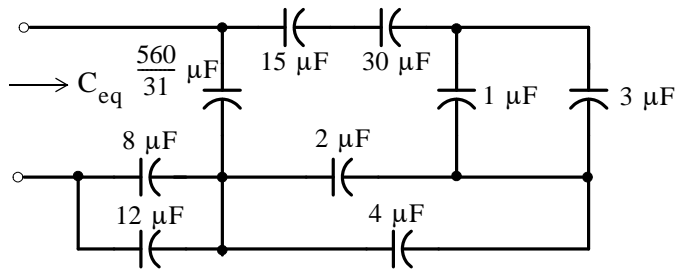


Figure 5.31. Network for Example 5.7

$$15 \mu\text{F in series with } 30\mu\text{F} = 10\mu\text{F}$$

$$8 \mu\text{F} \parallel 12 \mu\text{F} = 20 \mu\text{F}$$

The network then reduces to that shown in Figure 5.32.

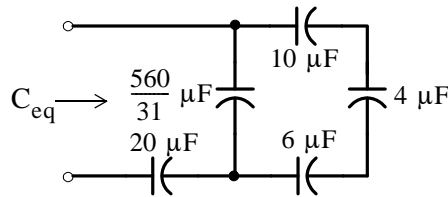


Figure 5.32. First step in combination of capacitances

Next, the series combination of 10, 4, and 6 μF capacitors yields  $60/31 \mu\text{F}$  and  $60/31 \mu\text{F} \parallel 560/31 \mu\text{F} = 20 \mu\text{F}$ . Finally, the series combination of 20 μF and 20 μF yields  $C_{eq} = 10 \mu\text{F}$  as shown in Figure 5.33.

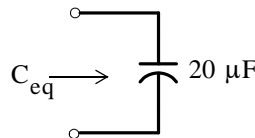


Figure 5.33. Network showing the equivalent inductance of Figure 5.16

## 5.8 Nodal and Mesh Equations in General Terms

In Examples 5.3 and 5.6 the voltage and current sources were constant and therefore, the steady-state circuit analysis could be performed by nodal, mesh or any other method of analysis as we learned in Chapter 3. However, if the voltage and current sources are time-varying quantities we must apply KCL or KVL in general terms as illustrated by the following example.

## Example 5.8

Write nodal and mesh equations for the circuit shown in Figure 5.34.

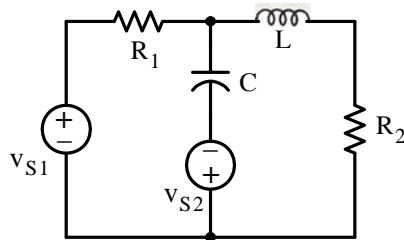


Figure 5.34. Circuit for Example 5.8

**Solution:**

### a. Nodal Analysis:

We assign nodes as shown in Figure 5.35. Thus, we need  $N - 1 = 5 - 1 = 4$  nodal equations.

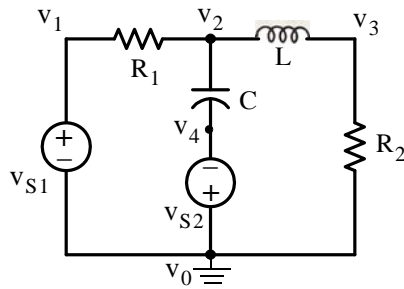


Figure 5.35. Nodal analysis for the circuit of Example 5.8

At Node 1:

$$v_1 = v_{S1}$$

At Node 2:

$$\frac{v_2 - v_1}{R_1} + C \frac{d}{dt}(v_2 - v_4) + \frac{1}{L} \int_{-\infty}^t (v_2 - v_3) dt = 0$$

At Node 3:

$$\frac{1}{L} \int_{-\infty}^t (v_3 - v_2) dt + \frac{v_3}{R_2} = 0$$

At Node 4:

$$v_4 = -v_{S2}$$

### b. Mesh Analysis:

We need  $M = B - 1 = 6 - 5 + 1 = 2$  mesh equations. Thus, we assign currents  $i_1$  and  $i_2$  as shown in Figure 5.36.

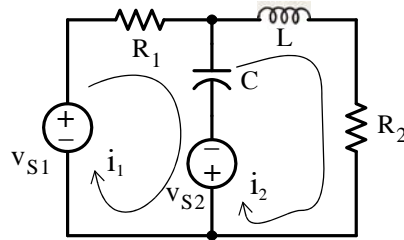


Figure 5.36. Mesh analysis for the circuit of Example 5.8

For Mesh 1:

$$R_1 i_1 + \frac{1}{C} \int_{-\infty}^t (i_1 - i_2) dt - v_{S1} - v_{S2} = 0$$

For Mesh 2:

$$L \frac{d}{dt} i_2 + R_2 i_2 + v_{S2} + \frac{1}{C} \int_{-\infty}^t (i_2 - i_1) dt = 0$$

---

In both the nodal and mesh equations, the initial conditions are included in the limits of integration. Alternately, we can add the initial condition terms and in the integrodifferential equations above, replace the lower limit of integration  $-\infty$  with zero.

## 5.9 Summary

- Inductance is associated with a magnetic field which is created whenever there is current flow.
- The magnetic field loops are circular in form and are called lines of magnetic flux. The unit of magnetic flux is the weber (Wb).
- The magnetic flux is denoted as  $\phi$  and, if there are  $N$  turns and we assume that the flux  $\phi$  passes through each turn, the total flux, denoted as  $\lambda$ , is called flux linkage. Then,  $\lambda = N\phi$
- For an inductor, the voltage–current relationship is  $v_L = L(di_L/dt)$
- The unit of inductance is the Henry abbreviated as H.
- Unlike the resistor which dissipates energy (in the form of heat), the (ideal) inductor is a physical device capable of storing energy in analogy to the potential energy of a stretched spring.
- The energy stored in an inductor is  $W_L(t) = (1/2)Li_L^2(t)$
- The current through an inductor cannot change instantaneously.
- In circuits where the applied voltage source or current source are constants, after steady–state conditions have been reached, an inductor behaves like a short circuit.
- Inductors in series combine as resistors in series do.
- Inductors in parallel combine as resistors in parallel do.
- Capacitance is associated with an electric field.
- A capacitor consists of two parallel metal plates separated by an air space or by a sheet of some type of insulating material called the dielectric.
- The charge  $q$  on the plates of a capacitor is directly proportional to the voltage between the plates and the capacitance  $C$  is the constant of proportionality. Thus,  $q = Cv$
- In a capacitor, the voltage–current relationship is  $i_C = C(dv_C/dt)$
- The unit of capacitance is the Farad abbreviated as F.
- Like an inductor, a capacitor is a physical device capable of storing energy.
- The energy stored in a capacitor is  $W_C(t) = (1/2)Cv_C^2(t)$
- The voltage across a capacitor cannot change instantaneously.
- In circuits where the applied voltage source or current source are constants, after steady–state conditions have been reached, a capacitor behaves like an open circuit.
- Capacitors in series combine as resistors in parallel do.

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## Chapter 5 Inductance and Capacitance

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- Capacitors in parallel combine as resistors in series do.
- In a circuit that contains inductors and/or capacitors, if the applied voltage and current sources are time-varying quantities, the nodal and mesh equations are, in general, integrodifferential equations.

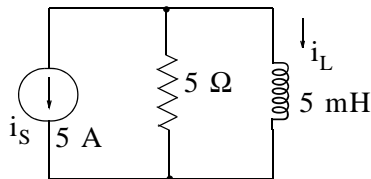
## 5.10 Exercises

### Multiple Choice

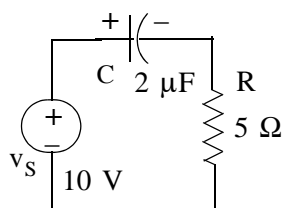
- The unit of inductance is the
  - Farad
  - Ohm
  - mH
  - Weber
  - None of the above
- The unit of capacitance is the
  - $\mu\text{F}$
  - Ohm
  - Farad
  - Coulomb
  - None of the above
- Faraday's law of electromagnetic induction states that
  - $\lambda = N\phi$
  - $\lambda = Li$
  - $v = L(di/dt)$
  - $v = d\lambda/dt$
  - None of the above
- In an electric field of a capacitor, the lines of force
  - are complete loops
  - start at the positive plate and end at the negative plate
  - start at the negative plate and end at the positive plate
  - are unpredictable
  - None of the above
- The energy in an inductor is
  - $(1/2)(Li^2)$
  - $(1/2)(Lv^2)$
  - $v_L i_L$
  - dissipated in the form of heat
  - None of the above

## Chapter 5 Inductance and Capacitance

6. The energy in a capacitor is
- A.  $(1/2)(Ci^2)$
  - B.  $(1/2)(Cv^2)$
  - C.  $v_C i_C$
  - D. dissipated in the form of heat
  - E. None of the above
7. In an inductor
- A. the voltage cannot change instantaneously
  - B. the current cannot change instantaneously
  - C. neither the voltage nor the current can change instantaneously
  - D. both the voltage and the current can change instantaneously
  - E. None of the above
8. In a capacitor
- A. the voltage cannot change instantaneously
  - B. the current cannot change instantaneously
  - C. neither the voltage nor the current can change instantaneously
  - D. both the voltage and the current can change instantaneously
  - E. None of the above
9. In the circuit below, after steady-state conditions have been established, the current  $i_L$  through the inductor will be



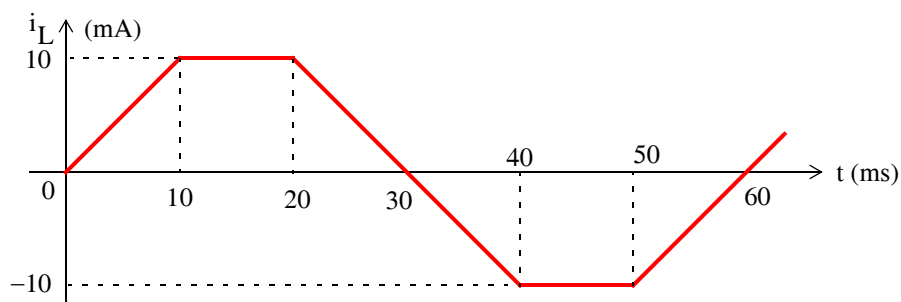
- A.  $0\text{ A}$
  - B.  $\infty\text{ A}$
  - C.  $2.5\text{ A}$
  - D.  $5\text{ A}$
  - E. None of the above
10. In the circuit below, after steady-state conditions have been established, the voltage  $v_C$  across the capacitor will be



- A. 0 V
- B.  $\infty$  V
- C. -10 V
- D. 10 V
- E. None of the above

### Problems

1. The current  $i_L$  flowing through a 10 mH inductor is shown by the waveform below.



- a. Compute and sketch the voltage  $v_L$  across this inductor for  $t > 0$
  - b. Compute the first time after  $t = 0$  when the power  $p_L$  absorbed by this inductor is  $p_L = 50 \mu\text{w}$ . Answer:  $t = 5 \text{ ms}$
  - c. Compute the first time after  $t = 0$  when the power  $p_L$  absorbed by this inductor is  $p_L = -50 \mu\text{w}$ . Answer:  $t = 25 \text{ ms}$
2. The current  $i_C$  flowing through a  $1 \mu\text{F}$  capacitor is given as  $i_C(t) = \cos 100t \text{ mA}$ , and it is known that  $v_C(0) = 0$
- a. Compute and sketch the voltage  $v_C$  across this capacitor for  $t > 0$
  - b. Compute the first time after  $t = 0$  when the power  $p_C$  absorbed by this capacitor is  $p_C = 5 \text{ mw}$ . Answer:  $7.85 \text{ ms}$
  - c. Compute the first time after  $t = 0$  when the power  $p_C$  absorbed by this capacitor is  $p_C = -5 \text{ mw}$ . Answer:  $23.56 \text{ ms}$

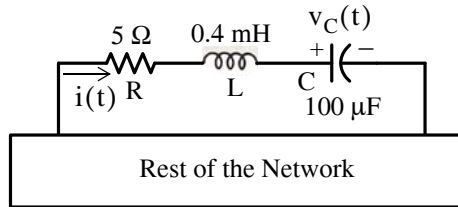


## Chapter 5 Inductance and Capacitance

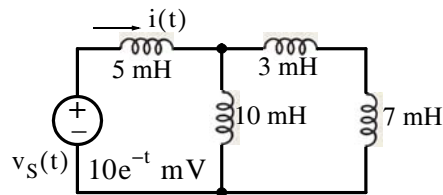
3. For the network below, compute the total energy stored in the series combination of the resistor, capacitor, and inductor at  $t = 10 \text{ ms}$  if:

a.  $i(t) = 0.1e^{-100t} \text{ mA}$  and it is known that  $v_C(0) = -10 \text{ V}$ . Answer:  $3.4 \text{ mJ}$

b.  $i(t) = 0.5\cos 5t \text{ mA}$  and it is known that  $v_C(0) = 0$ . Answer:  $50 \text{ }\mu\text{J}$

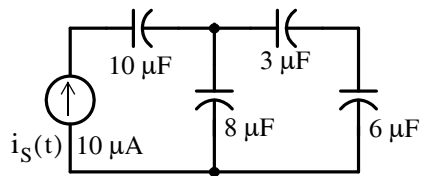


4. For the circuit below, compute the energy stored in the  $5 \text{ mH}$  inductor at  $t = 1 \text{ s}$  given that  $i(0) = 0$ . Answer:  $1 \text{ mJ}$

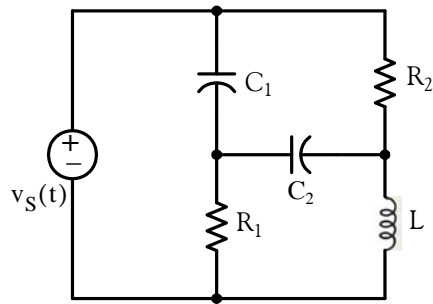


5. For the circuit below, replace all capacitors with an equivalent capacitance  $C_{eq}$  and then compute the energy stored in  $C_{eq}$  at  $t = 1 \text{ ms}$  given that  $v_C(0) = 0$  in all capacitors.

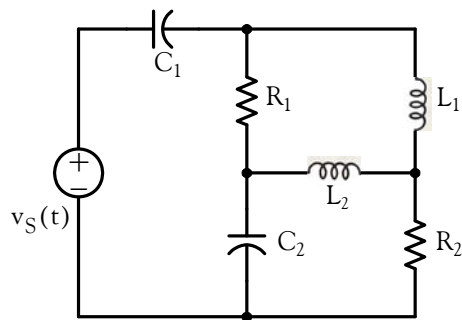
Answer:  $10 \text{ pJ}$



6. Write nodal equations for the circuit below.



7. Write mesh equations for the circuit below.



**5.11 Answers / Solutions to End-of-Chapter Exercises****Multiple Choice**

1. E Henry
2. C
3. D
4. B
5. A
6. B
7. B
8. A
9. E -5 A
10. D

**Problems**

1.
  - a. In an inductor the voltage and current are related by  $v_L = L(di_L/dt) = L \times \text{slope}$ . Thus, we need to compute the slope of each segment of the given waveform and multiply it by  $L$ .

$$v_L|_0^{10 \text{ ms}} = L \times \text{slope} = L \frac{\Delta i_L}{\Delta t} = 10 \times 10^{-3} \times \frac{10 \times 10^{-3} \text{ A}}{10 \times 10^{-3} \text{ s}} = 10 \text{ mV}$$

Likewise,

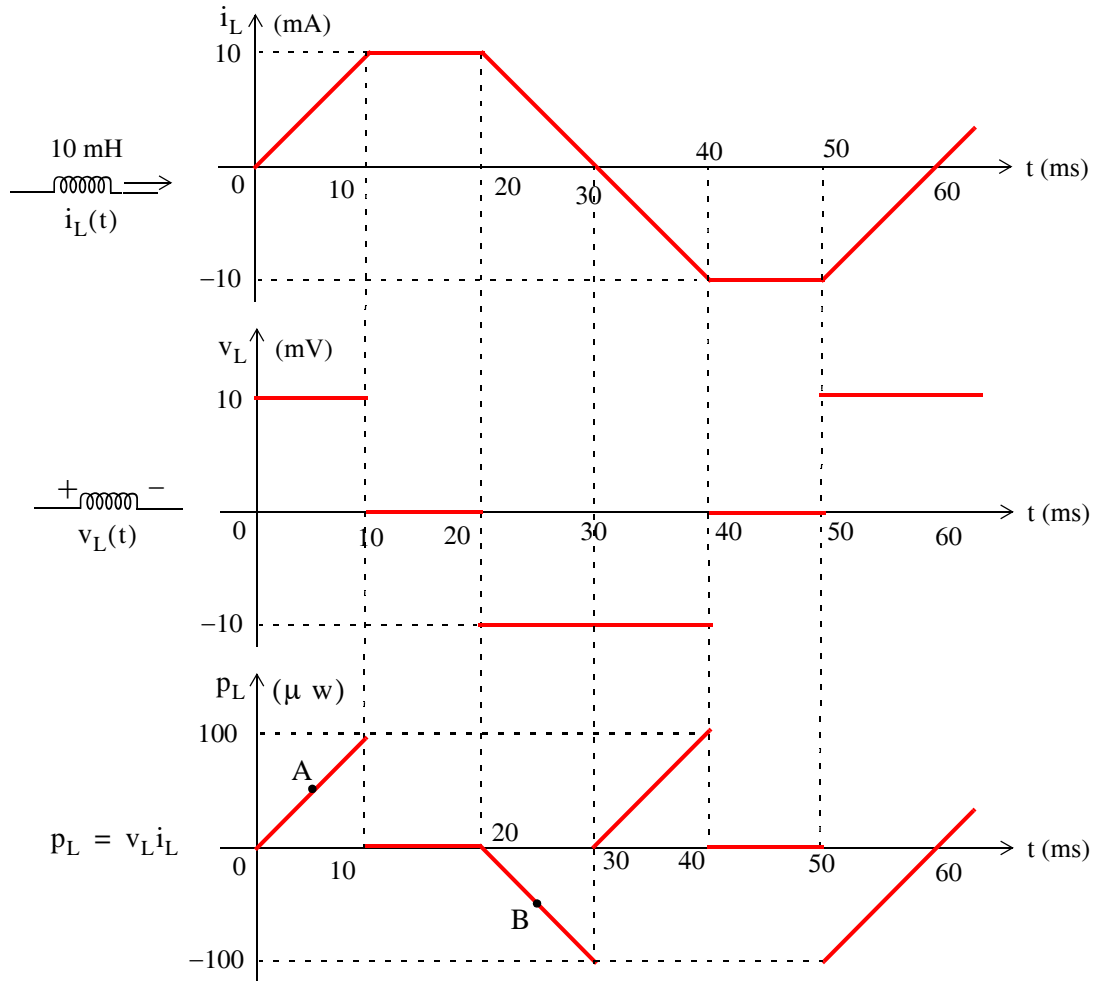
$$v_L|_{10}^{20 \text{ ms}} = L \times \text{slope} = L \times 0 = 0 \text{ mV}$$

$$v_L|_{20}^{40 \text{ ms}} = L \times \text{slope} = 10 \times 10^{-3} \times \frac{[-10 - (10)] \times 10^{-3} \text{ A}}{(40 - 20) \times 10^{-3} \text{ s}} = -10 \text{ mV}$$

$$v_L|_{40}^{50 \text{ ms}} = L \times \text{slope} = L \times 0 = 0 \text{ mV}$$

$$v_L|_0^{10 \text{ ms}} = L \times \text{slope} = 10 \times 10^{-3} \times \frac{[0 - (-10)] \times 10^{-3} \text{ A}}{(60 - 50) \times 10^{-3} \text{ s}} = 10 \text{ mV}$$

The current, voltage, and power waveforms are shown below.

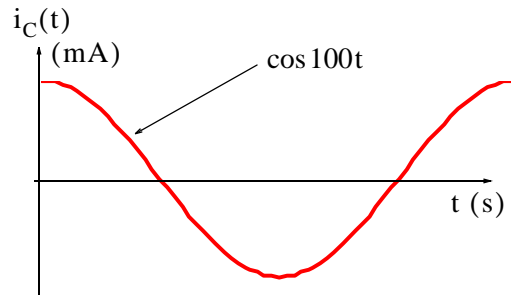


- b. From the power waveform above, we observe that  $p_L = v_L i_L = 50 \mu\text{w}$  occurs for the first time at point A where  $t = 5$  ms
- c. From the power waveform above, we observe that  $p_L = v_L i_L = -50 \mu\text{w}$  occurs for the first time at point A where  $t = 25$  ms

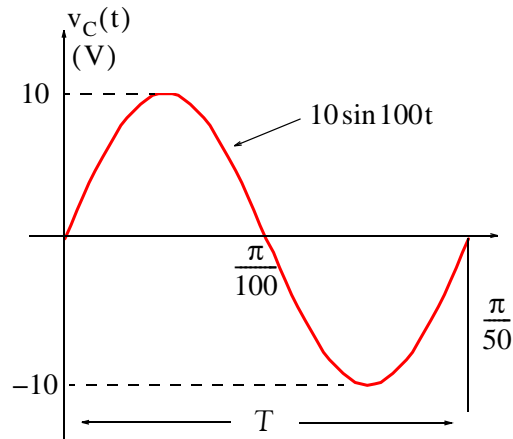
2.

- a. For this problem  $C = 1 \mu\text{F} = 10^{-6} \text{F}$  and the current  $i_C$  is a sinusoid given as  $i_C(t) = \cos 100t$  mA as shown below. The voltage  $v_C(t)$  across this capacitor is found from

$$\begin{aligned}
 v_C(t) &= \frac{1}{C} \int_0^t i_C \, d\tau + v_C(0) = 10^6 \int_0^t (10^{-3}) \cos 100\tau \, d\tau + 0 \\
 &= 10^3 \int_0^t \cos 100\tau \, d\tau = \frac{10^3}{100} \sin 100\tau \Big|_0^t = 10 \sin 100t
 \end{aligned}$$



and the waveform of  $v_C(t)$  is shown below.



Now,  $\omega T = 2\pi$  or  $\omega = 2\pi/T$ . Then,  $10 \sin \frac{2\pi}{T}t = 10 \sin 100t$  or  $2\pi/T = 100$  and  $T = 2\pi/100$  or  $T = \pi/50$ .

- b. Since  $v_C(t)$  is a sine function and  $i_C(t)$  a cosine function, the first time after zero that their product will be positive is in the interval  $0 < t < \pi/200$  where we want  $p_C = v_C i_C = 5 \text{ mw}$  or

$$p_C = (10 \sin 100t)(10^{-3} \cos 100t) = 5 \times 10^{-3} \text{ w}$$

or

$$p_C = (10 \sin 100t)(\cos 100t) = 5 \text{ w}$$

Recalling that

$$\sin 2x = 2 \sin x \cos x$$

it follows that

$$p_C = 5 \sin 200t = 5 \text{ w}$$

or

$$\sin 200t = 1$$

or

$$t = \frac{\sin^{-1} 1}{200} = \frac{\pi/2}{200} = \frac{\pi}{400} = 0.00785 \text{ s} = 7.85 \text{ ms}$$

- c. The time where  $p_C = -5 \text{ mw}$  will occur for the first time is 7.85 ms after  $t = \pi/200 \text{ s}$  or after  $t = 1000\pi/200 \text{ ms} = 5\pi \text{ ms}$ . Therefore,  $p_C = -5 \text{ mw}$  will occur for the first time at  $t = 7.85 + 5\pi = 7.85 + 15.71 = 23.56 \text{ ms}$

3. a. There is no energy stored in the resistor; it is dissipated in the form of heat. Thus, the total energy is stored in the capacitor and the inductor, that is,

$$W_T = W_L + W_C = \frac{1}{2} L i_L^2 + \frac{1}{2} C v_C^2$$

where

$$i_L = i(t) = 0.1e^{-100t}$$

and

$$\begin{aligned} v_C(t) &= \frac{1}{C} \int_0^t i_C \, d\tau + v_C(0) = 10^4 \int_0^t 0.1e^{-100\tau} \, d\tau - 10 \\ &= \frac{10^4 \times 0.1}{-100} e^{-100\tau} \Big|_0^t - 10 = 10e^{-100\tau} \Big|_t^0 = 10 - 10e^{-100t} - 10 \end{aligned}$$

or

$$v_C(t) = -10e^{-100t}$$

Then,

$$\begin{aligned} W_T \Big|_{t=10 \text{ ms}} &= \frac{1}{2} \times 0.4 \times 10^{-3} \times (0.1e^{-100t})^2 + \frac{1}{2} \times 10^{-4} \times (-10e^{-100t})^2 \\ &= 2.5 \times 10^{-4} [(0.1e^{-1})^2 + (-10e^{-1})^2] = 3.4 \text{ mJ} \end{aligned}$$

## Chapter 5 Inductance and Capacitance

We've used MATLAB as a calculator to obtain the answer, that is,

```
WT=2.5*10^(-4)*((0.1*exp(-1))^2+((-10)*exp(-1))^2);
fprintf(' \n'); fprintf('WT=%7.4f J',WT); fprintf(' \n')
```

```
WT= 0.0034 J
```

b. For this part,

$$i_L = i(t) = 0.5 \cos 5t \text{ mA}$$

and

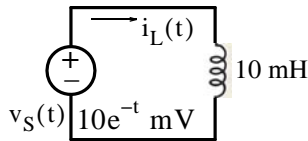
$$v_C(t) = \frac{1}{C} \int_0^t i_C \, d\tau + v_C(0) = 10^4 \times 10^{-4} \int_0^t 5 \cos 5\tau \, d\tau + 0 = \sin 5\tau \Big|_0^t = \sin 5t$$

Then,

$$\begin{aligned} W_T &= W_L + W_C = \frac{1}{2} \times 0.4 \times 10^{-4} \times (0.5 \cos 5t)^2 + \frac{1}{2} \times 10^{-4} \times (\sin 5t)^2 \\ &= 0.5 \times 10^{-4} \left[ \underbrace{\cos^2 5t + \sin^2 5t}_1 \right] = 0.05 \text{ mJ} = 50 \text{ } \mu\text{J} \end{aligned}$$

We observe that the total power is independent of time.

4. Beginning with the right side and proceeding to the left, the series-parallel combination of  $7 + 3 = 10$ ,  $10 \parallel 10 = 5$ , and  $5 + 5 = 10$  mH reduces the given circuit to the one shown below.



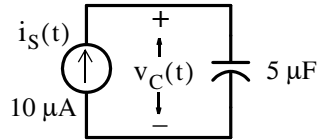
The current  $i_L(t)$  is

$$i_L(t) = \frac{1}{L} \int_0^t v_S \, d\tau + i_L(0) = \frac{1}{10 \times 10^{-3}} \times 10 \times 10^{-3} \int_0^t e^{-\tau} \, d\tau = -e^{-\tau} \Big|_0^t = e^{-\tau} \Big|_t^0 = 1 - e^{-t}$$

Then,

$$W_{5 \text{ mH}} \Big|_{t=1 \text{ s}} = \frac{1}{2} \times 5 \times 10^{-3} (1 - e^{-t})^2 \Big|_{t=1 \text{ s}} = 2.5 \times 10^{-3} \times (1 - e^{-1})^2 \approx 1 \text{ mJ}$$

5. Beginning with the right side and proceeding to the left, the series-parallel combination reduces the given circuit to the one shown below.



The current  $v_C(t)$  is

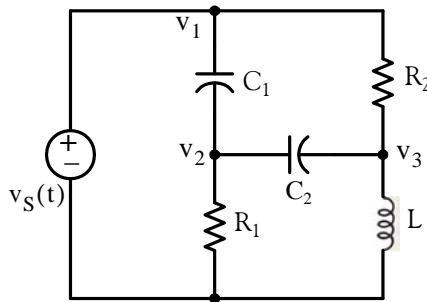
$$v_C(t) = \frac{1}{C} \int_0^t i_S \, d\tau + v_C(0) = \frac{1}{5 \times 10^{-6}} \times 10 \times 10^{-6} \int_0^t d\tau = 2\tau \Big|_0^t = 2t$$

Then,

$$W_{5 \, \mu\text{F}} \Big|_{t=1 \, \text{ms}} = \frac{1}{2} \times 5 \times 10^{-6} (2t)^2 \Big|_{t=1 \, \text{ms}} = 2.5 \times 10^{-6} \times 4 \times 10^{-6} = 10 \, \text{pJ}$$

6.

We assign node voltages  $v_1$ ,  $v_2$ , and  $v_3$  as shown below.



Then,

$$v_1 = v_S$$

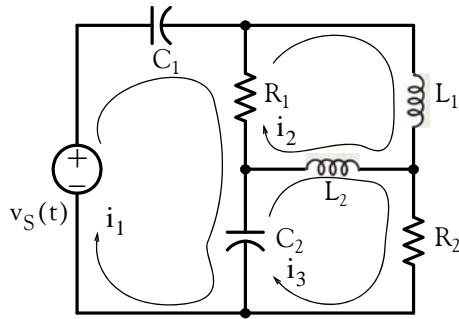
$$C_1 \frac{d}{dt}(v_2 - v_1) + C_2 \frac{d}{dt}(v_2 - v_3) + \frac{v_2}{R_1} = 0$$

$$\frac{v_3 - v_1}{R_2} + C_2 \frac{d}{dt}(v_3 - v_2) + \frac{1}{L} \int_{-\infty}^t v_3 \, d\tau = 0$$

7.

We assign mesh currents  $i_1$ ,  $i_2$ , and  $i_3$  as shown below.





Then,

$$\frac{1}{C_1} \int_{-\infty}^t i_1 \, d\tau + R_1(i_1 - i_2) + \frac{1}{C_2} \int_{-\infty}^t (i_1 - i_3) \, d\tau = v_S$$

$$R_1(i_2 - i_1) + L_1 \frac{di_2}{dt} + L_2 \frac{d}{dt}(i_2 - i_3) = 0$$

$$\frac{1}{C_2} \int_{-\infty}^t (i_3 - i_1) \, d\tau + L_2 \frac{d}{dt}(i_3 - i_2) + R_2 i_3 = 0$$

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# Chapter 6

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## Sinusoidal Circuit Analysis

This chapter is an introduction to circuits in which the applied voltage or current are sinusoidal. The time and frequency domains are defined and phasor relationships are developed for resistive, inductive and capacitive circuits. Reactance, susceptance, impedance and admittance are also defined. It is assumed that the reader is familiar with sinusoids and complex numbers. If not, it is strongly recommended that Appendix B is reviewed thoroughly before reading this chapter.

### 6.1 Excitation Functions

The applied voltages and currents in electric circuits are generally referred to as *excitations* or *driving functions*, that is, we say that a circuit is “excited” or “driven” by a constant, or a sinusoidal, or an exponential function of time. Another term used in circuit analysis is the word *response*; this may be the voltage or current in the “load” part of the circuit or any other part of it. Thus the response may be anything we define it as a response. Generally, the response is the voltage or current at the output of a circuit, but we need to specify what the output of a circuit is.

In Chapters 1 through 4 we considered circuits that consisted of excitations (active sources) and resistors only as the passive devices. We used various methods such as nodal and mesh analyses, superposition, Thevenin’s and Norton’s theorems to find the desired response such as the voltage and/or current in any particular branch. The circuit analysis procedure for these circuits is the same for DC and AC circuits. Thus, if the excitation is a constant voltage or current, the response will also be some constant value; if the excitation is a sinusoidal voltage or current, the response will also be sinusoidal with the same frequency but different amplitude and phase.

In Chapter 5 we learned that when the excitation is a constant and steady-state conditions are reached, an inductor behaves like a short circuit and a capacitor behaves like an open circuit. However, when the excitation is a time-varying function such as a sinusoid, inductors and capacitors behave entirely different as we will see in our subsequent discussion.

### 6.2 Circuit Response to Sinusoidal Inputs

We can apply the circuit analysis methods which we have learned in previous chapters to circuits where the voltage or current sources are sinusoidal. To find out how easy (or how difficult) the procedure becomes, we will consider the simple series circuit of Example 6.1.

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#### Example 6.1

For the circuit shown in Figure 6.1, derive an expression for  $v_C(t)$  in terms of  $V_p$ ,  $R$ ,  $C$ , and  $\omega$

## Chapter 6 Sinusoidal Circuit Analysis

where the subscript  $p$  is used to denote the *peak* or *maximum* value of a time varying function, and the sine symbol inside the circle denotes that the excitation is a sinusoidal function.

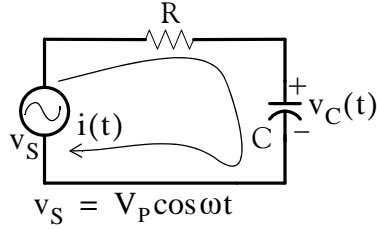


Figure 6.1. Circuit for Example 6.1

**Solution:**

By KVL,

$$v_R + v_C = v_S \quad (6.1)$$

where

$$v_R = Ri = Ri_C$$

and

$$i_C = C \frac{dv_C}{dt}$$

Then,

$$v_R = RC \frac{dv_C}{dt}$$

and by substitution into (6.1) we obtain

$$RC \frac{dv_C}{dt} + v_C = v_S = V_p \cos \omega t \quad (6.2)$$

As we know, differentiation (and integration) of a sinusoid of radian frequency  $\omega$  results in another sinusoid of the same frequency  $\omega$ . Accordingly, the solution of (6.2) must have the form

$$v_C(t) = A \cos(\omega t + \theta) \quad (6.3)$$

where the amplitude  $A$  and phase angle  $\theta$  are constants to be determined from the circuit parameters of  $V_p$ ,  $R$ ,  $C$ , and  $\omega$ . Substitution of (6.3) into (6.2) yields

$$-A\omega RC \sin(\omega t + \theta) + A \cos(\omega t + \theta) = V_p \cos \omega t \quad (6.4)$$

and recalling that

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

and

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

we rewrite (6.4) as

$$-A\omega RC \sin \omega t \cos \theta - A\omega RC \cos \omega t \sin \theta + A \cos \omega t \cos \theta - A \sin \omega t \sin \theta = V_p \cos \omega t$$

Collecting sine and cosine terms, equating like terms and, after some more tedious work, solving for amplitude  $A$  and phase angle  $\theta$  we obtain:

$$v_C(t) = \frac{V_P}{\sqrt{1 + (\omega RC)^2}} \cos(\omega t - \tan^{-1}(\omega RC)) \quad (6.5)$$

Obviously, analyzing circuits with sinusoidal excitations when they contain capacitors and/or inductors, using the above procedure is impractical. We will see on the next section that the *complex excitation function* greatly simplifies the procedure of analyzing such circuits. Complex numbers are discussed in Appendix B.

The complex excitation function does not imply complexity of a circuit; it just entails the use of complex numbers. We should remember also that when we say that the imaginary part of a complex number is some value, there is nothing “imaginary” about this value. In other words, the imaginary part is just as “real” as the real part of the complex number but it is defined on a different axis. Thus we display the real part of a complex function on the axis of the reals (usually the  $x$ -axis), and the imaginary part on the imaginary axis or the  $y$ -axis.

### 6.3 The Complex Excitation Function

We recall that the derivatives and integrals of sinusoids always produce sinusoids of the *same frequency* but different amplitude and phase since the cosine and sine functions are 90 degrees out-of-phase. Thus, if

$$v(t) = A \cos(\omega t + \theta)$$

then

$$\frac{dv}{dt} = -\omega A \sin(\omega t + \theta)$$

and if

$$i(t) = B \sin(\omega t + \phi)$$

then

$$\frac{di}{dt} = \omega B \cos(\omega t + \phi)$$

Let us consider the network of Figure 6.2 which consists of resistors, inductors and capacitors, and it is driven (excited) by a sinusoidal voltage source  $v_S(t)$ .

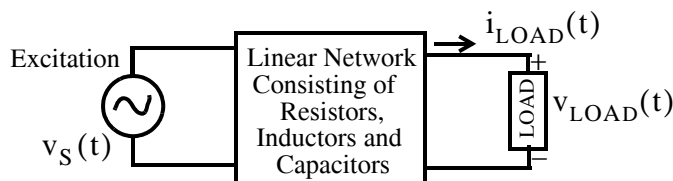


Figure 6.2. General presentation of a network showing excitation and load

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## Chapter 6 Sinusoidal Circuit Analysis

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Let us also define the voltage across the load as  $v_{\text{LOAD}}(t)$ \* as the response. As we know from Chapter 5, the nodal and mesh equations for such circuits are integrodifferential equations, and it is shown in differential equations textbooks† that the *forced response* or *particular solution* of these circuits have the form

$$v_{\text{LD}}(t) = A \cos(\omega t) + B \sin(\omega t)$$

We also know from Euler's identity that

$$\boxed{A \cos \omega t + jA \sin \omega t = A e^{j\omega t}} \quad (6.6)$$

and therefore, the real component is the response due to  $\cos \omega t$  and the imaginary component is the response to  $\sin \omega t$ . We will use Example 6.2 to illustrate the ease by which we can obtain the response of a circuit, which is excited by a sinusoidal source, using the complex function  $A e^{j\omega t}$  approach. In this text, we will represent all sinusoidal variations in terms of the cosine function.

---

### Example 6.2

Repeat Example 6.1, that is, find the capacitor voltage  $v_C(t)$  for the circuit of Figure 6.3 using the complex excitation method.

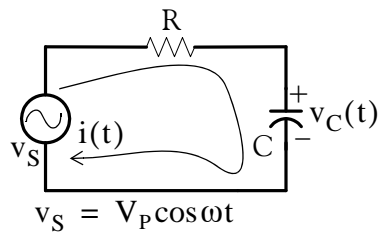


Figure 6.3. Circuit for Example 6.2

#### Solution:

Since

$$\cos \omega t = \text{Re}\{e^{j\omega t}\}$$

we let the excitation be

$$v_S(t) = V_p e^{j\omega t}$$

---

\* Some textbooks denote the voltage across and the current through the load as  $v_L$  and  $i_L$  respectively. As we stated previously, in this text, we use the  $v_{\text{LOAD}}$  and  $i_{\text{LOAD}}$  notations to avoid confusion with the voltage  $v_L$  across and the current  $i_L$  through an inductor.

† This topic is also discussed in *Circuit Analysis II with MATLAB Computing and Simulink / SimPowerSystems Modeling*, ISBN 978-1-934404-19-5.

and thus the response will have the form

$$v_C(t) = V_C e^{j(\omega t + \phi)}$$

As in Example 6.1,

$$RC \frac{dv_C}{dt} + v_C = V_p e^{j\omega t} \quad (6.7)$$

or

$$RC \frac{d}{dt}(V_C e^{j(\omega t + \phi)}) + V_C e^{j(\omega t + \phi)} = V_p e^{j\omega t}$$

or

$$(j\omega RC + 1)V_C e^{j(\omega t + \phi)} = V_p e^{j\omega t}$$

The last expression above shows that radian frequency  $\omega$  is the same for the response as it is for the excitation; therefore we only need to be concerned with the magnitude and the phase angle of the response. Accordingly, we can eliminate the radian frequency  $\omega$  by dividing both sides of that expression by  $e^{j\omega t}$  and thus the input–output (excitation–response) relation reduces to

$$(j\omega RC + 1)V_C e^{j\phi} = V_p$$

from which

$$V_C e^{j\phi} = \frac{V_p}{j\omega RC + 1} = \frac{V_p}{\sqrt{1 + \omega^2 R^2 C^2} e^{j[\tan^{-1}(\omega RC)]}} = \frac{V_p}{\sqrt{1 + \omega^2 R^2 C^2}} e^{-j[\tan^{-1}(\omega RC)]}$$

This expression above shows the response as a function of the maximum value of the excitation, its radian frequency and the circuit constants  $R$  and  $C$ .

If we wish to express the response in complete form, we simply multiply both sides by  $e^{j\omega t}$  and we obtain

$$V_C e^{j(\omega t + \phi)} = \frac{V_p}{\sqrt{1 + \omega^2 R^2 C^2}} e^{j[\omega t - \tan^{-1}(\omega RC)]}$$

Finally, since the excitation is the real part of the complex excitation, we use Euler's identity on both sides and equating reals parts, we obtain

$$v_C(t) = V_C \cos(\omega t + \phi) = \frac{V_p}{\sqrt{1 + \omega^2 R^2 C^2}} \cos[\omega t - \tan^{-1}(\omega RC)]$$


---

The first part of the above procedure where the excitation–response relation is simplified to amplitude and phase relationship is known as *time–domain to frequency–domain transformation*; the second part where the excitation–response is put back to its sinusoidal form is known as *fre-*

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## Chapter 6 Sinusoidal Circuit Analysis

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quency-domain to time-domain transformation. For brevity, we will denote the time domain as the  $t$ -domain, and the frequency domain as the  $j\omega$ -domain.

If a sinusoid is given in terms of the sine function, we must first convert it to a cosine function. Thus,

$$m(t) = A \sin(\omega t + \theta) = A \cos(\omega t + \theta - 90^\circ) \quad (6.8)$$

and in the  $j\omega$ -domain it is expressed as

$$\mathbf{M} = A e^{j(\theta - 90^\circ)} = A \angle(\theta - 90^\circ) \quad (6.9)$$

where  $\mathbf{M}$  represents a *phasor* (rotating vector) voltage  $\mathbf{V}$  or current  $\mathbf{I}$ .

In summary, the  $t$ -domain, to  $j\omega$ -domain transformation procedure is as follows:

1. Express the given sinusoid as a cosine function
2. Express the cosine function as the real part of the complex excitation using Euler's identity
3. Extract the magnitude and phase angle from it.

---

### Example 6.3

Transform the sinusoid  $v(t) = 10 \sin(100t - 60^\circ)$  to its equivalent  $j\omega$ -domain expression.

**Solution:**

For this example, we have

$$v(t) = 10 \sin(100t - 60^\circ) = 10 \cos(100t - 60^\circ - 90^\circ)$$

or

$$v(t) = 10 \cos(100t - 150^\circ) = \operatorname{Re} \left\{ 10 e^{j(100t - 150^\circ)} \right\}$$

Since the  $j\omega$ -domain contains only the amplitude and phase, we extract these from the bracketed term on the right side of the above expression, and we obtain the phasor  $\mathbf{V}$  as

$$\mathbf{V} = 10 e^{-j150^\circ} = 10 \angle -150^\circ$$

---

The  $j\omega$ -domain to  $t$ -domain transformation procedure is as follows:

1. Convert the given phasor from polar to exponential form
2. Add the radian frequency  $\omega$  multiplied by  $t$  to the exponential form
3. Extract the real part from it.

---

### Example 6.4

Transform the phasor  $\mathbf{I} = 120\angle-90^\circ$  to its equivalent time-domain expression.

**Solution:**

First, we express the given phasor in exponential form, that is,

$$\mathbf{I} = 120\angle-90^\circ = 120e^{-j90^\circ}$$

Next, adding the radian frequency  $\omega$  multiplied by  $t$  to the exponent of the above expression we obtain

$$i(t) = 120e^{j(\omega t - 90^\circ)}$$

and finally we extract the real part from it. Then,

$$i(t) = \operatorname{Re}\left\{120e^{j(\omega t - 90^\circ)}\right\} = 120\cos(\omega t - 90^\circ) = 120\sin\omega t$$

---

We can add, subtract, multiply and divide sinusoids of the same frequency using phasors as illustrated by the following example.

---

### Example 6.5

It is given that  $i_1(t) = 10\cos(120\pi t + 45^\circ)$  and  $i_2(t) = 5\sin(120\pi t - 45^\circ)$ . Compute the sum  $i(t) = i_1(t) + i_2(t)$ .

**Solution:**

As a first step, we express  $i_2(t)$  as a cosine function, that is,

$$i_2(t) = 5\sin(120\pi t - 45^\circ) = 5\cos(120\pi t - 45^\circ - 90^\circ) = 5\cos(120\pi t - 135^\circ)$$

Next, we perform the  $t$ -domain to  $j\omega$ -domain transformation and we obtain the phasors



$$\mathbf{I}_1 = 10\angle 45^\circ \quad \text{and} \quad \mathbf{I}_2 = 5\angle -135^\circ$$

and by addition,

$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 = 10\angle 45^\circ + 5\angle -135^\circ = 10\left(\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}\right) + 5\left(-\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2}\right)$$

or

$$\mathbf{I} = 5\left(\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}\right) = 5\angle 45^\circ$$

and finally transforming the phasor  $\mathbf{I}$  into the  $t$  – domain , we obtain

$$i(t) = 5 \cos(120\pi t + 45^\circ)$$

Also, for brevity, in our subsequent discussion we will designate resistive, inductive and capacitive circuits as  $R$  ,  $L$  , and  $C$  respectively.

### 6.4 Phasors in R, L, and C Circuits

The circuit analysis of circuits containing  $R$  ,  $L$  , and  $C$  devices, and which are excited by sinusoidal sources, is considerably simplified with the use of phasor voltages and phasor currents which we will represent by the boldface capital letters  $\mathbf{V}$  and  $\mathbf{I}$  respectively. We will now derive  $\mathbf{V}$  and  $\mathbf{I}$  phasor relationships in the  $j\omega$  – domain . *We must always remember that phasor quantities exist only in the  $j\omega$  – domain .*

#### 1. $\mathbf{V}$ and $\mathbf{I}$ phasor relationship in $R$ branches

Consider circuit 6.4 (a) below where the load is purely resistive.

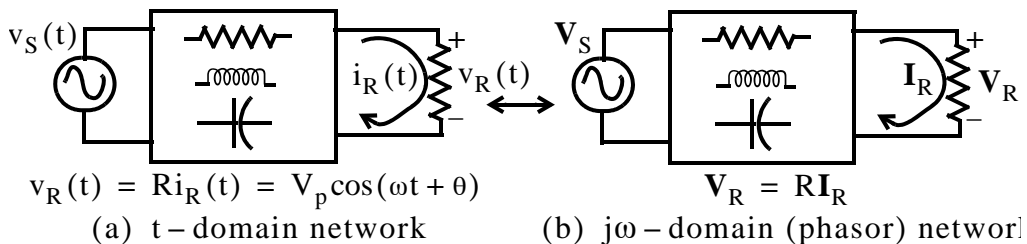


Figure 6.4. Voltage across a resistive load in  $t$  – domain and  $j\omega$  – domain

We know from Ohm’s law that  $v_R(t) = R i_R(t)$  where the resistance  $R$  is a constant. We will show that this relationship also holds for the phasors  $\mathbf{V}_R$  and  $\mathbf{I}_R$  shown in circuit 6.4 (b), that is, we will prove that

$$\mathbf{V}_R = R\mathbf{I}_R$$

**Proof:**

In circuit 6.4 (a) we let  $v_R(t)$  be a complex voltage, that is,

$$V_p e^{j(\omega t + \phi)} = V_p \cos(\omega t + \phi) + jV_p \sin(\omega t + \phi) \quad (6.10)$$

and since  $R$  is a constant, it will produce a current of the same frequency  $\omega$  and the same phase  $\phi^*$  whose form will be

$$I_p e^{j(\omega t + \phi)} = I_p \cos(\omega t + \phi) + jI_p \sin(\omega t + \phi)$$

and by Ohm's law,

$$V_p e^{j(\omega t + \phi)} = R I_p e^{j(\omega t + \phi)} \quad (6.11)$$

Transforming (6.11) to the  $j\omega$  – domain, we obtain the phasor relationship

$$V_p e^{j\phi} = R I_p e^{j\phi} \quad \text{or} \quad V_p \angle \phi = R I_p \angle \phi$$

Since the phasor current  $\mathbf{I}$  is *in-phase* with the voltage  $\mathbf{V}$  (both  $\mathbf{I}$  and  $\mathbf{V}$  have the same phase  $\phi$ ), we let

$$V_p \angle \phi = \mathbf{V}_R \quad \text{and} \quad I_p \angle \phi = \mathbf{I}_R$$

and it follows that

$$\mathbf{V}_R = R\mathbf{I}_R$$

Therefore, the phasor  $\mathbf{V}$  and  $\mathbf{I}$  relationship in resistors, obeys Ohm's law also, and the current through a resistor is always in-phase with the voltage across that resistor.

### Example 6.6

For the network in Figure 6.5, find  $i_R(t)$  when  $v_R(t) = 40 \sin(377t - 75^\circ)$ .

**Solution:**

We first perform the  $t$  – domain to  $j\omega$  – domain i.e.,  $v_R(t) \Leftrightarrow \mathbf{V}_R$  transformation as follows:

$$v_R(t) = 40 \sin(377t - 75^\circ) = 40 \cos(377t - 165^\circ) \Leftrightarrow \mathbf{V}_R = 40 \angle -165^\circ$$

\* The phase will be the same since neither differentiation nor integration is performed here.

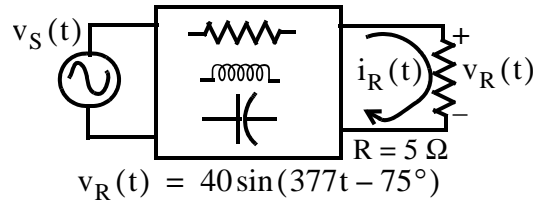


Figure 6.5. Voltage across the resistive load of Example 6.6

Then,

$$\mathbf{I}_R = \frac{\mathbf{V}_R}{R} = \frac{40\angle-165^\circ}{5} = 8\angle-165^\circ \text{ A}$$

Therefore,

$$\underbrace{\mathbf{I}_R = 8\angle-165^\circ \text{ A}}_{j\omega - \text{domain}} \Leftrightarrow \underbrace{i_R(t) = 8\cos(377t - 165^\circ) = 8\sin(377t - 75^\circ) \text{ A}}_{t - \text{domain}}$$

Alternately, since the resistance  $R$  is a constant, we can compute  $i_R(t)$  directly from the  $t$ -domain expression for  $v_R(t)$ , that is,

$$i_R(t) = \frac{v_R(t)}{R} = \frac{40\sin(377t - 75^\circ)}{5} = 8\sin(377t - 75^\circ) \text{ A}$$

## 2. V and I phasor relationship in L branches

Consider circuit 6.6 (a) below where the load is purely inductive.

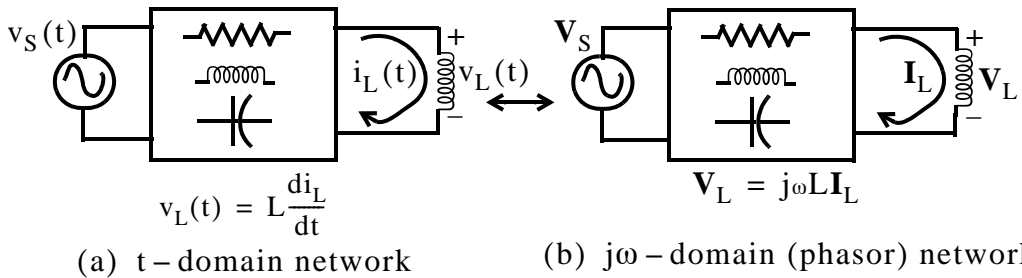


Figure 6.6. Voltage across an inductive load in  $t$ -domain and  $j\omega$ -domain

We will prove that the relationship between the phasors  $\mathbf{V}_L$  and  $\mathbf{I}_L$  shown in circuit 6.6 (b) is

$$\mathbf{V}_L = j\omega L \mathbf{I}_L \quad (6.12)$$

**Proof:**

In circuit 6.6 (a) we let  $v_L(t)$  be a complex voltage, that is,

$$V_p e^{j(\omega t + \phi)} = V_p \cos(\omega t + \phi) + jV_p \sin(\omega t + \phi) \quad (6.13)$$

and recalling that if  $x(t) = \sin(\omega t + \phi)$  then  $dx/dt = \omega \cos(\omega t + \phi)$ , that is, differentiation (or integration) does not change the radian frequency  $\omega$  or the phase angle  $\phi$ , the current through the inductor will have the form

$$I_p e^{j(\omega t + \phi)} = I_p \cos(\omega t + \phi) + jI_p \sin(\omega t + \phi) \quad (6.14)$$

and since

$$v_L(t) = L \frac{di_L}{dt}$$

then,

$$V_p e^{j(\omega t + \phi)} = L \frac{d}{dt}(I_p e^{j(\omega t + \phi)}) = j\omega L I_p e^{j(\omega t + \phi)} \quad (6.15)$$

Next, transforming (6.16) to the  $j\omega$  – domain, we obtain the phasor relationship

$$V_p e^{j\phi} = j\omega L I_p e^{j\phi} \quad \text{or} \quad V_p \angle \phi = j\omega L I_p \angle \phi$$

and letting

$$V_p \angle \phi = V_L \quad \text{and} \quad I_p \angle \phi = I_L$$

we obtain

$$\boxed{V_L = j\omega L I_L} \quad (6.16)$$

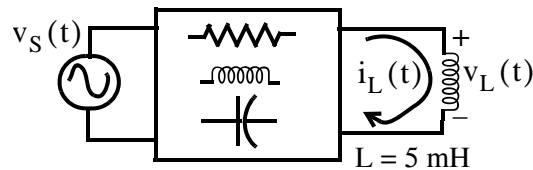
The presence of the  $j$  operator in (6.17) indicates that the voltage across an inductor leads the current through it by  $90^\circ$ .

### Example 6.7

For the network in Figure 6.7, find  $i_L(t)$  when  $v_L(t) = 40 \sin(2t - 75^\circ)$ .

#### Solution:

We first perform the  $t$  – domain to  $j\omega$  – domain i.e.,  $v_L(t) \Leftrightarrow V_L$  transformation as follows:



$$v_L(t) = 40 \sin(2t - 75^\circ)$$

Figure 6.7. Voltage across the inductive load in Example 6.7

$$v_L(t) = 40 \sin(2t - 75^\circ) = 40 \cos(2t - 165^\circ) \Leftrightarrow \mathbf{V}_L = 40 \angle -165^\circ \text{ mV}$$

and

$$\mathbf{I}_L = \frac{\mathbf{V}_L}{j\omega L} = \frac{(40 \angle -165^\circ) \times 10^{-3}}{j10 \times 10^{-3}} = \frac{40 \angle -165^\circ}{10 \angle 90^\circ} = 4 \angle -255^\circ = 4 \angle 105^\circ \text{ A}$$

Therefore,

$$\underbrace{\mathbf{I}_L = 4 \angle 105^\circ \text{ A}}_{j\omega\text{-domain}} \Leftrightarrow \underbrace{i_L(t) = 4 \cos(2t + 105^\circ) = 4 \sin(2t - 165^\circ) \text{ A}}_{t\text{-domain}}$$

### 3. V and I phasor relationship in C branches

Consider circuit 6.8 (a) below where the load is purely capacitive.

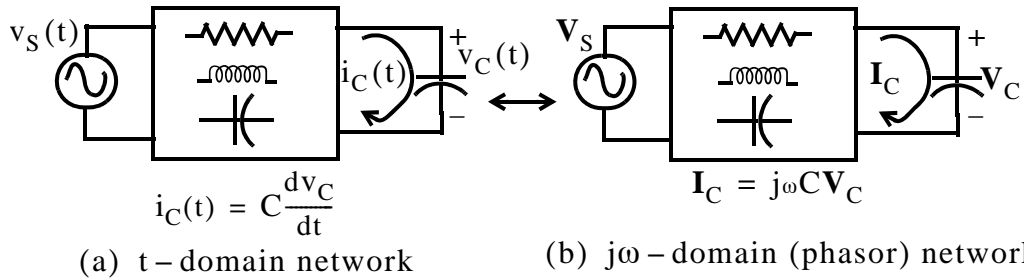


Figure 6.8. Voltage across a capacitive load in t-domain and jω-domain

We will prove that the relationship between the phasors  $\mathbf{V}_C$  and  $\mathbf{I}_C$  shown in the network in Figure 6.8 (b) is

$$\mathbf{I}_C = j\omega C \mathbf{V}_C \quad (6.17)$$

**Proof:**

In circuit 6.8 (a) we let  $v_C(t)$  be a complex voltage, that is,

$$V_p e^{j(\omega t + \phi)} = V_p \cos(\omega t + \phi) + jV_p \sin(\omega t + \phi)$$

then the current through the capacitor will have the form

$$I_p e^{j(\omega t + \phi)} = I_p \cos(\omega t + \phi) + jI_p \sin(\omega t + \phi)$$

and since

$$i_C(t) = \frac{dv_C}{dt}$$

It follows that

$$I_p e^{j(\omega t + \phi)} = C \frac{d}{dt} (V_p e^{j(\omega t + \phi)}) = j\omega C V_p e^{j(\omega t + \phi)} \quad (6.18)$$

Next, transforming (6.18) to the  $j\omega$  – domain, we obtain the phasor relationship

$$I_p e^{j\phi} = j\omega C V_p e^{j\phi} \quad \text{or} \quad I_p \angle\phi = j\omega C V_p \angle\phi$$

and letting

$$I_p \angle\phi = I_C \quad \text{and} \quad V_p \angle\phi = V_C$$

we obtain

$$\boxed{I_C = j\omega C V_C} \quad (6.19)$$

The presence of the  $j$  operator in (6.19) indicates that the current through a capacitor leads the voltage across it by  $90^\circ$ .

### Example 6.8

For the circuit shown below, find  $i_C(t)$  when  $v_C(t) = 170 \cos(60\pi t - 45^\circ)$ .

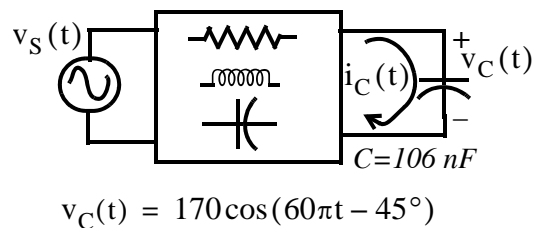


Figure 6.9. Voltage across the capacitive load of Example 6.8

#### Solution:

We first perform the  $t$  – domain to  $j\omega$  – domain i.e.,  $v_C(t) \Leftrightarrow V_C$  transformation as follows:

$$v_C(t) = 170 \cos(60\pi t - 45^\circ) \Leftrightarrow V_C = 170 \angle -45^\circ$$

Then,

$$\begin{aligned} I_C &= j\omega C V_C = j \times 60\pi \times 106 \times 10^{-9} \times 170 \angle -45^\circ = 1 \angle 90^\circ \times 3.4 \times 10^{-3} \times 1 \angle -45^\circ \\ &= 3.4 \times 10^{-3} \angle 45^\circ = 3.4 \angle 45^\circ \text{ mA} \end{aligned}$$

Therefore,

$$\underbrace{I_C = 3.4 \angle 45^\circ \text{ mA}}_{j\omega - \text{domain}} \Leftrightarrow \underbrace{i_C(t) = 3.4 \cos(60\pi t + 45^\circ) \text{ mA}}_{t - \text{domain}}$$

6.5 Impedance

Consider the  $t$  – domain circuit in Figure 6.10 (a) and its equivalent phasor circuit shown in 6.10 (b).

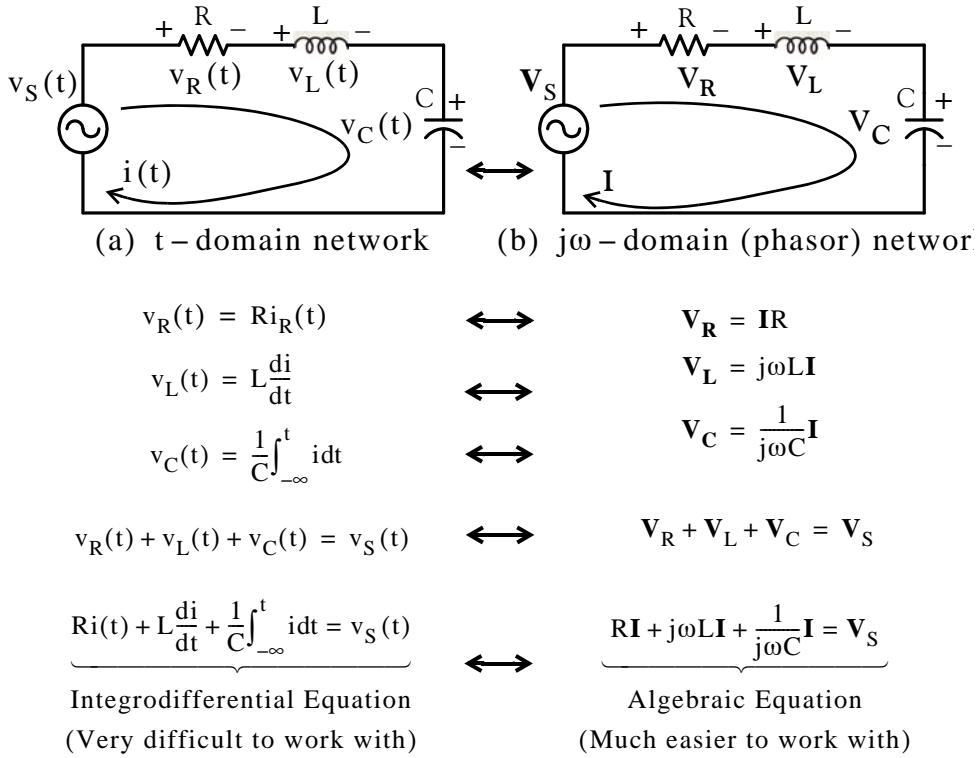


Figure 6.10. The  $t$  – domain and  $j\omega$  – domain relationships in a series RLC circuit

The last equation on the right side of the phasor circuit may be written as

$$\left( R + j\omega L + \frac{1}{j\omega C} \right) \mathbf{I} = \mathbf{V}_S \tag{6.20}$$

and dividing both sides of (6.20) by  $\mathbf{I}$  we obtain the *impedance* which, by definition, is

$$\boxed{\text{Impedance} = Z = \frac{\text{Phasor Voltage}}{\text{Phasor Current}} = \frac{\mathbf{V}_S}{\mathbf{I}} = R + j\omega L + \frac{1}{j\omega C}} \tag{6.21}$$

Expression (6.21) is referred to as *Ohm’s law for AC Circuits*.

Like resistance, the unit of impedance is the Ohm ( $\Omega$ ).

We can express the impedance  $Z$  as the sum of a real and an imaginary component as follows:

Since

$$\frac{1}{j} = \frac{1}{j} \cdot \frac{j}{j} = \frac{j}{j^2} = -j$$

it follows that

$$\frac{1}{j\omega C} = -j\frac{1}{\omega C}$$

and thus

$$\boxed{Z = R + j\left(\omega L - \frac{1}{\omega C}\right)} \quad (6.22)$$

We can also express (6.22) in polar form as

$$\boxed{Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \angle \tan^{-1}\left(\omega L - \frac{1}{\omega C}\right)/R} \quad (6.23)$$

We must remember that *the impedance is not a phasor*; it is a complex quantity whose real part is the resistance  $R$  and the imaginary part is  $\omega L - 1/\omega C$ , that is,

$$\operatorname{Re}\{Z\} = R \quad \text{and} \quad \operatorname{Im}\{Z\} = \omega L - \frac{1}{\omega C} \quad (6.24)$$

The imaginary part of the impedance  $Z$  is called *reactance* and it is denoted with the letter  $X$ . The two components of reactance are the *inductive reactance*  $X_L$  and the *capacitive reactance*  $X_C$ , i.e.,

$$X = X_L + X_C = \omega L - 1/\omega C \quad (6.25)$$

$$X_L = \omega L \quad (6.26)$$

$$X_C = 1/\omega C \quad (6.27)$$

The unit of the inductive and capacitive reactances is also the Ohm ( $\Omega$ ).

In terms of reactances, the impedance can be expressed as

$$\boxed{Z = R + jX = R + j(X_L - X_C) = \sqrt{R^2 + (X_L - X_C)^2} \angle \tan^{-1}[(X_L - X_C)/R]} \quad (6.28)$$

By a procedure similar to that of Chapter 2, we can show that impedances combine as resistances do.

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### Example 6.9

For the circuit in Figure 6.11, find the current  $i(t)$  given that  $v_s(t) = 100\cos(100t - 30^\circ)$ .



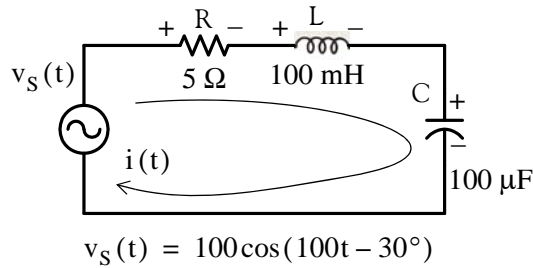


Figure 6.11. Circuit for Example 6.9

**Solution:**

If we attempt to solve this problem in the time-domain directly, we will need to solve an integro-differential equation. But as we now know, a much easier solution is with the transformation of the given circuit to a phasor circuit. Here,  $\omega = 100 \text{ rad/s}$  and thus

$$j\omega L = jX_L = j100 \times 0.1 = j10 \text{ } \Omega$$

and

$$\frac{1}{j\omega C} = -j\frac{1}{\omega C} = -jX_C = -j\frac{1}{10^2 \times 10^2 \times 10^{-6}} = -j100$$

Also,

$$\mathbf{V}_S = 100 \angle -30^\circ$$

and the phasor circuit is as shown in Figure 6.12.

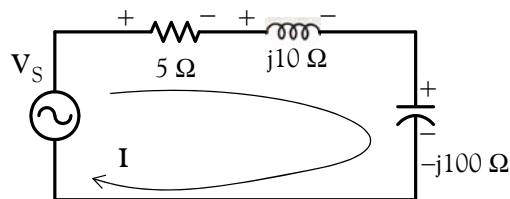


Figure 6.12. Phasor circuit for Example 6.9

From the phasor circuit in Figure 6.12,

$$\mathbf{Z} = 5 + j10 - j100 = 5 - j90 = \sqrt{5^2 + 90^2} \angle \tan^{-1}(-90/5) = 90.14 \angle -86.82^\circ$$

and

$$\mathbf{I} = \frac{\mathbf{V}_S}{\mathbf{Z}} = \frac{100 \angle -30^\circ}{90.14 \angle -86.82^\circ} = 1.11 \angle [-30^\circ - (-86.82^\circ)]$$

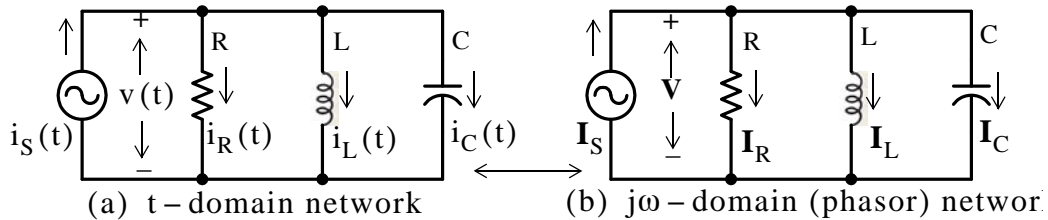
Therefore,

$$\mathbf{I} = 1.11 \angle 56.82^\circ \Leftrightarrow i(t) = 1.11 \cos(100t + 56.82)$$



## 6.6 Admittance

Consider the  $t$  – domain circuit in Figure 6.13 (a) and its equivalent phasor circuit shown in Figure 6.13 (b).



$i_G(t) = Gv(t)$	$\longleftrightarrow$	$I_G = GV$
$i_C(t) = C \frac{dv}{dt}$	$\longleftrightarrow$	$I_C = j\omega CV$
$i_L(t) = \frac{1}{L} \int_{-\infty}^t v dt$	$\longleftrightarrow$	$I_L = \frac{1}{j\omega L} V$
$i_G(t) + i_L(t) + i_C(t) = i_S(t)$	$\longleftrightarrow$	$I_R + I_L + I_C = I_S$
$Gv(t) + C \frac{dv}{dt} + \frac{1}{L} \int_{-\infty}^t v dt = i_S(t)$	$\longleftrightarrow$	$GV + j\omega CV + \frac{1}{j\omega L} V = I_S$
Integrodifferential Equation (Very difficult to work with)		Algebraic Equation (Much easier to work with)

Figure 6.13. The  $t$  – domain and  $j\omega$  – domain relationships in a parallel RLC circuit

The last equation of the phasor circuit may be written as

$$\left( G + \frac{1}{j\omega L} + j\omega C \right) \mathbf{V} = \mathbf{I}_S \quad (6.29)$$

Dividing both sides of (6.29) by  $\mathbf{V}$ , we obtain the *admittance*, that is, by definition

$$\boxed{\text{Admittance} = Y = \frac{\text{Phasor Current}}{\text{Phasor Voltage}} = \frac{\mathbf{I}_S}{\mathbf{V}} = G + \frac{1}{j\omega L} + j\omega C = \frac{1}{Z}} \quad (6.30)$$

Here we observe that the admittance  $Y$  is the reciprocal of the impedance  $Z$  as conductance  $G$  is the reciprocal of the resistance  $R$ .

Like conductance, the unit of admittance is the Siemens or mho ( $\Omega^{-1}$ ).

As with the impedance  $Z$ , we can express the admittance  $Y$  as the sum of a real component and an imaginary component as follows:

$$\boxed{Y = G + j\left(\omega C - \frac{1}{\omega L}\right)} \quad (6.31)$$

and in polar form

$$\boxed{Y = \sqrt{G^2 + \left(\omega C - \frac{1}{\omega L}\right)^2} \angle \tan^{-1}\left[\left(\omega C - \frac{1}{\omega L}\right)/G\right]} \quad (6.32)$$

Like the impedance  $Z$ , the admittance  $Y$  is not a phasor; it is a complex quantity whose real part is the conductance  $G$  and the imaginary part is  $\omega C - \frac{1}{\omega L}$ , that is,

$$\operatorname{Re}\{Y\} = G \quad \text{and} \quad \operatorname{Im}\{Y\} = \omega C - \frac{1}{\omega L} \quad (6.33)$$

The imaginary part of the admittance  $Y$  is called *susceptance* and it is denoted with the letter  $B$ . The two components of susceptance are the *capacitive susceptance*  $B_C$  and the *inductive susceptance*  $B_L$ , that is,

$$B = B_C + B_L = \omega C - 1/\omega L \quad (6.34)$$

$$B_C = \omega C \quad (6.35)$$

$$B_L = 1/\omega L \quad (6.36)$$

The unit of the susceptances  $B_C$  and  $B_L$  is also the Siemens ( $\Omega^{-1}$ ).

In terms of susceptances, the admittance  $Y$  can be expressed as

$$\boxed{Y = G + jB = G + j(B_C - B_L) = \sqrt{G^2 + (B_C - B_L)^2} \angle \tan^{-1}[(B_C - B_L)/G]} \quad (6.37)$$

By a procedure similar to that of Chapter 2, we can show that admittances combine as conductances do.

**Duality** is a term meaning that there is a similarity in which some quantities are related to others. The dual quantities we have encountered thus far are listed in Table 6.1.

TABLE 6.1 Dual quantities

Series	Parallel
Voltage	Current
Resistance	Conductance
Thevenin	Norton
Inductance	Capacitance
Reactance	Susceptance
Impedance	Admittance

**Example 6.10**

Consider the series and parallel networks shown in Figure 6.14. How should their real and imaginary terms be related so that they will be equivalent?

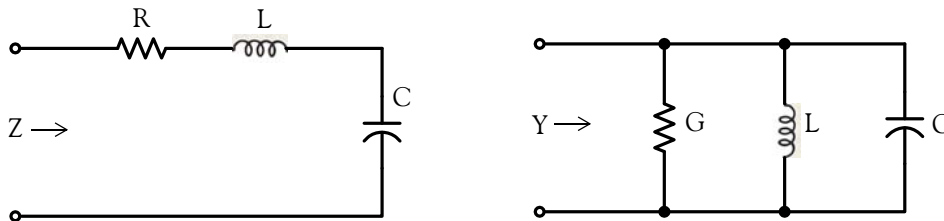


Figure 6.14. Networks for Example 6.10

**Solution:**

For these circuits to be equivalent, their impedances  $Z$  or admittances  $Y$  must be equal. Therefore,

$$Y = \frac{1}{Z} = \frac{1}{R + jX} = G + jB = \frac{1}{R + jX} \cdot \frac{R - jX}{R - jX} = \frac{R - jX}{R^2 + X^2} = \frac{R}{R^2 + X^2} - j \frac{X}{R^2 + X^2}$$

and equating reals and imaginaries we obtain

$$\boxed{G = \frac{R}{R^2 + X^2} \quad \text{and} \quad B = \frac{-X}{R^2 + X^2}} \quad (6.38)$$

Relation (6.38) is worth memorizing.

**Example 6.11**

Compute  $Z$  and  $Y$  for the network in Figure 6.15.

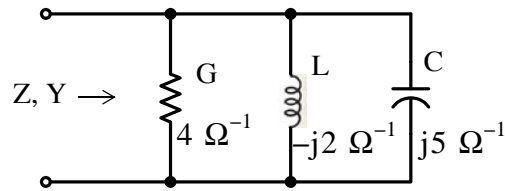


Figure 6.15. Network for Example 6.11

**Solution:**

Since this is a parallel network, it is easier to compute the admittance  $Y$  first. Thus,

$$Y = G + \frac{1}{j\omega L} + j\omega C = 4 - j2 + j5 = 4 + j3 = 5 \angle 36.9^\circ$$

Since the impedance is the reciprocal of admittance, it follows that

$$Z = \frac{1}{Y} = \frac{1}{5 \angle 36.9^\circ} = 0.2 \angle -36.9^\circ = 0.16 - j0.12$$

### Example 6.12

Compute  $Z$  and  $Y$  for the circuit shown below. Verify your answers with MATLAB.

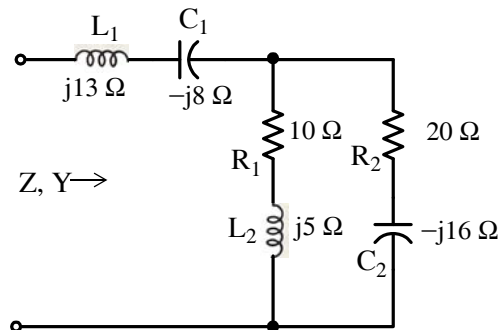


Figure 6.16. Network for Example 6.12

**Solution:**

Let the given network be represented as shown in Figure 6.17 where  $Z_1 = j13 - j8 = j5$ ,  $Z_2 = 10 + j5$ , and  $Z_3 = 20 - j16$

Then,

$$\begin{aligned} Z &= Z_1 + \frac{Z_2 Z_3}{Z_2 + Z_3} = j5 + \frac{(10 + j5)(20 - j16)}{10 + j5 + 20 - j16} = j5 + \frac{(11.18 \angle 26.6^\circ)(25.61 \angle -38.7^\circ)}{31.95 \angle -20.1^\circ} \\ &= j5 + 8.96 \angle 8^\circ = j5 + 8.87 + j1.25 = 8.87 + j6.25 = 10.85 \angle 35.2^\circ \end{aligned}$$

and

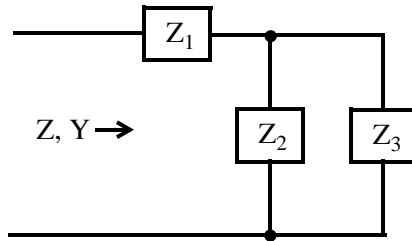


Figure 6.17. Simplified network for Example 6.12

$$Y = \frac{1}{Z} = \frac{1}{10.85 \angle 35.2^\circ} = 0.092 \angle -35.2^\circ = 0.0754 - j0.531$$

Check with MATLAB:

```
z1=j*5; z2=10+j*5; z3=20-j*16; z=z1+(z2*z3/(z2+z3)), y=1/z % Impedance z, Admittance y
```

```
z =  
8.8737 + 6.2537i
```

```
y =  
0.0753 -0.0531i
```

As we found out in Example 6.1, analyzing circuits with sinusoidal excitations when they contain capacitors and/or inductors, using the procedure in that example is impractical. However, we can use a Simulink / SimPowerSystems model to display sinusoidal voltages and currents in branches of a circuit as illustrated in the Example 6.13 below.

### Example 6.13

Create a Simulink / SimPowerSystems model to display the potential difference  $v_a - v_b$  in the circuit of Figure 6.18.

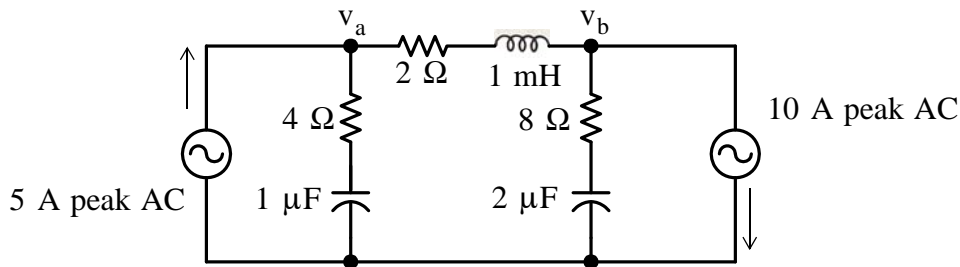


Figure 6.18. Circuit to be analyzed with a Simulink / SimPowerSystems model

**Solution:**

The model is shown in Figure 6.19, and the waveforms in Figure 6.20 where

## Chapter 6 Sinusoidal Circuit Analysis

$$v_a = 5 \sin \omega t, v_b = -15 \sin \omega t, v_{ab} = 20 \sin \omega t, \omega = 2\pi f = 0.4\pi$$

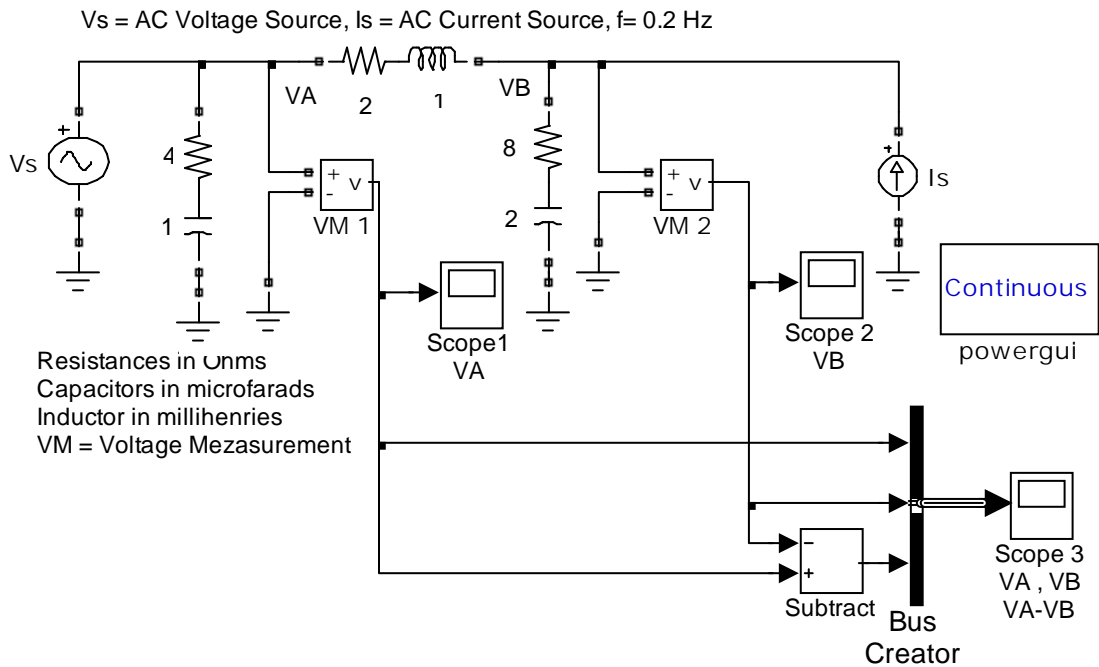


Figure 6.19. Simulink / SimPowerSystems model for the circuit in Example 6.13

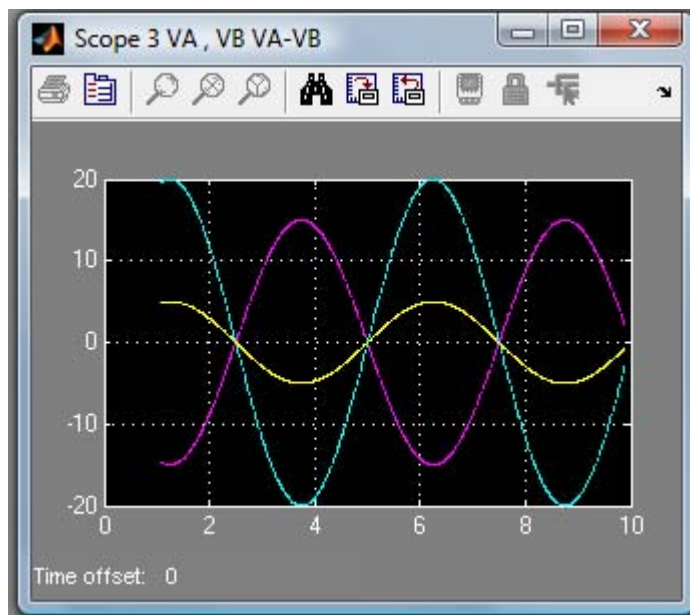


Figure 6.20. Waveforms for  $v_a$ ,  $v_b$ , and  $v_a - v_b$

## 6.7 Summary

- Excitations or driving functions refer to the applied voltages and currents in electric circuits.
- A response is anything we define it as a response. Typically response is the voltage or current in the “load” part of the circuit or any other part of it.
- If the excitation is a constant voltage or current, the response will also be some constant value.
- If the excitation is a sinusoidal voltage or current, in general, the response will also be sinusoidal with the same frequency but with different amplitude and phase.
- If the excitation is a time-varying function such as a sinusoid, inductors and capacitors do not behave like short circuits and open circuits respectively as they do when the excitation is a constant and steady-state conditions are reached. They behave entirely different.
- Circuit analysis in circuits where the excitation is a time-varying quantity such as a sinusoid is difficult and time consuming and thus impractical in the  $t$  – domain .
- The complex excitation function greatly simplifies the procedure of analyzing such circuits when excitation is a time-varying quantity such as a sinusoid.
- The procedure where the excitation–response relation is simplified to amplitude and phase relationship is known as time–domain to frequency–domain transformation.
- The procedure where the excitation–response is put back to its sinusoidal form is known as frequency–domain to time–domain transformation.
- For brevity, we denote the time domain as the  $t$  – domain , and the frequency domain as the  $j\omega$  – domain .
- If a sinusoid is given in terms of the sine function, it is convenient to convert it to a cosine function using the identity  $m(t) = A \sin(\omega t + \theta) = A \cos(\omega t + \theta - 90^\circ)$  before converting it to the  $j\omega$  – domain .
- The  $t$  – domain to  $j\omega$  – domain transformation procedure is as follows:
  1. Express the given sinusoid as a cosine function
  2. Express the cosine function as the real part of the complex excitation using Euler’s identity
  3. Extract the magnitude and phase angle from it.
- The  $j\omega$  – domain to  $t$  – domain transformation procedure is as follows:
  1. Convert the given phasor from polar to exponential form
  2. Add the radian frequency  $\omega$  multiplied by  $t$  to the exponential form



3. Extract the real part from it.

- The circuit analysis of circuits containing  $R$ ,  $L$ , and  $C$  devices, and which are excited by sinusoidal sources, is considerably simplified with the use of phasor voltages and phasor currents which we represent by the boldface capital letters  $\mathbf{V}$  and  $\mathbf{I}$  respectively.
- Phasor quantities exist only in the  $j\omega$  – domain
- In the  $j\omega$  – domain the current through a resistor is always in-phase with the voltage across that resistor
- In the  $j\omega$  – domain the current through an inductor lags the voltage across that inductor by  $90^\circ$
- In the  $j\omega$  – domain the current through a capacitor leads the voltage across that capacitor by  $90^\circ$
- In the  $j\omega$  – domain the impedance  $Z$  is defined as

$$\text{Impedance} = Z = \frac{\text{Phasor Voltage}}{\text{Phasor Current}} = \frac{\mathbf{V}_S}{\mathbf{I}} = R + j\omega L + \frac{1}{j\omega C}$$

- Like resistance, the unit of impedance is the Ohm ( $\Omega$ ).
- Impedance is a complex quantity whose real part is the resistance  $R$ , and the imaginary part is  $\omega L - 1/\omega C$ , that is,

$$\text{Re}\{Z\} = R \quad \text{and} \quad \text{Im}\{Z\} = \omega L - \frac{1}{\omega C}$$

- In polar form the impedance is expressed as

$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \angle \tan^{-1}\left(\omega L - \frac{1}{\omega C}\right) / R$$

- The imaginary part of the impedance  $Z$  is called reactance and it is denoted with the letter  $X$ . The two components of reactance are the inductive reactance  $X_L$  and the capacitive reactance  $X_C$ , i.e.,

$$X = X_L - X_C = \omega L - \frac{1}{\omega C}$$

- The unit of the inductive and capacitive reactances is also the Ohm ( $\Omega$ ).
- In the  $j\omega$  – domain the admittance  $Y$  is defined as

$$\text{Admittance} = Y = \frac{\text{Phasor Current}}{\text{Phasor Voltage}} = \frac{\mathbf{I}_S}{\mathbf{V}} = G + \frac{1}{j\omega L} + j\omega C = \frac{1}{Z}$$

- The admittance  $Y$  is the reciprocal of the impedance  $Z$  as conductance  $G$  is the reciprocal of the resistance  $R$ .
- The unit of admittance is the siemens or mho ( $\Omega^{-1}$ ).
- The admittance  $Y$  is a complex quantity whose real part is the conductance  $G$  and the imaginary part is  $\omega C - \frac{1}{\omega L}$ , that is,

$$\text{Re}\{Y\} = G \quad \text{and} \quad \text{Im}\{Y\} = \omega C - \frac{1}{\omega L}$$

- The imaginary part of the admittance  $Y$  is called susceptance and it is denoted with the letter  $B$ . The two components of susceptance are the capacitive susceptance  $B_C$  and the inductive susceptance  $B_L$ , that is,

$$B = B_C - B_L = \omega C - \frac{1}{\omega L}$$

- In polar form the admittance is expressed as

$$Y = \sqrt{G^2 + \left(\omega C - \frac{1}{\omega L}\right)^2} \angle \tan^{-1} \left[ \left(\omega C - \frac{1}{\omega L}\right) / G \right]$$

- The unit of the susceptances  $B_C$  and  $B_L$  is also the siemens ( $\Omega^{-1}$ ).
- Admittances combine as conductances do.
- In phasor circuit analysis, conductance is not necessarily the reciprocal of resistance, and susceptance is not the negative reciprocal of reactance. Whenever we deal with resistance and reactance we must think of devices in series, and when we deal with conductance and susceptance we must think of devices in parallel. However, the admittance is always the reciprocal of the impedance
- The ratio  $\mathbf{V}/\mathbf{I}$  of the phasor voltage to the phasor current exists only in the  $j\omega$  – domain and it is not the ratio  $v(t)/i(t)$  in the  $t$  – domain. Although the ratio  $v(t)/i(t)$  could yield some value, this value is not impedance. Similarly, the ratio  $i(t)/v(t)$  is not admittance.
- Duality is a term meaning that there is a similarity in which some quantities are related to others.

### 6.8 Exercises

#### Multiple Choice

1. Phasor voltages and phasor currents can be used in the  $t$  – domain if a circuit contains
  - A. independent and dependent sources with resistors only
  - B. independent and dependent sources with resistors and inductors only
  - C. independent and dependent sources with resistors and capacitors only
  - D. independent and dependent sources with resistors, inductors, and capacitors
  - E. none of the above
2. If the excitation in a circuit is a single sinusoidal source with amplitude  $A$ , radian frequency  $\omega$ , and phase angle  $\theta$ , and the circuit contains resistors, inductors, and capacitors, all voltages and all currents in that circuit will be of the same
  - A. amplitude  $A$  but different radian frequency  $\omega$  and different phase angle  $\theta$
  - B. radian frequency  $\omega$  but different amplitude  $A$  and different phase angle  $\theta$
  - C. phase angle  $\theta$  but different amplitude  $A$  and different radian frequency  $\omega$
  - D. amplitude  $A$  same radian frequency  $\omega$  and same phase angle  $\theta$
  - E. none of the above
3. The sinusoid  $v(t) = 120\sin(\omega t + 90^\circ)$  in the  $j\omega$  – domain is expressed as
  - A.  $\mathbf{V} = 120e^{j(\omega t + 90^\circ)}$
  - B.  $\mathbf{V} = 120e^{j\omega t}$
  - C.  $\mathbf{V} = 120e^{j90^\circ}$
  - D.  $\mathbf{V} = 120e^{j0^\circ}$
  - E. none of the above
4. A series RLC circuit contains two voltage sources with values  $v_1(t) = 100\cos(10t + 45^\circ)$  and  $v_2(t) = 200\sin(5t - 60^\circ)$ . We can transform this circuit to a phasor equivalent to find the current by first replacing these with a single voltage source  $v(t) = v_1(t) + v_2(t)$  whose value is
  - A.  $v(t) = 300\cos(15t - 15^\circ)$

- B.  $v(t) = 100\cos(5t + 105^\circ)$   
 C.  $v(t) = 150\cos(7.5t - 15^\circ)$   
 D.  $v(t) = 150\cos(7.5t + 15^\circ)$   
 E. none of the above

5. The equivalent impedance  $Z_{eq}$  of the network below is

- A.  $1 + j1$   
 B.  $1 - j1$   
 C.  $-j1$   
 D.  $2 + j0$   
 E. none of the above

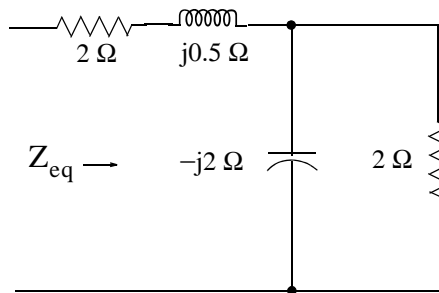


Figure 6.21. Network for Questions 5 and 6

6. The equivalent admittance  $Y_{eq}$  of the network in Figure 6.18 is

- A.  $4 - j1.5$   
 B.  $\frac{16}{73} + j\frac{6}{73}$   
 C.  $\frac{12}{37} + j\frac{2}{37}$   
 D.  $2 - j2$   
 E. none of the above

7. The resistance of a coil is  $R = 1.5 \Omega$  and the inductance of that coil is  $L = 5.3 \text{ mH}$ . If a current of  $i(t) = 4\cos\omega t \text{ A}$  flows through that coil and operates at the frequency of

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## Chapter 6 Sinusoidal Circuit Analysis

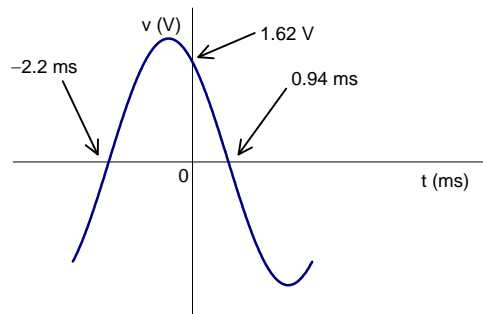
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- $f = 60\text{Hz}$ , the phasor voltage  $\mathbf{V}$  across that coil is
- A.  $10\angle 53.1^\circ \text{ V}$
  - B.  $6\angle 0^\circ \text{ V}$
  - C.  $5.3 \times 10^{-3}\angle 90^\circ \text{ V}$
  - D.  $6.8\angle 45^\circ \text{ V}$
  - E. none of the above
8. A resistor with value  $R = 5 \Omega$  is in series with a capacitor whose capacitive reactance at some particular frequency  $\omega$  is  $-jX_C = -5 \Omega$ . A phasor current with value  $\mathbf{I} = 8\angle 0^\circ \text{ A}$  is flowing through this series combination. The  $t$ -domain voltage across this series combination is
- A.  $80\cos\omega t$
  - B.  $80\sin\omega t$
  - C.  $56.6\cos(\omega t - 45^\circ)$
  - D.  $56.6\cos(\omega t + 45^\circ)$
  - E. none of the above
9. A conductance with value  $G = 0.3 \Omega^{-1}$  is in parallel with a capacitor whose capacitive susceptance at some particular frequency  $\omega$  is  $jB_C = j0.3 \Omega^{-1}$ . A phasor voltage with value  $\mathbf{V} = 10\angle 0^\circ$  is applied across this parallel combination. The  $t$ -domain total current through this parallel combination is
- A.  $3\cos\omega t + j3\sin\omega t$
  - B.  $3\cos\omega t - j3\sin\omega t$
  - C.  $5\sin(\omega t + 53.2^\circ)$
  - D.  $5\cos(\omega t + 53.2^\circ)$
  - E. none of the above
10. If the phasor  $\mathbf{I} = j e^{j(\pi/2)}$ , then in the  $t$ -domain  $i(t)$  is
- A.  $\cos(\omega t + \pi/2)$

- B.  $\sin(\omega t + \pi/2)$   
 C.  $-\cos \omega t$   
 D.  $-\sin \omega t$   
 E. none of the above

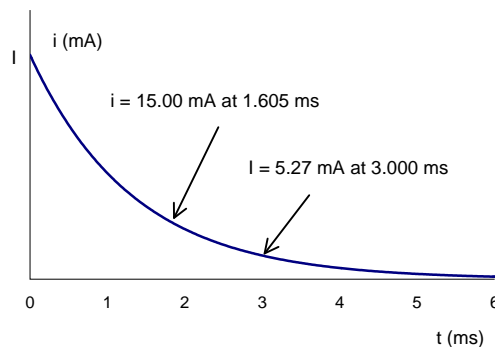
### Problems

1. Express the sinusoidal voltage waveform shown below as  $v(t) = A \cos(\omega t + \theta)$ , that is, find  $A$ ,  $\omega$ , and  $\theta$ . Answer:  $v(t) = 2 \cos(1000t + 36.1^\circ)$

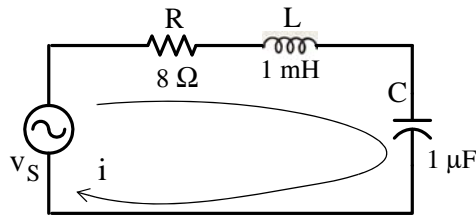


2. The current  $i(t)$  through a device decays exponentially as shown by the waveform below, and two values are known as indicated. Compute  $i(1)$ , that is, the current at  $t = 1$  ms.

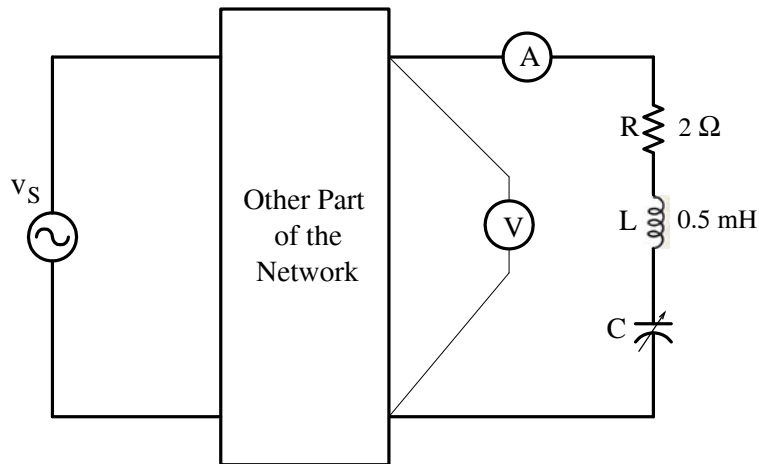
Answers:  $i(t) = 50e^{-750t}$  mA, 23.62 mA



3. At what frequency  $f$  is the network shown below operating if it is known that  $v_S = 120 \cos \omega t$  V and  $i = 12 \cos(\omega t - 36.9^\circ)$  A? Answer:  $f = 5.533$  KHz

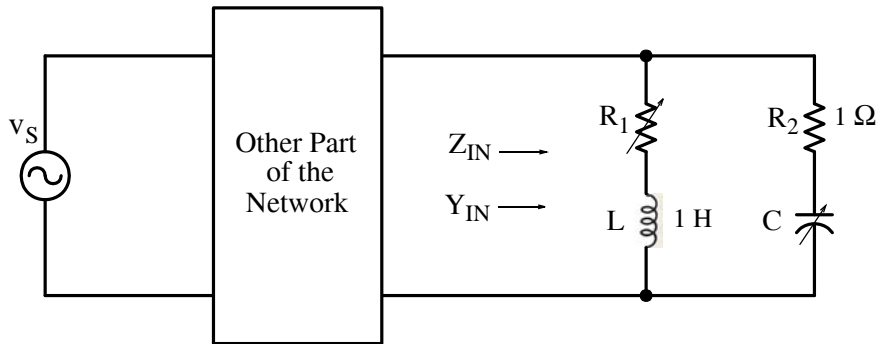


4. In the circuit below,  $v_S = V \cos(2000t + \theta)$  V and the symbols V and A inside the circles denote an AC *voltmeter*\* and *ammeter* respectively. Assume that the ammeter has negligible internal resistance. The variable capacitor C is adjusted until the voltmeter reads 25 V and the ammeter reads 5 A. Find the value of the capacitor. Answer:  $C = 89.6 \mu\text{F}$

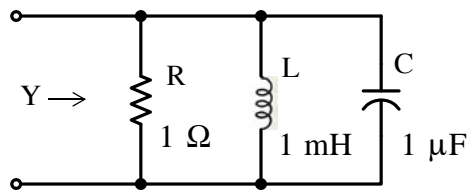


5. In the circuit shown below, is it possible to adjust the variable resistor  $R_1$  and the variable capacitor C so that  $Z_{IN}$  and  $Y_{IN}$  have the same numerical value regardless of the operating frequency? If so, what are these values? Answer: Yes, if  $C = 1 \text{ F}$  and  $R_1 = 1 \Omega$

\* Voltmeters and Ammeters are discussed in Chapter 8. For this exercise, it will suffice to say that these instruments indicate the magnitude (absolute) values of voltage and current.



6. Consider the parallel RLC circuit below. As we know, the capacitive susceptance  $B_C$  and the inductive susceptance  $B_L$  are functions of frequency, that is,  $|B_C| = 2\pi fC$ , and  $|B_L| = 1/(2\pi fL)$



Find the frequency\* at which the capacitive susceptance cancels the inductive susceptance, that is, the frequency at which the admittance  $Y$ , generally computed from the relation

$$|Y| = \sqrt{G^2 + (B_C - B_L)^2} \text{ is reduced to } Y = \sqrt{G^2} = G. \text{ Answer: } f \approx 5 \text{ KHz}$$

\* This frequency is known as the resonance frequency. It is discussed in detail in *Circuit Analysis II with MATLAB Computing and Simulink / SimPowerSystems Modeling*, ISBN 978-1-934404-19-5.



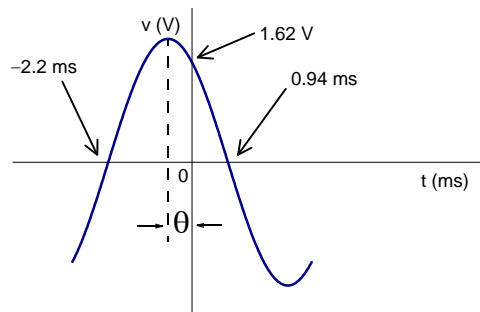
**6.9 Solutions to End-of-Chapter Exercises****Multiple Choice**

1. E Phasors exist in the  $j\omega$  – domain only
2. B
3. D
4. E The voltage sources  $v_1(t)$  and  $v_1(t)$  operate at different frequencies. Therefore, to find the current we must apply superposition.
5. E  $3 - j0.5$  This value is obtained with the MATLAB script `z1=2+0.5j; z2=2*(-2j)/(2-2j); z=z1+z2`  
$$z = 3.0000 - 0.5000i$$
6. C
7. A  $\omega = 2\pi f = 2\pi \times 60 = 377 \text{ r/s}$ ,  $jX_L = j\omega L = j \times 377 \times 5.3 \times 10^{-3} = j2 \ \Omega$   
$$Z = 1.5 + j2 = 2.5 \angle 53.13^\circ, \mathbf{V} = \mathbf{Z}\mathbf{I} = 2.5 \angle 53.13^\circ \times 4 \angle 0^\circ = 10 \angle 53.13^\circ$$
8. C
9. D
10. C

**Problems**

1. The  $t$  – axis crossings define half of the period  $T$ . Thus,  $T/2 = 2.2 + 0.94 = 3.14 \text{ ms}$ , and one period is  $T = 6.28 \text{ ms}$ . The frequency is  $f = 1/T = 10^3/6.28 = 10^3/2\pi$ . Then,  $\omega = 2\pi f$  or  $\omega = 2\pi \times 10^3/2\pi = 1000 \text{ r/s}$

Next, we find the phase angle  $\theta$  from the figure above observing that  $\pi/2 + \theta = 2.2 \text{ ms}$



or

$$\begin{aligned}\theta &= 2.2\text{ms} - \frac{\pi}{2} = 2.2 \times 10^{-3} \text{ s} \times \frac{2\pi \text{ rad}}{6.28 \times 10^{-3} \text{ s}} \times \frac{180^\circ}{\pi \text{ rad}} - \frac{\pi}{2} \\ &= \frac{2.2 \times 2 \times 180^\circ}{6.28} - \frac{\pi}{2} = 126.1^\circ - 90^\circ = 36.1^\circ\end{aligned}$$

Finally, we find the amplitude  $A$  by observing that at  $t = 0$ ,  $v = 1.62 \text{ V}$ , that is,

$$v(0) = 1.62 = A \cos(0 + 36.1^\circ)$$

or

$$A = \frac{1.62}{\cos 36.1^\circ} = 2 \text{ V}$$

Therefore,

$$v(t) = 2 \cos(1000t + 36.1^\circ)$$

2. The decaying exponential has the form  $i(t) = Ae^{-\alpha t}$  mA where the time is in ms and thus for this problem we need to compute the values of  $A$  and  $\alpha$  using the given values. Then,

$$i|_{t=1.605 \text{ ms}} = 15 \text{ mA} = Ae^{-(1.605 \times 10^{-3})\alpha}$$

and

$$i|_{t=3.000 \text{ ms}} = 5.27 \text{ mA} = Ae^{-(3.000 \times 10^{-3})\alpha}$$

Division of the first equation by the second yields

$$\frac{Ae^{-(1.605 \times 10^{-3})\alpha}}{Ae^{-(3.000 \times 10^{-3})\alpha}} = \frac{15 \text{ mA}}{5.27 \text{ mA}}$$

or

$$e^{-(1.605 \times 10^{-3})\alpha + (3.000 \times 10^{-3})\alpha} = \frac{15}{5.27}$$

or

$$e^{1.395 \times 10^{-3}\alpha} = \frac{15}{5.27}$$

or

$$\ln\left(\frac{15}{5.27}\right) = 1.395 \times 10^{-3}\alpha$$

or

$$\alpha = \frac{\ln(15/5.27) \times 10^3}{1.395} = 750$$

and thus

$$i(t) = Ae^{-750t} \text{ mA}$$

To find the value of  $A$  we make use of the fact that  $i|_{t=3 \text{ ms}} = 5.27 \text{ mA}$ . Then,

$$5.27 = Ae^{-750 \times 3 \times 10^{-3}}$$

or

$$A = \frac{5.27 \times 10^{-3}}{e^{-2.25}}$$

or

$$A = 0.050 \text{ A} = 50 \text{ mA}$$

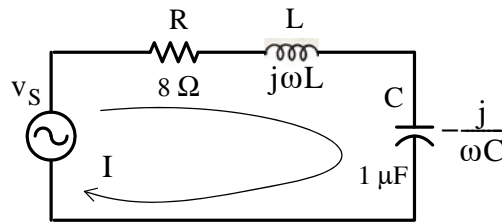
Therefore,

$$i(t) = 50e^{-750t} \text{ mA}$$

and

$$i|_{t=1 \text{ ms}} = 50e^{-750 \times 10^{-3}} = 23.62 \text{ mA}$$

3. The equivalent phasor circuit is shown below.



In the  $j\omega$ -domain  $V_S = 120\angle 0^\circ \text{ V}$ ,  $I = 12\angle -36.9^\circ \text{ A}$ ,  $j\omega L = j10^{-3}\omega$ , and  $-j/\omega C = -j10^6/\omega$

Then,

$$Z = \frac{V_S}{I} = \frac{120\angle 0^\circ \text{ V}}{12\angle -36.9^\circ \text{ A}} = 10\angle 36.9^\circ$$

and

$$|Z| = 10 = \sqrt{R^2 + (\omega L - 1/\omega C)^2}$$

or

$$R^2 + (\omega L - 1/\omega C)^2 = 100$$

or

$$8^2 + (\omega L - 1/\omega C)^2 = 100$$

or

$$(\omega L - 1/\omega C)^2 = 36$$

or

$$\omega L - 1/\omega C = 6$$

or

$$\omega^2 - \frac{6}{L}\omega - \frac{1}{LC} = 0$$

or

$$\omega^2 - 6 \times 10^3 \omega - 10^9 = 0$$

Solving for  $\omega$  and ignoring the negative value, we obtain

$$\omega = \frac{6 \times 10^3 + \sqrt{36 \times 10^6 + 4 \times 10^9}}{2} = 34,765 \text{ r/s}$$

and

$$f = \frac{\omega}{2\pi} = \frac{34,765 \text{ r/s}}{2\pi} = 5,533 \text{ Hz} = 5.533 \text{ KHz}$$

Check:  $j\omega L = j34.765$ ,  $-j/\omega C = -j28.765$

$$Z = R + j(\omega L - 1/(\omega C)) = 8 + j(34.765 - 28.765) = 8 + j6 = 10 \angle 36.9^\circ$$

and

$$I = \frac{120 \angle 0^\circ}{10 \angle 36.9^\circ} = 12 \angle -36.9^\circ$$

4. Since the instruments read absolute values, we are only need to be concerned the magnitudes of the phasor voltage, phasor current, and impedance. Thus,

$$|V| = |Z||I| = 25 = \sqrt{R^2 + (\omega L - 1/\omega C)^2} \times 5$$

or

$$\begin{aligned} |V|^2 &= 25^2 = [R^2 + (\omega L - 1/\omega C)^2] \times 25 = \left[4 + \left(1 - \frac{5 \times 10^{-4}}{C}\right)^2\right] \times 25 \\ &= 100 + 25 - \frac{250 \times 10^{-4}}{C} + \frac{625 \times 10^{-8}}{C^2} = 625 \end{aligned}$$

and after simplification we obtain

$$500C^2 + 250 \times 10^{-4}C - 625 \times 10^{-8} = 0$$

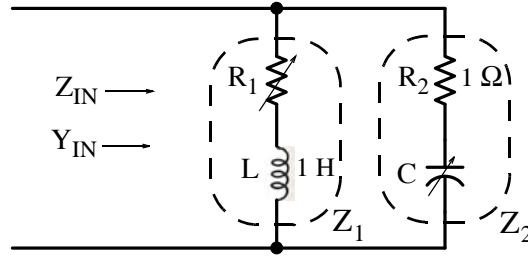
Using MATLAB, we obtain

$$p=[500 \ 250*10^{(-4)} \ -625*10^{(-8)}]; r=roots(p)$$

and this yields  $C = 89.6 \mu\text{F}$

The second root of this polynomial is negative and thus it is discarded.

5. We group the series devices as shown below.



Thus  $Z_1 = R_1 + j\omega$ ,  $Z_2 = 1 - j/(\omega C)$ , and

$$Z_{IN} = \frac{Z_1 \cdot Z_2}{Z_1 + Z_2} = \frac{(R_1 + j\omega)(1 - j/\omega C)}{R_1 + j\omega + 1 - j/\omega C}$$

and at any frequency  $\omega$

$$Y_{IN} = \frac{1}{Z_{IN}} = \frac{R_1 + j\omega + 1 - j/\omega C}{(R_1 + j\omega)(1 - j/\omega C)}$$

Therefore, if the condition  $Y_{IN} = Z_{IN}$  is to hold for all frequencies, the right sides of  $Z_{IN}$  and  $Y_{IN}$  must be equal, that is,

$$\frac{(R_1 + j\omega)(1 - j/\omega C)}{R_1 + j\omega + 1 - j/\omega C} = \frac{R_1 + j\omega + 1 - j/\omega C}{(R_1 + j\omega)(1 - j/\omega C)}$$

$$[(R_1 + j\omega)(1 - j/\omega C)]^2 = [R_1 + j\omega + 1 - j/\omega C]^2$$

$$(R_1 + j\omega)(1 - j/\omega C) = R_1 + j\omega + 1 - j/\omega C$$

$$R_1 - j\frac{R_1}{\omega C} + j\omega + \frac{1}{C} = R_1 + 1 + j\left(\omega - \frac{1}{\omega C}\right)$$

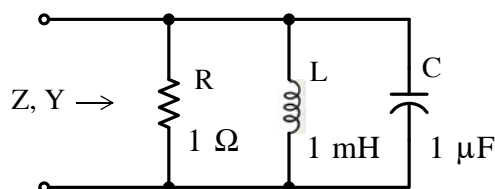
$$\left(R_1 + \frac{1}{C}\right) + j\left(\omega - \frac{R_1}{\omega C}\right) + j\omega = (R_1 + 1) + j\left(\omega - \frac{1}{\omega C}\right)$$

Equating reals and imaginaries we obtain

$$R_1 + \frac{1}{C} = R_1 + 1 \quad \omega - \frac{R_1}{\omega C} = \omega - \frac{1}{\omega C}$$

From the first equation above we obtain  $C = 1 \text{ F}$  and by substitution of this value into the second equation we obtain  $R_1 = 1 \ \Omega$ .

6.



The admittance  $|Y| = \sqrt{G^2 + (B_C - B_L)^2}$  is reduced to  $Y = \sqrt{G^2} = G$  when  $B_C - B_L = 0$ , or  $B_C = B_L$ , or  $2\pi fC = 1/(2\pi fL)$ , from which  $f = 1/(2\pi\sqrt{LC})$ , and with the given values,

$$f = \frac{1}{2\pi \cdot \sqrt{10^{-3} \times 10^{-6}}} \approx 5000 \text{ Hz}$$

and since the resistive branch is unity, at this frequency  $Z = Y = 1$  and the phase is zero degrees.

The magnitude and phase at other frequencies can be plotted with a spreadsheet or MATLAB, but it is easier with the Simulink / SimPowerSystems model shown in Figure 6.22.

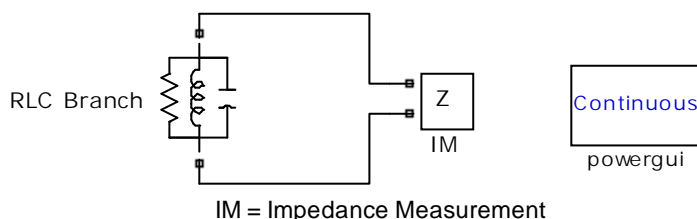


Figure 6.22. SimPowerSystems model for impedance measurement

After the simulation command is executed, we must click the **Powergui** block, and on the pop up window we must select the **Impedance vs Frequency Measurement** option to display the magnitude and phase of the impedance function shown in Figure 6.23.

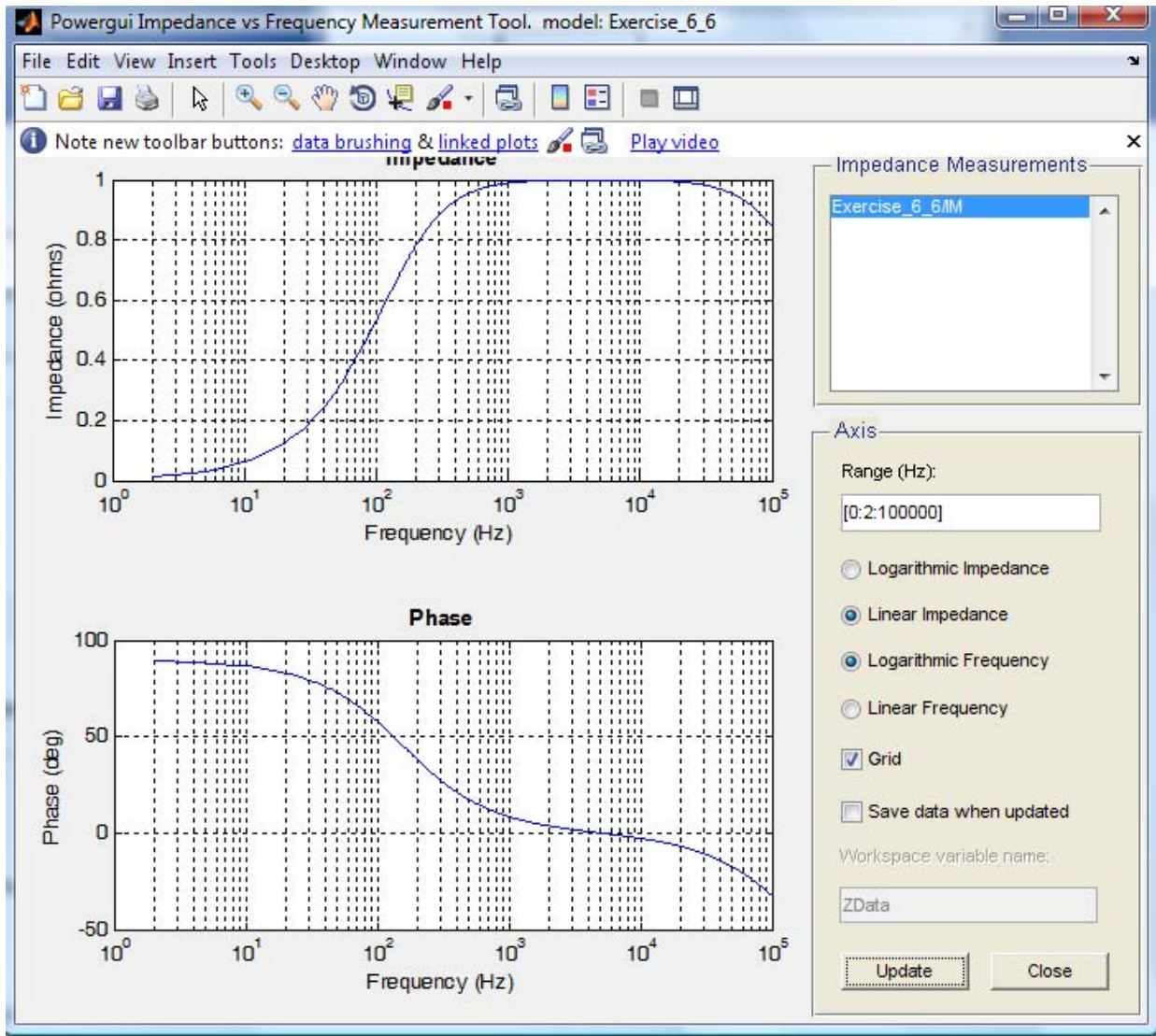


Figure 6.23. Magnitude and Phase plots for the SimPowerSystems model in Figure 6.22

We observe that the maximum value of the impedance, i.e.,  $1 \Omega$ , occurs at approximately 5 KHz , and at this frequency the phase is zero degrees.

# Chapter 7

## Phasor Circuit Analysis

This chapter begins with the application of nodal analysis, mesh analysis, superposition, and Thevenin's and Norton's theorems in phasor circuits. Then, phasor diagrams are introduced, and the input-output relationships for an RC low-pass filter and an RC high-pass filter are developed.

### 7.1 Nodal Analysis

The procedure of analyzing a phasor\* circuit is the same as in Chapter 3, except that in this chapter we will be using phasor quantities. The following example illustrates the procedure.

#### Example 7.1

Use nodal analysis to compute the phasor voltage  $\mathbf{V}_{AB} = \mathbf{V}_A - \mathbf{V}_B$  for the circuit of Figure 7.1.

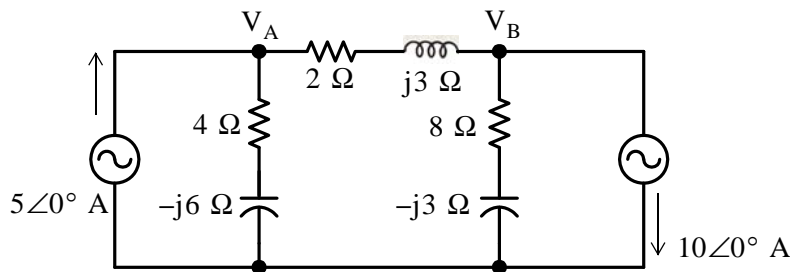


Figure 7.1. Circuit for Example 7.1

#### Solution:

As in Chapter 3, we choose a reference node as shown in Figure 7.2, and we write nodal equations at the other two nodes A and B. Also, for convenience, we designate the devices in series as  $Z_1$ ,  $Z_2$ , and  $Z_3$  as shown, and then we write the nodal equations in terms of these impedances.

$$Z_1 = 4 - j6 = 7.211 \angle -56.3^\circ$$

$$Z_2 = 2 + j3 = 3.606 \angle 56.3^\circ$$

$$Z_3 = 8 - j3 = 8.544 \angle -20.6^\circ$$

\* A phasor is a rotating vector



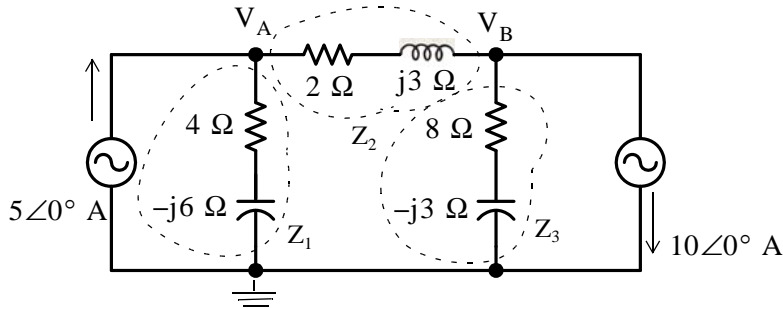


Figure 7.2. Nodal analysis for the circuit for Example 7.1

By application of KCL at  $\mathbf{V}_A$ ,

$$\frac{\mathbf{V}_A}{Z_1} + \frac{\mathbf{V}_A - \mathbf{V}_B}{Z_2} = 5\angle 0^\circ$$

$$\left(\frac{1}{Z_1} + \frac{1}{Z_2}\right)\mathbf{V}_A - \frac{1}{Z_2}\mathbf{V}_B = 5\angle 0^\circ \quad (7.1)$$

or

$$\left(\frac{Z_1 + Z_2}{Z_1 Z_2}\right)\mathbf{V}_A - \frac{1}{Z_2}\mathbf{V}_B = 5\angle 0^\circ$$

and by substitution for  $Z_1$  and  $Z_2$  we obtain

$$\frac{4 - j6 + 2 + j3}{(7.211\angle -56.3^\circ)(3.606\angle 56.3^\circ)}\mathbf{V}_A - \frac{1}{3.606\angle 56.3^\circ}\mathbf{V}_B = 5\angle 0^\circ$$

$$\frac{6 - j3}{26.0\angle 0^\circ}\mathbf{V}_A - (0.277\angle -56.3^\circ)\mathbf{V}_B = 5\angle 0^\circ$$

$$\frac{6.708\angle -26.6^\circ}{26\angle 0^\circ}\mathbf{V}_A - (0.277\angle -56.3^\circ)\mathbf{V}_B = 5\angle 0^\circ$$

$$(0.258\angle -26.6^\circ)\mathbf{V}_A - (0.277\angle -56.3^\circ)\mathbf{V}_B = 5\angle 0^\circ \quad (7.2)$$

Next, at  $\mathbf{V}_B$ :

$$\frac{\mathbf{V}_B - \mathbf{V}_A}{Z_2} + \frac{\mathbf{V}_B}{Z_3} = -10\angle 0^\circ$$

$$-\frac{1}{Z_2}\mathbf{V}_A + \left(\frac{1}{Z_2} + \frac{1}{Z_3}\right)\mathbf{V}_B = -10\angle 0^\circ \quad (7.3)$$

In matrix form (7.1) and (7.3) are written as follows:

$$\begin{bmatrix} \left(\frac{1}{Z_1} + \frac{1}{Z_2}\right) & -\frac{1}{Z_2} \\ -\frac{1}{Z_2} & \left(\frac{1}{Z_2} + \frac{1}{Z_3}\right) \end{bmatrix} \begin{bmatrix} \mathbf{V}_A \\ \mathbf{V}_B \end{bmatrix} = \begin{bmatrix} 5 \\ -10 \end{bmatrix} \quad (7.4)$$

We will follow a step-by-step procedure to solve these equations using Cramer's rule, and we will use MATLAB®\* to verify the results.

We rewrite (7.3) as

$$\begin{aligned} -\frac{1}{Z_2} \mathbf{V}_A + \left(\frac{Z_2 + Z_3}{Z_2 Z_3}\right) \mathbf{V}_B &= 10 \angle 180^\circ \\ -\frac{1}{3.606 \angle 56.3^\circ} \mathbf{V}_A + \frac{2 + j3 + 8 - j3}{(3.606 \angle 56.3^\circ)(8.544 \angle -20.6^\circ)} \mathbf{V}_B &= 10 \angle 180^\circ \\ -(0.277 \angle -56.3^\circ) \mathbf{V}_A + \frac{10}{30.810 \angle 35.7^\circ} \mathbf{V}_B &= 10 \angle 180^\circ \\ -(0.277 \angle -56.3^\circ) \mathbf{V}_A + (0.325 \angle -35.7^\circ) \mathbf{V}_B &= 10 \angle 180^\circ \end{aligned} \quad (7.5)$$

and thus with (7.2) and (7.5) the system of equations is

$$\begin{aligned} (0.258 \angle -26.6^\circ) \mathbf{V}_A - (0.277 \angle -56.3^\circ) \mathbf{V}_B &= 5 \angle 0^\circ \\ -(0.277 \angle -56.3^\circ) \mathbf{V}_A + (0.325 \angle -35.7^\circ) \mathbf{V}_B &= 10 \angle 180^\circ \end{aligned} \quad (7.6)$$

We find  $\mathbf{V}_A$  and  $\mathbf{V}_B$  from

$$\mathbf{V}_A = \frac{D_1}{\Delta} \quad (7.7)$$

and

$$\mathbf{V}_B = \frac{D_2}{\Delta} \quad (7.8)$$

The determinant  $\Delta$  is

$$\begin{aligned} \Delta &= \begin{vmatrix} (0.258 \angle -26.6^\circ) & -(0.277 \angle -56.3^\circ) \\ -(0.277 \angle -56.3^\circ) & (0.325 \angle -35.7^\circ) \end{vmatrix} \\ &= (0.258 \angle -26.6^\circ) \cdot (0.325 \angle -35.7^\circ) - (0.277 \angle -56.3^\circ) \cdot (-0.277 \angle -56.3^\circ) \\ &= (0.084 \angle -62.3^\circ) - (0.077 \angle -112.6^\circ) - (0.039 - j0.074) - (-0.023 - j0.071) \\ &= (0.062 - j0.003 = 0.062 \angle -2.8^\circ) \end{aligned}$$

Also,

---

\* If unfamiliar with MATLAB, please refer to Appendix A

$$\begin{aligned}
 D_1 &= \begin{vmatrix} 5\angle 0^\circ & -(0.277\angle -56.3^\circ) \\ 10\angle 180^\circ & (0.325\angle -35.7^\circ) \end{vmatrix} \\
 &= (5\angle 0^\circ)(0.325\angle -35.7^\circ) - (10\angle 180^\circ)[-(0.277\angle -56.3^\circ)] \\
 &= (1.625\angle -35.7^\circ + 2.770\angle 123.7^\circ) = 1.320 - j0.948 + (-1.537 + j2.305) \\
 &= -0.217 + j1.357 = 1.374\angle 99.1^\circ
 \end{aligned}$$

and

$$\begin{aligned}
 D_2 &= \begin{vmatrix} (0.258\angle -26.6^\circ) & 5\angle 0^\circ \\ -(0.277\angle -56.3^\circ) & 10\angle 180^\circ \end{vmatrix} \\
 &= (0.258\angle -26.6^\circ)(10\angle 180^\circ) - (-0.277\angle -56.3^\circ)(5\angle 0^\circ) \\
 &= 2.580\angle 153.4^\circ + 1.385\angle -56.3^\circ = (-2.307 + j1.155 + 0.769 - j1.152) \\
 &= -1.358 + j0.003 = 1.358\angle 179.9^\circ
 \end{aligned}$$

Therefore, by substitution into (7.7) and (7.8), we obtain

$$\mathbf{V}_A = \frac{D_1}{\Delta} = \frac{1.374\angle 99.1^\circ}{0.062\angle -2.8^\circ} = 22.161\angle 101.9^\circ = -4.570 + j21.685$$

and

$$\mathbf{V}_B = \frac{D_2}{\Delta} = \frac{1.358\angle 179.9^\circ}{0.062\angle -2.8^\circ} = 24.807\angle -177.3^\circ = -24.780 - j1.169$$

Finally,

$$\begin{aligned}
 \mathbf{V}_{AB} &= \mathbf{V}_A - \mathbf{V}_B = -4.570 + j21.685 - (-24.780 - j1.169) \\
 &= 20.21 + j22.85 = 30.5\angle 48.5^\circ
 \end{aligned}$$

Check with MATLAB:

```

z1=4-j*6; z2=2+j*3; z3=8-j*3;           % Define z1, z2 and z3
Z=[1/z1+1/z2 -1/z2; -1/z2 1/z2+1/z3];   % Elements of matrix Z
I=[5 -10]';                             % Column vector I
V=Z\I; Va=V(1,1); Vb=V(2,1); Vab=Va-Vb; % Va = V(1), Vb = V(2) are also acceptable
% With fprintf only the real part of each parameter is processed so we will use disp
fprintf(' \n'); disp('Va = '); disp(Va); disp('Vb = '); disp(Vb); disp('Vab = '); disp(Vab);
fprintf(' \n');

```

```

Va = -4.1379 + 19.6552i
Vb = -22.4138 - 1.0345i
Vab = 18.2759 + 20.6897i

```

These values differ by about 10% from the values we obtained with Cramer's rule where we rounded the values to three decimal places. MATLAB performs calculations with accuracy of 15 decimal places, although it only displays four decimal places in the short (default) number display format. Accordingly, we should accept the MATLAB values as more accurate.

## 7.2 Mesh Analysis

Again, the procedure of analyzing a phasor circuit is the same as in Chapter 3 except that in this chapter we will be using phasor quantities. The following example illustrates the procedure.

### Example 7.2

For the circuit of Figure 7.3, use mesh analysis to find the voltage  $V_{10A}$ , that is, the voltage across the  $10\angle 0^\circ$  current source.

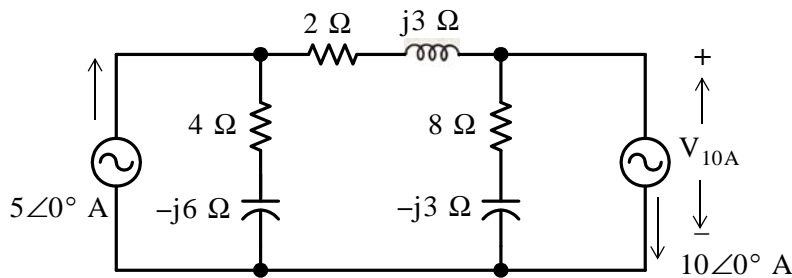


Figure 7.3. Circuit for Example 7.2

#### Solution:

As in the previous example, for convenience, we denote the passive devices in series as  $Z_1$ ,  $Z_2$ , and  $Z_3$ , and we write mesh equations in terms of these impedances. The circuit then is as shown in Figure 7.4 with the mesh currents assigned in a clockwise direction.

We observe that the voltage across the  $10\angle 0^\circ$  current source is the same as the voltage across the  $8\ \Omega$  and  $-j3\ \Omega$  series combination.

By inspection, for Mesh 1,

$$\mathbf{I}_1 = 5\angle 0^\circ \quad (7.9)$$

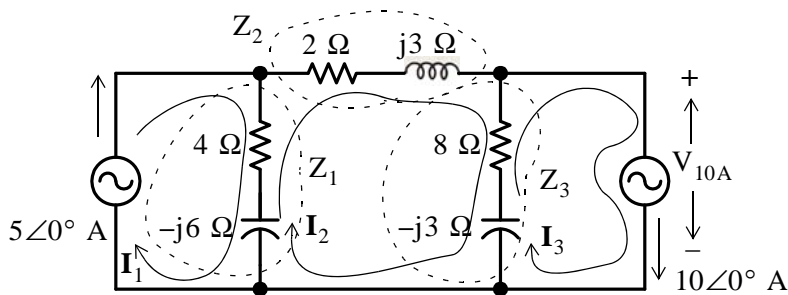


Figure 7.4. Mesh analysis for the circuit of Example 7.2

By application of KVL around Mesh 2,

$$\begin{aligned} -Z_1\mathbf{I}_1 + (Z_1 + Z_2 + Z_3)\mathbf{I}_2 - Z_3\mathbf{I}_3 &= 0 \\ -(4 - j6)\mathbf{I}_1 + (14 - j6)\mathbf{I}_2 - (8 - j3)\mathbf{I}_3 &= 0 \end{aligned} \quad (7.10)$$

Also, by inspection for Mesh 3,

$$\mathbf{I}_3 = 10\angle 0^\circ \quad (7.11)$$

and in matrix form, (7.9), (7.10), and (7.11) are written as

$$\begin{bmatrix} 1 & 0 & 0 \\ -(4 - j6) & (14 - j6) & -(8 - j3) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \\ \mathbf{I}_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 10 \end{bmatrix} \quad (7.12)$$

We use MATLAB for the solution of 7.12.\*

```
Z=[1 0 0; -(4-j*6) 14-j*6 -(8-j*3); 0 0 1];
V=[5 0 10]';
I=Z\V; i1=I(1); i2=I(2); i3=I(3); fprintf(' \n');
disp('i1 = '); disp(i1); disp('i2 = '); disp(i2); disp('i3 = '); disp(i3); fprintf(' \n');
i1 = 5      i2 = 7.5862 - 1.0345i      i3 = 10
```

Therefore, the voltage across the  $10\angle 0^\circ$  A current source is

$$\mathbf{V}_{10A} = Z_3(\mathbf{I}_2 - \mathbf{I}_3) = (8 - j3)(7.586 - j1.035 - 10) = -22.417 - j1.038$$

We observe that this is the same value as that of the voltage  $\mathbf{V}_B$  in the previous example.

---

### 7.3 Application of Superposition Principle

As we know from Chapter 3, the superposition principle is most useful when a circuit contains two or more independent voltage or current sources. The following example illustrates the application of the superposition principle in phasor circuits.

---

#### Example 7.3

Use the superposition principle to find the phasor voltage across capacitor  $C_2$  in the circuit of Figure 7.5.

---

\* As we experienced with Example 7.1, the computation of phasor voltages and currents becomes quite tedious. Accordingly, in our subsequent discussion we will use MATLAB for the solution of simultaneous equations with complex coefficients.

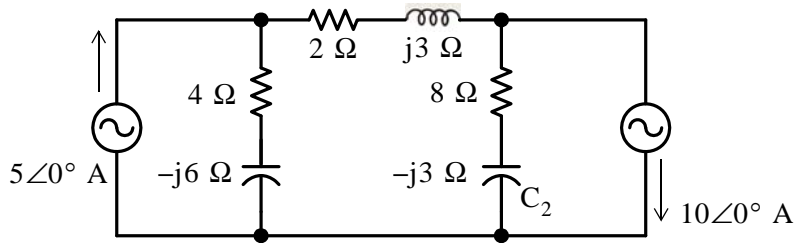


Figure 7.5. Circuit for Example 7.3

**Solution:**

Let the phasor voltage across  $C_2$  due to the  $5\angle 0^\circ$  A current source acting alone be denoted as  $\mathbf{V}'_{C_2}$ , and that due to the  $10\angle 0^\circ$  A current source as  $\mathbf{V}''_{C_2}$ . Then,

$$\mathbf{V}_{C_2} = \mathbf{V}'_{C_2} + \mathbf{V}''_{C_2}$$

With the  $5\angle 0^\circ$  A current source acting alone, the circuit reduces to that shown in Figure 7.6.

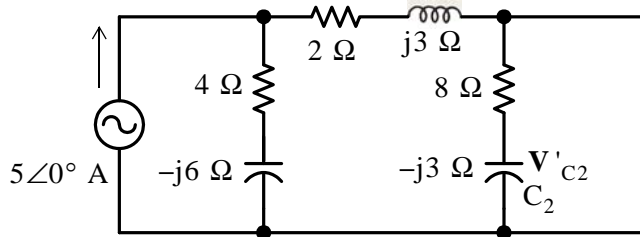


Figure 7.6. Circuit for Example 7.3 with the  $5\angle 0^\circ$  A current source acting alone

By application of the current division expression, the current  $\mathbf{I}'_{C_2}$  through  $C_2$  is

$$\mathbf{I}'_{C_2} = \frac{4 - j6}{4 - j6 + 2 + j3 + 8 - j3} 5\angle 0^\circ = \frac{7.211\angle -56.3^\circ}{15.232\angle -23.2^\circ} 5\angle 0^\circ = 2.367\angle -33.1^\circ$$

The voltage across  $C_2$  with the  $5\angle 0^\circ$  current source acting alone is

$$\begin{aligned} \mathbf{V}'_{C_2} &= (-j3)(2.367\angle -33.1^\circ) = (3\angle -90^\circ)(2.367\angle -33.1^\circ) \\ &= 7.102\angle -123.1^\circ = -3.878 - j5.949 \end{aligned} \tag{7.13}$$

Next, with the  $10\angle 0^\circ$  A current source acting alone, the circuit reduces to that shown in Figure 7.7.

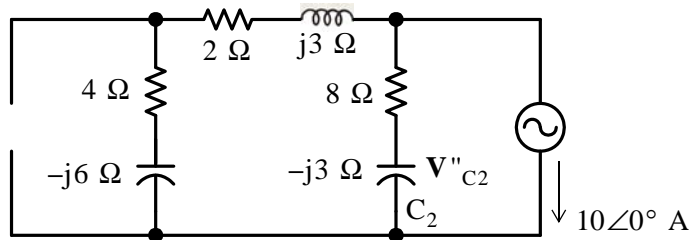


Figure 7.7. Circuit for Example 7.3 with the  $10\angle 0^\circ$  A current source acting alone and by application of the current division expression, the current  $\mathbf{I}''_{C_2}$  through  $C_2$  is

$$\begin{aligned} \mathbf{I}''_{C_2} &= \frac{4 - j6 + 2 + j3}{4 - j6 + 2 + j3 + 8 - j3} (-10\angle 0^\circ) \\ &= \frac{6.708\angle -26.6^\circ}{15.232\angle -23.2^\circ} 10\angle 180^\circ = 4.404\angle 176.6^\circ \end{aligned}$$

The voltage across  $C_2$  with the  $10\angle 0^\circ$  current source acting alone is

$$\begin{aligned} \mathbf{V}''_{C_2} &= (-j3)(4.404\angle 176.6^\circ) = (3\angle -90^\circ)(4.404\angle 176.6^\circ) \\ &= (13.213\angle 86.6^\circ = 0.784 + j13.189) \end{aligned} \quad (7.14)$$

Addition of (7.13) with (7.14) yields

$$\mathbf{V}_{C_2} = \mathbf{V}'_{C_2} + \mathbf{V}''_{C_2} = -3.878 - j5.949 + 0.784 + j13.189$$

or

$$\mathbf{V}_{C_2} = -3.094 + j7.240 = 7.873\angle 113.1^\circ \quad (7.15)$$

Check with MATLAB:

```
z1=4-6j; z2=2+3j; z3=8-3j; Is=5; i1=z1*Is/(z1+z2+z3);...
i1, magI1=abs(i1), phaseI1=angle(i1)*180/pi, v1=-3j*i1,...
magV1=abs(v1), phaseV1=angle(v1)*180/pi,...
Is2=-10; i2=(z1+z2)*Is2/(z1+z2+z3); magI2=abs(i2), phaseI2=angle(i2)*180/pi,...
v2=-3j*i2, magV2=abs(v2), phaseV2=angle(v2)*180/pi,...
vC=v1+v2, magvC=abs(vC), phasevC=angle(vC)*180/pi
```

```
i1 =
    1.9828 - 1.2931i
magI1 =
    2.3672
phaseI1 =
   -33.1113
v1 =
   -3.8793 - 5.9483i
```

```

magV1 =
    7.1015
phaseV1 =
   -123.1113
magI2 =
    4.4042
phaseI2 =
   176.6335
v2 =
    0.7759 +13.1897i
magV2 =
   13.2125
phaseV1 =
    86.6335
vC =
   -3.1034 + 7.2414i
magvC =
    7.8784
phasevC =
   113.1986
    
```

The Simulink models for the computation of  $V'_{C2}$  and  $V''_{C2}$  are shown in Figures 7.8 and 7.9 respectively.

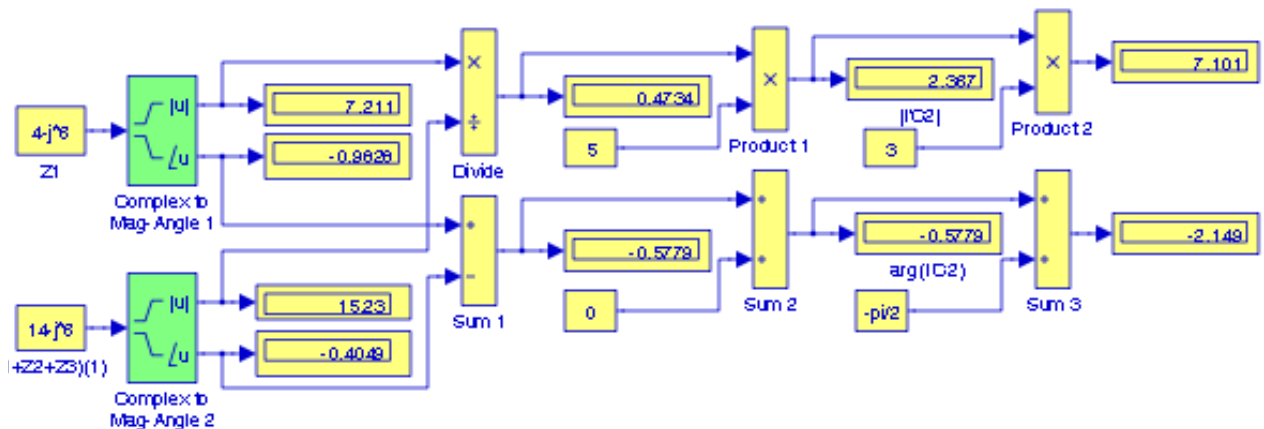


Figure 7.8. Model for the computation of  $V'_{C2}$ , Example 7.3

The final step is to add  $V'_{C2}$  with  $V''_{C2}$ . This addition is performed with the model of Figure 7.10 where the models of Figures 7.8 and 7.9 have been converted to Subsystems 1 and 2 respectively.



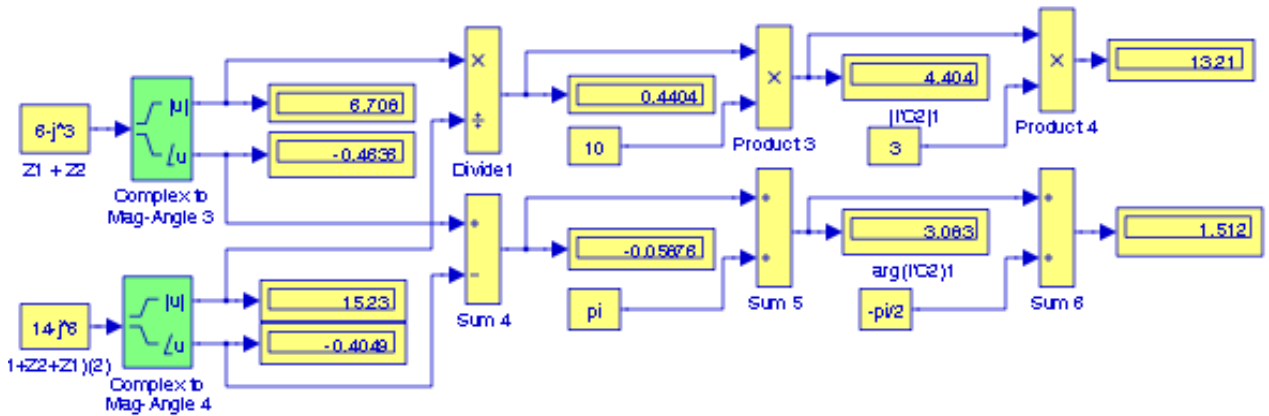


Figure 7.9. Model for the computation of  $V_{C2}$ , Example 7.3

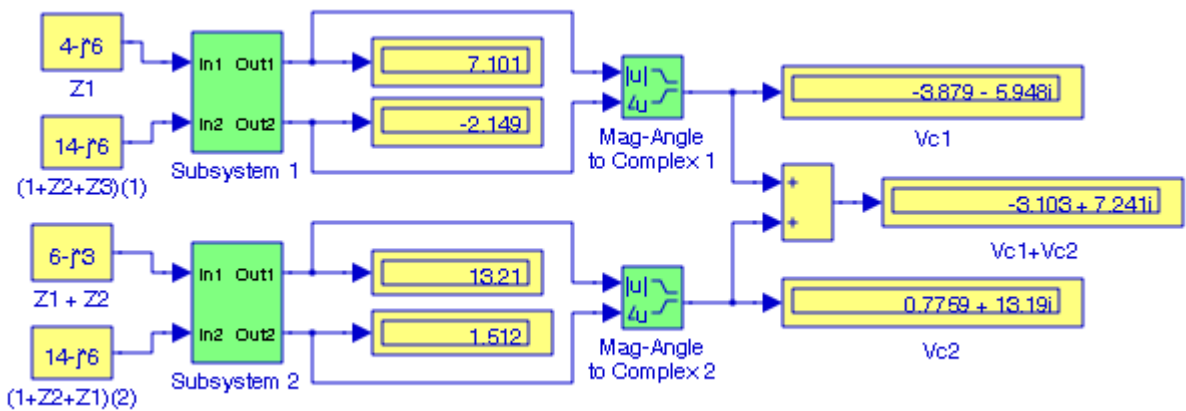


Figure 7.10. Model for the addition of  $V_{C2}$  with  $V_{C2}$ , Example 7.3

The model in Figure 7.10 can now be used with the circuit of Figure 7.5 for any values of the current sources and the impedances.

## 7.4 Thevenin's and Norton's Theorems

These two theorems also offer a very convenient method in analyzing phasor circuits as illustrated by the following example.

### Example 7.4

For the circuit of Figure 7.11, apply Thevenin's theorem to compute  $I_X$  and then draw Norton's equivalent circuit.

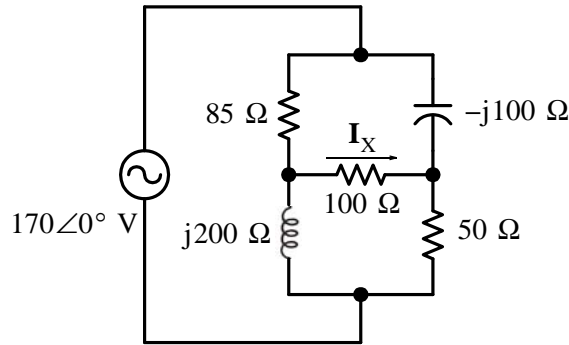


Figure 7.11. Circuit for Example 7.4

**Solution:**

With the 100 Ω resistor disconnected, the circuit reduces to that shown in Figure 7.12.

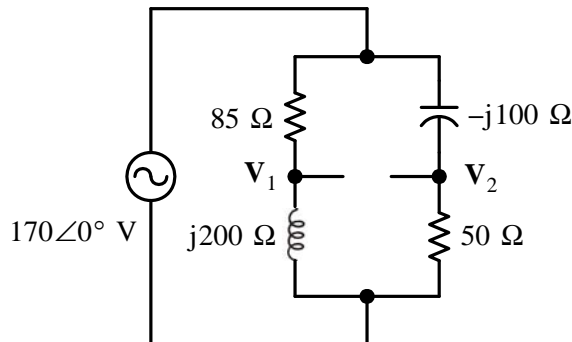


Figure 7.12. Circuit for Example 7.4 with the 100 Ω resistor disconnected

By application of the voltage division expression,

$$\mathbf{V}_1 = \frac{j200}{85 + j200} 170\angle 0^\circ = \frac{200\angle 90^\circ}{217.31\angle 67^\circ} 170\angle 0^\circ = 156.46\angle 23^\circ = 144 + j61.13 \quad (7.16)$$

and

$$\mathbf{V}_2 = \frac{50}{50 - j100} 170\angle 0^\circ = \frac{50}{111.8\angle (-63.4)^\circ} 170\angle 0^\circ = 76\angle 63.4^\circ = 34 + j68 \quad (7.17)$$

Then, from (7.16) and (7.17),

$$\begin{aligned} \mathbf{V}_{\text{TH}} = \mathbf{V}_{\text{OC}} = \mathbf{V}_{12} = \mathbf{V}_1 - \mathbf{V}_2 &= 144 + j61.13 - (34 + j68) \\ \mathbf{V}_{\text{TH}} &= 110 - j6.87 = 110.21\angle -3.6^\circ \end{aligned} \quad (7.18)$$

Next, we find the Thevenin equivalent impedance  $Z_{\text{TH}}$  by shorting the 170∠0° V voltage source. The circuit then reduces to that shown in Figure 7.13.

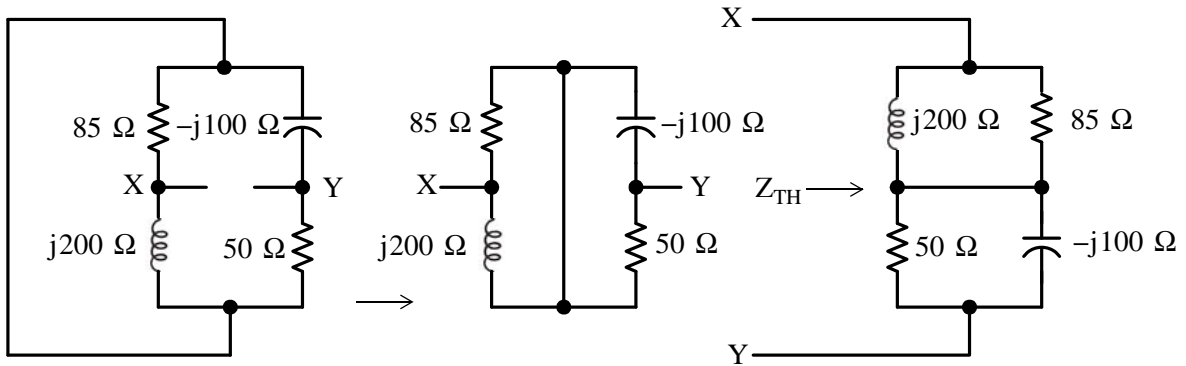


Figure 7.13. Circuit for Example 7.4 with the voltage source shorted

We observe that the parallel combinations  $j200 \parallel 85$  and  $50 \parallel j100$  are in series as shown in Figure 7.14.

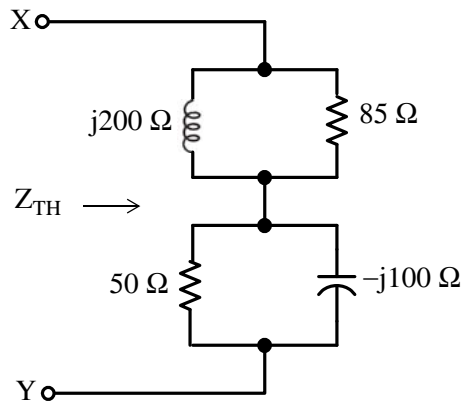


Figure 7.14. Network for the computation of  $Z_{TH}$  for Example 7.4

From Figure 7.14,

$$Z_{TH} = \frac{85 \times j200}{85 + j200} + \frac{50 \times (-j100)}{50 - j100}$$

and with MATLAB,

$$Z_{th} = 85 * 200j / (85 + 200j) + 50 * (-100j) / (50 - 100j)$$

$$Z_{th} = 1.1200e+002 + 1.0598e+001i$$

or

$$Z_{TH} = 112.0 + j10.6 = 112.5 \angle 5.4^\circ \Omega$$

The Thevenin equivalent circuit is shown in Figure 7.15.

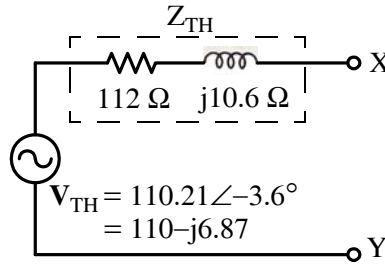


Figure 7.15. Thevenin equivalent circuit for Example 7.4

With the 100 Ω resistor connected at X–Y, the circuit becomes as shown in Figure 7.16.

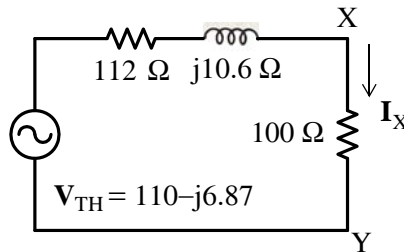


Figure 7.16. Simplified circuit for computation of  $I_X$  in Example 7.4

We find  $I_X$  using MATLAB:

```
Vth=110-6.87j; Zth=112+10.6j; Ix=Vth/(Zth+100);
fprintf(' \n'); disp('Ix = '); disp(Ix); fprintf(' \n');
```

```
Ix = 0.5160 - 0.0582i
```

that is,

$$I_X = \frac{V_{TH}}{Z_{TH} + 100 \Omega} = 0.516 - j0.058 = 0.519 \angle -6.4^\circ \text{ A} \quad (7.19)$$

The same answer is found in Example C.18 of Appendix C where we applied nodal analysis to find  $I_X$ .

Norton's equivalent is obtained from Thevenin's circuit by exchanging  $V_{TH}$  and its series  $Z_{TH}$  with  $I_N$  in parallel with  $Z_N$  as shown in Figure 7.14. Thus,

$$I_N = \frac{V_{TH}}{Z_{TH}} = \frac{110.21 \angle -3.6^\circ}{112.5 \angle 5.4^\circ} = 0.98 \angle -9^\circ \text{ A}$$

and

$$Z_N = Z_{TH} = 112.5 \angle 5.4^\circ \Omega$$

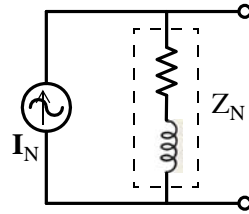


Figure 7.17. Norton equivalent circuit for Example 7.4

## 7.5 Phasor Analysis in Amplifier Circuits

Other circuits such as those who contain op amps and op amp equivalent circuits can be analyzed using any of the above methods.

### Example 7.5

Compute  $i_X(t)$  for the circuit in Figure 7.18 where  $v_{in}(t) = 2 \cos(30000\omega t)$  V.

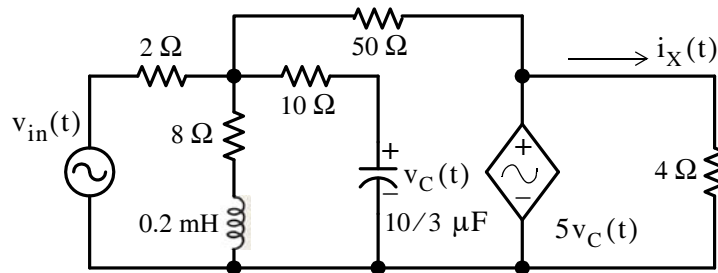


Figure 7.18. Circuit for Example 7.5

### Solution:

As a first step, we perform the  $t$ -domain, to  $j\omega$ -domain transformation. Thus,

$$jX_L = j\omega L = j0.2 \times 10^{-3} \times 30 \times 10^3 = j6$$

and

$$-jX_C = -j\frac{1}{\omega C} = -j\frac{1}{30 \times 10^3 \times \frac{10}{3} \times 10^{-6}} = -j10$$

Also,

$$\mathbf{V}_{IN} = 2\angle 0^\circ$$

and the phasor circuit is shown in Figure 5.19.

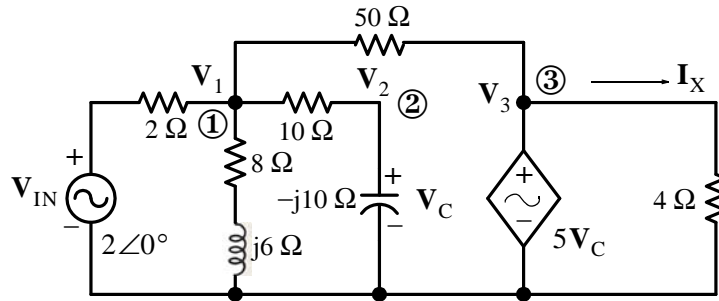


Figure 7.19. Phasor circuit for Example 7.5

At Node ①:

$$\frac{V_1 - 2\angle 0^\circ}{2} + \frac{V_1}{8 + j6} + \frac{V_1 - V_C}{10} + \frac{V_1 - 5V_C}{50} = 0 \quad (7.20)$$

and since

$$\frac{1}{8 + j6} = \frac{1}{8 + j6} \cdot \frac{8 - j6}{8 - j6} = \frac{8 - j6}{100} = \frac{4}{50} - j\frac{3}{50}$$

the nodal equation of (7.20) simplifies to

$$\left( \frac{35}{50} - j\frac{3}{50} \right) V_1 - \frac{1}{5} V_C = 1\angle 0^\circ \quad (7.21)$$

At Node ②:

$$\frac{V_C - V_1}{10} + \frac{V_C}{-j10} = 0$$

or

$$-\frac{1}{10} V_1 + \left( \frac{1}{10} + j\frac{1}{10} \right) V_C = 0 \quad (7.22)$$

At Node ③:

$$V_3 = 5V_C$$

We use MATLAB to solve (7.21) and (7.22).

```
G=[35/50 -j*3/50; -1/5 1/10+j*1/10]; I=[1 0]'; V=G\I;
Ix=5*V(2,1)/4; % Multiply Vc by 5 and divide by 4 to obtain current Ix
magIx=abs(Ix); theta=angle(Ix)*180/pi; % Convert current Ix to polar form
fprintf(' \n'); disp(' Ix = '); disp(Ix);...
fprintf('magIx = %4.2f A \t', magIx); fprintf('theta = %4.2f deg \t', theta);...
fprintf(' \n'); fprintf(' \n');
```

$I_x = 2.1176 - 1.7546i$        $\text{mag}I_x = 2.75 \text{ A}$        $\text{theta} = -39.64 \text{ deg}$

Therefore,

$$I = 2.75\angle -39.6^\circ \Leftrightarrow i(t) = 2.75 \cos(30000t - 39.6^\circ)$$

## Chapter 7 Phasor Circuit Analysis

### Example 7.6

Compute the phasor  $\mathbf{V}_{\text{out}}$  for the op amp circuit of Figure 7.20.

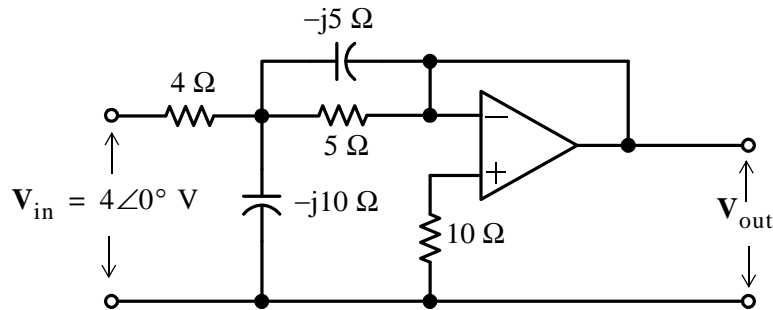


Figure 7.20. Circuit for Example 7.6

#### Solution:

We assign phasor voltages  $\mathbf{V}_1$  and  $\mathbf{V}^+$  as shown in Figure 7.21, and we apply KCL at these nodes, while observing that  $\mathbf{V}_{\text{out}} = \mathbf{V}^+$

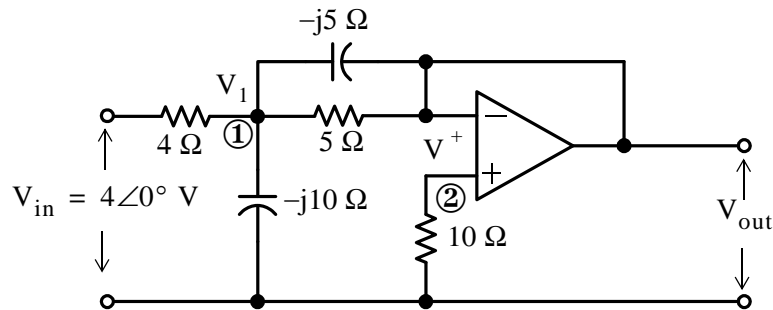


Figure 7.21. Application of KCL for the circuit of Example 7.6

At Node ①:

$$\frac{\mathbf{V}_1 - 4\angle 0^\circ}{4} + \frac{\mathbf{V}_1 - \mathbf{V}_{\text{out}}}{-j5} + \frac{\mathbf{V}_1 - \mathbf{V}_{\text{out}}}{5} + \frac{\mathbf{V}_1}{-j10} = 0$$

or

$$\left(\frac{9}{20} + j\frac{3}{10}\right)\mathbf{V}_1 - \left(\frac{1}{5} + j\frac{1}{5}\right)\mathbf{V}_{\text{out}} = 1\angle 0^\circ \quad (7.23)$$

At Node ②,

$$\mathbf{V}_2 = \mathbf{V}^+ = \mathbf{V}_{\text{out}}$$

and thus,

$$\frac{\mathbf{V}_{\text{out}}}{10} + \frac{\mathbf{V}_{\text{out}} - \mathbf{V}_1}{5} + \frac{\mathbf{V}_{\text{out}} - \mathbf{V}_1}{-j5} = 0$$

or

$$-\left(\frac{1}{5} + j\frac{1}{5}\right)\mathbf{V}_1 + \left(\frac{3}{10} + j\frac{1}{5}\right)\mathbf{V}_{\text{out}} = 0 \quad (7.24)$$

Solving (7.23) and (7.24) with MATLAB we obtain:

```
format rat
G=[9/20+j*3/10 -1/5-j*1/5; -1/5-j*1/5 3/10+j*1/5]; I=[1 0]'; V=G\I;
fprintf(' \n');disp('V1 = '); disp(V(1,1)); disp('Vout = '); disp(V(2,1));
format short
magV=abs(V(2,1)); thetaV=angle(V(2,1))*180/pi;
fprintf('magIx = %5.3f A \t', magIx); fprintf('theta = %4.2f deg \t', theta);...
fprintf(' \n'); fprintf(' \n')
```

```
V1 = 68/25 - 24/25i      Vout = 56/25 - 8/25i
magIx = 2.750 A        theta = -39.64 deg
```

Therefore,  $\mathbf{V}_{\text{out}} = 2.263 \angle -8.13^\circ$  (7.25)

## 7.6 Phasor Diagrams

A *phasor diagram* is a sketch showing the magnitude and phase relationships among the phasor voltages and currents in phasor circuits. The procedure is best illustrated with the examples below.

### Example 7.7

Compute and sketch all phasor quantities for the circuit of Figure 7.22.

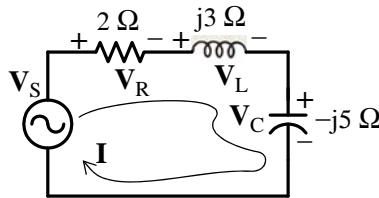


Figure 7.22. Circuit for Example 7.7

#### Solution:

Since this is a series circuit, the phasor current  $\mathbf{I}$  is common to all circuit devices. Therefore, we assign to this phasor current the value  $\mathbf{I} = 1 \angle 0^\circ$  and use it as our reference as shown in the phasor diagram of Figure 7.23 where:



$$\mathbf{V}_R = (2 \Omega)(1\angle 0^\circ) = 2\angle 0^\circ \text{ V}$$

$$\mathbf{V}_L = (j3 \Omega)(1\angle 0^\circ) = j3 = 3\angle 90^\circ \text{ V}$$

$$\mathbf{V}_C = (-j5 \Omega)(1\angle 0^\circ) = -j5 = 5\angle -90^\circ \text{ V}$$

$$\mathbf{V}_S = \mathbf{V}_R + (\mathbf{V}_L + \mathbf{V}_C) = 2 - j2 = 2\sqrt{2}\angle -45^\circ$$

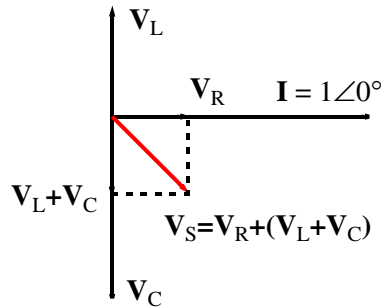


Figure 7.23. Phasor diagram for the circuit of Example 7.7

### Example 7.8

Compute and sketch all phasor quantities for the circuit of Figure 7.24.

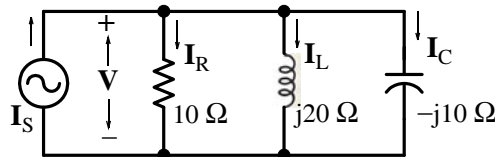


Figure 7.24. Circuit for Example 7.8

### Solution:

Since this is a parallel circuit, the phasor voltage  $\mathbf{V}$  is common to all circuit devices. Therefore let us assign this phasor voltage the value  $\mathbf{V} = 1\angle 0^\circ$  and use it as our reference phasor as shown in the phasor diagram of Figure 7.25 where:

$$\mathbf{I}_R = 1\angle 0^\circ / 10 = 100\angle 0^\circ \text{ mA}$$

$$\mathbf{I}_L = 1\angle 0^\circ / j20 = 1\angle 0^\circ / 20\angle 90^\circ = 50\angle -90^\circ \text{ m}$$

$$\mathbf{I}_C = 1\angle 0^\circ / (-j10) = 1\angle 0^\circ / 10\angle -90^\circ = 100\angle 90^\circ \text{ mA}$$

$$\mathbf{I}_C + \mathbf{I}_L = 50\angle 90^\circ \text{ mA}$$

$$\mathbf{I}_S = \mathbf{I}_R + (\mathbf{I}_C + \mathbf{I}_L) = 100 + j50 = 111.8\angle 26.6^\circ$$

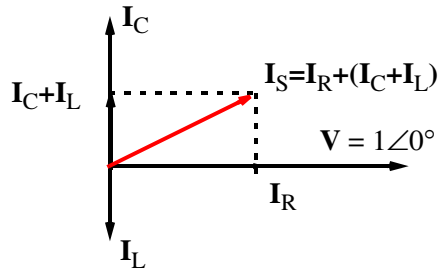


Figure 7.25. Phasor diagram for Example 7.8

We can draw a phasor diagram for other circuits that are neither series nor parallel by assigning any phasor quantity as a reference.

### Example 7.9

Compute and sketch all phasor voltages for the circuit of Figure 7.26. Then, use MATLAB to plot these quantities in the  $t$  – domain.

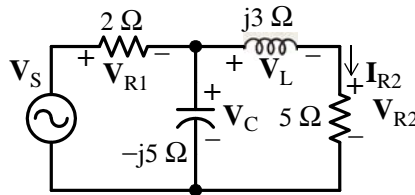


Figure 7.26. Circuit for Example 7.9

#### Solution:

We will begin by selecting  $I_{R2} = 1\angle 0^\circ$  A as our reference as shown on the phasor diagram of Figure 7.27. Then,

$$V_{R2} = 5 \Omega \times I_{R2} = 5 \times 1\angle 0^\circ = 5\angle 0^\circ$$

$$V_L = j3 \Omega \times I_{R2} = 3\angle 90^\circ \times 1\angle 0^\circ = 3\angle 90^\circ$$

$$V_C = V_L + V_{R2} = 5\angle 0^\circ + 3\angle 90^\circ = 5 + j3 = 5.83\angle 31^\circ$$

$$V_{R1} = 2 \Omega \times I_{R1} = 2(I_C + I_{R2}) = 2\left(\frac{V_C}{-j5} + I_{R2}\right) = 2\left(\frac{5.83\angle 31^\circ}{5\angle -90^\circ} + 5\angle 0^\circ\right)$$

$$= 2.33\angle 121^\circ + 10\angle 0^\circ = -1.2 + j2 + 10 = 8.8 + j2 = 9\angle 12.8^\circ$$

and

$$V_S = V_{R1} + V_C = 8.8 + j2 + 5 + j3 = 13.8 + j5 = 14.7\angle 20^\circ$$

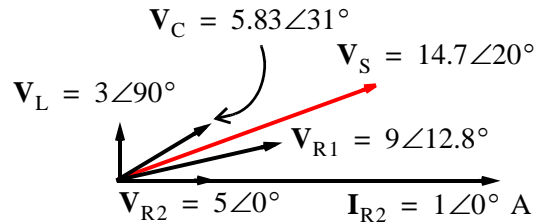


Figure 7.27. Phasor diagram for Example 7.9

Now, we can transform these phasors into time-domain quantities and use MATLAB to plot them. We will use the voltage source as a reference with the value  $\mathbf{V}_S = 1\angle 0^\circ$ , and we will apply nodal analysis with node voltages  $\mathbf{V}_1$ ,  $\mathbf{V}_2$ , and  $\mathbf{V}_3$  assigned as shown in Figure 7.28.

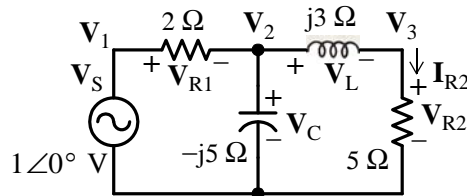


Figure 7.28. Circuit for Example 7.9 with the voltage source taken as reference

The node equations are shown below in matrix form.

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & \left(\frac{1}{2} + \frac{1}{-j5} + \frac{1}{j3}\right) & -\frac{1}{j3} \\ 0 & -\frac{1}{j3} & \left(\frac{1}{j3} + \frac{1}{5}\right) \end{bmatrix}}_G \underbrace{\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \\ \mathbf{V}_3 \end{bmatrix}}_V = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_I$$

The MATLAB script is as follows:

```
% Enter the non-zero values of the G matrix
G(1,1)=1;
G(2,1)=-1/2;
G(2,2)=1/2-1/5j+1/3j;
G(2,3)=-1/3j;
G(3,2)=-1/3j;
G(3,3)=1/3j+1/5;
%
% Enter all values of the I matrix
I=[1 0 0]';
%
% Compute node voltages
V=G\I;
%
```

```

VR1=V(1)-V(2);
VL=V(2)-V(3);
% Compute magnitudes and phase angles of voltages
magV1=abs(V(1)); magV2=abs(V(2)); magV3=abs(V(3));
phaseV1=angle(V(1))*180/pi; phaseV2=angle(V(2))*180/pi; phaseV3=angle(V(3))*180/pi;
magVR1=abs(VR1); phaseVR1=angle(VR1)*180/pi;
magVL=abs(VL); phaseVL=angle(VL)*180/pi;
%
% Denote radian frequency as w and plot wt for 0 to 2*pi range
wt=linspace(0,2*pi);
V1=magV1*cos(wt-phaseV1);
V2=magV2*cos(wt-phaseV2);
V3=magV3*cos(wt-phaseV3);
VR1t=magVR1*cos(wt-phaseVR1);
VLt=magVL*cos(wt-phaseVL);
%
% Convert wt to degrees
deg=wt*180/pi;
%
% Print phasor voltages, magnitudes, and phase angles
fprintf(' \n');
% With fprintf only the real part of each parameter is processed so we will use disp
disp('V1 = '); disp(V(1)); disp('V2 = '); disp(V(2)); disp('V3 = '); disp(V(3));
disp('VR1 = '); disp(VR1); disp('VL = '); disp(VL);
fprintf('magV1 = %4.2f V \t', magV1); fprintf('magV2 = %4.2f V \t', magV2);
fprintf('magV3 = %4.2f V', magV3); fprintf(' \n'); fprintf(' \n');
fprintf('phaseV1 = %4.2f deg \t', phaseV1);
fprintf('phaseV2 = %4.2f deg \t', phaseV2); fprintf('phaseV3 = %4.2f deg', phaseV3);
fprintf(' \n'); fprintf(' \n');
fprintf('magVR1 = %4.2f V \t', magVR1); fprintf('phaseVR1 = %4.2f deg ', phaseVR1);
fprintf(' \n'); fprintf(' \n');
fprintf('magVL = %4.2f V \t', abs(VL)); fprintf('phaseVL = %4.2f deg ', phaseVL);
fprintf(' \n');
%
plot(deg,V1,deg,V2,deg,V3,deg,VR1t,deg,VLt)
fprintf(' \n');

V1 = 1
V2 = 0.7503 - 0.1296i
V3 = 0.4945 - 0.4263i
VR1 = 0.2497 + 0.1296i
VL = 0.2558 + 0.2967i

magV1 = 1.00 V      magV2 = 0.76 V      magV3 = 0.65 V
phaseV1 = 0.00 deg  phaseV2 = -9.80 deg  phaseV3 = -40.76 deg
magVR1 = 0.28 V     phaseVR1 = 27.43 deg
magVL = 0.39 V      phaseVL = 49.24 deg

```

and with these values we have

$$v_s(t) = v_1(t) = \cos\omega t \quad v_2(t) = 0.76\cos(\omega t - 9.8^\circ) \quad v_3(t) = 0.65\cos(\omega t - 40.8^\circ)$$

$$v_{R1}(t) = 0.28 \cos(\omega t + 27.4^\circ) \quad v_L(t) = 0.39 \cos(\omega t + 49.2^\circ)$$

These are plotted with MATLAB as shown in Figure 7.29.

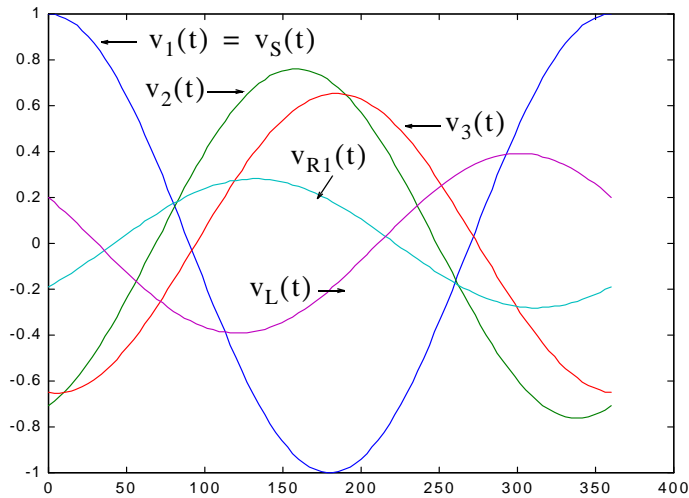


Figure 7.29. The  $t$ -domain plots for Example 7.9

### 7.7 Electric Filters

The characteristics of electric filters were introduced in Chapter 4 but are repeated below for convenience.

*Analog filters* are defined over a continuous range of frequencies. They are classified as *low-pass*, *high-pass*, *band-pass* and *band-elimination (stop-band)*. Another, less frequently mentioned filter, is the *all-pass* or *phase shift* filter. It has a constant amplitude response but its phase varies with frequency. This is discussed in *Signals and Systems with MATLAB Computing and Simulink Modeling*, ISBN 978-1-934404-11-9.

The ideal amplitude characteristics of each are shown in Figure 7.30. The ideal characteristics are not physically realizable; we will see that practical filters can be designed to approximate these characteristics. In this section we will derive the passive RC low and high-pass filter characteristics and those of an active low-pass filter using phasor analysis.

A *digital filter*, in general, is a computational process, or algorithm that converts one sequence of numbers representing the input signal into another sequence representing the output signal. Accordingly, a digital filter can perform functions as differentiation, integration, estimation, and, of course, like an analog filter, it can filter out unwanted bands of frequency. Digital filters are discussed in *Signals and Systems with MATLAB Computing and Simulink Modeling*, ISBN 978-1-934404-11-9.

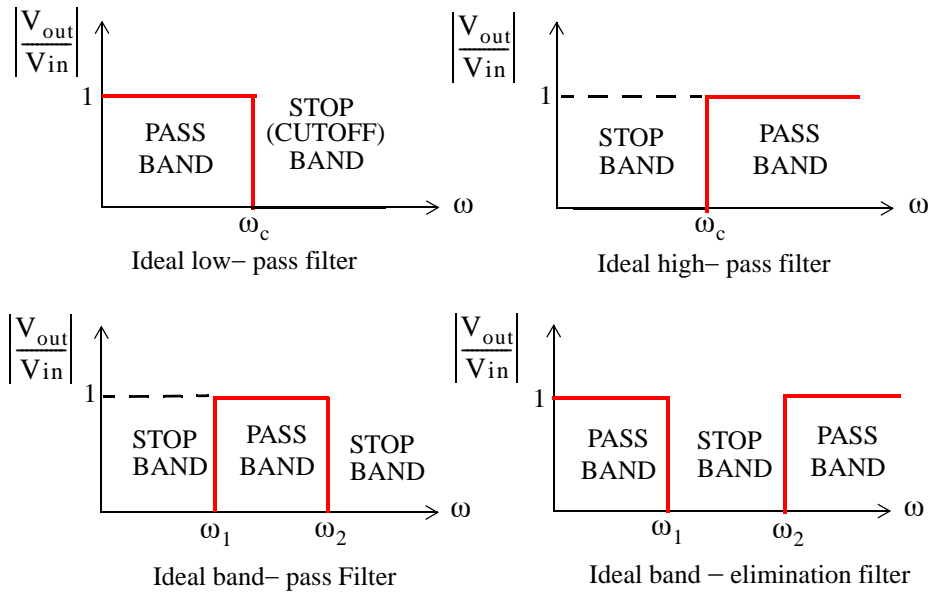


Figure 7.30. Amplitude characteristics of the types of filters

## 7.8 Basic Analog Filters

An analog filter can also be classified as *passive* or *active*. Passive filters consist of passive devices such as resistors, capacitors and inductors. Active filters are, generally, operational amplifiers with resistors and capacitors connected to them externally. We can find out whether a filter, passive or active, is a low-pass, high-pass, etc., from its the frequency response that can be obtained from its transfer function. The procedure is illustrated with the examples that follow.

### Example 7.10

Derive expressions for the magnitude and phase responses of the series RC network of Figure 7.31, and sketch their characteristics.

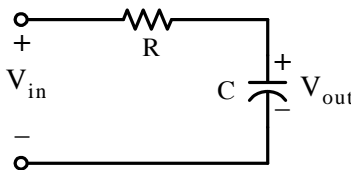


Figure 7.31. Series RC network for Example 7.10

#### Solution:

By the voltage division expression,

$$V_{\text{out}} = \frac{1/j\omega C}{R + 1/j\omega C} V_{\text{in}}$$

and denoting the ratio  $V_{\text{out}}/V_{\text{in}}$  as  $G(j\omega)$ , we obtain

$$G(j\omega) = \frac{V_{\text{out}}}{V_{\text{in}}} = \frac{1}{1 + j\omega RC} = \frac{1}{(\sqrt{1 + \omega^2 R^2 C^2}) \angle \text{atan}(\omega RC)} = \frac{1}{\sqrt{1 + \omega^2 R^2 C^2}} \angle -\text{atan}(\omega RC) \quad (7.26)$$

The magnitude of (7.26) is

$$|G(j\omega)| = \left| \frac{V_{\text{out}}}{V_{\text{in}}} \right| = \frac{1}{\sqrt{1 + \omega^2 R^2 C^2}} \quad (7.27)$$

and the phase angle  $\theta$ , also known as the *argument*, is

$$\theta = \arg\{G(j\omega)\} = \arg\left(\frac{V_{\text{out}}}{V_{\text{in}}}\right) = -\text{atan}(\omega RC) \quad (7.28)$$

We can obtain a quick sketch for the magnitude  $|G(j\omega)|$  versus  $\omega$  by evaluating (7.27) at  $\omega = 0$ ,  $\omega = 1/RC$ , and  $\omega \rightarrow \infty$ . Thus,

$$\text{as } \omega \rightarrow 0, |G(j\omega)| \cong 1$$

$$\text{for } \omega = 1/RC, |G(j\omega)| = 1/\sqrt{2} = 0.707$$

$$\text{and as } \omega \rightarrow \infty, |G(j\omega)| \cong 0$$

The magnitude, indicated as  $|G(j\omega)|$  versus radian frequency for several values of  $\omega$  is shown in Figure 7.32 where, for convenience, we have let  $RC = 1$ . The plot shows that this circuit is an approximation, although not a good one, to the amplitude characteristics of a low-pass filter.

We can also obtain a quick sketch for the phase angle, i.e.,  $\theta = \arg\{G(j\omega)\}$  versus  $\omega$  by evaluating of (11.3) at  $\omega = 0$ ,  $\omega = 1/RC$ ,  $\omega = -1/RC$ ,  $\omega \rightarrow -\infty$  and  $\omega \rightarrow \infty$ . Thus,

$$\text{as } \omega \rightarrow 0, \theta \cong -\text{atan}0 \cong 0^\circ$$

$$\text{for } \omega = 1/RC, \theta = -\text{atan}1 = -45^\circ$$

$$\text{for } \omega = -1/RC, \theta = -\text{atan}(-1) = 45^\circ$$

$$\text{as } \omega \rightarrow -\infty, \theta = -\text{atan}(-\infty) = 90^\circ$$

$$\text{and as } \omega \rightarrow \infty, \theta = -\text{atan}(\infty) = -90^\circ$$

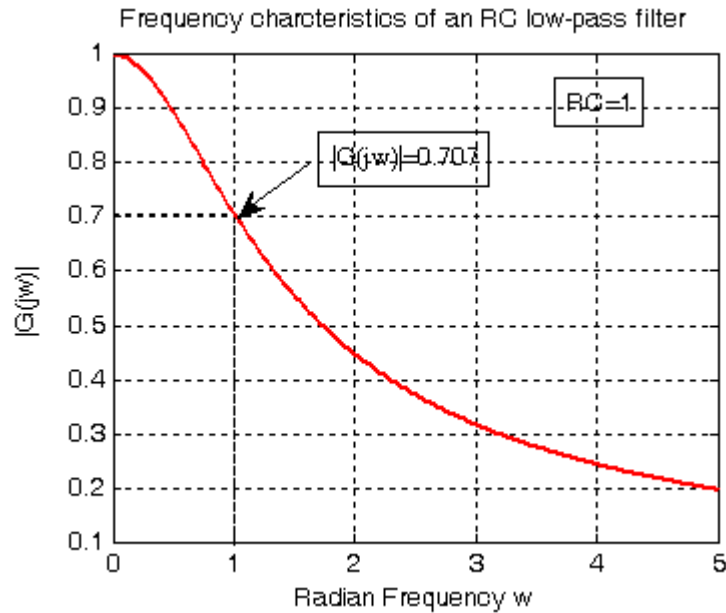


Figure 7.32. Amplitude characteristics of a series RC low-pass filter

Figure 7.33 shows the phase characteristic of an RC low-pass filter where, again for convenience, we have let  $RC = 1$ .

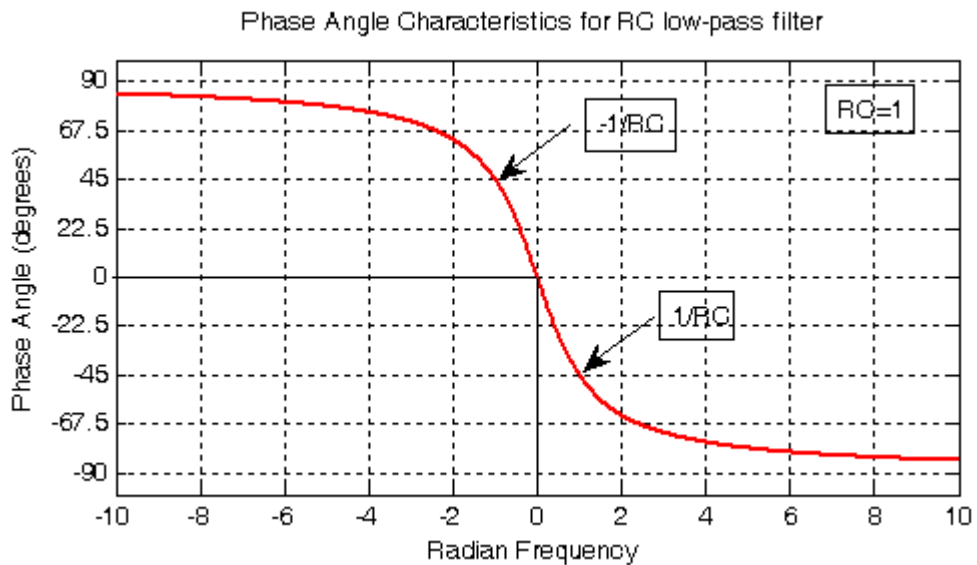


Figure 7.33. Phase characteristics of a series RC low-pass filter



### Example 7.11

The network of Figure 7.31 is also a series RC circuit, where the positions of the resistor and capacitor have been interchanged. Derive expressions for the magnitude and phase responses, and sketch their characteristics.

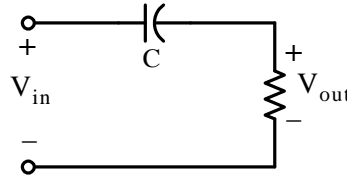


Figure 7.34. RC network for Example 7.11

**Solution:**

$$V_{\text{out}} = \frac{R}{R + 1/j\omega C} V_{\text{in}}$$

or

$$\begin{aligned} G(j\omega) &= \frac{V_{\text{out}}}{V_{\text{in}}} = \frac{j\omega RC}{1 + j\omega RC} = \frac{j\omega RC + \omega^2 R^2 C^2}{1 + \omega^2 R^2 C^2} = \frac{\omega RC(j + \omega RC)}{1 + \omega^2 R^2 C^2} \\ &= \frac{\omega RC \sqrt{1 + \omega^2 R^2 C^2} \angle \text{atan}(1/\omega RC)}{1 + \omega^2 R^2 C^2} = \frac{1}{\sqrt{1 + 1/(\omega^2 R^2 C^2)}} \angle \text{atan}\left(\frac{1}{\omega RC}\right) \end{aligned} \quad (7.29)$$

The magnitude of (7.29) is

$$\boxed{|G(j\omega)| = \frac{1}{\sqrt{1 + 1/(\omega^2 R^2 C^2)}}} \quad (7.30)$$

and the phase angle or argument, is

$$\boxed{\theta = \arg\{G(j\omega)\} = \text{atan}\left(\frac{1}{\omega RC}\right)} \quad (7.31)$$

We can obtain a quick sketch for the magnitude  $|G(j\omega)|$  versus  $\omega$  by evaluating (7.30) at  $\omega = 0$ ,  $\omega = 1/RC$ , and  $\omega \rightarrow \infty$ . Thus,

$$\text{as } \omega \rightarrow 0, |G(j\omega)| \cong 0$$

$$\text{for } \omega = 1/RC, |G(j\omega)| = 1/\sqrt{2} = 0.707$$

$$\text{and as } \omega \rightarrow \infty, |G(j\omega)| \cong 1$$

Figure 7.35 shows  $|G(j\omega)|$  versus radian frequency for several values of  $\omega$  where  $RC = 1$ . The plot shows that this circuit is an approximation, although not a good one, to the amplitude characteristics of a high-pass filter.

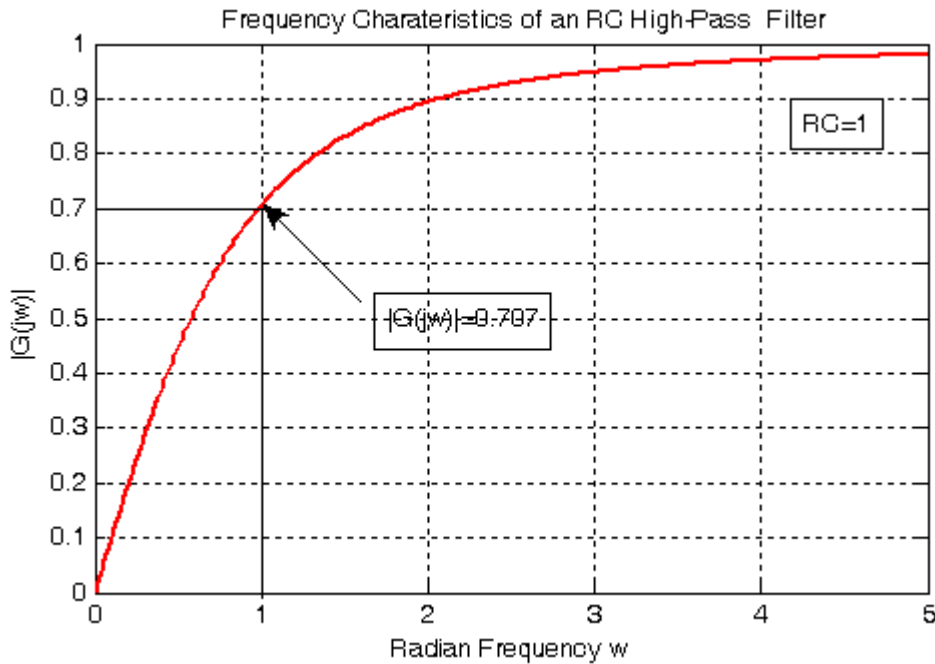


Figure 7.35. Amplitude characteristics of a series RC high-pass filter

We can also obtain a quick sketch for the phase angle, i.e.,  $\theta = \arg\{G(j\omega)\}$  versus  $\omega$ , by evaluating (7.31) at  $\omega = 0$ ,  $\omega = 1/RC$ ,  $\omega = -1/RC$ ,  $\omega \rightarrow -\infty$ , and  $\omega \rightarrow \infty$ . Thus,

as  $\omega \rightarrow 0$ ,  $\theta \cong -\text{atan}0 \cong 0^\circ$

for  $\omega = 1/RC$ ,  $\theta = -\text{atan}1 = -45^\circ$

for  $\omega = -1/RC$ ,  $\theta = -\text{atan}(-1) = 45^\circ$

as  $\omega \rightarrow -\infty$ ,  $\theta = -\text{atan}(-\infty) = 90^\circ$

and as  $\omega \rightarrow \infty$ ,  $\theta = -\text{atan}(\infty) = -90^\circ$

Figure 7.36 shows the phase angle  $\theta$  versus radian frequency for several values of  $\omega$ , where  $RC = 1$ .

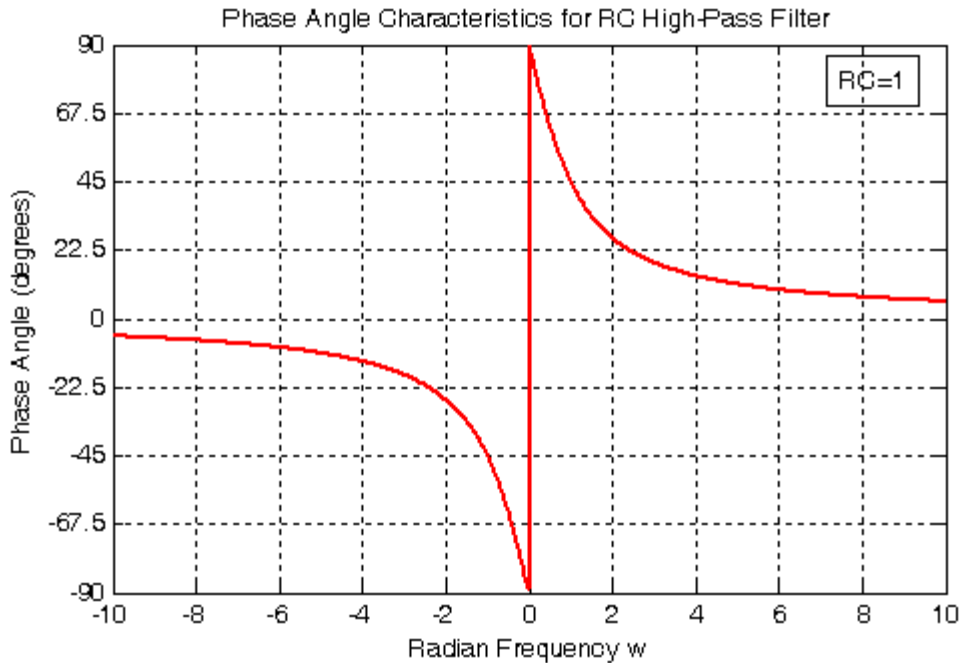


Figure 7.36. Phase characteristics of an RC high-pass filter

We should remember that the RC low-pass filter in Figure 7.31 and the RC high-pass filter in Figure 7.34 behave as filters only when the excitation (input voltage) is sinusoidal at some frequency. If the excitation is any input, the RC network in Figure 7.28 behaves as an integrator provided that  $v_{\text{OUT}} \ll v_{\text{IN}}$ , while the RC network in Figure 7.31 behaves as a differentiator provided that  $v_{\text{OUT}} \ll v_{\text{IN}}$ . The proofs are left as exercises for the reader at the end of this chapter.

### 7.9 Active Filter Analysis

We can analyze active filters, such as those we discussed in Chapter 4, using phasor circuit analysis.

#### Example 7.12

Compute the approximate cut-off frequency of the circuit of Figure 7.37 which is known as a *Multiple Feed Back (MFB) active low-pass filter*.

#### Solution:

We assign two nodes as shown in Figure 7.38, and we write the phasor circuit nodal equations as follows:

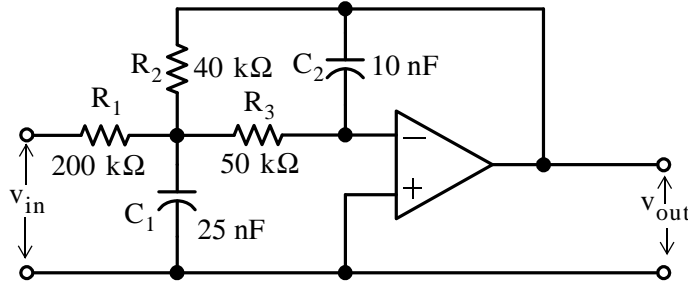


Figure 7.37. Low-pass filter for Example 7.12

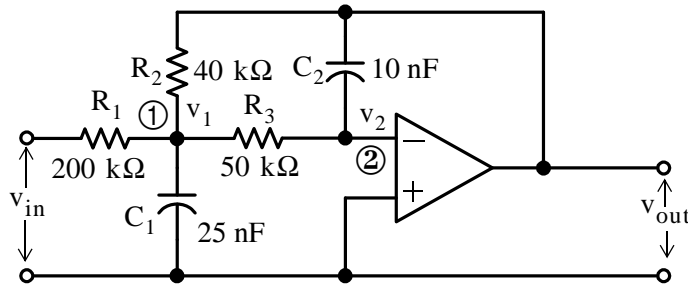


Figure 7.38. Circuit for nodal analysis, Example 7.12

At Node ①:

$$\frac{v_1 - v_{in}}{R_1} + \frac{v_1}{1/(j\omega C_1)} + \frac{v_1 - v_{out}}{R_2} + \frac{v_1 - v_2}{R_3} = 0 \quad (7.32)$$

At node ②:

$$\frac{v_2 - v_1}{R_3} = \frac{C_2}{1/(j\omega C_2)} \quad (7.33)$$

and since  $v_2 = 0$  (virtual ground), relation (7.33) reduces to

$$v_1 = (-j\omega R_3 C_2) v_{out} \quad (7.34)$$

and by substitution of (7.34) into (7.32), rearranging, and collecting like terms, we obtain:

$$\left[ \left( \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + j\omega C_1 \right) (-j\omega R_3 C_2) - \frac{1}{R_2} \right] v_{out} = \frac{1}{R_1} v_{in} \quad (7.35)$$

or

$$\frac{v_{out}}{v_{in}} = \frac{1}{R_1 \left[ \left( \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + j\omega C_1 \right) (-j\omega R_3 C_2) - \frac{1}{R_2} \right]} \quad (7.36)$$

By substitution of given values of resistors and capacitors, we obtain

$$\frac{V_{out}}{V_{in}} = \frac{1}{2 \times 10^5 \left[ \left( \frac{1}{20 \times 10^3} + j2.5 \times 10^{-8} \omega \right) (-j5 \times 10^4 \times 10^{-8} \omega) - \frac{1}{4 \times 10^4} \right]}$$

or

$$|G(j\omega)| = \frac{V_{out}}{V_{in}} = \frac{-1}{2.5 \times 10^{-6} \omega^2 - j5 \times 10^{-3} \omega + 5} \quad (7.37)$$

and now we can use MATLAB to find and plot the magnitude of (7.37) with the following script.

```
w=1:10:10000; Gjw=-1./(2.5.*10.^(-6).*w.^2-5.*j.*10.^(-3).*w+5);
semilogx(w,abs(Gjw)); grid; hold on
xlabel('Radian Frequency w'); ylabel('|Vout/Vin|');
title('Magnitude Vout/Vin vs. Radian Frequency')
```

The plot is shown in Figure 7.39 where we see that the cutoff frequency occurs at about 700 rad/s. We observe that the half-power point for this plot is  $0.2 \times 0.707 = 0.141$ .

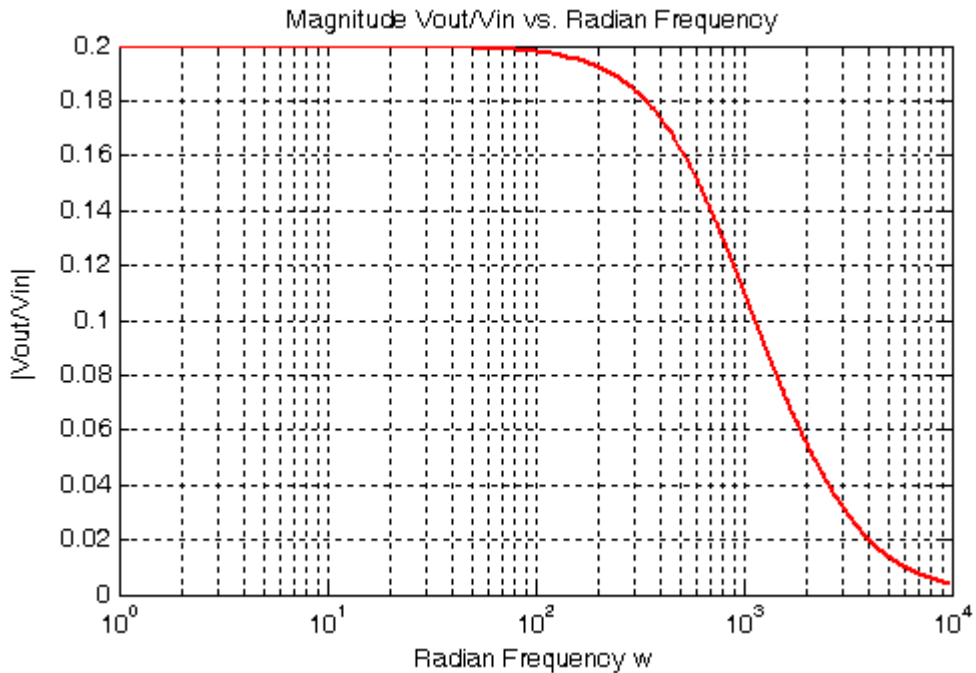


Figure 7.39. Plot for the magnitude of the low-pass filter circuit of Example 7.12

## 7.10 Summary

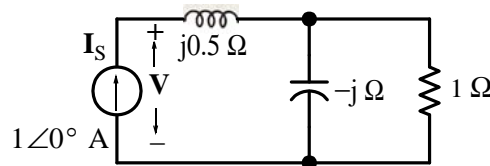
- In Chapter 3 we were concerned with constant voltage and constant current sources, resistances and conductances. In this chapter we were concerned with alternating voltage and alternating current sources, impedances, and admittances.
- Nodal analysis, mesh analysis, the principle of superposition, Thevenin's theorem, and Norton's theorem can also be applied to phasor circuits.
- The use of complex numbers make the phasor circuit analysis much easier.
- MATLAB can be used very effectively to perform the computations since it does not require any special procedures for manipulation of complex numbers.
- Whenever a branch in a circuit contains two or more devices in series or two or more devices in parallel, it is highly recommended that they are grouped and denoted as  $z_1$ ,  $z_2$ , and so on before writing nodal or mesh equations.
- Phasor diagrams are sketches that show the magnitude and phase relationships among several phasor voltages and currents. When constructing a phasor diagram, the first step is to select one phasor as a reference, usually with zero phase angle, and all other phasors must be drawn with the correct relative angles.
- The RC low-pass and RC high-pass filters are rudimentary types of filters and are not used in practice. They serve as a good introduction to electric filters.

## 7.11 Exercises

### Multiple Choice

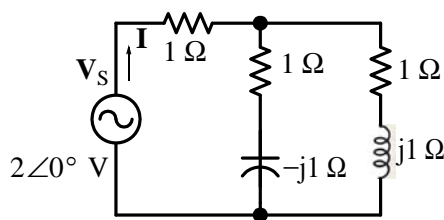
1. In the circuit below the phasor voltage  $\mathbf{V}$  is

- A.  $2 + j0$  V
- B.  $1 + j0$  V
- C.  $1 - j0$  V
- D.  $1 + j$  V
- E. none of the above



2. In the circuit below the phasor current  $\mathbf{I}$  is

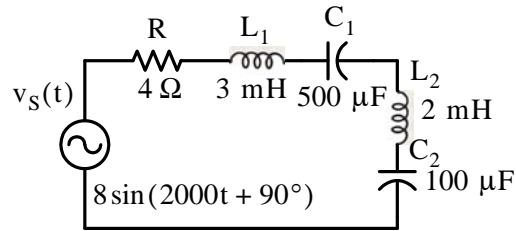
- A.  $0 + j2$  A
- B.  $0 - j2$  A
- C.  $1 + j0$  A
- D.  $2 + j2$  A
- E. none of the above



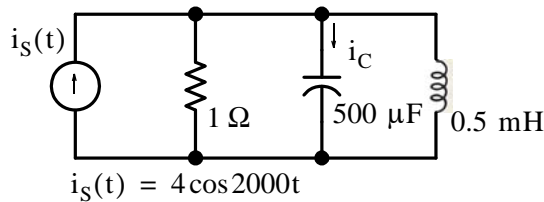
3. In the circuit below the voltage across the capacitor  $C_2$  is

- A.  $8 \times 10^{-4} \sin(2000t + 90^\circ)$  V
- B.  $\sqrt{50} \cos(2000t - 45^\circ)$  V
- C.  $\sqrt{50} \cos(2000t + 45^\circ)$  V

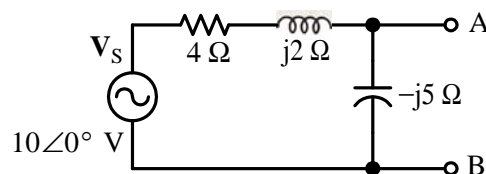
- D.  $\sqrt{50} \cos(2000t + 90^\circ)$  V
- E. none of the above



4. In the circuit below the current  $i_C(t)$  through the capacitor is
- A.  $4 \sin 2000t$
  - B.  $4 \sin(2000t + 180^\circ)$
  - C.  $\sqrt{32} \cos(2000t - 45^\circ)$
  - D.  $\sqrt{32} \cos(2000t + 90^\circ)$
  - E. none of the above



5. The Thevenin equivalent voltage  $V_{TH}$  at terminals A and B in the circuit below is
- A.  $10 \angle -90^\circ$  V
  - B.  $10 \angle -53.13^\circ$  V
  - C.  $10 \angle 53.13^\circ$  V
  - D.  $10 \angle -45^\circ$  V
  - E. none of the above





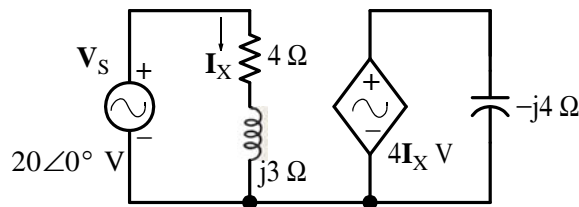
## Chapter 7 Phasor Circuit Analysis

6. The Thevenin equivalent impedance  $Z_{TH}$  at terminals A and B in the circuit above is

- A.  $2 + j4 \ \Omega$
- B.  $4 + j2 \ \Omega$
- C.  $4 - j2 \ \Omega$
- D.  $-j5 \ \Omega$
- E. none of the above

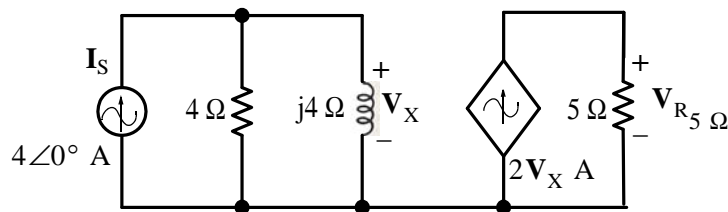
7. In the circuit below the phasor voltage  $V_C$  is

- A.  $5 \angle -90^\circ \text{ V}$
- B.  $5 \angle -45^\circ \text{ V}$
- C.  $4 \angle -53.1^\circ \text{ V}$
- D.  $4 \angle 53.1^\circ \text{ V}$
- E. none of the above



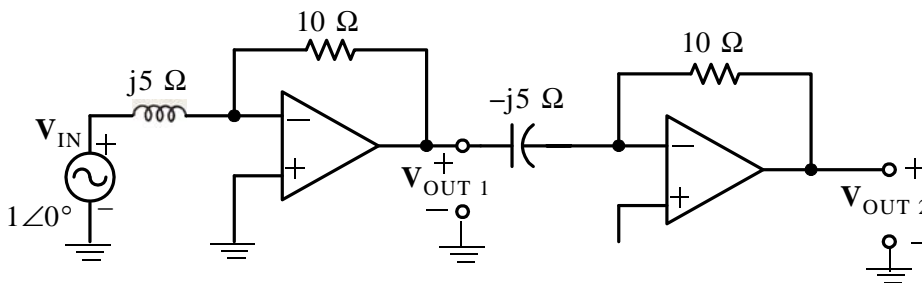
8. In the circuit below the phasor voltage  $V_{R_5}$  is

- A.  $20 + j0 \text{ V}$
- B.  $0 + j20 \text{ V}$
- C.  $20 + j20 \text{ V}$
- D.  $80 - j80 \text{ V}$
- E. none of the above



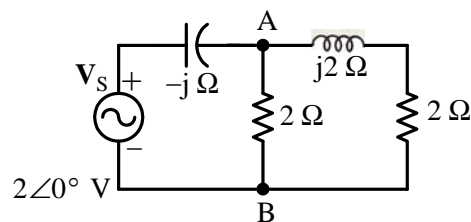
9. In the circuit below the phasor voltage  $V_{OUT 2}$  is

- A.  $2 + j0$  V
- B.  $4 + j0$  V
- C.  $4 - j0$  V
- D.  $1 + j1$  V
- E. none of the above



10. In the circuit below the t – domain voltage  $v_{AB}(t)$  is

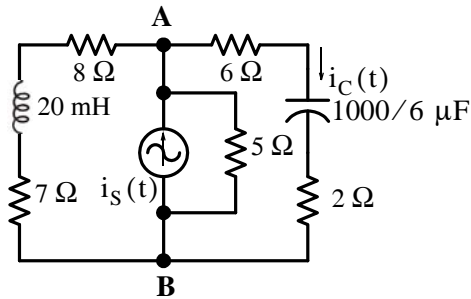
- A.  $1.89 \cos(\omega t + 45^\circ)$  V
- B.  $0.53 \cos(\omega t - 45^\circ)$  V
- C.  $2 \cos \omega t$  V
- D.  $0.5 \cos(\omega t + 53.1^\circ)$  V
- E. none of the above



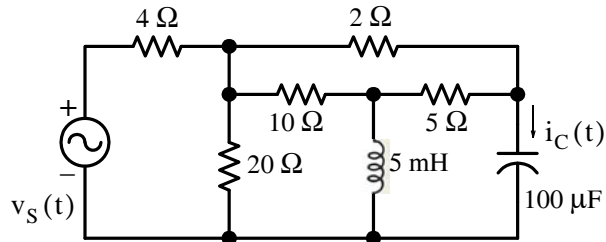
## Chapter 7 Phasor Circuit Analysis

### Problems

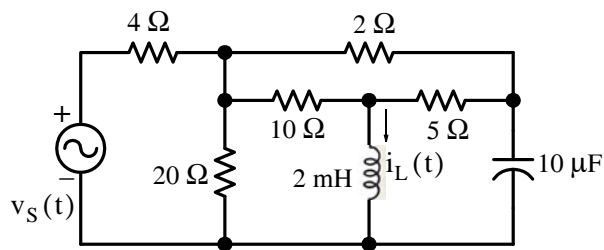
1. For the circuit below  $i_S(t) = 2 \cos 1000t$  A. Compute  $v_{AB}(t)$  and  $i_C(t)$ .



2. Write nodal equations and use MATLAB to compute  $i_C(t)$  for the circuit below given that  $v_S(t) = 12 \cos(1000t + 45^\circ)$  V.



3. Write mesh equations and use MATLAB to compute  $i_L(t)$  for the circuit below given that  $v_S(t) = 100 \cos(10000t + 60^\circ)$  V.

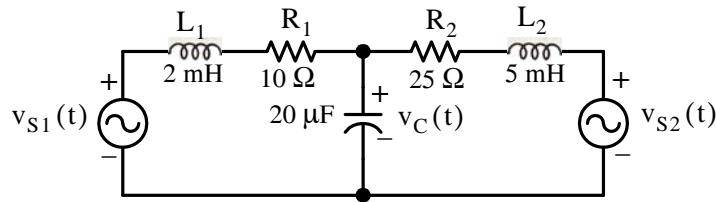


4. For the circuit below it is given that

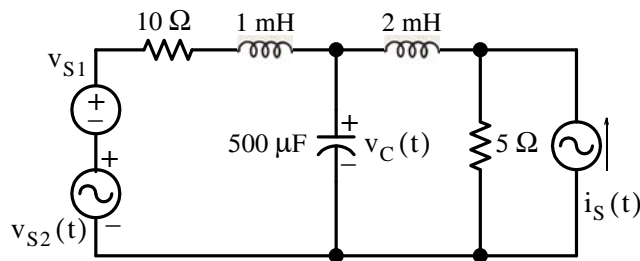
$$v_{S1}(t) = 40 \cos(5000t + 60^\circ) \text{ V}$$

and

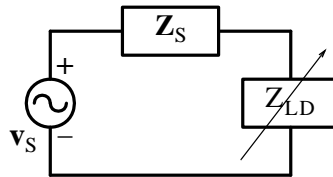
$$v_{S2}(t) = 60 \sin(5000t + 60^\circ) \text{ V. Use superposition to find } v_C(t)$$



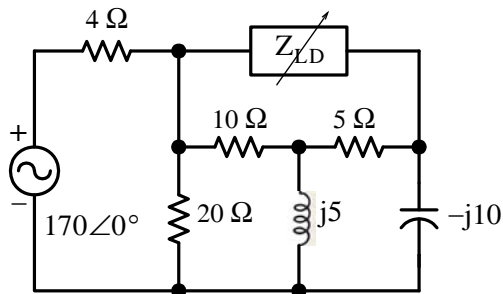
5. For the circuit below find  $v_C(t)$  if  $v_{S1} = 15 \text{ V}$ ,  $v_{S2}(t) = 20 \cos 1000t \text{ V}$ , and  $i_S(t) = 4 \cos 2000t \text{ A}$ . Plot  $v_C(t)$  using MATLAB or Excel.



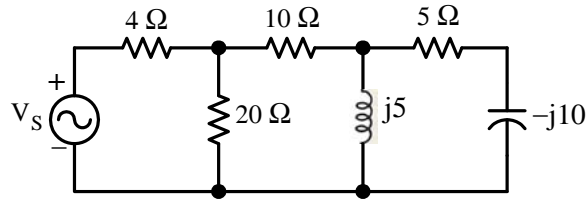
6. For the circuit below find the value of  $Z_{LD}$  which will receive maximum power.



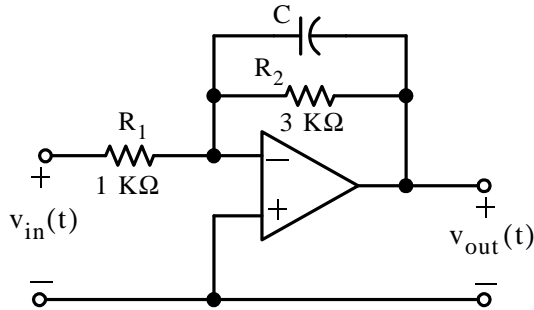
7. For the circuit below, to what value should the load impedance  $Z_{LD}$  be adjusted so that it will receive maximum power from the voltage source?



8. For the circuit below draw a phasor diagram that shows the voltage and current in each branch.

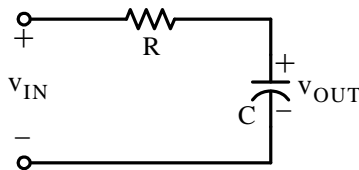


9. For the op amp circuit below  $v_{in}(t) = 3 \cos 1000t$  V. Find  $v_{out}(t)$ .



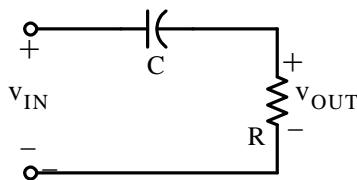
10. Prove that the RC network below, for any input it behaves as an integrator if  $v_{OUT} \ll v_{IN}$ , that is, show that

$$v_{OUT} \approx -\frac{1}{RC} \int v_{IN}$$



11. Prove that the RC network below, for any input it behaves as a differentiator if  $v_{OUT} \ll v_{IN}$ , that is, show that

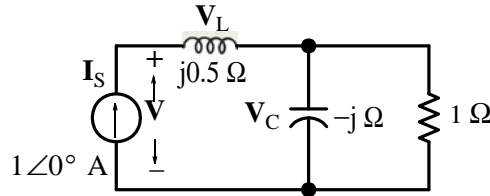
$$v_{OUT} = -RC \frac{d}{dt}(v_{IN})$$



7.12 Answers to End-of-Chapter Exercises

Multiple Choice

1. E

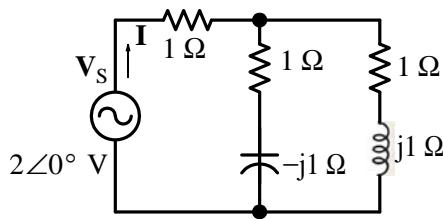


$V = V_L + V_C$  where  $V_L = 1\angle 0^\circ \times j1/2 = j1/2$  V and  $V_C$  is found from the nodal equation  $\frac{V_C}{1} + \frac{V_C}{-j} = 1 + j0$  or  $(1 + j)V_C = 1$  or  $V_C = \frac{1}{1 + j} \times \frac{1 - j}{1 - j} = \frac{1 - j}{2} = \frac{1}{2} - j\frac{1}{2}$  V.

Therefore,

$$V = j1/2 + 1/2 - j1/2 = 1/2 + j0 \text{ V}$$

2. C



Denoting the resistor in series with the voltage source as  $z_1$ , the resistor in series with the capacitor as  $z_2$ , and the resistor in series with the inductor as  $z_3$ , the equivalent impedance is

$$Z_{eq} = z_1 + \frac{z_2 \cdot z_3}{z_2 + z_3} = 1 + \frac{(1 - j1)(1 + j1)}{1 - j1 + 1 + j1} = 1 + \frac{2}{2} = 2 + j0$$

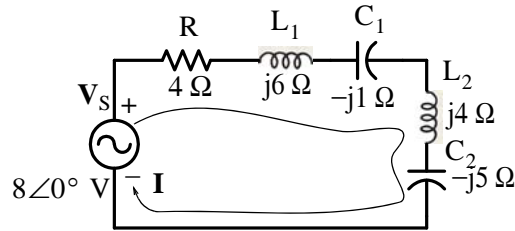
and

$$I = \frac{V_s}{Z} = \frac{2 + j0}{2 + j0} = 1 + j0 \text{ A}$$

3. B

$8 \sin(2000t + 90^\circ) = 8 \cos 2000t \Leftrightarrow 8\angle 0^\circ$  V,  $j\omega L_1 = j6$ ,  $j\omega L_2 = j4$ ,  $-j/\omega C_1 = -j1$ ,  $-j/\omega C_2 = -j1$  and the phasor equivalent circuit is shown below.

## Chapter 7 Phasor Circuit Analysis

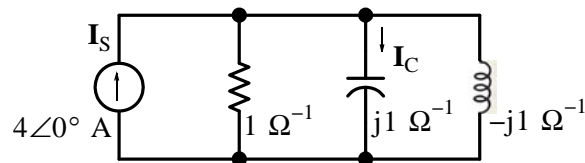


$$Z = 4 + j6 - j1 + j4 - j5 = 4 + j4, \quad \mathbf{I} = \frac{\mathbf{V}_S}{Z} = \frac{8 + j0}{4 + j4} = \frac{8 + j0}{4 + j4} \cdot \frac{4 - j4}{4 - j4} = \frac{32 - j32}{32} = 1 - j1, \text{ and}$$

$$\text{thus } \mathbf{V}_{C_2} = -j5 \times (1 - j) = 5 - j5 = \sqrt{50} \angle -45^\circ \Leftrightarrow \sqrt{50} \cos(2000t - 45^\circ) \text{ V}$$

4. D

$4 \cos 2000t \Leftrightarrow 4 \angle 0^\circ$ ,  $G = 1/R = 1 \Omega^{-1}$ ,  $j\omega C = j1 \Omega^{-1}$ ,  $-j/\omega L = -j1 \Omega^{-1}$ , and the phasor equivalent circuit is shown below.



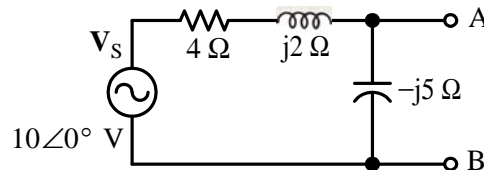
Denoting the parallel combination of the conductance and inductance as  $Y_1 = 1 - j1$  and using the *current division expression for admittances* we obtain

$$\mathbf{I}_C = \frac{j\omega C}{j\omega C + Y_1} \cdot \mathbf{I}_S = \frac{j1}{j1 + 1 - j1} = j1 \times 4 \angle 0^\circ = 1 \angle 90^\circ \times 4 \angle 0^\circ = 4 \angle 90^\circ \text{ A}$$

and thus

$$i_s(t) = 4 \cos(2000t + 90^\circ) \text{ A}$$

5. B



By the voltage division expression

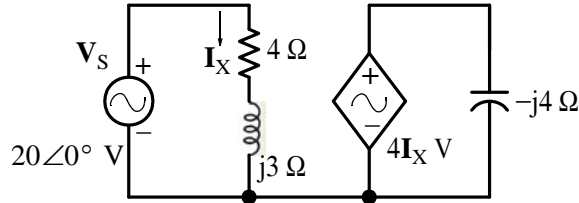
$$\mathbf{V}_{\text{TH}} = \mathbf{V}_{\text{AB}} = \frac{-j5}{4 + j2 - j5} \cdot 10 \angle 0^\circ = \frac{5 \angle -90^\circ \times 10 \angle 0^\circ}{4 - j3} = \frac{50 \angle -90^\circ}{5 \angle -36.9^\circ} = 10 \angle -53.1^\circ \text{ V}$$

6. C

We short the voltage source and looking to the left of points A and B we observe that the capacitor is in parallel with the series combination of the resistance and inductance. Thus,

$$Z_{\text{TH}} = \frac{(-j5)(4 + j2)}{4 + j2 - j5} = \frac{10 - j20}{4 - j3} = \frac{10 - j20}{4 - j3} \cdot \frac{4 + j3}{4 + j3} = \frac{100 - j50}{25} = 4 - j2$$

7. D

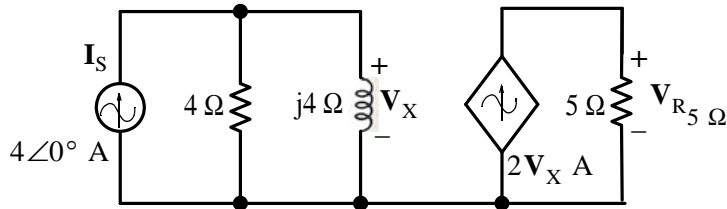


$$\mathbf{I}_X = \frac{20\angle 0^\circ}{4 + j3} = \frac{20\angle 0^\circ}{5\angle 36.9^\circ} = 4\angle -36.9^\circ, \quad 4\mathbf{I}_X = 16\angle -36.9^\circ$$

and

$$\mathbf{I}_C = \frac{4\mathbf{I}_X}{-j4} = \frac{16\angle -36.9^\circ}{4\angle -90^\circ} = 4\angle 53.1^\circ$$

8. E



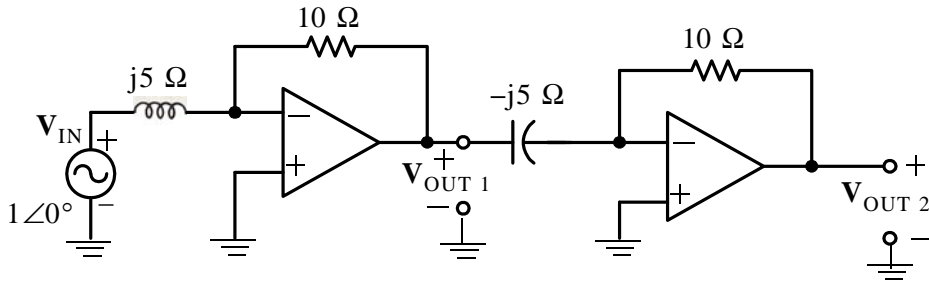
$$\mathbf{V}_X = \frac{4}{4 + j4} \times 4\angle 0^\circ \times j4 = \frac{64\angle 90^\circ}{\sqrt{32}\angle 45^\circ} = \frac{64 \times \sqrt{32}}{32} \angle 45^\circ = 2\sqrt{32} \angle 45^\circ$$

and

$$\mathbf{V}_{R_5} = 2\mathbf{V}_X \times 5 = 20 \times \sqrt{32} \angle 45^\circ = 20\sqrt{32} \left( \frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2} \right) = 80 + j80$$



9. B

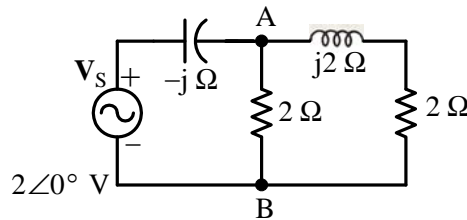


$$\mathbf{V}_{\text{OUT}1} = -\frac{10}{j5} \times 1\angle 0^\circ = j2 \times 1\angle 0^\circ = 2\angle 90^\circ \times 1\angle 0^\circ = 2\angle 90^\circ$$

and

$$\mathbf{V}_{\text{OUT}2} = -\frac{10}{-j5} \times \mathbf{V}_{\text{OUT}1} = -j2 \times 2\angle 0^\circ = 2\angle -90^\circ \times 2\angle 90^\circ = 4\angle 0^\circ = 4 + j0$$

10. A



We write the nodal equation at Node A for  $\mathbf{V}_{\text{AB}}$  as

$$\frac{\mathbf{V}_{\text{AB}} - 2\angle 0^\circ}{-j} + \frac{\mathbf{V}_{\text{AB}}}{2} + \frac{\mathbf{V}_{\text{AB}}}{2 + j2} = 0$$

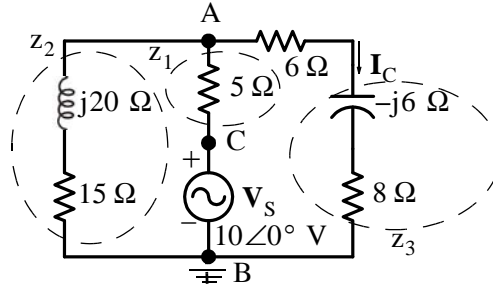
$$\left( \frac{1}{2} + j + \frac{1}{2 + j2} \right) \mathbf{V}_{\text{AB}} = 2\angle 90^\circ$$

$$\mathbf{V}_{\text{AB}} = \frac{2\angle 90^\circ}{1/2 + j + 1/4 - j/4} = \frac{2\angle 90^\circ}{3/4 + j3/4} = \frac{2\angle 90^\circ}{1.06\angle 45^\circ}$$

or  $\mathbf{V}_{\text{AB}} = 1.89\angle 45^\circ$  and in the  $t$ -domain  $v_{\text{AB}}(t) = 1.89 \cos(\omega t + 45^\circ)$

Problems

1. We transform the current source and its parallel resistance to a voltage source series resistance, we combine the series resistors, and we draw the phasor circuit below.



For this phasor circuit,  $\mathbf{V}_S = 2\angle 0^\circ \times 5 = 10\angle 0^\circ \text{ V}$ ,  $j\omega L = j10^3 \times 20 \times 10^{-3} = j20 \Omega$  and  $-j/\omega C = -j/(10^3 \times 10^3/6 \times 10^{-6}) = -j6$ ,  $z_1 = 5 \Omega$ ,  $z_2 = (15 + j20) \Omega$ , and  $z_3 = (8 - j6) \Omega$

We observe that  $\mathbf{V}_A = \mathbf{V}_{AB} = \mathbf{V}_{AC} + \mathbf{V}_{CB} = \mathbf{V}_{AC} + 10\angle 0^\circ \text{ V}$  and  $\mathbf{V}_B = 0$ . At Node A,

$$\frac{\mathbf{V}_A - \mathbf{V}_B}{z_2} + \frac{\mathbf{V}_A - 10\angle 0^\circ}{z_1} + \frac{\mathbf{V}_A - \mathbf{V}_B}{z_3} = 0$$

$$\left( \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} \right) \mathbf{V}_A = \frac{10\angle 0^\circ}{z_1}$$

$$\left( \frac{1}{5} + \frac{1}{15 + j20} + \frac{1}{8 - j6} \right) \mathbf{V}_A = \frac{10\angle 0^\circ}{5} = 2\angle 0^\circ$$

and

$$\begin{aligned} \mathbf{V}_A &= \frac{2\angle 0^\circ}{0.2 + \frac{1}{25\angle 53.1^\circ} + \frac{1}{10\angle -36.9^\circ}} = \frac{2\angle 0^\circ}{0.2 + 0.04\angle -53.1^\circ + 0.1\angle 36.9^\circ} \\ &= \frac{2\angle 0^\circ}{0.2 + 0.04 \cos 53.1^\circ - j0.04 \sin 53.1^\circ + 0.1 \cos 36.9^\circ + j0.1 \sin 36.9^\circ} \\ &= \frac{2\angle 0^\circ}{0.2 + 0.04 \times 0.6 - j0.04 \times 0.8 + 0.1 \times 0.8 + j0.1 \times 0.6} = \frac{2\angle 0^\circ}{0.304 + j0.028} \\ &= \frac{2\angle 0^\circ}{0.305\angle 5.26^\circ} = 6.55\angle -5.26^\circ \end{aligned}$$

Then, in the  $t$ -domain  $v_{AB}(t) = 6.55 \cos(1000t + 5.26^\circ)$ .

Also,

## Chapter 7 Phasor Circuit Analysis

$$\mathbf{I}_C = \frac{\mathbf{V}_A}{z_3} = \frac{6.55 \angle -5.26^\circ}{10 \angle -36.9^\circ} = 0.655 \angle 31.7^\circ$$

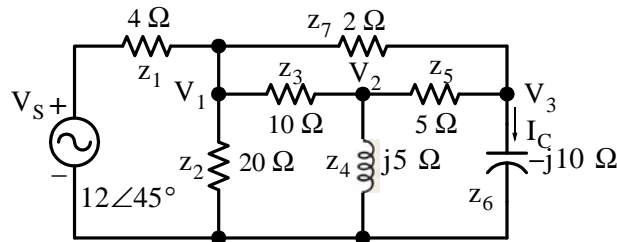
and

$$i_C(t) = 0.655 \cos(1000 + 31.7^\circ)$$

Check with MATLAB:

```
z1=5; z2=15+20j; z3=8-6j; VA=(10+0j)/(z1*(1/z1+1/z2+1/z3)); fprintf('\n');...
fprintf('magVA = %5.2f V \t',abs(VA));...
fprintf('phaseVA = %5.2f deg \t',angle(VA)*180/pi); fprintf('\n'); fprintf('\n');
magVA = 6.55 V   phaseVA = -5.26 deg
```

2. The equivalent phasor circuit is shown below where  $j\omega L = j10^3 \times 5 \times 10^{-3} = j5$  and  $-j/\omega C = -j/(10^3 \times 10^{-4}) = -j10$



Node  $\mathbf{V}_1$ :

$$\frac{\mathbf{V}_1 - \mathbf{V}_S}{z_1} + \frac{\mathbf{V}_1 - \mathbf{V}_2}{z_3} + \frac{\mathbf{V}_1}{z_2} + \frac{\mathbf{V}_1 - \mathbf{V}_3}{z_7}$$

or

$$\left( \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} + \frac{1}{z_7} \right) \mathbf{V}_1 - \frac{1}{z_3} \mathbf{V}_2 - \frac{1}{z_7} \mathbf{V}_3 = \frac{1}{z_1} \mathbf{V}_S$$

Node  $\mathbf{V}_2$ :

$$\frac{\mathbf{V}_2 - \mathbf{V}_1}{z_3} + \frac{\mathbf{V}_2}{z_4} + \frac{\mathbf{V}_2 - \mathbf{V}_3}{z_5} = 0$$

or

$$-\frac{1}{z_3} \mathbf{V}_1 + \left( \frac{1}{z_3} + \frac{1}{z_4} + \frac{1}{z_5} \right) \mathbf{V}_2 - \frac{1}{z_5} \mathbf{V}_3 = 0$$

Node  $\mathbf{V}_3$ :

$$\frac{\mathbf{V}_3 - \mathbf{V}_2}{z_5} + \frac{\mathbf{V}_3 - \mathbf{V}_1}{z_7} + \frac{\mathbf{V}_2 - \mathbf{V}_3}{z_6} = 0$$

or

$$-\frac{1}{z_7} \mathbf{V}_1 - \frac{1}{z_5} \mathbf{V}_2 + \left( \frac{1}{z_5} + \frac{1}{z_6} + \frac{1}{z_7} \right) \mathbf{V}_3 = 0$$

and in matrix form

$$\begin{bmatrix} \left(\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} + \frac{1}{z_7}\right) & -\frac{1}{z_3} & -\frac{1}{z_7} \\ -\frac{1}{z_3} & \left(\frac{1}{z_3} + \frac{1}{z_4} + \frac{1}{z_5}\right) & -\frac{1}{z_5} \\ -\frac{1}{z_7} & -\frac{1}{z_5} & \left(\frac{1}{z_5} + \frac{1}{z_6} + \frac{1}{z_7}\right) \end{bmatrix} \cdot \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \\ \mathbf{V}_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{z_1} \mathbf{V}_S \\ 0 \\ 0 \end{bmatrix}$$

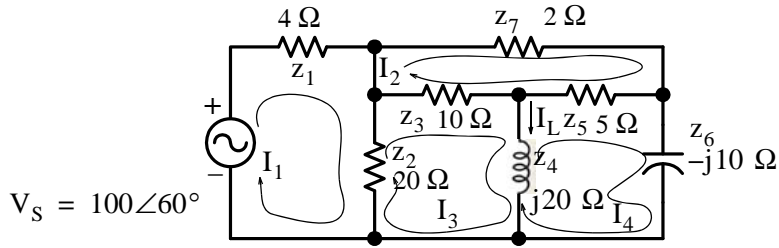
Shown below is the MATLAB script to solve this system of equations.

```
Vs=12*(cos(pi/4)+j*sin(pi/4)); % Express Vs in rectangular form
z1=4; z2=20; z3=10; z4=5j; z5=5; z6=-10j; z7=2;...
Y=[1/z1+1/z2+1/z3+1/z7 -1/z3 -1/z7;...
-1/z3 1/z3+1/z4+1/z5 -1/z5;...
-1/z7 -1/z5 1/z5+1/z6+1/z7];...
I=[Vs/z1 0 0]'; V=Y\I; Ic=V(3)/z6;...
magIc=abs(Ic); phaselc=angle(Ic)*180/pi;...
disp('V1='); disp(V(1)); disp('V2='); disp(V(2));...
disp('V3='); disp(V(3)); disp('Ic='); disp(Ic);...
format bank % Display magnitude and angle values with two decimal places
disp('magIc='); disp(magIc); disp('phaselc='); disp(phaselc);...
fprintf(' \n');
```

V1 = 5.9950 - 4.8789i  
V2 = 5.9658 - 0.5960i  
V3 = 5.3552 - 4.4203i  
Ic = 0.4420 + 0.5355i  
magIc = 0.69  
phaseIc = 50.46

Therefore,  $\mathbf{I}_C = 0.69 \angle 50.46^\circ \Leftrightarrow i_C(t) = 0.69 \cos(1000t + 50.46^\circ) \text{ A}$

3. The equivalent phasor circuit is shown below where  $j\omega L = j10^4 \times 2 \times 10^{-3} = j20$  and  $-j/\omega C = -j/(10^4 \times 10 \times 10^{-6}) = -j10$



Mesh  $I_1$  :

$$(z_1 + z_2)I_1 - z_2I_3 = V_S$$

Mesh  $I_2$  :

$$(z_1 + z_2 + z_7)I_2 - z_3I_3 - z_5I_4 = 0$$

Mesh  $I_3$  :

$$-z_2I_1 - z_3I_2 + (z_2 + z_3 + z_4)I_3 - z_4I_4 = 0$$

Mesh  $I_4$  :

$$-z_5I_2 - z_4I_3 + (z_4 + z_5 + z_6)I_4 = 0$$

and in matrix form

$$\begin{bmatrix} z_1 + z_2 & 0 & -z_2 & 0 \\ 0 & z_1 + z_2 + z_7 & -z_3 & -z_5 \\ -z_2 & -z_3 & z_2 + z_3 + z_4 & -z_4 \\ 0 & -z_5 & -z_4 & z_4 + z_5 + z_6 \end{bmatrix} \cdot \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} V_S \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Shown below is the MATLAB script to solve this system of equations.

```
Vs=100*(cos(pi/3)+j*sin(pi/3)); % Express Vs in rectangular form
z1=4; z2=20; z3=10; z4=20j; z5=5; z6=-10j; z7=2;...
Z=[z1+z2 0 -z2 0;...
0 z3+z5+z7 -z3 -z5;...
-z2 -z3 z2+z3+z4 -z4;...
0 -z5 -z4 z4+z5+z6];...
V=[Vs 0 0 0]'; I=Z\V; IL=I(3)-I(4);...
magIL=abs(IL); phaselL=angle(IL)*180/pi;...
disp('I1='); disp(I(1)); disp('I2='); disp(I(2));...
disp('I3='); disp(I(3)); disp('I4='); disp(I(4));...
disp('IL='); disp(IL);...
format bank % Display magnitude and angle values with two decimal places
disp('magIL='); disp(magIL); disp('phaselL='); disp(phaselL);...
fprintf('\n');
```

I1 = 5.4345 - 3.4110i  
I2 = 4.5527 + 0.7028i

$$I_3 = 4.0214 + 0.2369i$$

$$I_4 = 7.4364 + 1.9157i$$

$$I_L = -3.4150 - 1.6787i$$

$$\text{mag}I_L = 3.81$$

$$\text{phase}I_L = -153.82$$

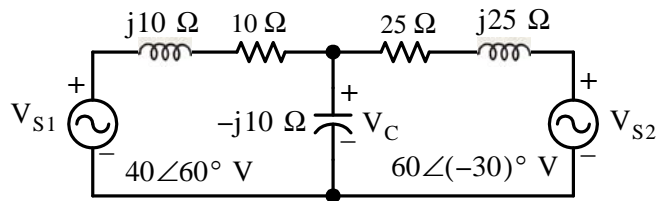
Therefore,  $I_L = 3.81 \angle -153.82^\circ \Leftrightarrow i_L(t) = 3.81 \cos(10^4 t - 153.82^\circ)$

4. The equivalent phasor circuit is shown below where

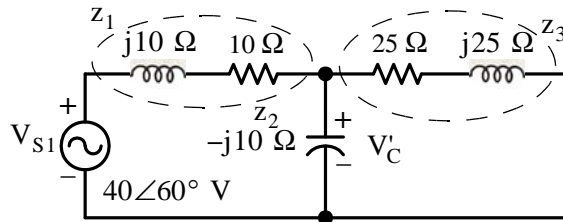
$$j\omega L_1 = j5 \times 10^3 \times 2 \times 10^{-3} = j10$$

$$j\omega L_2 = j5 \times 10^3 \times 5 \times 10^{-3} = j25$$

$$-j/\omega C = -j/(5 \times 10^3 \times 20 \times 10^{-6}) = -j10$$



We let  $V_C = V'_C + V''_C$  where  $V'_C$  is the capacitor voltage due to  $V_{S1}$  acting alone, and  $V''_C$  is the capacitor voltage due to  $V_{S2}$  acting alone. With  $V_{S1}$  acting alone the circuit reduces to that shown below.



By KCL

$$\frac{V'_C - V_{S1}}{z_1} + \frac{V'_C}{z_2} + \frac{V'_C}{z_3} = 0$$

$$\left( \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} \right) V'_C = \frac{V_{S1}}{z_1}$$

## Chapter 7 Phasor Circuit Analysis

$$\mathbf{V}'_C = \frac{\mathbf{V}_{S1}}{z_1 \cdot \left( \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} \right)} = \frac{\mathbf{V}_{S1}}{\left( 1 + \frac{z_1}{z_2} + \frac{z_1}{z_3} \right)}$$

and with MATLAB,

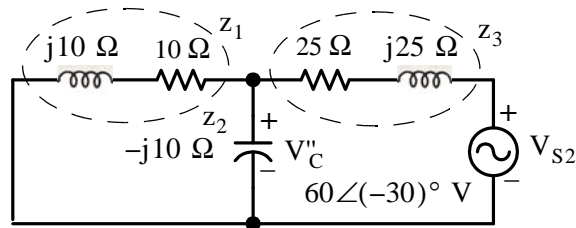
$$\mathbf{V}_{s1} = 40 \cdot (\cos(\pi/3) + j \sin(\pi/3)); \quad z_1 = 10 + 10j; \quad z_2 = -10j; \quad z_3 = 25 + 25j; \quad \mathbf{V}_{1c} = \mathbf{V}_{s1} / (1 + z_1/z_2 + z_1/z_3)$$

$$\mathbf{V}_{1c} = 36.7595 - 5.2962i$$

Therefore,

$$\mathbf{V}'_C = 36.76 - j5.30 \text{ V}$$

Next, with  $\mathbf{V}_{S2}$  acting alone the circuit reduces to that shown below.



By KCL

$$\frac{\mathbf{V}''_C}{z_1} + \frac{\mathbf{V}''_C}{z_2} + \frac{\mathbf{V}''_C - \mathbf{V}_{S2}}{z_3} = 0$$

$$\left( \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} \right) \mathbf{V}''_C = \frac{\mathbf{V}_{S2}}{z_3}$$

$$\mathbf{V}''_C = \frac{\mathbf{V}_{S2}}{z_3 \cdot \left( \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} \right)} = \frac{\mathbf{V}_{S2}}{\left( \frac{z_3}{z_1} + \frac{z_3}{z_2} + 1 \right)}$$

and with MATLAB

```
Vs2=60*(cos(pi/6)-j*sin(pi/6));...
z1=10+10j; z2=-10j; z3=25+25j; V1c=36.7595-5.2962j;...
V2c=Vs2/(z3/z1+z3/z2+1); Vc=V1c+V2c; fprintf(' \n');...
disp('V1c = '); disp(V1c); disp('V2c = '); disp(V2c);...
disp('Vc=V1c+V2c'); fprintf(' \n'); disp('Vc = '); disp(Vc);...
fprintf('magVc = %4.2f V \t',abs(Vc));...
fprintf('phaseVc = %4.2f deg \t',angle(Vc)*180/pi);...
fprintf(' \n'); fprintf(' \n');
```

$$\mathbf{V}_{1c} = 36.7595 - 5.2962i$$

$$V_{2C} = -3.1777 - 22.0557i$$

$$V_C = V_{1C} + V_{2C}$$

$$V_C = 33.5818 - 27.3519i$$

$$\text{mag}V_C = 43.31 \text{ V} \quad \text{phase}V_C = -39.16 \text{ deg}$$

Then,

$$\mathbf{V}_C = \mathbf{V}'_C + \mathbf{V}''_C = 33.58 - j27.35 = 43.31 \angle 27.35^\circ$$

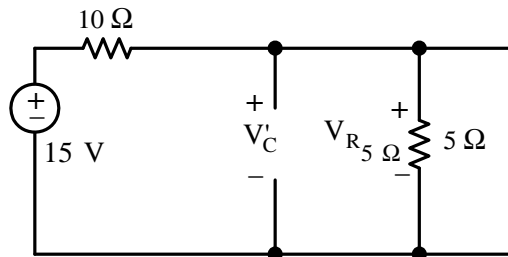
and

$$v_C(t) = 43.31 \cos(5000t - 27.35^\circ)$$

5. This circuit is excited by a DC (constant) voltage source, an AC (sinusoidal) voltage source, and an AC current source of different frequency. Therefore, we will apply the superposition principle.

Let  $\mathbf{V}'_C$  be the capacitor voltage due to  $v_{S1}$  acting alone,  $\mathbf{V}''_C$  the capacitor voltage due to  $v_{S2}(t)$  acting alone, and  $\mathbf{V}'''_C$  the capacitor voltage due to  $i_S(t)$  acting alone. Then, the capacitor voltage due to all three sources acting simultaneously will be  $\mathbf{V}_C = \mathbf{V}'_C + \mathbf{V}''_C + \mathbf{V}'''_C$

With the DC voltage source acting alone, after steady-state conditions have been reached the inductors behave like short circuits and the capacitor as an open circuit and thus the circuit is simplified as shown below.



By the voltage division expression

$$\mathbf{V}'_C = \mathbf{V}_{R_{5\Omega}} = \frac{5}{10 + 5} \cdot 15 = 5 \text{ V DC}$$

and

$$v'_C(t) = 5 \text{ V DC}$$

Next, with the sinusoidal voltage source  $v_{S2}(t)$  acting alone the reactances are

$$j\omega_1 L_1 = j10^3 \times 1 \times 10^{-3} = j1 \Omega$$

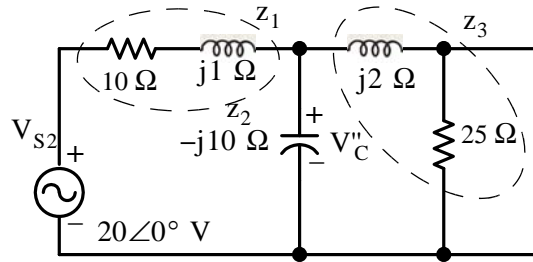
$$j\omega_1 L_2 = j10^3 \times 2 \times 10^{-3} = j2 \Omega$$



## Chapter 7 Phasor Circuit Analysis

$$-j/\omega_1 C = -j/(10^3 \times 5 \times 10^{-4}) = -j2 \Omega$$

and the equivalent phasor circuit is as shown below.



By KCL

$$\frac{V_C'' - V_{S2}}{z_1} + \frac{V_C''}{z_2} + \frac{V_C''}{z_3} = 0$$

$$\left( \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} \right) V_C'' = \frac{V_{S2}}{z_1}$$

$$V_C'' = \frac{V_{S2}}{z_1 \cdot \left( \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} \right)} = \frac{V_{S2}}{\left( 1 + \frac{z_1}{z_2} + \frac{z_1}{z_3} \right)}$$

and with MATLAB

```
Vs2=20+0j; z1=10+j; z2=-2j; z3=5+2j; V2c=Vs2/(1+z1/z2+z1/z3); fprintf('\n');...
disp('V2c = '); disp(V2c); fprintf('magV2c = %4.2f V \t',abs(V2c));...
fprintf('phaseV2c = %4.2f deg \t',angle(V2c)*180/pi); fprintf('\n'); fprintf('\n');
```

```
V2c = 1.8089 - 3.5362i
```

```
magV2c = 3.97 V phaseV2c = -62.91 deg
```

Then,

$$V_C'' = 1.81 - j3.54 = 3.97 \angle -62.9^\circ$$

and

$$v_C''(t) = 3.97 \cos(1000t - 62.9^\circ)$$

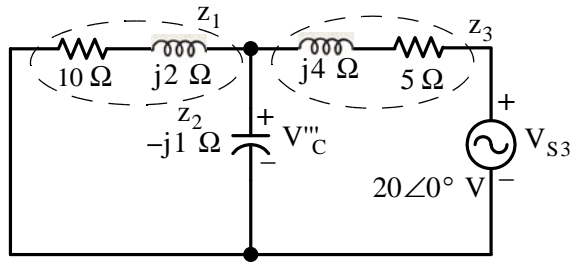
Finally, with the sinusoidal current source  $i_s(t)$  acting alone the reactances are

$$j\omega_2 L_1 = j2 \times 10^3 \times 1 \times 10^{-3} = j2 \Omega$$

$$j\omega_2 L_2 = j2 \times 10^3 \times 2 \times 10^{-3} = j4 \Omega$$

$$-j/\omega_2 C = -j/(2 \times 10^3 \times 5 \times 10^{-4}) = -j1 \Omega$$

and the equivalent phasor circuit is as shown below where the current source and its parallel resistance have been replaced with a voltage source with a series resistor.



By KCL

$$\frac{V_C}{z_1} + \frac{V_C}{z_2} + \frac{V_C - V_{S3}}{z_3} = 0$$

$$\left( \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} \right) V_C = \frac{V_{S3}}{z_3}$$

$$V_C = \frac{V_{S3}}{z_3 \cdot \left( \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} \right)} = \frac{V_{S3}}{\left( \frac{z_3}{z_1} + \frac{z_3}{z_2} + 1 \right)}$$

and with MATLAB

```
Vs3=20+0j; z1=10+2j; z2=-j; z3=5+4j; V3c=Vs3/(z3/z1+z3/z2+1); fprintf(' \n');...
disp('V3c = '); disp(V3c); fprintf('magV3c = %4.2f V \t',abs(V3c));...
fprintf('phaseV3c = %4.2f deg \t',angle(V3c)*180/pi); fprintf(' \n'); fprintf(' \n');
```

```
V3c = -1.4395 - 3.1170i
```

```
magV3c = 3.43 V phaseV3c = -114.79 deg
```

Then,

$$V_C = -1.44 - j3.12 = 3.43 \angle -114.8^\circ$$

or

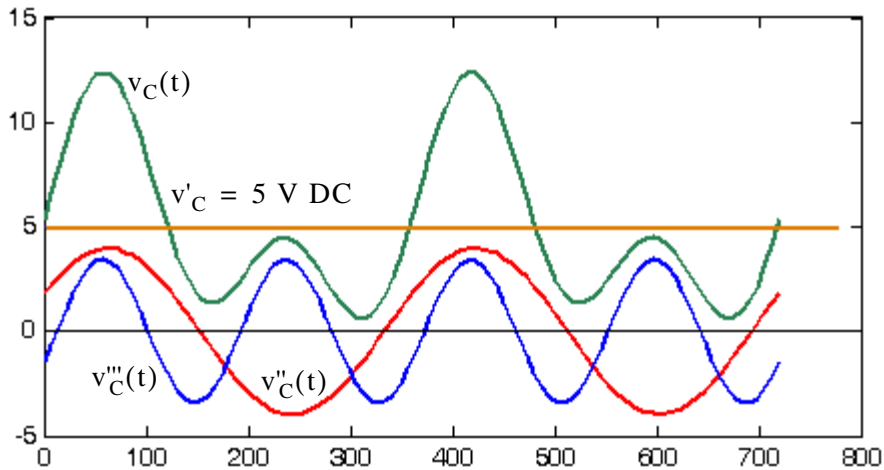
$$v_C(t) = 3.43 \cos(2000t - 114.8^\circ)$$

and

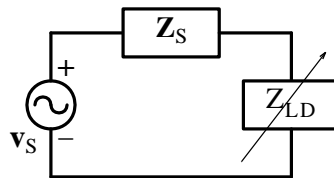
$$v_C(t) = v_C + v_C''(t) + v_C'''(t) = 5 + 3.97 \cos(1000t - 62.9^\circ) + 3.43 \cos(2000t - 114.8^\circ)$$

These waveforms are plotted below using the following MATLAB script:

```
wt=linspace(0,2*2*pi); deg=wt*180/pi; V1c=5;
V2c=3.97.*cos(wt-62.9.*pi./180);
V3c=3.43.*cos(2.*wt-114.8.*pi./180); plot(deg,V1c,deg,V2c,deg,V3c, deg,V1c+V2c+V3c)
```



6.



Since  $Z_S$  and  $Z_{LD}$  are complex quantities, we will express them as  $Z_S = \text{Re}\{Z_S\} + j\text{Im}\{Z_S\}$  and  $Z_{LD} = \text{Re}\{Z_{LD}\} + j\text{Im}\{Z_{LD}\}$  where  $\text{Re}$  and  $\text{Im}$  denote the real and imaginary components respectively.

We want to maximize the expression

$$\begin{aligned}
 p_{LD} &= i_{LD}^2 \cdot Z_{LD} = \frac{v_S^2}{(Z_S + Z_{LD})^2} \cdot Z_{LD} \\
 &= \frac{v_S^2 \cdot Z_{LD}}{[\text{Re}\{Z_S\} + j\text{Im}\{Z_S\} + j(\text{Re}\{Z_{LD}\} + j\text{Im}\{Z_{LD}\})]^2}
 \end{aligned}$$

The only quantities that can vary are  $\text{Re}\{Z_{LD}\}$  and  $\text{Im}\{Z_{LD}\}$  and we must consider them independently from each other.

From the above expression we observe that  $p_{LD}$  will be maximum when the denominator is minimum and this occurs when  $\text{Im}\{Z_{LD}\} = -\text{Im}\{Z_S\}$ , that is, when the imaginary parts of  $Z_{LD}$  and  $Z_S$  cancel each other. Under this condition,  $p_{LD}$  simplifies to

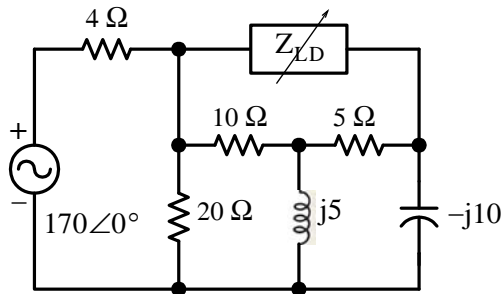
$$P_{LD} = \frac{v_S^2 \cdot R_{LD}}{(R_S + R_{LD})^2}$$

and, as we found in Chapter 3, for maximum power transfer  $R_{LD} = R_S$ . Therefore, the load impedance  $Z_{LD}$  will receive maximum power when

$$Z_{LD} = Z_S^*$$

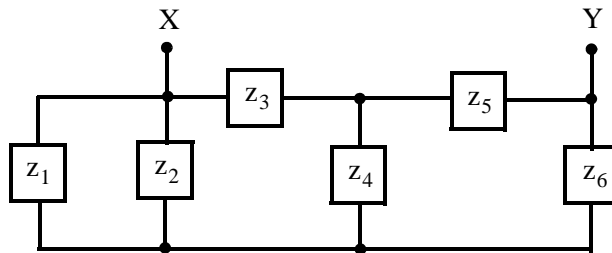
that is, when  $Z_{LD}$  is adjusted to be equal to the complex conjugate of  $Z_S$ .

7.

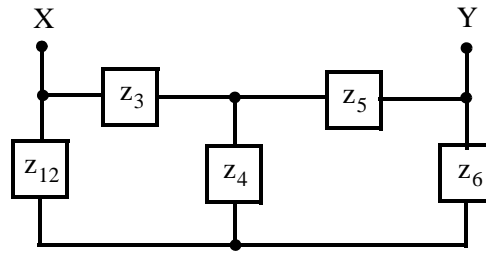


For this, and other similar problems involving the maximum power transfer theorem, it is best to replace the circuit with its Thevenin equivalent. Moreover, we only need to compute  $Z_{TH}$ .

For this problem, to find  $Z_{TH}$  we remove  $Z_{LD}$  and we short the voltage source. The remaining circuit then is as shown below.

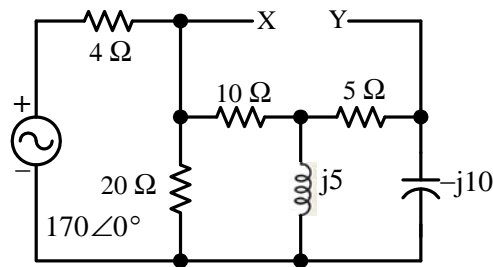


We observe that  $z_1$  is in parallel with  $z_2$  and this combination is shown as  $z_{12}$  in the simplified circuit below.

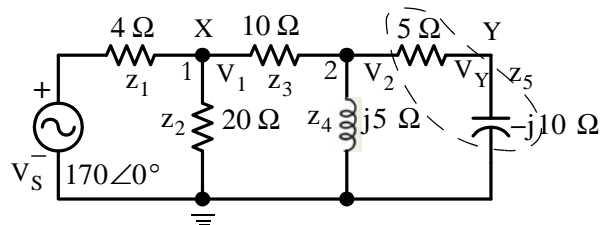


But this circuit cannot be simplified further unless we perform Wye to Delta transformation which we have not discussed. This and the Delta to Wye transformation are very useful in three-phase circuits and are discussed in *Circuit Analysis II with MATLAB Applications*, ISBN 978-1-934404-19-5. Therefore, we will compute  $Z_{TH}$  using the relation  $Z_{TH} = V_{OC}/I_{SC}$  where  $V_{OC}$  is the open circuit voltage, that is,  $V_{TH}$  and  $I_{SC}$  is the current that would flow between the terminals when the load is replaced by a short. Thus, we will begin our computations with the Thevenin voltage.

We disconnect  $Z_{LD}$  from the circuit at points X and Y as shown below.



We will replace the remaining circuit with its Thevenin equivalent. Thus, with  $Z_{LD}$  disconnected the circuit simplifies to that shown below.



Now, we will find

$$V_{TH} = V_{XY} = V_X - V_Y = V_1 - (V_2 - V_{R_{5\Omega}})$$

At Node 1:

$$\frac{V_1 - V_S}{z_1} + \frac{V_1}{z_2} + \frac{V_1 - V_2}{z_3} = 0$$

$$\left(\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right)\mathbf{V}_1 - \frac{1}{z_3}\mathbf{V}_2 = \frac{\mathbf{V}_S}{z_1}$$

At Node 2:

$$\frac{\mathbf{V}_2 - \mathbf{V}_1}{z_3} + \frac{\mathbf{V}_2}{z_4} + \frac{\mathbf{V}_2}{z_5} = 0$$

$$-\frac{1}{z_3}\mathbf{V}_1 + \left(\frac{1}{z_3} + \frac{1}{z_4} + \frac{1}{z_5}\right)\mathbf{V}_2$$

and with MATLAB,

```
Vs=170; z1=4; z2=20; z3=10; z4=5j; z5=5-10j;...
Y=[1/z1+1/z2+1/z3 -1/z3; -1/z3 1/z3+1/z4+1/z5]; I=[Vs/z1 0]'; V=Y\I; V1=V(1); V2=V(2);...
VX=V1; VY=(5/z5)*V2; VTH=VX-VY; fprintf(' \n');...
disp('V1 = '); disp(V1); disp('V2 = '); disp(V2);...
disp('VTH = '); disp(VTH); fprintf('magVTH = %4.2f V ',abs(VTH));...
fprintf('phaseVTH = %4.2f deg ',angle(VTH)*180/pi); fprintf(' \n'); fprintf(' \n');
```

```
V1 = 1.1731e+002 + 1.1538e+001i
```

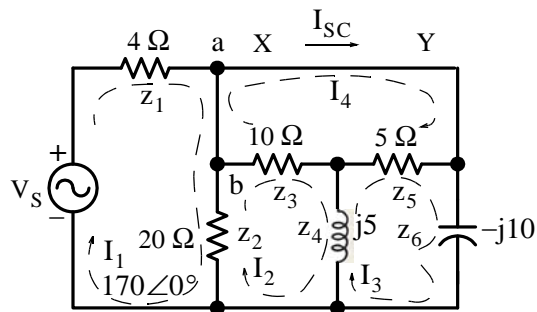
```
V2 = 44.2308+46.1538i
```

```
VTH = 1.2692e+002 - 1.5385e+001i
```

```
magVTH = 127.85 V phaseVTH = -6.91 deg
```

Thus,  $\mathbf{V}_{TH} = 127.85\angle-6.91^\circ$

Next, we must find  $\mathbf{I}_{SC}$  from the circuit shown below.



We will write four mesh equations as shown above but we only are interested in phasor current  $\mathbf{I}_4$ . Observing that  $a$  and  $b$  are the same point the mesh equations are

## Chapter 7 Phasor Circuit Analysis

$$\begin{aligned}(z_1 + z_2) \mathbf{I}_1 - z_2 \mathbf{I}_2 &= \mathbf{V}_S \\ -z_2 \mathbf{I}_1 + (z_2 + z_3 + z_4) \mathbf{I}_2 - z_4 \mathbf{I}_3 - z_3 \mathbf{I}_4 &= 0 \\ -z_4 \mathbf{I}_2 + (z_4 + z_5 + z_6) \mathbf{I}_3 - z_5 \mathbf{I}_4 &= 0 \\ -z_3 \mathbf{I}_2 - z_5 \mathbf{I}_3 + (z_3 + z_5) \mathbf{I}_4 &= 0\end{aligned}$$

and in matrix form

$$\begin{bmatrix} z_1 + z_2 & -z_2 & 0 & 0 \\ -z_2 & z_2 + z_3 + z_4 & -z_4 & -z_3 \\ 0 & -z_4 & z_4 + z_5 + z_6 & -z_5 \\ 0 & -z_3 & -z_5 & z_3 + z_5 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \\ \mathbf{I}_3 \\ \mathbf{I}_4 \end{bmatrix} = \begin{bmatrix} \mathbf{V}_S \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

With MATLAB,

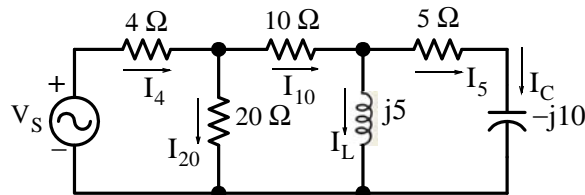
```
Vs=170; VTH=126.92-15.39j; z1=4; z2=20; z3=10; z4=5j; z5=5; z6=-10j;...
Z=[z1+z2 -z2 0 0; -z2 z2+z3+z4 -z4 -z3; 0 -z4 z4+z5+z6 -z5; 0 -z3 -z5 z3+z5];...
V=[Vs 0 0 0]'; I=Z\V; I1=I(1); I2=I(2); I3=I(3); I4=I(4);...
ZTH=VTH/I4; fprintf(' \n'); disp('I1 = '); disp(I1); disp('I2 = '); disp(I2);...
disp('I3 = '); disp(I3); disp('I4 = '); disp(I4); disp('ZTH = '); disp(ZTH); fprintf(' \n');
```

```
I1 = 15.6745 - 2.6300i
I2 = 10.3094 - 3.1559i
I3 = -1.0520 + 10.7302i
I4 = 6.5223 + 1.4728i
ZTH = 18.0084 - 6.4260i
```

Thus,  $Z_{TH} = 18.09 - j6.43 \Omega$  and by Problem 6, for maximum power transfer there must be  $Z_{LD} = Z_{TH}^*$  or

$$Z_{LD} = 18.09 + j6.43 \Omega$$

8. We assign phasor currents as shown below.



We choose  $\mathbf{I}_5$  as a reference, that is, we let

$$\mathbf{I}_5 = 1\angle 0^\circ \text{ A}$$

Then,

$$\mathbf{V}_5 = 5\angle 0^\circ \text{ V}$$

and since  $\mathbf{I}_C = \mathbf{I}_5$

$$\mathbf{V}_C = \mathbf{I}_C / (-j10) = 1\angle 0^\circ \cdot 10\angle -90^\circ = 10\angle -90^\circ \text{ V}$$

Next,

$$\mathbf{V}_L = \mathbf{V}_5 + \mathbf{V}_C = 5\angle 0^\circ + 10\angle -90^\circ = 5 + (-j10) = 5 - j10 = 11.18\angle -63.4^\circ \text{ V}$$

and

$$\mathbf{I}_L = \mathbf{V}_L / j5 = (11.18\angle -63.4^\circ) / (5\angle 90^\circ) = 2.24\angle -153.4^\circ = -2 - j \text{ A}$$

Now,

$$\mathbf{I}_{10} = \mathbf{I}_L + \mathbf{I}_5 = -2 - j + 1 = -1 - j = \sqrt{2}\angle -135^\circ \text{ A}$$

and

$$\mathbf{V}_{10} = 10 \times \sqrt{2}\angle -135^\circ = 10 \times (-1 - j) = -10 - j10 \text{ V}$$

Continuing we find that

$$\mathbf{V}_{20} = \mathbf{V}_{10} + \mathbf{V}_L = -10 - j10 + 5 - j10 = -5 - j20 \text{ V}$$

and

$$\mathbf{I}_{20} = \mathbf{V}_{20} / 20 = (-5 - j20) / 20 = -0.25 - j \text{ A}$$

Also,

$$\mathbf{I}_4 = \mathbf{I}_{20} + \mathbf{I}_{10} = -0.25 - j - 1 - j = -1.25 - j2 \text{ A}$$

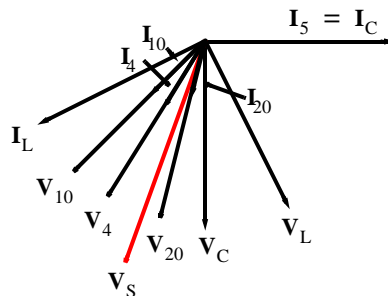
and

$$\mathbf{V}_4 = 4\mathbf{I}_4 = 4 \times (-1.25 - j2) = -5 - j8 \text{ V}$$

Finally,

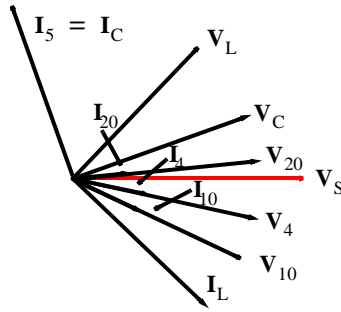
$$\mathbf{V}_S = \mathbf{V}_4 + \mathbf{V}_{20} = -5 - j8 - 5 - j20 = -10 - j28 = 29.73\angle -109.7^\circ \text{ V}$$

The magnitudes (not to scale) and the phase angles are shown below.

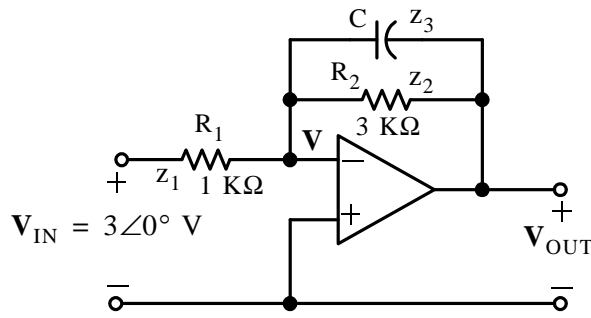


The phasor diagram above is acceptable. However, it would be more practical if we rotate it by  $109.7^\circ$  to show the voltage source  $\mathbf{V}_S$  as reference at  $0^\circ$  as shown below.





9. The equivalent phasor circuit is shown below where  $z_1 = R_1 = 1 \text{ K}\Omega$ ,  $z_2 = R_2 = 3 \text{ K}\Omega$ , and  $z_3 = -j/\omega C = -j/(10^3 \times 0.25 \times 10^{-6}) = -j4 \text{ K}\Omega$



Application of KCL at the inverting input yields

$$\frac{\mathbf{V} - \mathbf{V}_{\text{IN}}}{z_1} + \frac{\mathbf{V} - \mathbf{V}_{\text{OUT}}}{z_2} + \frac{\mathbf{V} - \mathbf{V}_{\text{OUT}}}{z_3} = 0$$

and since  $\mathbf{V} = 0$  the above relation reduces to

$$\left( \frac{1}{z_2} + \frac{1}{z_3} \right) \mathbf{V}_{\text{OUT}} = \frac{-\mathbf{V}_{\text{IN}}}{z_1}$$

or

$$\mathbf{V}_{\text{OUT}} = \frac{-\mathbf{V}_{\text{IN}}}{z_1 \cdot \left( \frac{1}{z_2} + \frac{1}{z_3} \right)} = \frac{-\mathbf{V}_{\text{IN}}}{\left( \frac{z_1}{z_2} + \frac{z_1}{z_3} \right)}$$

and with MATLAB,

```
Vin=3; z1=1000; z2=3000; z3=-4000j; Vout=-Vin/(z1/z2+z1/z3);...
fprintf(' \n'); disp('Vout = '); disp(Vout); fprintf('magVout = %5.2f V \t',abs(Vout));...
fprintf('phaseVout = %5.2f deg \t',angle(Vout)*180/pi); fprintf(' \n'); fprintf(' \n');
```

```
Vout = -5.7600 + 4.3200i
```

$$\text{mag}V_{\text{out}} = 7.20 \text{ V} \quad \text{phase}V_{\text{out}} = 143.13 \text{ deg}$$

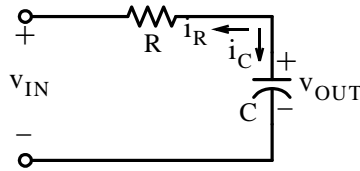
Thus,

$$\mathbf{V}_{\text{OUT}} = -5.76 + j4.32 = 7.2 \angle 143.13^\circ \text{ V}$$

and

$$v_{\text{out}}(t) = 7.2 \cos(1000t + 143.13^\circ) \text{ V}$$

10.



$$i_C = i_R$$

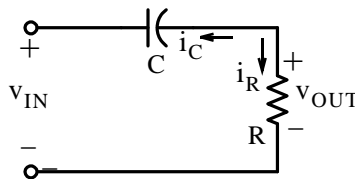
$$C \frac{dv_C}{dt} = \frac{v_{\text{OUT}} - v_{\text{IN}}}{R}$$

$$\frac{dv_{\text{OUT}}}{dt} = \frac{v_{\text{OUT}} - v_{\text{IN}}}{RC}$$

and since  $v_{\text{OUT}} \ll v_{\text{IN}}$ , by integrating both sides of the expression above, we obtain

$$v_{\text{OUT}} \approx -\frac{1}{RC} \int v_{\text{IN}}$$

11.



$$i_R = i_C$$

$$\frac{v_{\text{OUT}}}{R} = C \frac{d}{dt} (v_{\text{OUT}} - v_{\text{IN}})$$

and since  $v_{\text{OUT}} \ll v_{\text{IN}}$ , we obtain

$$v_{\text{OUT}} = -RC \frac{d}{dt} (v_{\text{IN}})$$

---

# Chapter 8

---

## Average and RMS Values, Complex Power, and Instruments

This chapter defines average and effective values of voltages and currents, instantaneous and average power, power factor, the power triangle, and complex power. It also discusses electrical instruments that are used to measure current, voltage, resistance, power, and energy.

### 8.1 Periodic Time Functions

A periodic time function satisfies the expression

$$f(t) = f(t + nT) \quad (8.1)$$

where  $n$  is a positive integer and  $T$  is the period of the periodic time function. The sinusoidal and sawtooth waveforms of Figure 8.1 are examples of periodic functions of time.

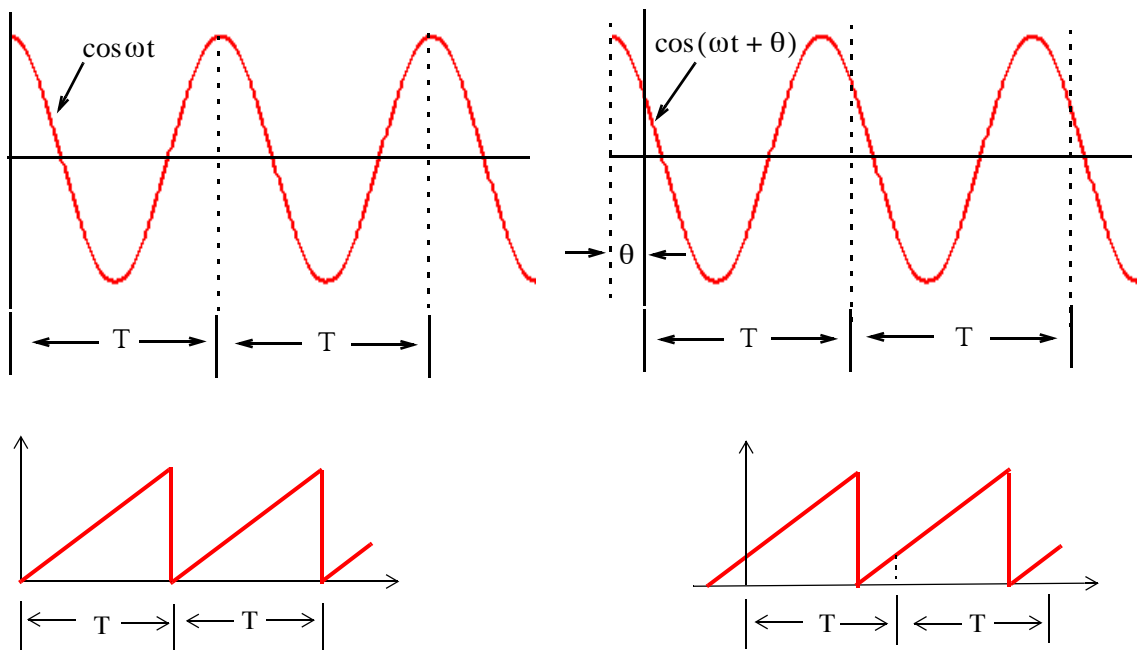


Figure 8.1. Examples of periodic functions of time

Other periodic functions of interest are the square and the triangular waveforms.

## 8.2 Average Values

The *average value* of any continuous function  $f(t)$  such as that shown in Figure 8.2 over an interval  $a \leq t \leq b$ ,

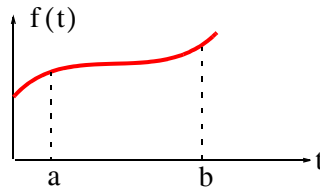


Figure 8.2. A continuous time function  $f(t)$

is defined as

$$f(t)_{\text{ave}} = \frac{1}{b-a} \int_a^b f(t) dt = \frac{1}{b-a} (\text{area})_a^b \quad (8.2)$$

The *average value of a periodic time function*  $f(t)$  is defined as the average of the function over one period.

---

### Example 8.1

Compute the average value of the sinusoid shown in Figure 8.3, where  $V_p$  denotes the peak (maximum) value of the sinusoidal voltage.

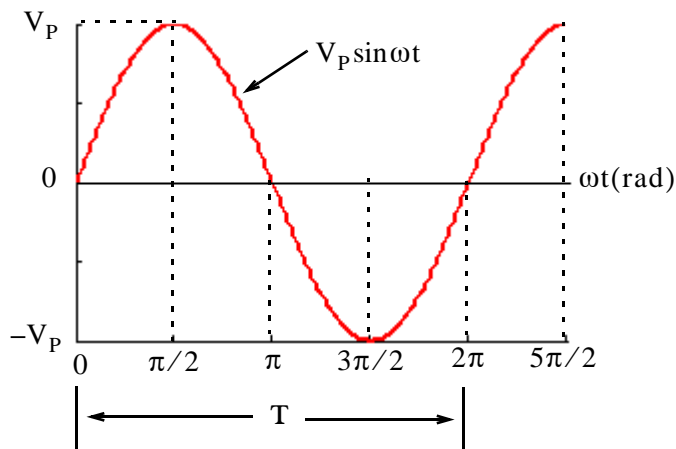


Figure 8.3. Waveform for Example 8.1

**Solution:**

By definition,

$$\begin{aligned}
 V_{\text{ave}} &= \frac{1}{T} \int_0^T v(t) dt = \frac{1}{\omega T} \int_0^{\omega T} V_p \sin \omega t d(\omega t) = \frac{V_p}{2\pi} \int_0^{2\pi} \sin \omega t d(\omega t) \\
 &= \frac{V_p}{2\pi} (-\cos \omega t) \Big|_0^{2\pi} = \frac{V_p}{2\pi} (\cos \omega t) \Big|_{2\pi}^0 = \frac{V_p}{2\pi} (1-1) = 0
 \end{aligned}$$

as expected since the net area of the positive and negative half cycles is zero.

### Example 8.2

Compute the average value of the *half-wave rectification* waveform shown in Figure 8.4.

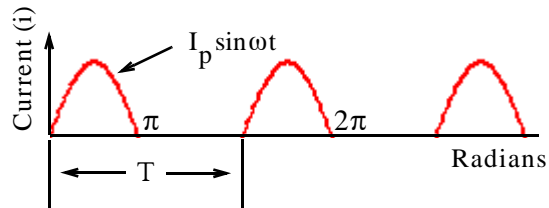


Figure 8.4. Waveform for Example 8.2

#### Solution:

This waveform is defined as

$$i(t) = \begin{cases} I_p \sin \omega t & 0 < \omega t < \pi \\ 0 & \pi < \omega t < 2\pi \end{cases} \quad (8.3)$$

Then, its average value is found from

$$\begin{aligned}
 I_{\text{ave}} &= \frac{1}{2\pi} \int_0^{2\pi} I_p \sin \omega t d(\omega t) = \frac{1}{2\pi} \left[ \int_0^{\pi} I_p \sin \omega t d(\omega t) + \int_{\pi}^{2\pi} 0 d(\omega t) \right] \\
 &= \frac{I_p}{2\pi} (-\cos \omega t) \Big|_0^{\pi} = \frac{I_p}{2\pi} \cos \omega t \Big|_{\pi}^0 = \frac{I_p}{2\pi} [1 - (-1)] = \frac{I_p}{\pi}
 \end{aligned} \quad (8.4)$$

In other words, the average value of the half-wave rectification waveform is equal to its peak value divided by  $\pi$ .

### 8.3 Effective Values

The *effective current*  $I_{\text{eff}}$  of a periodic current waveform  $i(t)$  is defined as the current which produces heat in a given resistance  $R$  at the same average rate as a direct (constant) current  $I_{\text{dc}}$ , that is,

$$\text{Average Power} = P_{\text{ave}} = RI_{\text{eff}}^2 = RI_{\text{dc}}^2 \quad (8.5)$$

Also, in a periodic current waveform  $i(t)$ , the instantaneous power is

$$p(t) = Ri^2(t) \quad (8.6)$$

and

$$P_{\text{ave}} = \frac{1}{T} \int_0^T p(t) dt = \frac{1}{T} \int_0^T Ri^2 dt = \frac{R}{T} \int_0^T i^2 dt \quad (8.7)$$

Equating (8.5) with (8.7) we obtain

$$RI_{\text{eff}}^2 = \frac{R}{T} \int_0^T i^2 dt$$

or

$$I_{\text{eff}}^2 = \frac{1}{T} \int_0^T i^2 dt$$

or

$$I_{\text{eff}} = \sqrt{\frac{1}{T} \int_0^T i^2 dt} = I_{\text{Root Mean Square}} = I_{\text{RMS}} = \sqrt{\text{Ave}(i^2)}$$

(8.8)

#### Caution 1:

In general,  $\text{ave}(i^2) \neq (i_{\text{ave}})^2$  since the expression  $\text{ave}(i^2)$  implies that the function  $i$  must first be squared and the average of the squared value is then to be found. On the other hand,  $(i_{\text{ave}})^2$  implies that the average value of the function must first be found and then the average must be squared. The waveforms in Figure 8.5 illustrate this point.

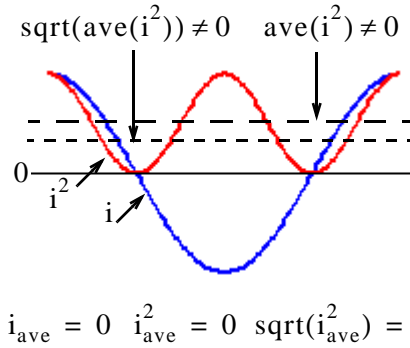


Figure 8.5. Waveforms to illustrate that  $\text{ave}(i^2) \neq (i_{\text{ave}})^2$

**Caution 2:**

In general,  $P_{\text{ave}} \neq V_{\text{ave}} \times I_{\text{ave}}$ . For example, if  $v(t) = V_p \cos \omega t$  and  $i(t) = I_p \cos(\omega t + \phi)$ , then  $V_{\text{ave}} = 0$ , and also  $I_{\text{ave}} = 0$ . Thus,  $P_{\text{ave}} = 0$ . However,

$$P_{\text{ave}} = \frac{1}{T} \int_0^T p(t) dt = \frac{1}{T} \int_0^T v(t)i(t) dt = \frac{1}{T} \int_0^T (V_p \cos \omega t) [I_p \cos(\omega t + \phi)] dt \neq 0$$

**8.4 Effective (RMS) Value of Sinusoids**

Now, we will derive an expression for the *Root Mean Square (RMS)* value of a sinusoid in terms of its peak (maximum) value. We will denote the peak values of voltages and currents as  $V_p$  and  $I_p$  respectively. The value from positive to negative peak will be denoted as  $V_{p-p}$  and  $I_{p-p}$ , and the RMS values as  $V_{\text{RMS}}$  and  $I_{\text{RMS}}$ . Their notations and relationships are shown in Figure 8.6.

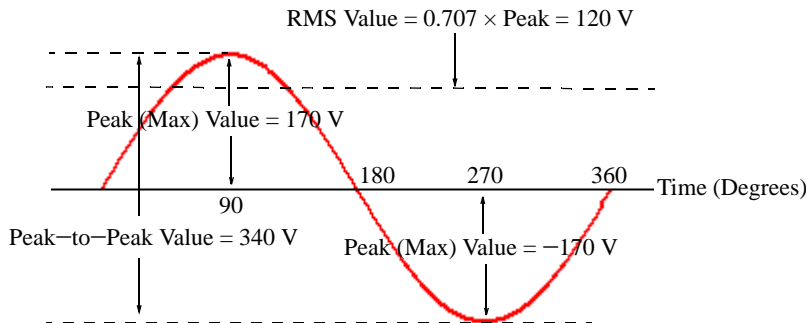


Figure 8.6. Definitions of  $V_{p-p}$ ,  $I_{p-p}$ ,  $V_{\text{RMS}}$ , and  $I_{\text{RMS}}$  in terms of  $V_p$  and  $I_p$

Let

$$i = I_p \cos(\omega t - \theta)$$

then,

$$I_{\text{RMS}}^2 = \frac{1}{T} \int_0^T i^2 dt = \frac{1}{2\pi} \int_0^{2\pi} I_p^2 \cos^2(\omega t - \theta) d(\omega t)$$

and using the identity

$$\cos^2 \phi = \frac{1}{2}(\cos 2\phi + 1)$$

we obtain

$$\begin{aligned} I_{\text{RMS}}^2 &= \frac{I_p^2}{4\pi} \left[ \int_0^{2\pi} \cos(2\omega t - \theta) d(\omega t) + \int_0^{2\pi} d(\omega t) \right] \\ &= \frac{I_p^2}{4\pi} \left[ \frac{\sin(2\omega t - \theta)}{2} \Big|_0^{2\pi} + (\omega t) \Big|_0^{2\pi} \right] = \frac{I_p^2}{4\pi} \left[ \frac{\sin(4\pi - \theta) - \sin(-\theta)}{2} + 2\pi \right] \end{aligned} \quad (8.9)$$

Using the trigonometric identities

$$\sin(x - y) = \sin x \cos y - \cos x \sin y$$

and

$$-\sin(-\alpha) = \sin \alpha$$

by substitution into (8.9), we obtain

$$I_{\text{RMS}}^2 = \frac{I_p^2}{4\pi} \left[ \frac{\sin 4\pi \cos \theta - \cos 4\pi \sin \theta + \sin \theta}{2} + 2\pi \right] = \frac{I_p^2}{4\pi} (2\pi) = \frac{I_p^2}{2}$$

and therefore,

$$I_{\text{RMS}} = \frac{I_p}{\sqrt{2}} = 0.707 I_p$$

**FOR SINUSOIDS ONLY**

(8.10)

We observe that the RMS value of a sinusoid is independent of the frequency and phase angle, in other words, it is dependent on the amplitude of the sinusoid only.

### Example 8.3

Compute the  $I_{\text{ave}}$  and  $I_{\text{RMS}}$  for the sawtooth waveform shown in Figure 8.7.



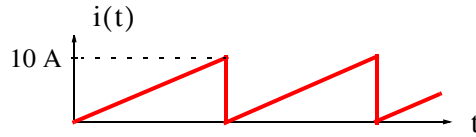


Figure 8.7. Waveform for Example 8.3

### Solution:

By inspection, the period  $T$  is as shown in Figure 8.8.

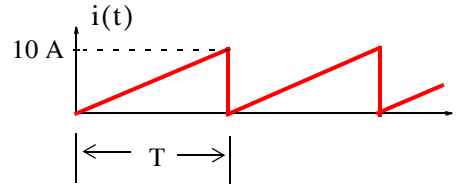


Figure 8.8. Defining the period for the waveform of Example 8.3

The average value is

$$I_{\text{ave}} = \frac{\text{Area}}{\text{Period}} = \frac{(1/2) \times 10 \times T}{T} = 5 \text{ A}$$

To find  $I_{\text{RMS}}$  we cannot use (8.10); this is for sinusoids only. Accordingly, we must use the definition of the RMS value as derived in (8.8). Then,

$$I_{\text{RMS}}^2 = \frac{1}{T} \int_0^T i^2(t) dt = \frac{1}{T} \int_0^T \left(\frac{10}{T}t\right)^2 dt = \frac{1}{T} \left(\frac{100}{T^2} \cdot \frac{t^3}{3}\right) \Bigg|_0^T = \frac{1}{T} \left(\frac{100}{T^2} \cdot \frac{T^3}{3}\right) = \frac{100}{3}$$

or

$$I_{\text{RMS}} = \sqrt{\frac{100}{3}} = \frac{10}{\sqrt{3}} = 5.77 \text{ A}$$

## 8.5 RMS Values of Sinusoids with Different Frequencies

The RMS value of a waveform which consists of a sum of sinusoids of different frequencies, is equal to the square root of the sum of the squares of the RMS values of each sinusoid. Thus, if

$$i = I_0 + I_1 \cos(\omega_1 t \pm \theta_1) + I_2 \cos(\omega_2 t \pm \theta_2) + \dots + I_N \cos(\omega_N t \pm \theta_N) \quad (8.11)$$

where  $I_0$  represents a constant current, and  $I_1, I_2, \dots, I_N$  represent the amplitudes of the sinusoids. Then, the RMS value of  $i$  is found from

$$I_{\text{RMS}} = \sqrt{I_0^2 + I_{1 \text{ RMS}}^2 + I_{2 \text{ RMS}}^2 + \dots + I_{N \text{ RMS}}^2} \quad (8.12)$$

or

$$I_{\text{RMS}} = \sqrt{I_0^2 + \frac{1}{2}I_{1 \text{ p}}^2 + \frac{1}{2}I_{2 \text{ p}}^2 + \dots + \frac{1}{2}I_{N \text{ p}}^2} \quad (8.13)$$

**Example 8.4**

Find the  $I_{\text{RMS}}$  value of the square waveform of Figure 8.9 by application of (8.12)

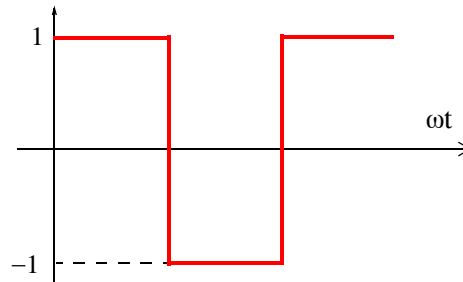


Figure 8.9. Waveform for Example 8.4

**Solution:**

By inspection, the period  $T = 2\pi$  is as shown in Figure 8.10.

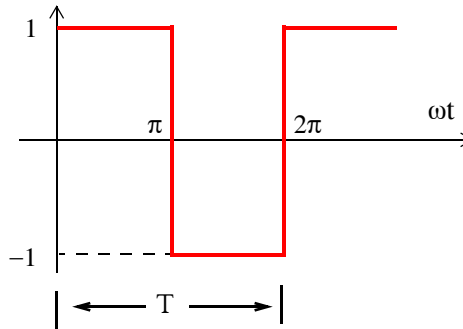


Figure 8.10. Determination of the period to the waveform of Example 8.4

Then,

a.

$$\begin{aligned} I_{\text{RMS}}^2 &= \frac{1}{T} \int_0^T i^2 dt = \frac{1}{2\pi} \int_0^{2\pi} i^2 d(\omega t) = \frac{1}{2\pi} \left[ \int_0^{\pi} 1^2 d(\omega t) + \int_{\pi}^{2\pi} (-1)^2 d(\omega t) \right] \\ &= \frac{1}{2\pi} [\omega t]_0^{\pi} + \omega t \Big|_{\pi}^{2\pi} = \frac{1}{2\pi} [\pi + 2\pi - \pi] = 1 \end{aligned}$$

or

$$I_{\text{RMS}} = 1 \tag{8.14}$$

b. Fourier series analysis textbooks\* show that the square waveform above can be expressed as

$$i(t) = \frac{4}{\pi} \left( \sin \omega t + \frac{1}{3} \sin 3\omega t + \frac{1}{5} \sin 5\omega t + \dots \right) \tag{8.15}$$

and as we know, the RMS value of a sinusoid is a real number independent of the frequency and the phase angle, and it is equal to 0.707 times its peak value, that is,  $I_{\text{RMS}} = 0.707 \times I_p$ . Then from (8.12) and (8.15),

$$I_{\text{RMS}} = \frac{4}{\pi} \sqrt{0 + \frac{1}{2}(1)^2 + \frac{1}{2}\left(\frac{1}{3}\right)^2 + \frac{1}{2}\left(\frac{1}{5}\right)^2 + \dots} = 0.97 \tag{8.16}$$

The numerical accuracy of (8.16) is good considering that higher harmonics have been neglected.

## 8.6 Average Power and Power Factor

Consider the network shown in Figure 8.11.

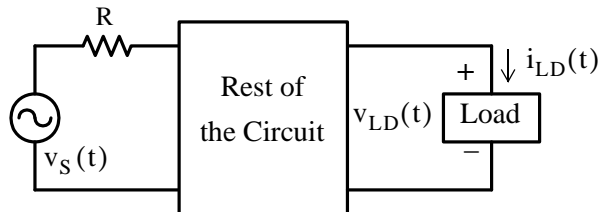


Figure 8.11. Network where it is assumed that  $i_{\text{LD}}(t)$  and  $v_{\text{LD}}(t)$  are out-of-phase

We will assume that the load current  $i_{\text{LD}}(t)$  is  $\theta$  degrees out-of-phase with the voltage  $v_{\text{LD}}(t)$ , i.e., if  $v_{\text{LD}}(t) = V_p \cos \omega t$ , then  $i_{\text{LD}}(t) = I_p \cos(\omega t + \theta)$ . We want to find an expression for the average power absorbed by the load.

We know that

$$p = vi$$

that is,

$$\text{instantaneous power} = \text{instantaneous voltage} \times \text{instantaneous current}$$

and the instantaneous power  $p_{\text{LD}}(t)$  absorbed by the load is

\* Refer to *Signals and Systems with MATLAB Computing and Simulink Modeling*, ISBN 978-0-9744239-9-8.

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## Chapter 8 Average and RMS Values, Complex Power, and Instruments

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$$p_{LD}(t) = v_{LD}(t) \times i_{LD}(t) = V_p I_p \cos \omega t \times \cos(\omega t + \theta) \quad (8.17)$$

Using the trigonometric identity

$$\cos x \cos y = \frac{1}{2} [\cos(x + y) + \cos(x - y)]$$

we express (8.17) as

$$p_{LOAD}(t) = \frac{V_p I_p}{2} [\cos(2\omega t + \theta) + \cos \theta] \quad (8.18)$$

and the average power is

$$\begin{aligned} P_{ave LD} &= \frac{1}{T} \int_0^T p_{LD} dt = \frac{1}{T} \int_0^T \left( \frac{V_p I_p}{2} [\cos(2\omega t + \theta) + \cos \theta] \right) dt \\ &= \frac{V_p I_p}{2T} \int_0^T ([\cos(2\omega t + \theta)]) dt + \frac{V_p I_p}{2T} \int_0^T \cos \theta dt \end{aligned} \quad (8.19)$$

We observe that the first integral on the right side of (8.19) is zero, and the second integral, being a constant, has an average value of that constant. Then,

$$\boxed{P_{ave LD} = \frac{V_p I_p}{2} \cos \theta} \quad (8.20)$$

and using the relations

$$V_{RMS} = \frac{V_p}{\sqrt{2}}$$

and

$$I_{RMS} = \frac{I_p}{\sqrt{2}}$$

we can express (8.19) as

$$\boxed{P_{ave LD} = V_{RMS LD} I_{RMS LD} \cos \theta} \quad (8.21)$$

and it is imperative that we remember that *these relations are valid for circuits with sinusoidal excitations.*

The term  $\cos \theta$  in (8.20) and (8.21) is known as the *power factor* and thus

$$\boxed{\text{Power Factor}_{LD} = PF_{LD} = \cos \theta_{LD} = \frac{P_{ave LD}}{V_{RMS LD} I_{RMS LD}}} \quad (8.22)$$

## 8.7 Average Power in a Resistive Load

The voltage and current in a resistive branch of a circuit are always in phase, that is, the phase angle  $\theta = 0^\circ$ . Therefore, denoting that resistive branch with the subscript R we have:

$$P_{\text{ave R}} = V_{\text{RMS R}} I_{\text{RMS R}} \cos 0^\circ = V_{\text{RMS R}} I_{\text{RMS R}} \quad (8.23)$$

or

$$P_{\text{ave R}} = \frac{V_{\text{RMS R}}^2}{R} = I_{\text{RMS R}}^2 R = \frac{1}{2} \frac{V_m^2}{R} = \frac{1}{2} I_m^2 R \quad (8.24)$$

## 8.8 Average Power in Inductive and Capacitive Loads

With inductors and capacitors there is a  $90^\circ$  phase difference between the voltage and current, that is,  $\theta = 90^\circ$  and therefore, denoting that inductive or capacitive branch with the subscript X we obtain:

$$P_{\text{ave X}} = V_{\text{RMS X}} I_{\text{RMS X}} \cos 90^\circ = 0$$

Of course, the instantaneous power is zero only at specific instants.

Obviously, if the load of a circuit contains resistors, inductors and capacitors, the phase angle  $\theta$  between  $V_{\text{RMS LOAD}}$  and  $I_{\text{RMS LOAD}}$  will be within  $0 \leq \theta \leq 90^\circ$ , and the power factor  $\cos \theta$  will be within  $0 \leq \cos \theta \leq 1$ .

### Example 8.5

For the circuit of Figure 8.11, find the average power supplied by the voltage source, the average power absorbed by the resistor, the inductor, and the capacitor.

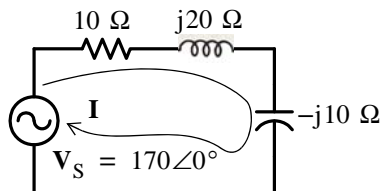


Figure 8.12. Circuit for Example 8.5

#### Solution:

Since this is a series circuit, we need to find the current  $I$  and its phase relation to the source voltage  $V_S$ . Then,

$$\mathbf{I} = \frac{\mathbf{V}_S}{\mathbf{Z}} = \frac{170\angle 0^\circ}{10 + j20 - j10} = \frac{170\angle 0^\circ}{10 + j10} = \frac{170\angle 0^\circ}{10\sqrt{2}\angle 45^\circ} = 12\angle -45^\circ \quad (8.25)$$

Relation (8.25) indicates that  $I_p = 12$  A,  $\theta = -45^\circ$ , and the power factor is

$$\cos\theta = \cos(-45^\circ) = 0.707$$

Therefore, using (8.24) we find that the average power absorbed by the resistor is

$$P_{\text{ave R}} = \frac{1}{2} I_p^2 R = \frac{1}{2} (12)^2 10 = 720 \text{ w} \quad (8.26)$$

The average power absorbed by the inductor and the capacitor is zero since the voltages and currents in these devices are  $90^\circ$  out-of-phase with each other.

**Check:** The average power delivered by the voltage source is

$$P_{\text{ave SOURCE}} = \frac{V_p I_p}{2} \cos\theta = \frac{(170)(12)}{2} 0.707 = 721 \text{ w} \quad (8.27)$$

and we observe that (8.26) and (8.27) are in close agreement.

### Example 8.6

For the circuit of Figure 8.13, find the power absorbed by each resistor, and the power supplied (or absorbed) by the current sources.

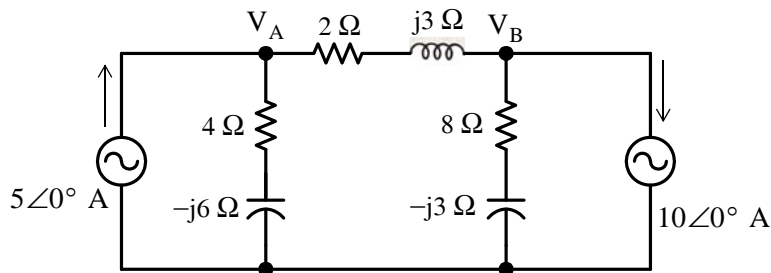


Figure 8.13. Circuit for Example 8.6

**Solution:**

This is the same circuit as in Example 7.1 where we found that

$$\mathbf{V}_A = -4.138 + j19.655 = 20.086\angle 101.9^\circ \quad (8.28)$$

and

$$\mathbf{V}_B = -22.414 - j1.035 = 22.440\angle -177.4^\circ \quad (8.29)$$

Then,

$$I_{2\ \Omega} = \frac{\mathbf{V}_A - \mathbf{V}_B}{2 + j3} = \frac{18.276 + j20.690}{3.61 \angle 56.3^\circ} = \frac{32.430 \angle 145.0^\circ}{3.61 \angle 56.3^\circ} = 8.983 \angle 88.7^\circ$$

and

$$P_{\text{ave } 2\ \Omega} = \frac{1}{2} I_{p\ 2\ \Omega}^2 (2\ \Omega) = \frac{1}{2} \times 8.983^2 \times 2 = 80.70\ \text{w} \quad (8.30)$$

Also,

$$I_{4\ \Omega} = \frac{\mathbf{V}_A}{4 - j6} = \frac{20.086 \angle 101.9^\circ}{7.21 \angle -56.3^\circ} = 2.786 \angle 158.2^\circ$$

and

$$P_{\text{ave } 4\ \Omega} = \frac{1}{2} I_{p\ 4\ \Omega}^2 (4\ \Omega) = \frac{1}{2} \times 2.786^2 \times 4 = 15.52\ \text{w} \quad (8.31)$$

Likewise,

$$I_{8\ \Omega} = \frac{\mathbf{V}_B}{8 - j3} = \frac{22.440 \angle -177.4^\circ}{8.54 \angle -20.6^\circ} = 2.627 \angle -156.7^\circ$$

and

$$P_{\text{ave } 8\ \Omega} = \frac{1}{2} I_{p\ 8\ \Omega}^2 (8\ \Omega) = \frac{1}{2} \times 2.627^2 \times 8 = 27.61\ \text{w} \quad (8.32)$$

The voltages across the current sources are the same as  $\mathbf{V}_A$  and  $\mathbf{V}_B$  but they are  $101.9^\circ$  and  $-177.4^\circ$  out-of-phase respectively with the current sources as shown by (8.28) and (8.29). Therefore, we let  $\theta_1 = 101.9^\circ$  and  $\theta_2 = -177.4^\circ$ . Then, the power absorbed by the 5 A source is

$$\begin{aligned} P_{\text{ave } 5\ \text{A}} &= \frac{V_p I_p}{2} \cos \theta_1 = \frac{|\mathbf{V}_A| |5\ \text{A}|}{2} \cos(101.9^\circ) \\ &= \frac{20.086 \times 5}{2} \times (-0.206) = -10.35\ \text{w} \end{aligned} \quad (8.33)$$

and the power absorbed by the 10 A source is

$$\begin{aligned} P_{\text{ave } 10\ \text{A}} &= \frac{V_p I_p}{2} \cos \theta_2 = \frac{|\mathbf{V}_B| |10\ \text{A}|}{2} \cos(-177.4^\circ) \\ &= \frac{22.440 \times 10}{2} \times (-0.999) = -112.08\ \text{w} \end{aligned} \quad (8.34)$$

The negative values in (8.33) and (8.34) indicate that both current sources supply power to the rest of the circuit.

**Check:** Total average power absorbed by resistors is

$$80.70 + 15.52 + 27.61 = 123.83\ \text{w}$$

and the total average power supplied by current sources is

$$112.08 + 10.35 = 122.43\ \text{w}$$

Thus, the total average power supplied by the current sources is equal to the total average power absorbed by the resistors. The small difference is due to rounding of fractional numbers.

### 8.9 Average Power in Non-Sinusoidal Waveforms

If the excitation in a circuit is non-sinusoidal, we can compute the average power absorbed by a resistor from the relations

$$P_{\text{ave}} = \frac{1}{T} \int_0^T p \, dt = \frac{1}{T} \int_0^T \frac{v^2}{R} \, dt = \frac{1}{T} \int_0^T i^2 R \, dt \quad (8.35)$$

#### Example 8.7

Compute the average power absorbed by a  $5 \, \Omega$  resistor when the voltage across it is the half-wave rectification waveform shown in Figure 8.14.

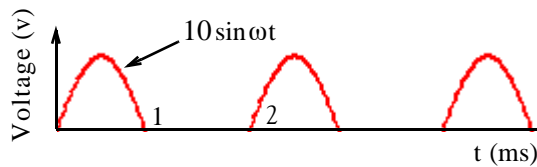


Figure 8.14. Waveform for Example 8.7

#### Solution:

We first need to find the numerical value of  $\omega$ . It is found as follows:

$$T = 2 \text{ ms} = 2 \times 10^{-3} \text{ s} \quad \omega T = 2\pi \quad \omega = \frac{2\pi}{T} = 10^3 \pi$$

and thus

$$10 \sin \omega t = 10 \sin 10^3 \pi t$$

Then,

$$P_{\text{ave}} = \frac{1}{T} \int_0^T \frac{v^2}{R} \, dt = \frac{1}{2 \times 10^{-3}} \left[ \int_0^{10^{-3}} \frac{10^2 \sin^2 10^3 \pi t}{5} \, dt + \int_{10^{-3}}^{2 \times 10^{-3}} 0 \, dt \right]$$

or

$$\begin{aligned} P_{\text{ave}} &= \frac{100}{10 \times 10^{-3}} \left[ \int_0^{10^{-3}} \frac{1}{2} (1 - \cos 2 \times 10^3 \pi t) \, dt \right] = 5 \times 10^3 \left[ \int_0^{10^{-3}} dt - \int_0^{10^{-3}} \cos 2 \times 10^3 \pi t \, dt \right] \\ &= \left( 5 \times 10^3 \left[ t - \frac{\sin 2 \times 10^3 \pi t}{2 \times 10^3 \pi} \right] \right) \Bigg|_0^{10^{-3}} = 5 \times 10^3 \left[ 10^{-3} - \frac{\sin 2 \times 10^3 \pi \times 10^{-3}}{2 \times 10^3 \pi} \right] \end{aligned}$$



and since  $\sin 2n\pi = 0$  for  $n = \text{integer}$ , the last term of the expression above reduces to

$$P_{\text{ave}} = 5 w$$

---

### 8.10 Lagging and Leading Power Factors

By definition an *inductive load* is said to have a *lagging power factor*. This refers to the phase angle of the current through the load with respect to the voltage across this load as shown in Figure 8.15.

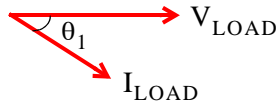


Figure 8.15. Lagging power factor

In Figure 8.15, the cosine of the angle  $\theta_1$ , that is,  $\cos\theta_1$  is referred to as *lagging power factor* and it is denoted as *pf lag*.

The term “inductive load” means that the load is more “inductive” (with some resistance) than it is “capacitive”. But in a “purely inductive load”  $\theta_1 = 90^\circ$  and thus the power factor is

$$\cos\theta_1 = \cos 90^\circ = 0$$

By definition a *capacitive load* is said to have a *leading power factor*. Again, this refers to the phase angle of the current through the load with respect to the voltage across this load as shown in Figure 8.16.

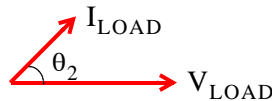


Figure 8.16. Leading power factor

In Figure 8.16, the cosine of the angle  $\theta_2$ , that is,  $\cos\theta_2$  is referred to as *leading power factor* and it is denoted as *pf lead*.

The term “capacitive load” means that the load is more “capacitive” (with some resistance) than it is “inductive”. But in a “purely capacitive load”  $\theta_2 = 90^\circ$  and thus the power factor is

$$\cos\theta_2 = \cos 90^\circ = 0$$

8.11 Complex Power – Power Triangle

We recall that

$$P_{ave} = \frac{1}{2} V_p I_p \cos\theta = V_{RMS} I_{RMS} \cos\theta \tag{8.36}$$

This relation can be represented by the so-called *power triangle*. Figure 8.17 (a) shows the power triangle of an inductive load, and Figure 8.16 (b) shows the power triangle for both a capacitive load.



(a) Power Triangle for Inductive Load      (b) Power Triangle for Capacitive Load

Figure 8.17. Power triangles for inductive and capacitive loads

In a power triangle, the product  $V_{RMS} \times I_{RMS}$  is referred to as the *apparent power*, and it is denoted as  $P_a$ . The apparent power is expressed in volt – amperes or VA. The product  $V_{RMS} \times I_{RMS} \times \sin\theta$  is referred to as the *reactive power*, and it is denoted as  $Q$ . The reactive power is expressed in volt – amperes reactive or VAR. Thus, for either triangle of Figure 8.17,

$$P_{real} = P_{ave} = V_{RMS} I_{RMS} \cos\theta = \quad (\text{in watts}) \tag{8.37}$$

$$Q = \text{Reactive Power} = V_{RMS} I_{RMS} \sin\theta = \quad (\text{in VARs}) \tag{8.38}$$

$$P_a = \text{Apparent Power} = V_{RMS} I_{RMS} \quad (\text{in VAs}) \tag{8.39}$$

The apparent power  $P_a$  is the vector sum of the real and reactive power components, that is,

$$P_a = P_{real\ power} \pm jQ = P_{ave} \pm jQ \tag{8.40}$$

where the (+) sign is used for inductive loads and the (-) sign for capacitive loads. Because relation of (8.40) consists of a real part and an imaginary part, it is known as the *complex power*.

Example 8.8

For the circuit shown in Figure 8.18, find:

- the average power delivered to the load
- the average power absorbed by the line
- the apparent power supplied by the voltage source
- the power factor of the load
- the power factor of the line plus the load

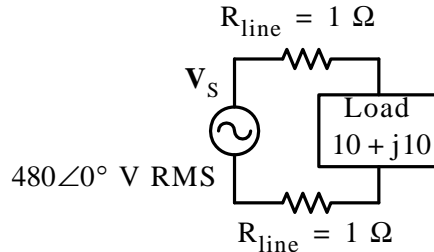


Figure 8.18. Circuit for Example 8.8

**Solution:**

For simplicity, we redraw the circuit as shown in Figure 8.19 where the line resistances have been combined into a single  $2\ \Omega$  resistor.

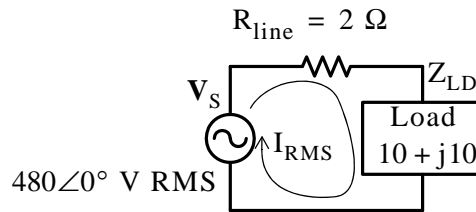


Figure 8.19. Circuit for Example 8.8 with the line resistances combined

From the circuit of Figure 8.19, we find that

$$I_{\text{RMS}} = \frac{V_{\text{S RMS}}}{R_{\text{line}} + Z_{\text{LD}}} = \frac{480\angle 0^\circ}{2 + 10 + j10} = \frac{480\angle 0^\circ}{15.62\angle 39.8^\circ} = 30.73\angle -39.8^\circ$$

and therefore, the current lags the voltage as shown on the phasor diagram of Figure 8.20.

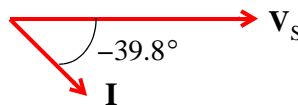


Figure 8.20. Phasor diagram for the circuit of Example 8.8

Then,

- The average power delivered to the load is

$$P_{\text{ave LD}} = I_{\text{RMS}}^2 \operatorname{Re}\{Z_{\text{LD}}\} = (30.73)^2 \times 10 = 9443 \text{ w} = 9.443 \text{ Kw}$$

b. The average power absorbed by the line is

$$P_{\text{ave line}} = I_{\text{RMS}}^2 R_{\text{line}} = (30.73)^2 \times 2 = 1889 \text{ w} = 1.889 \text{ Kw}$$

c. The apparent power supplied by the voltage source is

$$P_{\text{a source}} = V_{\text{S RMS}} I_{\text{RMS}} = 480 \times 30.73 = 14750 \text{ w} = 14.750 \text{ Kw}$$

d. The power factor of the load is

$$\begin{aligned} \text{pf}_{\text{LD}} = \cos\theta_{\text{LD}} &= \frac{P_{\text{ave LD}}}{P_{\text{a LD}}} = \frac{9443}{|V_{\text{RMS LD}}||I_{\text{RMS}}|} \\ &= \frac{9443}{|(480\angle 0^\circ - 2(30.73\angle -39.8^\circ))| \times 30.73} = \frac{9443}{(434.56)(30.73)} = \frac{9443}{13354} = 0.707 \end{aligned}$$

e. The power factor of the line plus the load is

$$\text{pf}_{(\text{line} + \text{LD})} = \cos\theta_{(\text{line} + \text{LD})} = \frac{P_{\text{ave total}}}{P_{\text{a source}}} = \frac{P_{\text{ave line}} + P_{\text{ave LD}}}{P_{\text{a source}}} = \frac{1889 + 9443}{14750} = 0.77$$

---

## 8.12 Power Factor Correction

The consumer pays the electric utility company for the average or real power, not the apparent power and, as we have seen, a low power factor (larger angle  $\theta$ ) demands more current. This additional current must be furnished by the utility company which must provide larger current-carrying capacity if the voltage must remain constant. Moreover, this additional current creates larger  $i^2R$  losses in the utility's transmission and distribution system. For this reason, electric utility companies impose a penalty on industrial facility customers who operate at a low power factor, typically lower than 0.85. Accordingly, facility engineers must install the appropriate equipment to raise the power factor.

The power factor correction procedure is illustrated with the following example.

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### Example 8.9

In the circuit shown in Figure 8.21, the resistance of the lines between the voltage source and the load and the internal resistance of the source are considered small, and thus can be neglected.

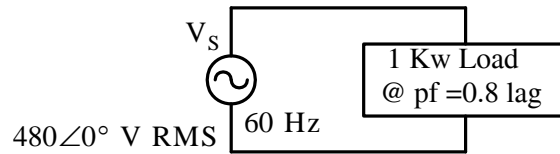


Figure 8.21. Circuit for Example 8.9

It is desired to “raise” the power factor of the load to 0.95 lagging. Compute the size and the rating of a capacitor which, when added across the load, will accomplish this.

**Solution:**

The power triangles for the existing and desired power factors are shown in Figure 8.22.

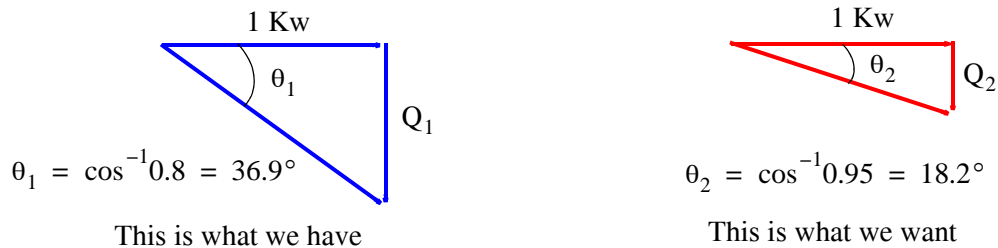


Figure 8.22. Power triangles for existing and desired power factors

Since the voltage across the given load must not change (otherwise it will affect the operation of it), it is evident that a load, say  $Q_3$ , in opposite direction of  $Q_1$  must be added, and must be connected in parallel with the existing load. Obviously, the  $Q_3$  load must be capacitive. Accordingly, the circuit of Figure 8.21 must be modified as shown in Figure 8.23.

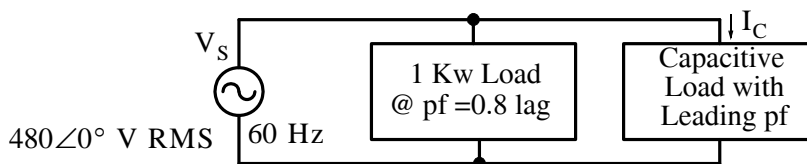


Figure 8.23. Circuit for power factor correction

For the existing load,

$$Q_1 = (1 \text{ Kw})\tan 36.9^\circ = 750 \text{ VAR}$$

and for the desired  $\text{pf} = \cos\theta_2 = 0.95$ , the VAR value of  $Q_2$  must be reduced to

$$Q_2 = (1 \text{ Kw})\tan 18.2^\circ = 329 \text{ VAR}$$

Therefore, the added capacitive load must be a vector  $Q_3$  such that

$$Q_3 = Q_1 - Q_2 = 750 - 329 = 421 \text{ VAR}$$

The current  $I_C$  through the capacitive load is found from

$$Q_3 = I_C V_C = I_C V_S$$

Then,

$$I_C = \frac{Q_3}{V_{S \text{ RMS}}} = \frac{421}{480} = 0.88 \text{ A}$$

and

$$X_C = \frac{V_C}{I_C} = \frac{480}{0.88} = 547 \text{ } \Omega$$

Therefore, the capacitive load must consist of a capacitor with the value

$$C = \frac{1}{\omega X_C} = \frac{1}{2\pi f X_C} = \frac{1}{2\pi(60)(547)} = 4.85 \text{ } \mu\text{F}$$

However, not any 4.85  $\mu\text{F}$  capacitor will do; the capacitor must be capable of withstanding a maximum voltage of

$$V_{C \text{ max}} = \sqrt{2} \times 480 = 679 \text{ V}$$

and for all practical purposes, we can choose a 5  $\mu\text{F}$  capacitor rated at 700 volts or higher.

---

### 8.13 Instruments

*Ammeters* are electrical instruments used to measure current in electric circuits, *voltmeters* measure voltage, *ohmmeters* measure resistance, *wattmeters* measure power, and *watt-hour meters* measure electric energy. Voltmeters, Ohmmeters, and Milliammeters (ammeters which measure current in milliamperes) are normally combined into one instrument called VOM. Figure 8.24 shows a typical analog type VOM, and Figure 8.25 shows a typical digital type VOM. We will see how a digital VOM can be constructed from an analog VOM equivalent at the end of this section. An *oscilloscope* is an electronic instrument that produces an instantaneous trace on the screen of a cathode-ray tube corresponding to oscillations of voltage and current. A typical oscilloscope is shown in Figure 8.26. DC ammeters and voltmeters read average values whereas AC ammeters and voltmeters read RMS values.

The basic meter movement consists of a permanent horse shoe magnet, an electromagnet which typically is a metal cylinder with very thin wire wound around it which is referred to as the coil, and a control spring. The coil is free to move on pivots, and when there is current in the coil, a torque is produced that tends to rotate the coil. Rotation of the coil is restrained by a helical spring so that the motion of the coil and the pointer which is attached to it, is proportional to the current in the coil.



Figure 8.24. The Triplet Analog Multimeter Model 60



Figure 8.25. The Votcraft Model 3850 Digital Multimeter



Figure 8.26. The Agilent Technologies Series 5000 Portable Oscilloscope

An ammeter measures current in amperes. For currents less than one ampere, a *milliammeter* or *microammeter* may be used where the former measures current in milliamperes and the latter in microamperes.

Ammeters, milliammeters, and microammeters must always be connected in series with the circuits in which they are used.

Often, the electric current to be measured, exceeds the range of the instrument. For example, we cannot directly measure a current of 5 to 10 milliamperes with a milliammeter whose range is 0 to 1 milliamperes. In such a case, we can use a low range milliammeter with a *shunt* (parallel) resistor as shown in Figure 8.27, where the circle with mA represents an ideal milliammeter (a milliammeter with zero resistance). In Figure 8.27  $I_T$  is the total current to be measured,  $I_M$  is the current through the meter,  $I_S$  is the current through the shunt resistor,  $R_M$  is the milliammeter internal resistance, and  $R_S$  is the shunt resistance.

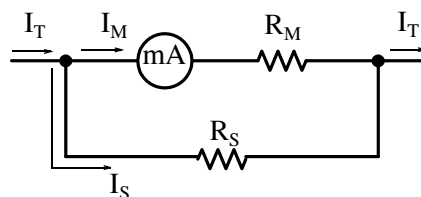




Figure 8.27. Milliammeter with shunt resistor  $R_S$ 

From the circuit of Figure 8.27, we observe that the sum of the current flowing through the milliammeter  $I_M$  and the current  $I_S$  through the shunt resistor is equal to the total current  $I_T$ , that is,

$$I_T = I_M + I_S \quad (8.41)$$

Also, the shunt resistor  $R_S$  is in parallel with the milliammeter branch; therefore, the voltages across these parallel branches are equal, that is,

$$R_M I_M = R_S I_S$$

and since we normally need to calculate the shunt resistor, then

$$R_S = \frac{I_M}{I_S} R_M \quad (8.42)$$

### Example 8.10

In the circuit of Figure 8.28, the total current entering the circuit is 5 mA and the milliammeter range is 0 to 1 milliamperes, that is, the milliammeter has a full-scale current  $I_{fs}$  of 1 mA, and its internal resistance is  $40 \Omega$ . Compute the value of the shunt resistor  $R_S$ .

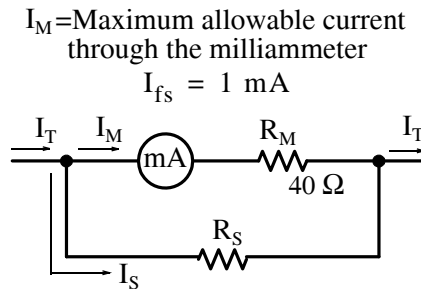


Figure 8.28. Circuit for Example 8.10

#### Solution:

The maximum current that the milliammeter can allow to flow through it is 1 mA and since the total current is 5 milliamperes, the remaining 4 milliamperes must flow through the shunt resistor, that is,

$$I_S = I_T - I_M = 5 - 1 = 4 \text{ mA}$$

The required value of the shunt resistor is found from (8.42), i.e.,

$$R_S = \frac{I_M}{I_S} R_M = \frac{1}{4} \times 40 = 10 \, \Omega$$

**Check:** The calculated value of the shunt resistor is  $10 \, \Omega$ ; this is one-fourth the value of the milliammeter internal resistor of  $40 \, \Omega$ . Therefore, the  $10 \, \Omega$  resistor will allow four times as much current as the milliammeter to flow through it.

A *multi-range ammeter/milliammeter* is an instrument with two or more scales. Figure 8.29 shows the circuit of a typical multi-range ammeter/milliammeter.

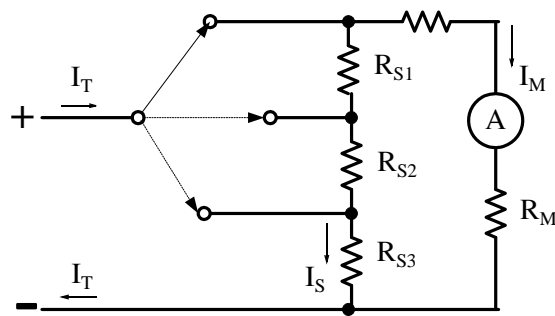


Figure 8.29. Circuit for a multi-range ammeter/milliammeter

A voltmeter, as stated earlier, measures voltage in volts. Typically, a voltmeter is a modified milliammeter where an external resistor  $R_V$  is connected in series with the milliammeter as shown in Figure 8.30 where

$I$  = current through circuit

$R_M$  = internal resistance of milliammeter

$R_V$  = external resistor in series with  $R_M$

$V_M$  = voltmeter full scale reading

$R_V =$  Voltmeter internal resistance

$V_M =$  Voltmeter range

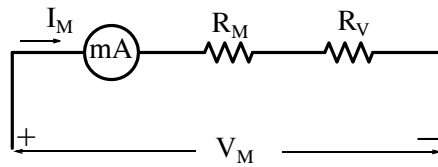


Figure 8.30. Typical voltmeter circuit

For the circuit of Figure 8.30,

$$I_M(R_M + R_V) = V_M$$

or

$$R_V = \frac{V_M}{I_M} - R_M \quad (8.43)$$

Voltmeters must always be connected in parallel with those devices of the circuit whose voltage is to be measured.

### Example 8.11

Design a voltmeter which will have a 1 volt full-scale using a milliammeter with 1 milliampere full-scale and internal resistance  $100 \Omega$ .

#### Solution:

The voltmeter circuit consists of the milliammeter circuit and the external resistance  $R_V$  as shown in Figure 8.31.

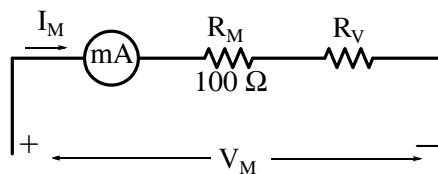


Figure 8.31. Circuit for Example 8.11

Here, we only need to compute the value of the external resistor  $R_V$  so that the voltage across the series combination will be 1 volt full scale. Then, from (8.43),

$$R_V = \frac{V_M}{I_M} - R_M = \frac{1}{10^{-3}} - 100 = 1000 - 100 = 900 \Omega \quad (8.44)$$

## Chapter 8 Average and RMS Values, Complex Power, and Instruments

Therefore, to convert a 1 milliamper full-scale milliammeter with an internal resistance of  $100\ \Omega$  to a 1 volt full-scale voltmeter, we only need to attach a  $900\ \Omega$  resistor in series with that milliammeter.

Figure 8.32 shows a typical multi-range voltmeter.

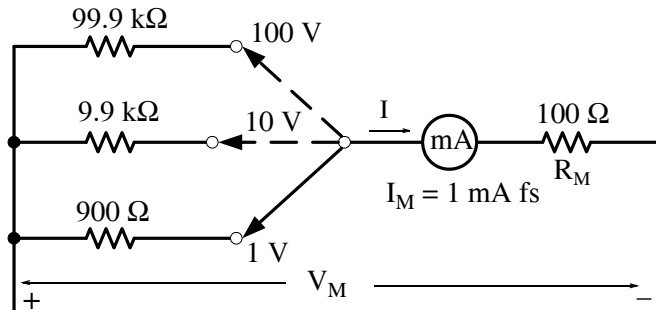


Figure 8.32. Circuit for a multi-range voltmeter

An *Ohmmeter* measures resistance in Ohms. In the series type Ohmmeter, the resistor  $R_X$  whose resistance is to be measured, is connected in series with the Ohmmeter circuit shown in Figure 8.33.

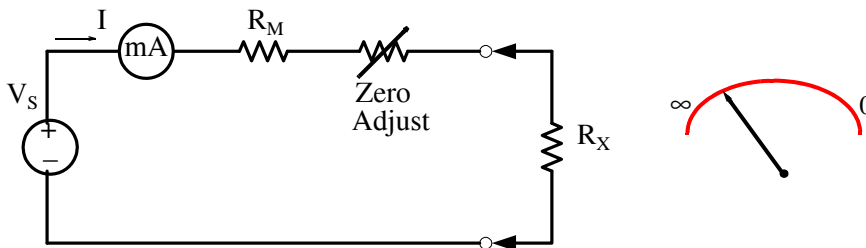


Figure 8.33. Circuit for a series type Ohmmeter

We observe from Figure 8.33 that for the series type Ohmmeter, the current  $I$  is maximum when the resistor  $R_X$  is zero (short circuit), and the current is zero when  $R_X$  is infinite (open circuit). For this reason, the 0 (zero) point appears on the right-most point of the Ohmmeter scale, and the infinity symbol appears on the left-most point of the scale.

Figure 8.34 shows the circuit of a shunt (parallel) type Ohmmeter where the resistor  $R_X$  whose value is to be measured, is in parallel with the Ohmmeter circuit.

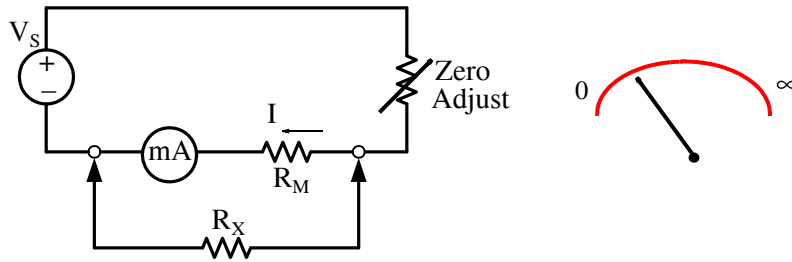


Figure 8.34. Circuit for a parallel type Ohmmeter

From Figure 8.34 we see that, for the shunt type Ohmmeter, the current through the milliammeter circuit is zero when the resistor  $R_X$  is zero (short circuit) since all current flows through that short. However, when  $R_X$  is infinite (open circuit), the current through the milliammeter branch is maximum. For this reason, the 0 (zero) point appears on the left-most point of the Ohmmeter scale, and the infinity symbol appears on the right-most point of the scale.

An instrument which can measure unknown resistance values very accurately is the *Wheatstone Bridge* shown in Figure 8.35.

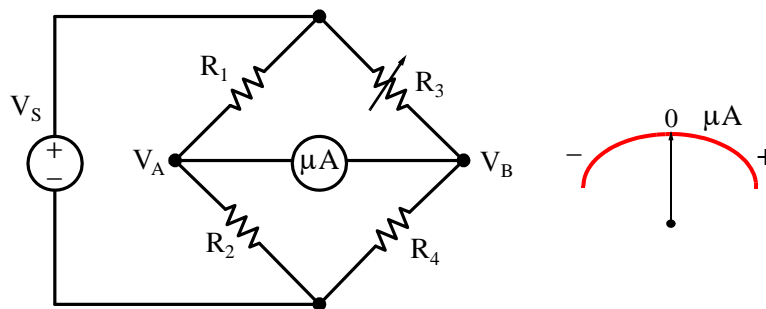


Figure 8.35. Wheatstone Bridge Circuit

One of the resistors, say  $R_4$ , is the unknown resistor whose value is to be measured, and another resistor, say  $R_3$  is adjusted until the bridge is balanced, that is, until there is no current flow through the meter of this circuit. This balance occurs when

$$\frac{R_1}{R_2} = \frac{R_3}{R_4}$$

from which the value of the unknown resistor is found from

$$R_4 = \frac{R_2}{R_1} R_3 \quad (8.45)$$

### Example 8.12

In the Wheatstone Bridge circuit of Figure 8.36, resistor  $R_3$  is adjusted until the meter reads zero, and when this occurs, its value is  $120\ \Omega$ . Compute the value of the unknown resistor  $R_4$ .

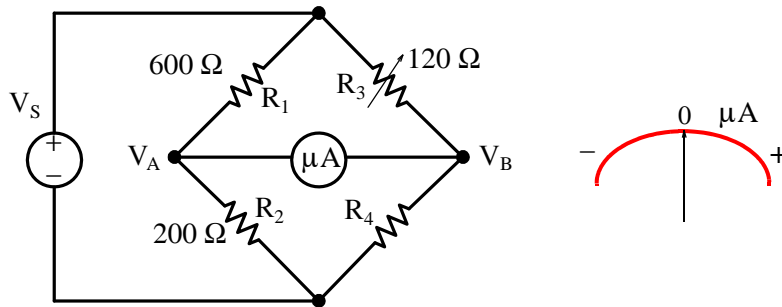


Figure 8.36. Circuit for Example 8.12

### Solution:

When the bridge is balanced, that is, when the current through the meter is zero, relation (8.45) holds. Then,

$$R_4 = \frac{R_2}{R_1} R_3 = \frac{200}{600} \times 120 = 40\ \Omega$$

When measuring resistance values, the voltage sources in the circuit to which the unknown resistance is connected must be turned off, and one end of the resistor whose value is to be measured must be disconnected from the circuit.

Because of their great accuracy, Wheatstone Bridges are also used to accept or reject resistors whose values exceed a given tolerance.

A *wattmeter* is an instrument which measures power in watts or kilowatts. It is constructed with two sets of coils, a current coil and a voltage coil where the interacting magnetic fields of these coils produce a torque which is proportional to the  $V \times I$  product.

A *watt-hour meter* is an instrument which measures electric energy  $W$ , where  $W$  is the product of the average power  $P$  in watts and time  $t$  in hours, that is,  $W = Pt$  in *watt-hours*. Electric utility companies use kilowatt-hour meters to bill their customers for the use of electricity.

Digital meters include an additional circuit called *analog-to-digital converter* (ADC). There are different types of analog-to-digital converters such as the *flash converter*, the *time-window converter*, *slope converter* and *tracking converter*. Shown in Figure 8.37 is a flash converter ADC.

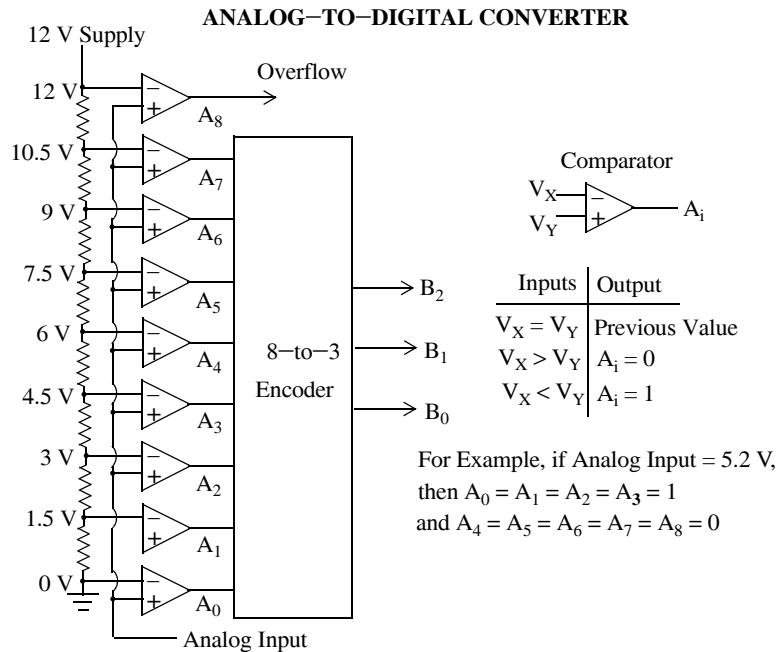


Figure 8.37. Typical analog-to-digital converter

Analog Input	A8	A7	A6	A5	A4	A3	A2	A1	A0	B2	B1	B0
Less than 0 V	0	0	0	0	0	0	0	0	0	x	x	x <sup>†</sup>
0 to less than 1.5 V	0	0	0	0	0	0	0	0	1	0	0	0
1.5 to less than 3.0 V	0	0	0	0	0	0	0	1	1	0	0	1
3.0 to less than 4.5 V	0	0	0	0	0	0	1	1	1	0	1	0
4.5 to less than 6.0 V	0	0	0	0	0	1	1	1	1	0	1	1
6.0 to less than 7.5 V	0	0	0	0	1	1	1	1	1	1	0	0
7.5 to less than 9.0 V	0	0	0	1	1	1	1	1	1	1	0	1
9.0 to less than 10.5 V	0	0	1	1	1	1	1	1	1	1	1	0
10.5 to 12 V	0	1	1	1	1	1	1	1	1	1	1	1
Greater than 12 V	1	1	1	1	1	1	1	1	1	x	x	x <sup>‡</sup>

† Underflow

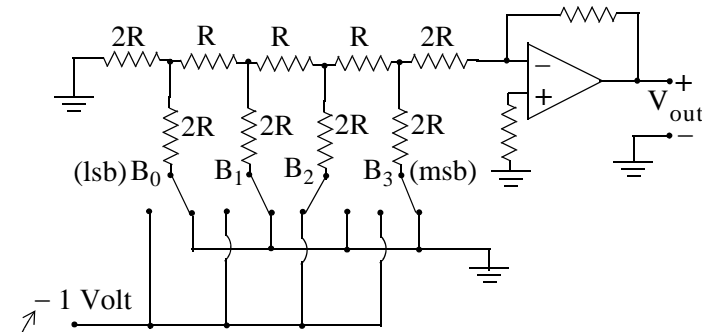
‡ Overflow

As shown in Figure 8.37, the flash type ADC consists of a resistive network, comparators (denoted as triangles), and an eight-to-three line encoder.

A digital-to-analog converter (DAC) performs the inverse operation, that is, it converts digital values to equivalent analog values.

Figure 8.38 shows a four-bit R-2R ladder network and an op-amp connected to form a DAC.

DIGITAL-TO-ANALOG CONVERTER



Switch Settings: For Logic "0" (ground) positioned to the right  
 For Logic "1" (+5 V) positioned to the left

With the switches positioned as shown,  $B_3 B_2 B_1 B_0 = 0100$

Negative reference voltage is used so that the inverting op amp's output will be positive.

lsb = least significant bit

msb = most significant bit

Figure 8.38. A typical digital-to-analog converter



## 8.14 Summary

- A periodic time function is one which satisfies the relation  $f(t) = f(t + nT)$  where  $n$  is a positive integer and  $T$  is the period of the periodic time function.
- The average value of any continuous function  $f(t)$  over an interval  $a \leq t \leq b$ , is defined as

$$f(t)_{\text{ave}} = \frac{1}{b-a} \int_a^b f(t) dt = \frac{1}{b-a} (\text{area})_a^b$$

- The average value of a periodic time function  $f(t)$  is defined as the average of the function over one period.
- A half-wave rectification waveform is defined as

$$f(t) = \begin{cases} A \sin \omega t & 0 < \omega t < \pi \\ 0 & \pi < \omega t < 2\pi \end{cases}$$

- The effective current  $I_{\text{eff}}$  of a periodic current waveform  $i(t)$  is defined as

$$I_{\text{eff}} = \sqrt{\frac{1}{T} \int_0^T i^2 dt} = I_{\text{Root Mean Square}} = I_{\text{RMS}} = \sqrt{\text{Ave}(i^2)}$$

- For sinusoids only,  $I_{\text{RMS}} = I_p / \sqrt{2} = 0.707 I_p$
- For sinusoids of different frequencies,  $I_{\text{RMS}} = \sqrt{I_0^2 + I_{1 \text{ RMS}}^2 + I_{2 \text{ RMS}}^2 + \dots + I_{N \text{ RMS}}^2}$
- For circuits with sinusoidal excitations the average power delivered to a load is

$$P_{\text{ave LD}} = \frac{V_p I_p}{2} \cos \theta = V_{\text{RMS LD}} I_{\text{RMS LD}} \cos \theta$$

where  $\theta$  is the phase angle between  $V_{\text{LD}}$  and  $I_{\text{LD}}$  and it is within the range  $0 \leq \theta \leq 90^\circ$ , and  $\cos \theta$  is known as the power factor defined within the range  $0 \leq \cos \theta \leq 1$ .

- The average power in a resistive load is

$$P_{\text{ave R}} = \frac{V_{\text{RMS R}}^2}{R} = I_{\text{RMS R}}^2 R$$

- The average power in inductive and capacitive loads is

$$P_{\text{ave X}} = V_{\text{RMS X}} I_{\text{RMS X}} \cos 90^\circ = 0$$

---

## Chapter 8 Average and RMS Values, Complex Power, and Instruments

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- If the excitation in a circuit is non-sinusoidal, we can compute the average power absorbed by a resistor from the relations

$$P_{\text{ave}} = \frac{1}{T} \int_0^T p \, dt = \frac{1}{T} \int_0^T \frac{v^2}{R} \, dt = \frac{1}{T} \int_0^T i^2 R \, dt$$

- An inductive load is said to have a lagging power factor and a capacitive load is said to have a leading power factor.
- In a power triangle

$$P_{\text{real}} = P_{\text{ave}} = V_{\text{RMS}} I_{\text{RMS}} \cos \theta \quad (\text{in watts})$$

$$Q = \text{Reactive Power} = V_{\text{RMS}} I_{\text{RMS}} \sin \theta \quad (\text{in VARs})$$

$$P_a = \text{Apparent Power} = V_{\text{RMS}} I_{\text{RMS}} \quad (\text{in VAs})$$

- The apparent power  $P_a$ , also known as complex power, is the vector sum of the real and reactive power components, that is,

$$P_a = P_{\text{real power}} \pm jQ = P_{\text{ave}} \pm jQ$$

where the (+) sign is used for inductive loads and the (-) sign for capacitive loads.

- A power factor can be corrected by placing a capacitive load in parallel with the load of the circuit.
- Ammeters are instruments used to measure current in electric circuits. Ammeters, milliammeters, and microammeters must always be connected in series with the circuits in which they are used.
- Voltmeters are instruments used to measure voltage. Voltmeters must always be connected in parallel with those devices of the circuit whose voltage is to be measured.
- Ohmmeters are instruments used to measure resistance. When measuring resistance values, the voltage sources in the circuit to which the unknown resistance is connected must be turned off, and one end of the resistor whose value is to be measured must be disconnected from the circuit.
- A Wheatstone Bridge is an instrument which can measure unknown resistance values very accurately.
- Voltmeters, Ohmmeters, and Milliammeters (ammeters which measure current in milliamperes) are normally combined into one instrument called VOM.
- Wattmeters are instruments used to measure power.

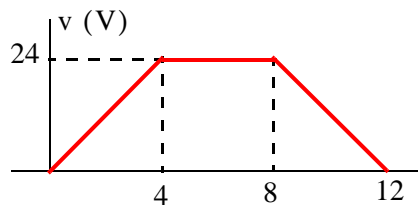
- Watt–Hour meters are instruments used to measure energy.
- An oscilloscope is an electronic instrument that produces an instantaneous trace on the screen of a cathode–ray tube corresponding to oscillations of voltage and current.
- DC ammeters and DC voltmeters read average values
- AC ammeters and AC voltmeters read RMS values.
- Digital meters include an additional circuit called analog–to–digital converter (ADC).

### 8.15 Exercises

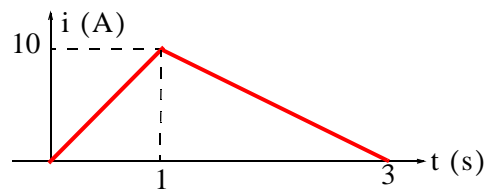
#### Multiple Choice

- The average value of a constant (DC) voltage of 12 V is
  - 6 V
  - 12 V
  - $12/\sqrt{2}$  V
  - $12 \times \sqrt{2}$  V
  - none of the above
- The average value of  $i = 5 + \cos 100t$  A is
  - $5 + \sqrt{2}/2$  A
  - $5 \times \sqrt{2}$  A
  - $5/\sqrt{2}$  A
  - 5 A
  - none of the above
- The RMS value of a constant (DC) voltage of 12 V is
  - $12/\sqrt{2}$  V
  - $6 \times \sqrt{2}/2$  V
  - 12 V
  - $12 \times \sqrt{2}$  V
  - none of the above
- The RMS value of  $i = 5 + \cos 100t$  A is
  - $5 + \sqrt{2}/2$  A
  - $5 \times \sqrt{2}$  A
  - $5/\sqrt{2}$  A
  - 5 A

- E. none of the above
5. The voltage across a load whose impedance is  $Z = 75 + j38 \Omega$  is 115 V RMS. The average power absorbed by that load is
- A. 176.33 w  
B. 157.44 w  
C. 71.3 w  
D. 352.67 w  
E. none of the above
6. The average value of the waveform below is



- A. 24 V  
B. 16 V  
C. 12 V  
D. 6 V  
E. none of the above
7. The RMS value of the waveform below is



- A.  $10/\sqrt{2}$  V  
B.  $10 \times \sqrt{2}$  V  
C.  $10/\sqrt{3}$  V  
D.  $10 \times \sqrt{3}$  V

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## Chapter 8 Average and RMS Values, Complex Power, and Instruments

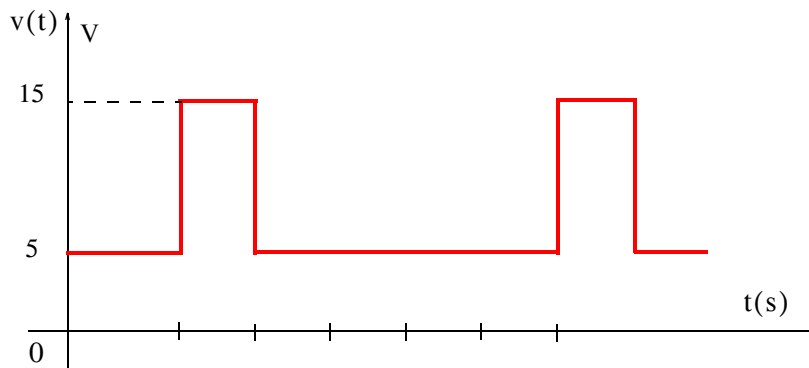
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- E. none of the above
8. A current with a value of  $i = 5 \cos 10000t$  A is flowing through a load that consists of the series combination of  $R = 2 \Omega$ ,  $L = 1$  mH, and  $C = 10 \mu\text{F}$ . The average power absorbed by this load is
- A. 25 w  
B. 10 w  
C. 5 w  
D. 0 w  
E. none of the above
9. If the average power absorbed by a load is 500 watts and the reactive power is 500 VAR, the apparent power is
- A. 0 VA  
B. 500 VA  
C. 250 VA  
D.  $500 \times \sqrt{2}$  VA  
E. none of the above
10. A load with a leading power factor of 0.60 can be corrected to a lagging power factor of 0.85 by adding
- A. a capacitor in parallel with the load  
B. an inductor in parallel with the load  
C. an inductor in series with the load  
D. a capacitor in series with the load  
E. none of the above

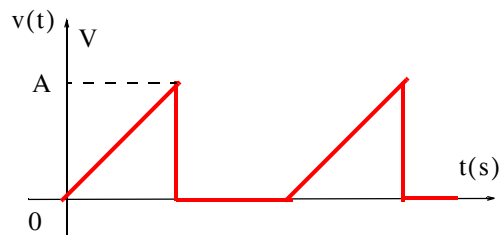
### Problems

1. The current  $i_L(t)$  through a 0.5 H inductor is given as  $i_L(t) = 5 + 10 \sin t$  A. Compute:
- a. The average values of the current, voltage and power for this inductor.  
b. The RMS values of the current and voltage.

2. Compute the average and RMS values of the voltage waveform below.



3. Compute the RMS value of the voltage waveform below.

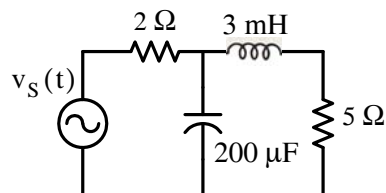


4. Compute the RMS value of  $i(t) = 10 + 2 \cos 100t + 5 \sin 200t$ .

5. A radar transmitter sends out periodic pulses. It transmits for  $5 \mu\text{s}$  and then rests. It sends out one of these pulses every  $1 \text{ ms}$ . The average output power of this transmitter is  $750 \text{ w}$ . Compute:

- The energy transmitted in each pulse.
- The power output during the transmission of a pulse.

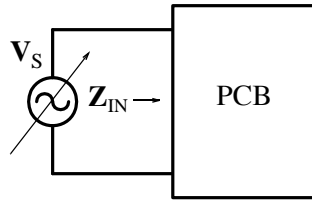
6. For the circuit below,  $v_s(t) = 100 \cos 1000t \text{ V}$ . Compute the average power delivered (or absorbed) by each device.



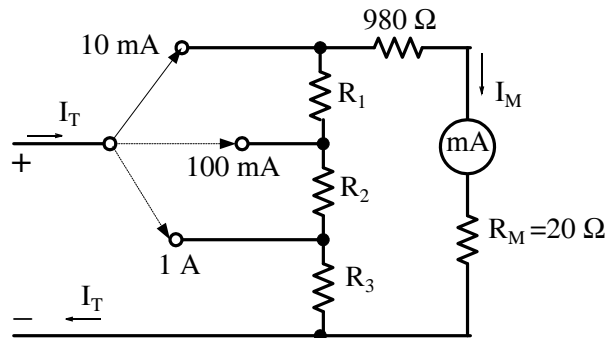
7. For the circuit below, the input impedance of the PCB (Printed Circuit Board) is  $Z_{\text{IN}} = 100 - j100 \Omega$  and the board must not absorb more than  $200 \text{ mw}$  of power; otherwise it

## Chapter 8 Average and RMS Values, Complex Power, and Instruments

will be damaged. Compute the largest RMS value that the variable voltage source  $V_S$  can be adjusted to.




8. For the multi-range ammeter/milliammeter shown below, the meter full scale is 1 mA. Compute the values of  $R_1$ ,  $R_2$ ,  $R_3$ , and  $R_4$  so that the instrument will display the indicated values.

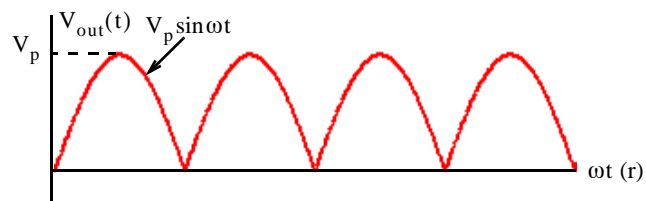
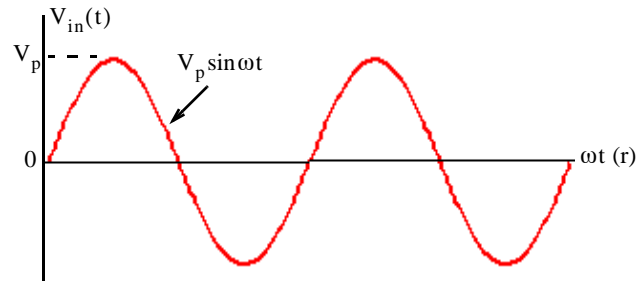
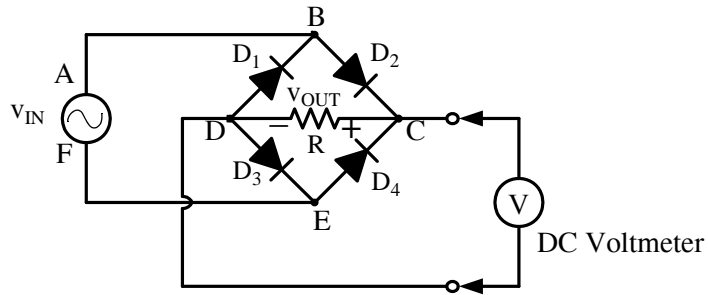


9. The circuit below is known as *full-wave rectifier*. The input and output voltage waveforms are shown in Figure 8.47. During the positive input half cycle, current flows from point A to point B, through  $D_2$  to point C, through the resistor  $R$  to point D, through diode  $D_3$  to point E, and returns to the other terminal point F of the input voltage source. During the negative input half cycle, current flows from point F to point E, through diode  $D_4$  to point C, through the resistor  $R$  to point D, through the diode  $D_1$  to point B, and returns to the other terminal point A of the input voltage source. There is a small voltage drop  $v_D$  across each diode\* but it can be neglected if  $v_{in} \gg v_D$ . Compute the value indicated by the DC voltmeter.

\* For silicon type diodes, the voltage drop is approximately 0.7 volt.



 Diode – Allows current to flow in the indicated direction only



**8.16 Answers / Solutions to End-of-Chapter Exercises****Multiple Choice**

1. B
2. D
3. C
4. A
5. E

$Z = 75 + j38 = 84.08 \angle 26.87^\circ$ ,  $I_{\text{RMS}} = 115 \angle 0^\circ / 84.08 \angle 26.87^\circ = 1.37 \angle -26.87^\circ$  and thus

$$P_{\text{ave}} = V_{\text{RMS}} \cdot I_{\text{RMS}} \cos \theta = 115 \times 1.37 \times \cos(-26.87^\circ) = 140.54 \text{ w}$$

6. B
7. C
8. A
9. D
10. B

**Problems**

1.  $i_L = 5 + 10 \sin t$ ,  $v_L = L \frac{di_L}{dt} = 0.5 \frac{d}{dt}(5 + 10 \sin t) = 5 \cos t$

a.

$$i_{L \text{ ave}} = \frac{1}{T} \int_0^T i_L dt = \frac{1}{T} \int_0^T (5 + 10 \sin t) dt$$

and since

$$\frac{1}{T} \int_0^T 10 \sin t dt = 0$$

it follows that

$$\frac{1}{T} \int_0^T 5 dt = \frac{1}{T} 5T = 5 \text{ A}$$

Likewise,

$$v_{L \text{ ave}} = \frac{1}{T} \int_0^T 5 \cos t dt = 0$$

Also,

$$P_{L \text{ ave}} = \frac{1}{T} \int_0^T p_L dt = \frac{1}{T} \int_0^T v_L i_L dt = \frac{1}{T} \int_0^T 5 \cos t (5 + 10 \sin t) dt = \frac{1}{T} \int_0^T (25 \cos t + 50 \sin t \cos t) dt$$

and using  $\sin 2x = 2 \sin x \cos x$  it follows that  $50 \sin t \cos t = 25 \sin 2t$  and thus

$$P_{L \text{ ave}} = \frac{1}{T} \int_0^T (25 \cos t + 25 \sin 2t) dt = 0$$

b.

$$\begin{aligned} I_{L \text{ RMS}}^2 &= \frac{1}{T} \int_0^T i_L^2 dt = \frac{1}{T} \int_0^T (5 + 10 \sin t)^2 dt \\ &= \frac{1}{T} \int_0^T [5(1 + 2 \sin t)]^2 dt = \frac{25}{T} \int_0^T (1 + 4 \sin t + 4 \sin^2 t) dt \end{aligned}$$

Using  $\sin^2 x = \frac{1 - \cos 2x}{2}$  and observing that  $\frac{1}{T} \int_0^T 4 \sin t dt = 0$  and  $\frac{1}{T} \int_0^T \cos 2t dt = 0$  we obtain

$$I_{L \text{ RMS}}^2 = \frac{25}{T} \left( t \Big|_0^T + \frac{4}{2} t \Big|_0^T \right) = \frac{25}{T} (T + 2T) = 75$$

and

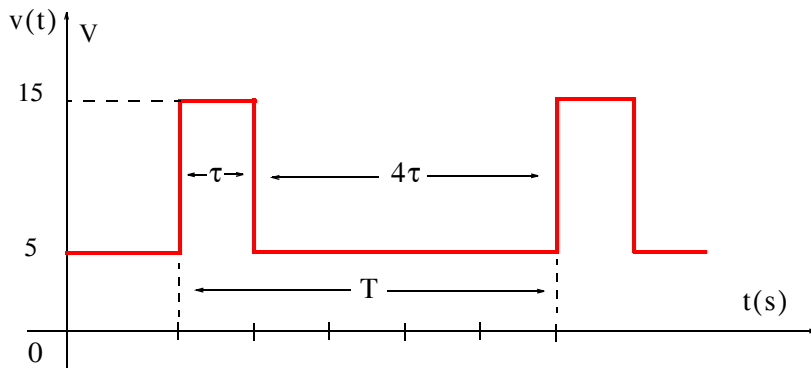
$$I_{L \text{ RMS}} = \sqrt{75} = 8.66 \text{ A}$$

For sinusoids  $V_{\text{RMS}} = V_p / (\sqrt{2}) = 0.707 V_p$  and since  $V_p = 5$  it follows that

$$V_{\text{RMS}} = 0.707 \times 5 = 3.54 \text{ V}$$

2. From the waveform below we observe that Period =  $T = 5\tau$  and since

$$V_{\text{ave}} = \text{Area/Period} = (15\tau + 20\tau) / 5\tau = 7 \text{ V}$$



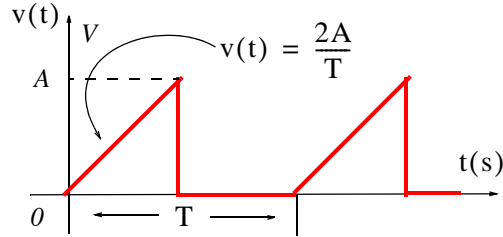
Also,

$$V_{\text{RMS}}^2 = \frac{1}{T} \int_0^T v^2 dt = \frac{1}{5\tau} \left[ \int_0^\tau (15)^2 dt + \int_\tau^{5\tau} (5)^2 dt \right] = \frac{1}{5\tau} (225\tau + 125\tau - 25\tau) = 65$$

and thus

$$V_{\text{RMS}} = \sqrt{65} = 8.06 \text{ V}$$

3. We choose the period  $T$  as shown below.



Using the straight line equation  $y = mx + b$  we find that for  $0 < t < T/2$ ,  $v(t) = \frac{2A}{T}t$ . Then,

$$\begin{aligned} V_{\text{RMS}}^2 &= \frac{1}{T} \int_0^T v^2 dt = \frac{1}{T} \int_0^{T/2} \left( \frac{2A}{T}t \right)^2 dt + \frac{1}{T} \int_{T/2}^T 0 \cdot dt = \frac{4A^2}{T^3} \int_0^{T/2} t^2 dt \\ &= \frac{4A^2}{3T^3} t^3 \Big|_0^{T/2} = \frac{4A^2}{24} = A^2/6 \end{aligned}$$

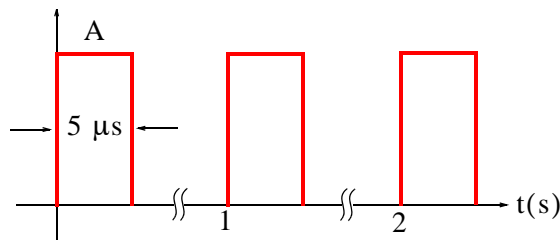
and

$$V_{\text{RMS}} = \sqrt{A^2/6} = \frac{\sqrt{6}}{6}A = 0.41A$$

4. The effective (RMS) value of a sinusoid is a real number that is independent of frequency and phase angle and for current it is equal to  $I_{\text{RMS}} = I_p / \sqrt{2}$ . The RMS value of sinusoids with different frequencies is given by (8.13). For this problem

$$I_{\text{RMS}} = \sqrt{10^2 + \frac{1}{2}2^2 + \frac{1}{2}5^2} = \sqrt{100 + 2 + 12.5} = 10.7 \text{ A}$$

5. The waveform representing the transmitter output pulses is shown below.



For this problem we do not know the amplitude  $A$  of each  $5 \mu\text{s}$  pulse but we know the average power of one period  $T = 1 \text{ s}$ . Since

$$P_{\text{ave}} = 750 \text{ W} = \frac{\text{Area}}{\text{Period}} = \frac{\text{Area}}{1 \text{ s}}$$

it follows that:

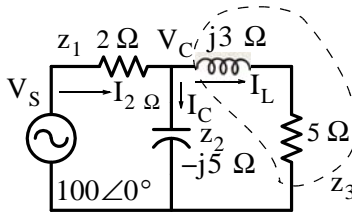
a. Energy transmitted during each pulse is

$$\text{Area of each pulse} = 750 \text{ W} \cdot \text{s}$$

b. The power during the transmission of a pulse is

$$P = W/t = 750 \text{ W} \cdot \text{s} / 5 \mu\text{s} = 750 \text{ W} \cdot \text{s} / 5 \times 10^{-6} = 150 \times 10^6 \text{ W} = 150 \text{ MW}$$

6. The phasor equivalent circuit is shown below where  $j\omega L = j10^3 \times 3 \times 10^{-3} = j3 \Omega$  and  $-j/\omega C = -j/10^3 \times 2 \times 10^{-4} = -j5 \Omega$



By application of KCL

$$\frac{V_C - V_S}{z_1} + \frac{V_C}{z_2} + \frac{V_C}{z_2} = 0$$

$$\left( \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} \right) V_C = \frac{V_S}{z_1}$$

$$V_C = \frac{V_S}{(1 + z_1/z_2 + z_1/z_3)}$$

Also,

$$I_2 = \frac{V_S - V_C}{z_1} \quad I_C = \frac{V_C}{z_2} \quad I_L = \frac{V_C}{z_3}$$

and with MATLAB

```
Vs=100; z1=2; z2=-5j; z3=5+3j;...
Vc=Vs/(1+z1/z2+z1/z3); I2=(Vs-Vc)/z1; Ic=Vc/z2; IL=Vc/z3; fprintf(' \n');...
disp('Vc = '); disp(Vc); disp('magVc = '); disp(abs(Vc));...
disp('phaseVc = '); disp(angle(Vc)*180/pi);...
disp('I2 = '); disp(I2); disp('magI2 = '); disp(abs(I2));...
disp('phaseI2 = '); disp(angle(I2)*180/pi);...
```

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```
disp('Ic = '); disp(Ic); disp('magIc = '); disp(abs(Ic));...
disp('phaseIc = '); disp(angle(Ic)*180/pi);...
disp('IL = '); disp(IL); disp('magIL = '); disp(abs(IL));...
disp('phaseIL = '); disp(angle(IL)*180/pi);
```

```
Vc = 75.0341-12.9604i
```

```
magVc = 76.1452
```

```
phaseVc = -9.7998
```

```
I2 = 12.4829 + 6.4802i
```

```
magI2 = 14.0647
```

```
phaseI2 = 27.4350
```

```
Ic = 2.5921 + 15.0068i
```

```
magIc = 15.2290
```

```
phaseIc = 80.2002
```

```
IL = 9.8909 - 8.5266i
```

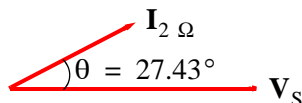
```
magIL = 13.0588
```

```
phaseIL = -40.7636
```

The average power delivered by the voltage source  $\mathbf{V}_S$  is computed from the relation

$$P_{\text{ave}} = V_{\text{RMS}} I_{\text{RMS}} \cos \theta = \frac{1}{2} V_p I_p \cos \theta$$

where  $\theta = 27.43^\circ$  as shown by the phasor diagram below.



Therefore,

$$P_{S \text{ ave}} = \frac{1}{2} \times |\mathbf{V}_S| \times |\mathbf{I}_2| \cos \theta = 0.5 \times 100 \times 14.07 \times \cos 27.43^\circ = 624.4 \text{ w}$$

Also,

$$P_{2 \Omega \text{ ave}} = \frac{1}{2} I_p^2 R_{2 \Omega} = 0.5 \times (14.07)^2 \times 2 = 197.97 \text{ w}$$

and

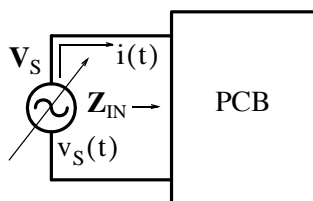
$$P_{5 \Omega \text{ ave}} = \frac{1}{2} I_L^2 R_{5 \Omega} = 0.5 \times (13.06)^2 \times 5 = 426.41 \text{ w}$$

**Check:**

$$P_{2 \Omega \text{ ave}} + P_{5 \Omega \text{ ave}} = 197.97 + 426.41 = P_{S \text{ ave}} = 624.4 \text{ w}$$

The average power in the capacitor and the inductor is zero since  $\theta = 90^\circ$  and  $\cos \theta = 0$ .

7. Let us consider the  $t$ -domain network below.



Let

$$v_S = V_p \cos \omega t$$

and

$$i = I_p \cos(\omega t + \theta)$$

Then,

$$p = v_S i = V_p I_p \cos \omega t \cdot \cos(\omega t + \theta)$$

and using

$$\cos x \cdot \cos y = \frac{1}{2} [\cos(x + y) + \cos(x - y)]$$

we obtain

$$p = \frac{V_p I_p}{2} [\cos(2\omega t + \theta) + \cos \theta]$$

We require that the power  $p$  does not exceed 200 mw or 0.2 w, that is, we must satisfy the condition

$$p = \frac{V_p I_p}{2} [\cos(2\omega t + \theta) + \cos \theta] \leq 0.2 \text{ w}$$

and therefore we must find the phase angle  $\theta$ . Since  $\theta$  appears also in the  $j\omega$ -domain, we can find its value from the given input impedance, that is,  $Z_{IN} = 100 - j100 \Omega$  or

$$Z_{IN} = |Z_{IN}| \angle \theta = \sqrt{100^2 + 100^2} \angle \tan^{-1} \frac{-100}{100} = 100\sqrt{2} \angle -45^\circ$$

and in the  $t$ -domain

$$p = \frac{V_p I_p}{2} [\cos(2\omega t - 45^\circ) + \cos(-45^\circ)]$$

The maximum power  $p$  occurs when  $\cos(2\omega t - 45^\circ) = 1$ , that is,

$$p_{\max} = \frac{V_p I_p}{2} \left(1 + \frac{\sqrt{2}}{2}\right) = 0.2 \text{ w}$$

## Chapter 8 Average and RMS Values, Complex Power, and Instruments

Then,

$$V_p I_p = 0.4 / 1.707$$

and now we can express  $I_p$  in terms of  $V_p$  using the relation  $|Z_{IN}| = 100\sqrt{2}$  and  $I_p = V_p / |Z_{IN}|$  and by substitution

$$V_p^2 = \frac{0.4 \times 100\sqrt{2}}{1.707} = 33.14$$

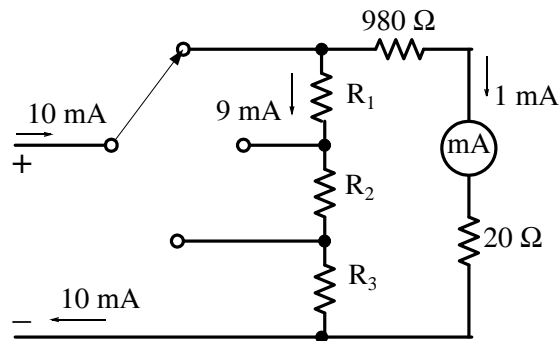
or

$$V_p = \sqrt{33.14} = 5.76$$

and

$$V_{RMS} = \frac{V_p}{\sqrt{2}} = \frac{5.76}{1.414} = 4.07 \text{ V}$$

8. With the switch at the 10 mA position, the circuit is as shown below.



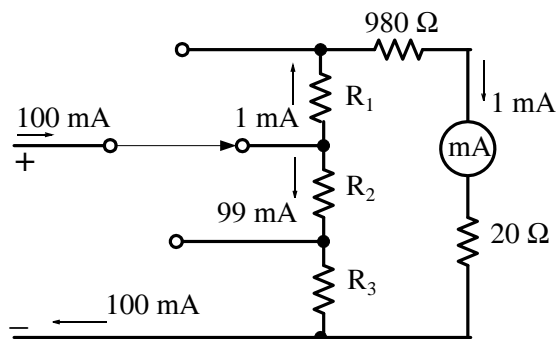
Then,

$$9 \times 10^{-3}(R_1 + R_2 + R_3) = (980 + 20) \times 10^{-3}$$

or

$$R_1 + R_2 + R_3 = \frac{1000}{9} \quad (1)$$

With the switch at the 100 mA position, the circuit is as shown below.





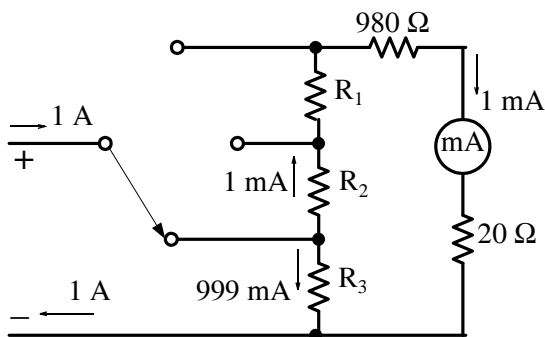
Then,

$$99 \times 10^{-3}(R_2 + R_3) = (R_1 + 980 + 20) \times 10^{-3}$$

or

$$-R_1 + 99R_2 + 99R_3 = 1000 \quad (2)$$

With the switch at the 1 A position, the circuit is as shown below.



Then,

$$999 \times 10^{-3}R_3 = (R_1 + R_2 + 980 + 20) \times 10^{-3}$$

or

$$-R_1 - R_2 + 999R_3 = 1000 \quad (3)$$

Addition of (1) and (3) yields

$$1000R_3 = \frac{1000}{9} + 1000 = \frac{10000}{9}$$

or

$$R_3 = \frac{10}{9} \Omega \quad (4)$$

Addition of (1) and (2) yields

$$100R_2 + 100R_3 = \frac{1000}{9} + 1000 = \frac{10000}{9}$$

or

$$R_2 + R_3 = \frac{100}{9} \quad (5)$$

Substitution of (4) into (5) yields

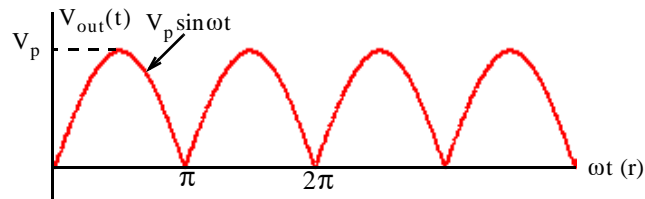
$$R_2 = 10 \Omega \quad (6)$$

and substitution of (4) and (6) into (1) yields

$$R_1 = 100 \Omega \quad (7)$$

9. DC instruments indicate average values. Therefore, the DC voltmeter will read the average value of the voltage  $v_{\text{OUT}}$  across the resistor. The period of the full-wave rectifier waveform is taken as  $\pi$ .

Then,



$$\begin{aligned}v_{\text{OUT ave}} &= \frac{1}{\pi} \int_0^{\pi} V_p \sin \omega t d(\omega t) = \frac{V_p}{\pi} (-\cos \omega t) \Big|_{\omega t = 0}^{\pi} \\ &= \frac{V_p}{\pi} \cos \omega t \Big|_{\pi}^0 = \frac{V_p}{\pi} (1 + 1) = \frac{2V_p}{\pi}\end{aligned}$$

As expected, this average is twice the average value of the half-wave rectifier waveform in Example 8.2.

This chapter discusses *the natural response* of electric circuits. The term natural implies that there is no excitation in the circuit, that is, the circuit is source-free, and we seek the circuit's natural response. The natural response is also referred to as the *transient response*.

### 9.1 Natural Response of a Series RL circuit

Let us find the natural response of the circuit of Figure 9.1 where the desired response is the current  $i$ , and it is given that at  $t = 0$ ,  $i = I_0$ , that is, the initial condition is  $i(0) = I_0$ .

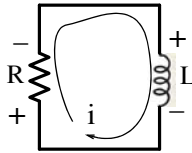


Figure 9.1. Circuit for determining the natural response of a series RL circuit

Application of KVL yields

$$v_L + v_R = 0$$

or

$$L \frac{di}{dt} + Ri = 0 \quad (9.1)$$

Here, we seek a value of  $i$  which satisfies the differential equation of (9.1), that is, we need to find the natural response which in differential equations terminology is the *complementary function*. As we know, two common methods are the separation of variables method and the assumed solution method. We will consider both.

#### 1. Separation of Variables Method

Rearranging (9.1), so that the variables  $i$  and  $t$  are separated, we obtain

$$\frac{di}{i} = -\frac{R}{L} dt$$

Next, integrating both sides and using the initial condition, we obtain

$$\int_{I_0}^i \frac{1}{i} di = -\frac{R}{L} \int_0^t d\sigma$$

---

## Chapter 9 Natural Response

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where  $\sigma$  is a dummy variable. Integration yields

$$\ln i \Big|_{I_0}^i = -\frac{R}{L} \sigma \Big|_0^t$$

or

$$\ln i - \ln I_0 = -\frac{R}{L} t$$

or

$$\ln \frac{i}{I_0} = -\frac{R}{L} t$$

Recalling that  $x = \ln y$  implies  $y = e^x$ , we obtain

$$\boxed{i(t) = I_0 e^{-(R/L)t}} \quad (9.2)$$

Substitution of (9.2) into (9.1) yields  $0 = 0$  and that at  $t = 0$ ,  $i(0) = I_0$ . Thus, both the differential equation and the initial condition are satisfied.

### 2. Assumed Solution Method

Relation (9.1) indicates that the solution must be a function which, when added to its first derivative will become zero. An exponential function will accomplish that and therefore, we assume a solution of the form

$$i(t) = Ae^{st} \quad (9.3)$$

where  $A$  and  $s$  are constants to be determined. Now, if (9.3) is a solution, it must satisfy the differential equation (9.1). Then, by substitution, we obtain:

$$RAe^{st} + sLAe^{st} = 0$$

or

$$\left(s + \frac{R}{L}\right) Ae^{st} = 0$$

The left side of the last expression above will be zero if  $A = 0$ , or if  $s = -\infty$ , or if  $s = -R/L$ . But, if  $A = 0$  or  $s = -\infty$ , then every response is zero and this represents a trivial solution. Therefore,  $s = -R/L$  is the only logical solution, and by substitution into (9.3) we obtain

$$i(t) = Ae^{-(R/L)t}$$

We must now evaluate the constant  $A$ . This is done with the use of the initial condition  $i(0) = I_0$ . Thus,  $I_0 = Ae^0$  or  $A = I_0$  and therefore,

$$i(t) = I_0 e^{-(R/L)t}$$

as before. Next, we rewrite it as

$$\frac{i(t)}{I_0} = e^{-(R/L)t} \quad (9.4)$$

and sketch it as shown in Figure 9.2.

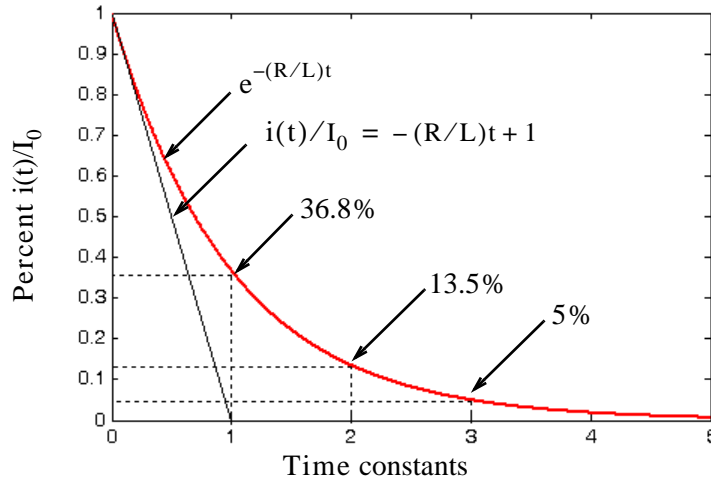


Figure 9.2. Plot for  $i(t)/I_0$  in a series RL circuit

From Figure 9.2 we observe that at  $t = 0$ ,  $i/I_0 = 1$ , and  $i \rightarrow 0$  as  $t \rightarrow \infty$ .

The initial rate (slope) of decay is found from the derivative of  $i/I_0$  evaluated at  $t = 0$ , that is,

$$\left. \frac{d}{dt} \left( \frac{i}{I_0} \right) \right|_{t=0} = \left. -\frac{R}{L} e^{-(R/L)t} \right|_{t=0} = -\frac{R}{L}$$

and thus the slope of the initial rate of decay is  $-R/L$

Next, we define the *time constant*  $\tau$  as the time required for  $i/I_0$  to drop from unity to zero assuming that the initial rate of decay remains constant. This constant rate of decay is represented by the straight line equation

$$\frac{i(t)}{I_0} = -\frac{R}{L}t + 1$$

and at  $t = \tau$ ,  $i/I_0 = 0$ . Then,

$$0 = -\frac{R}{L}\tau + 1$$

or

$\tau = \frac{L}{R}$
Time Constant for RL Circuit

(9.5)

## Chapter 9 Natural Response

Evaluating (9.4) at  $t = \tau = L/R$ , we obtain

$$\frac{i(\tau)}{I_0} = e^{-(R/L)\tau} = e^{-(R/L)(L/R)} = e^{-1} = 0.368$$

or

$$\boxed{i(\tau) = 0.368I_0} \quad (9.6)$$

Therefore, *in one time constant, the response has dropped to approximately 36.8% of its initial value.*

If we express the rate of decay in time constant intervals as shown in Figure 9.2, we find that  $i(t)/I_0 \approx 0$  after  $t = 5\tau$ , that is, it reaches its final value after five time constants.

### Example 9.1

For the circuit shown in Figure 9.3, in how many seconds after  $t = 0$  has the

- current  $i(t)$  has reached  $\frac{1}{2}$  of its initial value?
- energy stored in  $L$  has reached  $\frac{1}{4}$  of its initial value?
- power dissipated in  $R$  has reached  $\frac{3}{4}$  of its initial value?

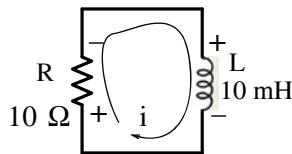


Figure 9.3. Circuit for Example 9.1

**Solution:**

From (9.2),

$$i(t) = I_0 e^{-(R/L)t}$$

where  $I_0 = i_L(0)$ . Then,

- The current  $i(t)$  will have reached  $\frac{1}{2}$  of its initial value when

$$0.5I_0 = I_0 e^{-(10/10 \times 10^{-3})t} = I_0 e^{-1000t}$$

or

$$e^{-1000t} = 0.5$$

or

$$-1000t = \ln(0.5) = -0.693$$

and therefore,

$$t = 693 \mu\text{s}$$

b. To find the energy stored in L which reaches  $\frac{1}{4}$  of its initial value, we begin with

$$W_L(t) = \frac{1}{2}Li^2(t)$$

and at  $t = 0$ ,  $I_0 = i_L(0)$ . Then,

$$W_L(0) = \frac{1}{2}LI_0^2$$

and

$$\frac{1}{4}W_L(0) = \frac{1}{4}\left(\frac{1}{2}LI_0^2\right)$$

Therefore,

$$\frac{1}{4}W_L(t) = \frac{1}{2}Li^2(t) = \frac{1}{2}L(I_0 e^{-(R/L)t})^2 = \frac{1}{4}\left(\frac{1}{2}LI_0^2\right)$$

or

$$e^{-2(R/L)t} = 1/4$$

$$e^{-2000t} = 1/4$$

$$-2000t = \ln(0.25) = -1.386$$

and

$$t = 693 \mu\text{s}$$

This is the same answer as in part (a) since the energy is proportional to the square of the current.

c. To find the power dissipated in R when it reaches  $\frac{3}{4}$  of its initial value, we start with the fact that the instantaneous power absorbed by the resistor is  $p_R = i_R^2 R$ , and since for the given circuit

$$i(t) = i_R(t) = I_0 e^{-(R/L)t}$$

then,

$$p_R = I_0^2 R e^{-2(R/L)t}$$

and the energy dissipated (in the form of heat) in the resistor is

$$W_R = \int_0^\infty p_R dt = I_0^2 R \int_0^\infty e^{-2(R/L)t} dt = I_0^2 R \left(-\frac{L}{2R}\right) e^{-2(R/L)t} \Bigg|_0^\infty = \frac{1}{2}LI_0^2$$

Also, from part (b) above,

$$W_L(0) = \frac{1}{2}LI_0^2$$

and thus

## Chapter 9 Natural Response

$$\frac{3}{4}W_R = \frac{3}{4}W_L(0) = \frac{1}{2}Li^2(t) = \frac{1}{2}L(I_0e^{-(R/L)t})^2 = \frac{3}{4}\left(\frac{1}{2}LI_0^2\right)$$

or

$$e^{-2(R/L)t} = 3/4$$

$$e^{-2000t} = 3/4$$

$$-2000t = \ln(0.75) = -0.288$$

and

$$t = 144 \mu\text{s}$$

In some examples and exercises that follow, the initial condition may not be given directly but it can be found from the fact that the current through an inductor cannot change instantaneously and therefore,

$$i_L(0^-) = i_L(0) = i_L(0^+) \quad (9.7)$$

where  $i_L(0^-)$  will be used to denote the time just before a switch is opened or closed, and  $i_L(0^+)$  will be used to denote the time just after the change has occurred.

Also, in our subsequent discussion, the expression “long time” will mean that sufficient time has elapsed so that the circuit has reached its steady-state conditions. As we know from Chapter 5, when the excitations are constant, at steady state conditions the inductor behaves as a short circuit, and the capacitor behaves as an open circuit.

### Example 9.2

In the circuit of Figure 9.4, the switch  $S$  has been in the closed position for a long time and opens at  $t = 0$ . Find  $i_L(t)$  for  $t > 0$ ,  $v_R(0^-)$ , and  $v_R(0^+)$

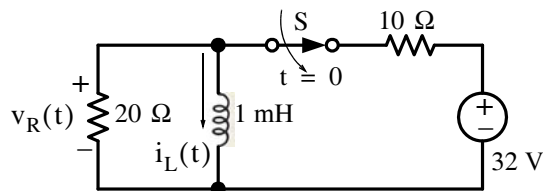


Figure 9.4. Circuit for Example 9.2

#### Solution:

We are not given an initial condition for this example; however, at  $t = 0^-$  the inductor acts as a short thereby shorting also the  $20 \Omega$  resistor. The circuit then is as shown in Figure 9.5.



Circuit at  $t = 0^-$

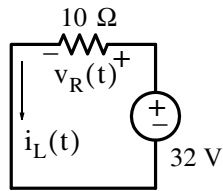


Figure 9.5. Circuit for Example 9.2 at  $t = 0^-$

From the circuit of Figure 9.5, we observe that

$$i_L(0^-) = i_L(0) = i_L(0^+) = 32/10 = 3.2 \text{ A}$$

and thus the initial condition has now been established as  $I_0 = 3.2 \text{ A}$ . We also observe that

$$v_R(0^-) = 0$$

At  $t = 0^+$ , the  $32 \text{ V}$  source and the  $10 \Omega$  resistor are disconnected from the circuit which now is as shown in Figure 9.6.

Circuit at  $t = 0^+$

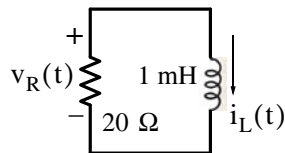


Figure 9.6. Circuit for Example 9.2 at  $t = 0^+$

For the circuit of Figure 9.6,

$$i_L(t) = I_0 e^{-(R/L)t} = 3.2 e^{-(20/10^{-3})t}$$

or

$$i_L(t) = 3.2 e^{-20000t}$$

and

$$v_R(0^+) = 20(-I_0) = 20(-3.2)$$

or

$$v_R(0^+) = -64 \text{ V}$$

We observe that  $v_R(0^+) \neq v_R(0^-)$

## Chapter 9 Natural Response

### Example 9.3

In the circuit shown in Figure 9.7, the switch  $S$  has been closed for a long time and opens at  $t = 0$ . Find:

- $i_L(t)$  for  $t > 0$
- $i_{60}(t)$  at  $t = 100 \mu\text{s}$
- $i_{48}(t)$  at  $t = 200 \mu\text{s}$

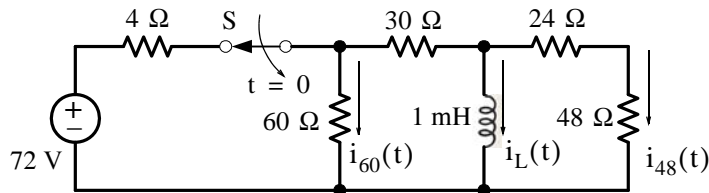


Figure 9.7. Circuit for Example 9.3

**Solution:**

- At  $t = 0^-$  the inductor acts as a short thereby shorting also the  $24 \Omega$  and  $48 \Omega$  resistors. The circuit then is as shown in Figure 9.8.

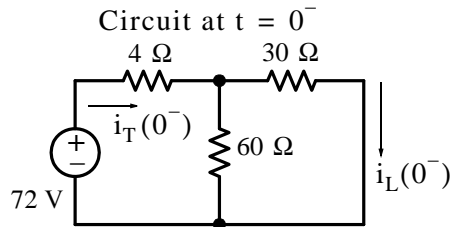


Figure 9.8. Circuit for Example 9.3 at  $t = 0^-$

Then,

$$i_T(0^-) = \frac{72 \text{ V}}{4 + 60 \parallel 30} = \frac{72 \text{ V}}{4 + 20} = 3 \text{ A}$$

and by the current division expression,

$$i_L(0^-) = \frac{60}{30 + 60} \cdot i_T(0^-) = \frac{6}{9} \times 3 = 2 \text{ A}$$

and thus the initial condition has been established as  $I_0 = 2 \text{ A}$

At  $t = 0^+$ , the  $72 \text{ V}$  source and the  $4 \Omega$  resistor are disconnected from the circuit which now is as shown in Figure 9.9.

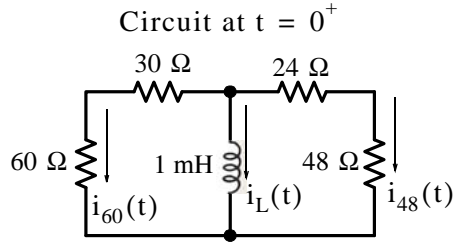


Figure 9.9. Circuit for Example 9.3 at  $t = 0^+$

From (9.2),

$$i_L(t) = I_0 e^{-(R_{eq}/L)t}$$

where

$$R_{eq} = (60 + 30) \parallel (24 + 48) = 40 \Omega$$

and thus

$$i_L(t) = 2e^{-(40/10^{-3})t}$$

or

$$i_L(t) = 2e^{-40000t}$$

Also,

$$i_{60}(t) \Big|_{t=100 \mu s} = \frac{(24 + 48)}{(30 + 60) + (24 + 48)} [-i_L(t)] \Big|_{t=100 \mu s}$$

or

$$i_{60}(t) \Big|_{t=100 \mu s} = \frac{12}{27} (-2e^{-40000t}) \Big|_{t=100 \mu s} = -\frac{8}{9} e^{-4} = -16.3 \text{ mA}$$

and

$$i_{48}(t) \Big|_{t=200 \mu s} = \frac{(30 + 60)}{(30 + 60) + (24 + 48)} [-i_L(t)] \Big|_{t=200 \mu s}$$

or

$$i_{48}(t) \Big|_{t=200 \mu s} = \frac{15}{27} (-2e^{-40000t}) \Big|_{t=200 \mu s} = -\frac{10}{9} e^{-8} = -0.373 \text{ mA}$$

## 9.2 Natural Response of a Series RC Circuit

In this section, we will find the natural response of the RC circuit shown in Figure 9.10 where the desired response is the capacitor voltage  $v_C$ , and it is given that at  $t = 0$ ,  $v_C = V_0$ , that is, the initial condition is  $v(0) = V_0$ .

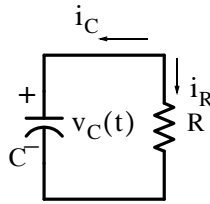


Figure 9.10. Circuit for determining the natural response of a series RC circuit

By KCL,

$$i_C + i_R = 0 \quad (9.8)$$

and with

$$i_C = C \frac{dv_C}{dt}$$

and

$$i_R = \frac{v_C}{R}$$

by substitution into (9.8), we obtain the differential equation

$$\frac{dv_C}{dt} + \frac{v_C}{RC} = 0 \quad (9.9)$$

As before, we assume a solution of the form

$$v_C(t) = Ae^{st}$$

and by substitution into (9.9)

$$Ase^{st} + \frac{Ae^{st}}{RC} = 0$$

or

$$\left(s + \frac{1}{RC}\right)Ae^{st} = 0 \quad (9.10)$$

Following the same reasoning as with the RL circuit, (9.10) will be satisfied when  $s = -1/RC$  and therefore,

$$v_C(t) = Ae^{-(1/RC)t}$$

The constant  $A$  is evaluated from the initial condition, i.e.,  $v_C(0) = V_0 = Ae^0$  or  $A = V_0$ . Therefore, the natural response of the RC circuit is

$$\boxed{v_C(t) = V_0 e^{-(1/RC)t}} \quad (9.11)$$

We express (9.11) as

$$\frac{v_C(t)}{V_0} = e^{-(1/RC)t}$$

and we sketch it as shown in Figure 9.11.

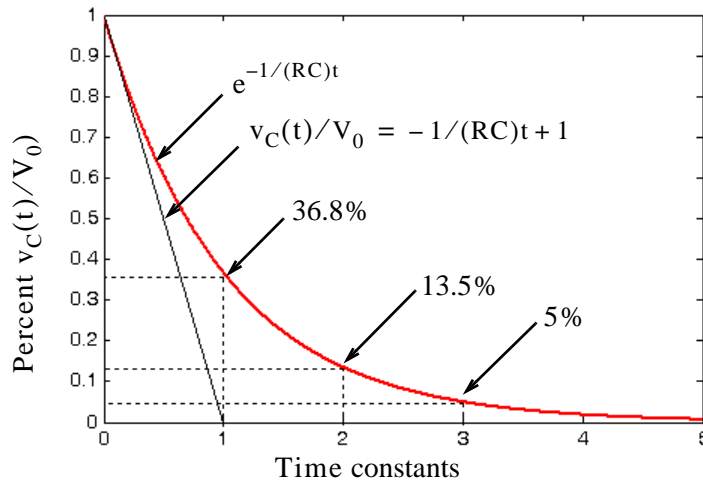


Figure 9.11. Circuit for determining the natural response of a series RC circuit

From Figure 9.11 we observe that at  $t = 0$ ,  $v_C/V_0 = 1$ , and  $i \rightarrow 0$  as  $t \rightarrow \infty$

The initial rate (slope) of decay is found from the derivative of  $v_C(t)/V_0$  evaluated at  $t = 0$ , that is,

$$\left. \frac{d}{dt} \left( \frac{v_C}{V_0} \right) \right|_{t=0} = \left. -\frac{1}{RC} e^{-(1/RC)t} \right|_{t=0} = -\frac{1}{RC}$$

and thus the slope of the initial rate of decay is  $-1/(RC)$

Next, we define the *time constant*  $\tau$  as the time required for  $v_C(t)/V_0$  to drop from unity to zero assuming that the initial rate of decay remains constant. This constant rate of decay is represented by the straight line equation

$$\frac{v_C(t)}{V_0} = -\frac{1}{RC}t + 1 \quad (9.12)$$

and at  $t = \tau$ ,  $v_C(t)/V_0 = 0$ . Then,

$$0 = -\frac{1}{RC}\tau + 1$$

or

$\tau = RC$
Time Constant for RC Circuit

(9.13)

Evaluating (9.11) at  $t = \tau = RC$ , we obtain

## Chapter 9 Natural Response

$$\frac{v_C(\tau)}{V_0} = e^{-\tau/RC} = e^{-RC/RC} = e^{-1} = 0.368$$

or

$$v_C(\tau) = 0.368V_0 \quad (9.14)$$

Therefore, in one time constant, the response has dropped to approximately 36.8% of its initial value.

If we express the rate of decay in time constant intervals as shown in Figure 9.11, we find that  $v_C(t)/V_0 \approx 0$  after  $t = 5\tau$ , that is, it reaches its final value after five time constants.

In the examples that follow, we will make use of the fact that

$$v_C(0^-) = v_C(0) = v_C(0^+) \quad (9.15)$$

### Example 9.4

In the circuit of Figure 9.12, the switch  $S$  has been in the closed position for a long time, and opens at  $t = 0$ . Find  $v_C(t)$  for  $t > 0$ ,  $i(0^-)$ , and  $i(0^+)$ .

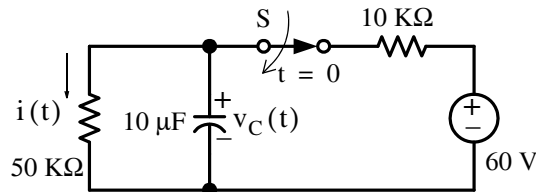


Figure 9.12. Circuit for Example 9.4

**Solution:**

At  $t = 0^-$  the capacitor acts as an open. The circuit then is as shown in Figure 9.13.

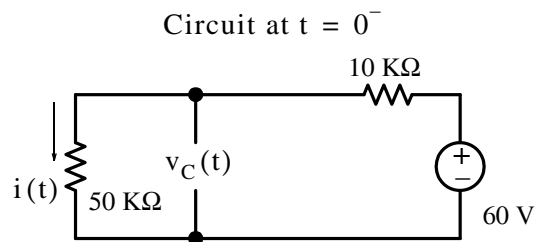


Figure 9.13. Circuit for Example 9.4 at  $t = 0^-$

From the circuit of Figure 9.13 we observe that

$$v_C(0^-) = v_C(0^+) = 50 \text{ K}\Omega \times i(0^-) = 50 \times \frac{60 \text{ V}}{10 \text{ K}\Omega + 50 \text{ K}\Omega} = 50 \text{ V}$$

and thus the initial condition has been established as  $V_0 = 50 \text{ V}$ . We also observe that

$$i(0^-) = \frac{60 \text{ V}}{10 \text{ K}\Omega + 50 \text{ K}\Omega} = 1 \text{ mA}$$

At  $t = 0^+$  the  $60 \text{ V}$  source and the  $10 \text{ K}\Omega$  resistor are disconnected from the circuit which now is as shown in Figure 9.14.

From (9.11),

$$v_C(t) = V_0 e^{-(1/RC)t}$$

Circuit at  $t = 0^+$

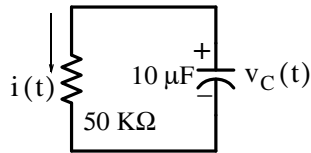


Figure 9.14. Circuit for Example 9.4 at  $t = 0^+$

where

$$RC = 50 \times 10^3 \times 10 \times 10^{-6} = 0.5$$

Then,

$$v_C(t) = 50 e^{-(1/0.5)t} = 50 e^{-2t}$$

and

$$i(0^+) = \frac{V_0}{R} = \frac{50 \text{ V}}{50 \text{ K}\Omega} = 1 \text{ mA}$$

We observe that  $i(0^+) = i(0^-)$ . This is true because the voltage across the capacitor cannot change instantaneously; hence, the voltage across the resistor must be the same at  $t = 0^-$  and at  $t = 0^+$ .

### Example 9.5

In the circuit of Figure 9.15, the switch  $S$  has been in the closed position for a long time and opens at  $t = 0$ . Find:

- a.  $v_C(t)$  for  $t > 0$
- b.  $v_{60}(t)$  at  $t = 100 \mu\text{s}$

## Chapter 9 Natural Response

c.  $v_{10}(t)$  at  $t = 200 \mu\text{s}$

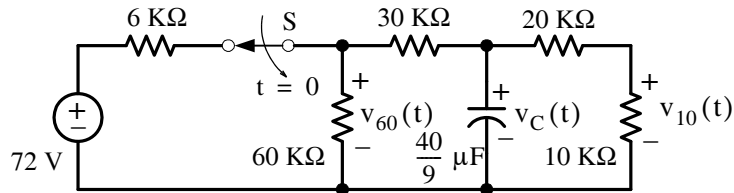


Figure 9.15. Circuit for Example 9.5

**Solution:**

a. At  $t = 0^-$  the capacitor acts as an open and the circuit then is as shown in Figure 9.16.

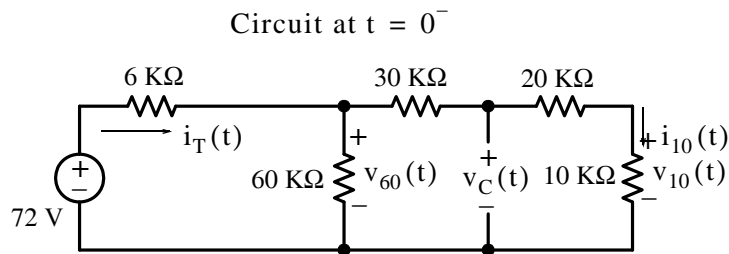


Figure 9.16. Circuit for Example 9.5 at  $t = 0^-$

From the circuit of Figure 9.16,

$$i_T(0^-) = \frac{72 \text{ V}}{6 \text{ k}\Omega + 60 \text{ k}\Omega \parallel 60 \text{ k}\Omega} = \frac{72 \text{ V}}{6 \text{ k}\Omega + 30 \text{ k}\Omega} = 2 \text{ mA}$$

and using the current division expression, we obtain

$$i_{10}(0^-) = \frac{60 \text{ k}\Omega}{60 \text{ k}\Omega + 60 \text{ k}\Omega} \cdot i_T(0^-) = \frac{1}{2} \times 2 = 1 \text{ mA}$$

Then,

$$v_C(0^-) = (20 \text{ k}\Omega + 10 \text{ k}\Omega) \cdot i_{10}(0^-) = 30 \text{ V}$$

and thus the initial condition has been established as  $V_0 = 30 \text{ V}$ .

At  $t = 0^+$ , the 72 V source and the 6 kΩ resistor are disconnected from the circuit which now is as shown in Figure 9.17.



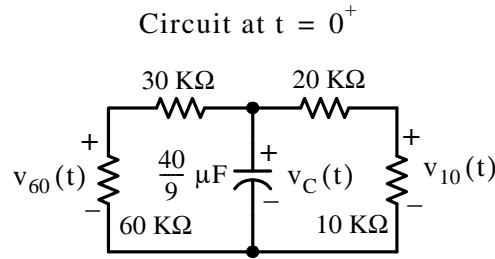


Figure 9.17. Circuit for Example 9.5 at  $t = 0^+$

From (9.11),

$$v_C(t) = V_0 e^{-(1/R_{eq}C)t}$$

where

$$R_{eq} = (60 \text{ K}\Omega + 30 \text{ K}\Omega) // (20 \text{ K}\Omega + 10 \text{ K}\Omega) = 22.5 \text{ K}\Omega$$

Then,

$$R_{eq}C = 22.5 \times 10^3 \times \frac{40}{9} \times 10^{-6} = 0.1$$

and

$$v_C(t) = 30e^{-(1/0.1)t} = 30e^{-10t}$$

b.

$$v_{60}(t) \Big|_{t=100 \text{ ms}} = \frac{60 \text{ K}\Omega}{30 \text{ K}\Omega + 60 \text{ K}\Omega} \cdot v_C(t) \Big|_{t=100 \text{ ms}}$$

or

$$v_{60}(t) \Big|_{t=100 \text{ ms}} = \frac{2}{3}(30e^{-10t}) \Big|_{t=100 \text{ ms}} = 20e^{-1} = 7.36 \text{ V}$$

c.

$$v_{10}(t) \Big|_{t=200 \text{ ms}} = \frac{10 \text{ K}\Omega}{10 \text{ K}\Omega + 20 \text{ K}\Omega} \cdot v_C(t) \Big|_{t=200 \text{ ms}}$$

or

$$v_{10}(t) \Big|_{t=200 \text{ ms}} = \frac{1}{3}(30e^{-10t}) \Big|_{t=200 \text{ ms}} = 10e^{-2} = 1.35 \text{ V}$$

### Example 9.6

For the circuit of Figure 9.18, it is known that  $v_C(0) = V_0 = 25 \text{ V}$ .

- To what value should the resistor  $R$  be adjusted so that the initial rate of change would be  $-200 \text{ V/s}$ ?
- What would then the energy in the capacitor be after two time constants?

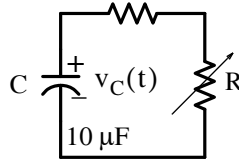


Figure 9.18. Circuit for Example 9.6

**Solution:**

a. The capacitor voltage decays exponentially as

$$v_C(t) = V_0 e^{-(1/RC)t}$$

and with the given values,

$$v_C(t) = 25 e^{-(100000/R)t}$$

Now, if the initial rate (slope) is to be  $-200$  V/s, then

$$\left. \frac{dv_C}{dt} \right|_{t=0} = \left( -\frac{100000}{R} \right) 25 e^{-(100000/R)t} \Big|_{t=0} = -\frac{2.5 \times 10^6}{R} = -200$$

and solving for R we obtain  $R = 12.5 \text{ K}\Omega$

b. After two time constants the capacitor voltage will drop to the value of

$$v_C(2\tau) = 25 e^{-(1/RC)2\tau} = 25 e^{-(2RC/RC)} = 25 e^{-2} = 3.38 \text{ V}$$

Therefore, the energy after two time constants will be

$$W_C|_{t=2\tau} = \frac{1}{2} C v_C^2 = 5 \times 10^{-6} \times 3.38^2 = 57.2 \text{ }\mu\text{J}$$

### 9.3 Summary

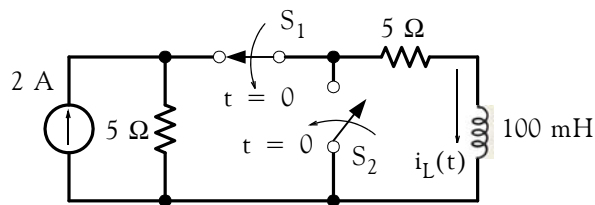
- The natural response of the inductor current  $i_L(t)$  in a simple RL circuit has the form  $i_L(t) = I_0 e^{-(R/L)t}$  where  $I_0$  denotes the value of the current in the inductor at  $t = 0$
- In a simple RL circuit the time constant  $\tau$  is the time required for  $i_L(t)/I_0$  to drop from unity to zero assuming that the initial rate of decay remains constant, and its value is  $\tau = L/R$
- In one time constant the natural response of the inductor current in a simple RL circuit has dropped to approximately 36.8% of its initial value.
- The natural response of the inductor current in a simple RL circuit reaches its final value, that is, it decays to zero, after approximately 5 time constants.
- The initial condition  $I_0$  can be established from the fact that the current through an inductor cannot change instantaneously and thus  $i_L(0^-) = i_L(0) = i_L(0^+)$
- The natural response of the capacitor voltage  $v_C(t)$  in a simple RC circuit has the form  $v_C(t) = V_0 e^{-(1/RC)t}$  where  $V_0$  denotes the value of the voltage across the capacitor at  $t = 0$
- In a simple RC circuit the time constant  $\tau$  is the time required for  $v_C(t)/V_0$  to drop from unity to zero assuming that the initial rate of decay remains constant, and its value is  $\tau = RC$
- In one time constant the natural response of the capacitor voltage in a simple RC circuit has dropped to approximately 36.8% of its initial value.
- The natural response of capacitor voltage in a simple RC circuit reaches its final value, that is, it decays to zero after approximately 5 time constants.
- The initial condition  $V_0$  can be established from the fact that the voltage across a capacitor cannot change instantaneously and thus  $v_C(0^-) = v_C(0) = v_C(0^+)$

## Chapter 9 Natural Response

### 9.4 Exercises

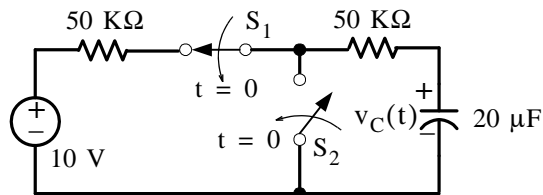
#### Multiple Choice

- In a simple RL circuit the unit of the time constant  $\tau$  is
  - dimensionless
  - the millisecond
  - the microsecond
  - the reciprocal of second, i.e.,  $s^{-1}$
  - none of the above
- In a simple RC circuit the unit of the term  $1/RC$  is
  - the second
  - the reciprocal of second, i.e.,  $s^{-1}$
  - the millisecond
  - the microsecond
  - none of the above
- In the circuit below switch  $S_1$  has been closed for a long time while switch  $S_2$  has been open for a long time. At  $t = 0$ , switch  $S_1$  opens and switch  $S_2$  closes. The current  $i_L(t)$  for all  $t > 0$  is
  - 2 A
  - $2e^{-100t}$  A
  - $2e^{-50t}$  A
  - $e^{-50t}$  A
  - none of the above



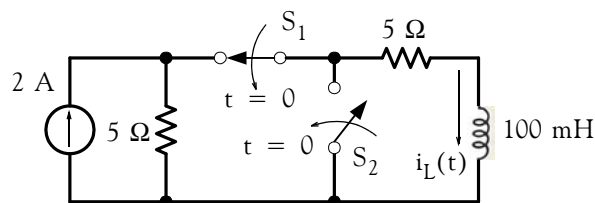
4. In the circuit below switch  $S_1$  has been closed for a long time while switch  $S_2$  has been open for a long time. At  $t = 0$ , switch  $S_1$  opens and switch  $S_2$  closes. The voltage  $v_C(t)$  for all  $t > 0$  is

- A. 10 V
- B.  $10e^{-10t}$  V
- C.  $10e^{-t}$  V
- D.  $10e^{-0.1t}$  V
- E. none of the above



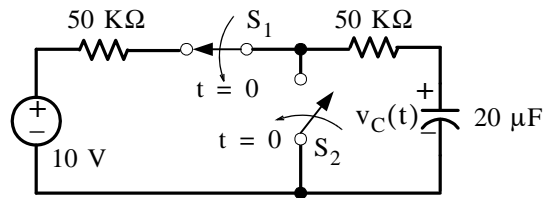
5. In the circuit below switch  $S_1$  has been closed for a long time while switch  $S_2$  has been open for a long time. At  $t = 0$ , switch  $S_1$  opens and switch  $S_2$  closes. The power absorbed by the inductor at  $t = +\infty$  will be

- A. 0 w
- B. 1 w
- C. 2 w
- D. 0.2 w
- E. none of the above



## Chapter 9 Natural Response

6. In the circuit below, switch  $S_1$  has been closed for a long time while switch  $S_2$  has been open for a long time. At  $t = 0$ , switch  $S_1$  opens and switch  $S_2$  closes. The power absorbed by the capacitor at  $t = +\infty$  will be
- A. 0 w
  - B. 10 w
  - C. 5 w
  - D. 10 mw
  - E. none of the above



7. In a simple RL circuit where  $R = 10 \text{ M}\Omega$  and  $L = 10 \text{ }\mu\text{H}$  the time constant  $\tau$  is
- A. 1 s
  - B. 100 s
  - C.  $10^{12}$  s
  - D.  $10^{-12}$  s
  - E. none of the above
8. In a simple RC circuit where  $R = 10 \text{ M}\Omega$  and  $C = 10 \text{ }\mu\text{F}$  the time constant  $\tau$  is
- A. 100 s
  - B. 0.01 s
  - C. 100 μs
  - D. 0.01 μs
  - E. none of the above

9. In a simple RL circuit the condition(s) \_\_\_ are always true.

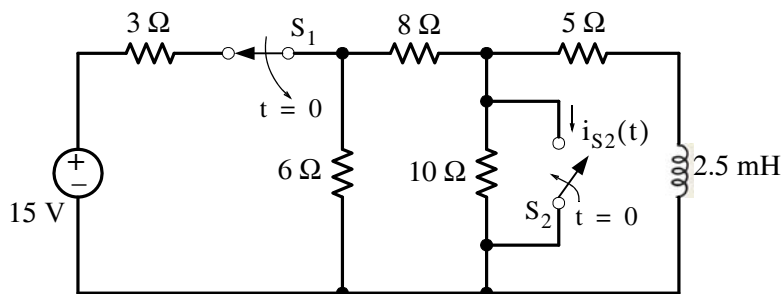
- A.  $i_L(0^-) = i_L(0) = i_L(0^+)$  and  $v_L(0^-) = v_L(0) = v_L(0^+)$
- B.  $i_L(0^-) = i_L(0) = i_L(0^+)$  and  $i_R(0^-) = i_R(0) = i_R(0^+)$
- C.  $i_L(0^-) = i_L(0) = i_L(0^+)$  and  $v_R(0^-) = v_R(0) = v_R(0^+)$
- D.  $i_L(0^-) = i_L(0) = i_L(0^+)$
- E. none of the above.

10. In a simple RC circuit the condition(s) \_\_\_ are always true.

- A.  $v_C(0^-) = v_C(0) = v_C(0^+)$  and  $i_C(0^-) = i_C(0) = i_C(0^+)$
- B.  $v_C(0^-) = v_C(0) = v_C(0^+)$  and  $v_R(0^-) = v_R(0) = v_R(0^+)$
- C.  $v_C(0^-) = v_C(0) = v_C(0^+)$
- D.  $v_C(0^-) = v_C(0) = v_C(0^+)$  and  $i_R(0^-) = i_R(0) = i_R(0^+)$
- E. none of the above.

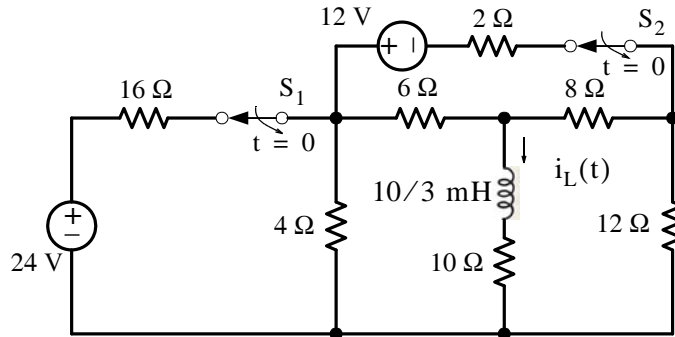
### Problems

1. In the circuit below, switch  $S_1$  has been closed for a long time and switch  $S_2$  has been open for a long time. Then, at  $t = 0$  switch  $S_1$  opens while  $S_2$  closes. Compute the current  $i_{S_2}(t)$  through switch  $S_2$  for  $t > 0$ .

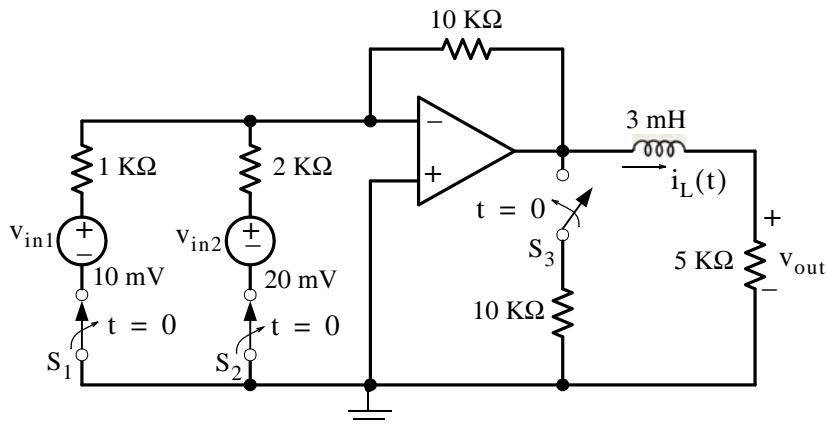


2. In the circuit below, both switches  $S_1$  and  $S_2$  have been closed for a long time and both are opened at  $t = 0$ . Compute and sketch the current  $i_L(t)$  for the time interval  $0 \leq t \leq 1$  ms

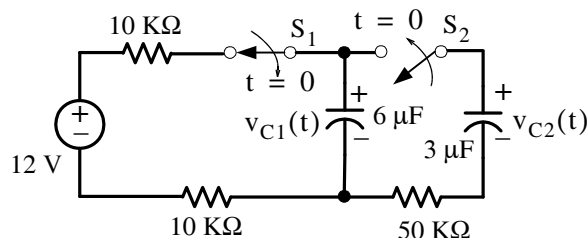
## Chapter 9 Natural Response



3. In a series RL circuit, the voltage  $v_L$  across the inductor is  $v_L = 0.2e^{-2000t}$  V and the current  $i_L$  at  $t = 0$  is  $i_L(0) = 10$  mA. Compute the values of R and L for that circuit.
4. In the circuit below both switches  $S_1$  and  $S_2$  have been closed for a long time, while switch  $S_3$  has been open for a long time. At  $t = 0$   $S_1$  and  $S_2$  are opened and  $S_3$  is closed. Compute the current  $i_L(t)$  for  $t > 0$ .

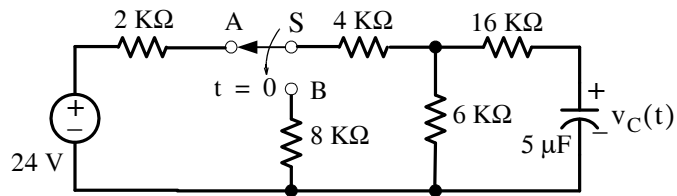


5. In the circuit below switch  $S_1$  has been closed and  $S_2$  has been open for a long time. At  $t = 0$  switch  $S_1$  is opened and  $S_2$  is closed. Compute the voltage  $v_{C2}(t)$  for  $t > 0$ .

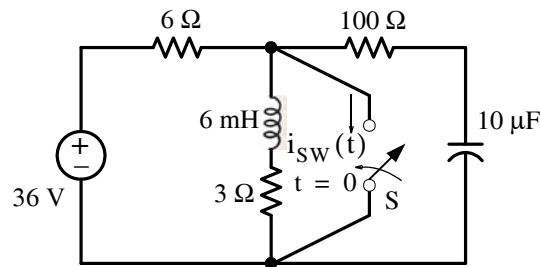




6. In the circuit below switch  $S$  has been in the A position for a long time and at  $t = 0$  is thrown in the B position. Compute the voltage  $v_C(t)$  across the capacitor for  $t > 0$ , and the energy stored in the capacitor at  $t = 1$  ms.

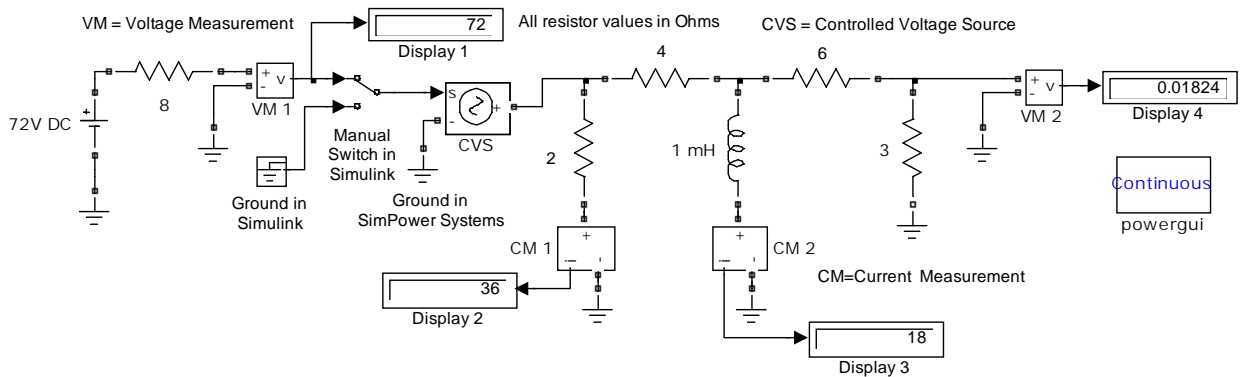


7. In the circuit below switch  $S$  has been open for a long time and closes at  $t = 0$ . Compute  $i_{sw}(t)$  for  $t > 0$ .

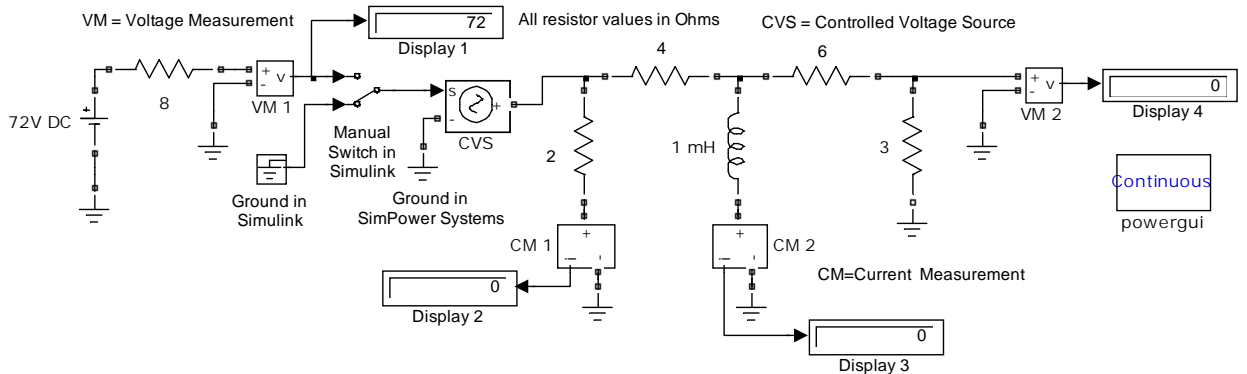


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8. a. In the Simulink / SimPowerSystems model shown below, are the values shown in the **Display** blocks justified after the simulation command is issued?



- b. When the **Manual Switch** block is double-clicked the model is as shown below. Are the values shown in the **Display** blocks justified after the simulation command is issued?



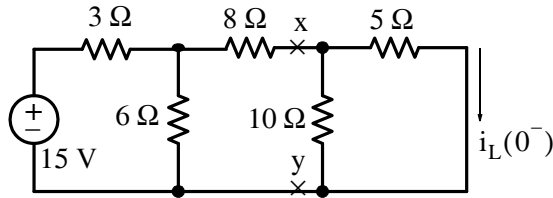
## 9.5 Answers / Solutions to End-of-Chapter Exercises

### Multiple Choice

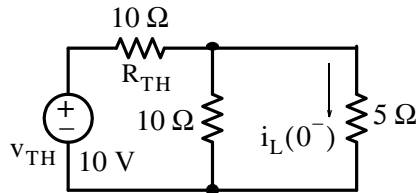
1. E  $\tau = L/R = [\text{volt}/(\text{ampere}/\text{second})]/[\text{volt}/\text{ampere}] = \text{second (s)}$
2. B
3. D
4. C
5. A
6. A
7. D
8. B
9. D
10. C

### Problems

1. The circuit at  $t = 0^-$  is as shown below.

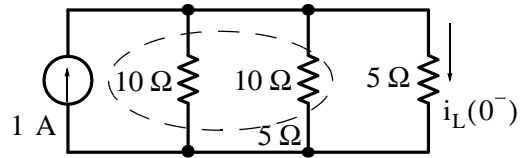


Replacing the circuit above with its Thevenin equivalent to the left of points  $x$  and  $y$  we find that  $v_{TH} = \frac{6}{3+6} \cdot 15 = 10 \text{ V}$  and  $R_{TH} = \frac{3 \times 6}{3+6} + 8 = 10 \Omega$  and attaching the rest of the circuit to it we obtain the circuit below.



By voltage–source to current–source transformation we obtain the circuit below.

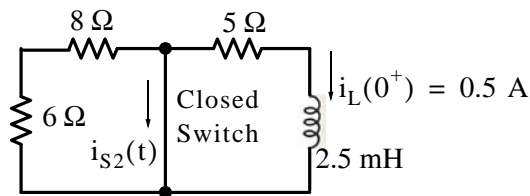
## Chapter 9 Natural Response



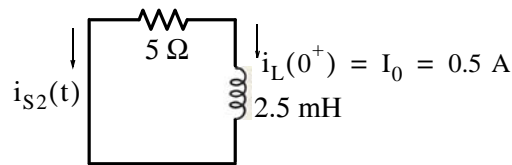
and by inspection,  $i_L(0^-) = 0.5 \text{ A}$ , that is, the initial condition has been established as

$$i_L(0^-) = i_L(0) = i_L(0^+) = I_0 = 0.5 \text{ A}$$

The circuit at  $t = 0^+$  is as shown below.



We observe that the closed shorts out the  $6 \Omega$  and  $8 \Omega$  resistors and the circuit simplifies to that shown below.

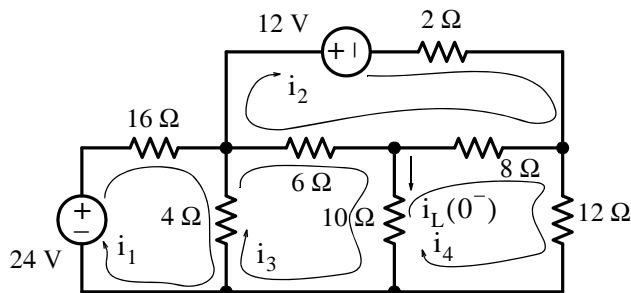


Thus for  $t > 0$ ,  $i_{S2}(t) = -i_L(t) = -I_0 e^{-(R/L)t} = -0.5 e^{-(5/2.5 \times 10^{-3})t} = -0.5 e^{-2000t} \text{ A}$

2. The circuit at  $t = 0^-$  is as shown below and the mesh equations are

$$\begin{aligned} 20i_1 - 4i_3 &= 24 \\ 16i_2 - 6i_3 - 8i_4 &= -12 \\ -4i_1 - 6i_2 + 20i_3 - 10i_4 &= 0 \\ 8i_2 - 10i_3 + 30i_4 &= 0 \end{aligned}$$

$$\text{Then, } i_L(0^-) = i_3 - i_4$$



and with MATLAB

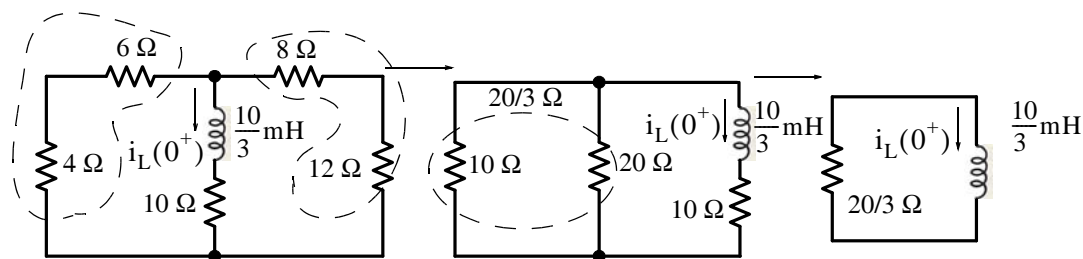
```
R=[20 0 -4 0; 0 16 -6 -8; -4 -6 20 -10; 0 -8 -10 30];...
V=[24 -12 0 0]'; I=R\V; iL0=I(3)-I(4); fprintf('\n');...
fprintf('i1 = %4.2f A \t', I(1)); fprintf('i2 = %4.2f A \t', I(2));...
fprintf('i3 = %4.2f A \t', I(3)); fprintf('i4 = %4.2f A \t', I(4));...
fprintf('iL0 = %4.2f A \t', I(3)-I(4)); fprintf('\n'); fprintf('\n');
```

$i_1 = 1.15 \text{ A}$   $i_2 = -1.03 \text{ A}$   $i_3 = -0.26 \text{ A}$   $i_4 = -0.36 \text{ A}$   $i_{L0} = 0.10 \text{ A}$

Therefore,

$$i_L(0^-) = i_L(0) = i_L(0^+) = I_0 = 0.1 \text{ A}$$

Shown below is the circuit at  $t = 0^+$  and the steps of simplification.



Thus for  $t > 0$ ,

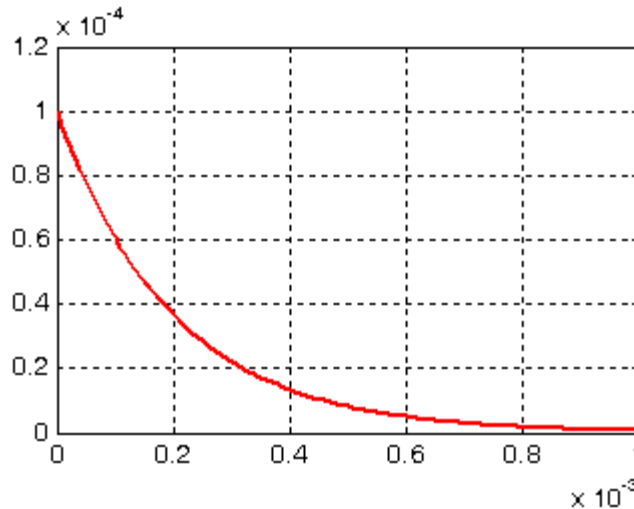
$$i_L(t) = I_0 e^{-(R/L)t} = 0.1 e^{-5000t} \text{ A}$$

and

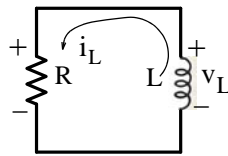
$$i_L|_{t=0.4 \text{ ms}} = 0.1 e^{-2} = 0.0137 \text{ A} = 13.7 \text{ mA}$$

To compute and sketch the current  $i_L(t)$  for the time interval  $0 \leq t \leq 1 \text{ ms}$  we use MATLAB as shown below.

```
t=(0:0.01:1)*10^(-3);...
iLt=0.1.*10.^(-3).*exp(-5000.*t);...
plot(t,iLt); grid
```



3. From the figure below  $v_L = Ri_L = 0.2e^{-2000t}$  for  $t > 0$



and with  $i_L(0) = 10 \text{ mA}$ , by substitution  $R(10 \times 10^{-3}) = 0.2e^{-0} = 0.2$

or  $R = 0.2/10^{-2} = 20 \text{ } \Omega$  Also, from  $R/L = 2000$ ,  $L = 20/2000 = 0.01 = 10 \text{ mH}$

4. The circuit at  $t = 0^-$  is as shown below and using the relation

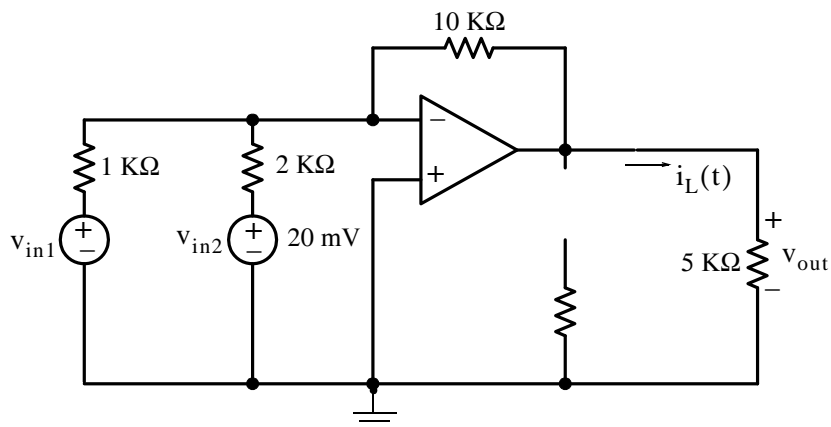
$$v_{\text{out}} = -R_f \left( \frac{v_{\text{in1}}}{R_{\text{in1}}} + \frac{v_{\text{in2}}}{R_{\text{in2}}} \right)$$

that was developed in Example 4.11 we have

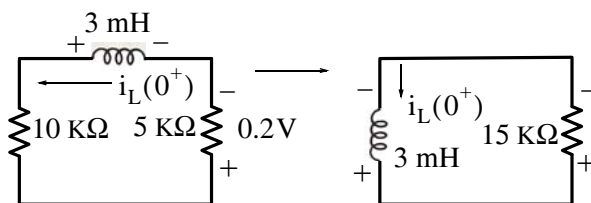
$$v_{\text{out}} = -10 \text{ K}\Omega \left( \frac{10^{-2}}{1 \text{ K}\Omega} + \frac{2 \times 10^{-2}}{2 \text{ K}\Omega} \right) = -10 \times 2 \times 10^{-2} = -0.2 \text{ V}$$

and

$$i_L(0^-) = I_0 = i_{5 \text{ K}\Omega} = \frac{-0.2 \text{ V}}{5 \text{ K}\Omega} = -40 \times 10^{-6} \text{ A} = -40 \text{ } \mu\text{A}$$



The circuit at  $t = 0^+$  is as shown below where  $i_L(0^+) = 40 \mu\text{A}$  with the direction shown.

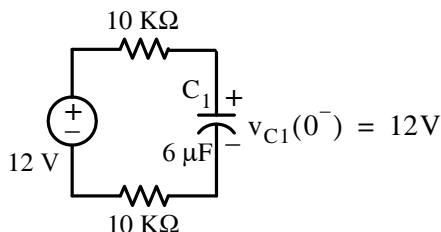


Then for  $t > 0$

$$i_L(t) = I_0 e^{(R/L)t} = 40 \times 10^{-6} e^{(15 \times 10^3 / 3 \times 10^{-3})t} = 40 e^{-5 \times 10^6 t} \mu\text{A}$$

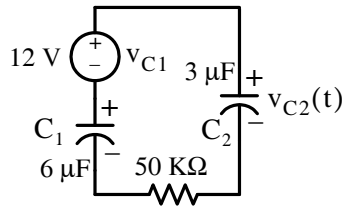
with the direction shown.

5. The circuit at  $t = 0^-$  is as shown below. As we've learned in Chapter 5, when a circuit is excited by a constant (DC) source, after sufficient time has elapsed the capacitor behaves as an open and thus the voltage across the capacitor  $C_1$  is 12 V as shown.



The circuit at  $t = 0^+$  is as shown below where the 12 V represents the voltage across capacitor  $C_1$ .

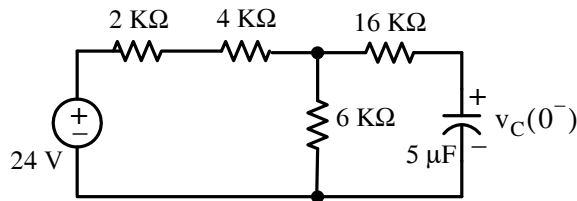
## Chapter 9 Natural Response



Now,  $v_{C2}(t) = v_{C1}e^{-(1/RC_{eq})}$  where  $v_{C1} = 12 \text{ V}$  and  $C_{eq} = \frac{C_1 \cdot C_2}{C_1 + C_2} = \frac{6 \times 3}{6 + 3} = 2 \mu\text{F}$

Then,  $1/RC_{eq} = 1/(5 \times 10^4 \times 2 \times 10^{-6}) = 10$  and thus  $v_{C2}(t) = 12e^{-10t}$

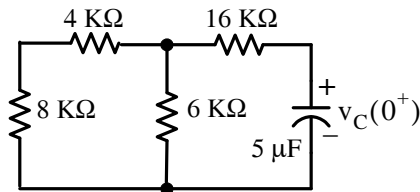
6. The circuit at  $t = 0^-$  is as shown below.



Because the capacitor behaves as an open, there is no current in the  $16 \text{ k}\Omega$ . Then,

$$v_C(0^-) = V_0 = v_{6\text{k}\Omega} = \frac{6 \text{ k}\Omega}{6 \text{ k}\Omega + 6 \text{ k}\Omega} \times 24 = 12 \text{ V}$$

The circuit at  $t = 0^+$  is as shown below where  $v_C(0^+) = 12 \text{ V}$ .

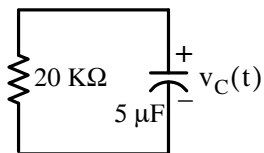


Series and parallel resistances reduction yields

$$R_{eq} = [(8 \text{ k}\Omega + 4 \text{ k}\Omega) \parallel 6 \text{ k}\Omega] + 16 \text{ k}\Omega = \frac{12 \times 6}{12 + 6} + 16 = 4 + 16 = 20 \text{ k}\Omega$$

and the circuit for  $t > 0$  reduces to the one shown below.

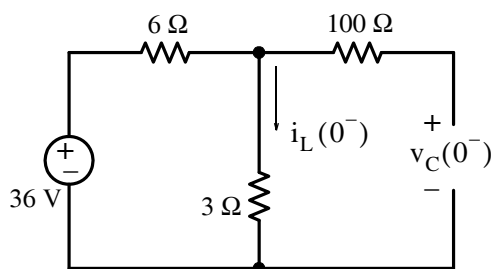




Now,  $R_{eq}C = 2 \times 10^4 \times 5 \times 10^{-6} = 0.1$ ,  $1/R_{eq}C = 10$  and  $v_C(t) = V_0 e^{-10t} = 12e^{-10t}$ . Also,

$$\begin{aligned} W_C|_{1 \text{ ms}} &= \frac{1}{2} C v_C^2(t) \Big|_{1 \text{ ms}} = 0.5 \times 5 \times 10^{-6} \times 144e^{-20t} \Big|_{t=1 \text{ ms}} \\ &= 360 \times 10^{-6} e^{-20t} \Big|_{t=1 \text{ ms}} = 0.35 \text{ mJ} \end{aligned}$$

7. The circuit at  $t = 0^-$  is as shown below.



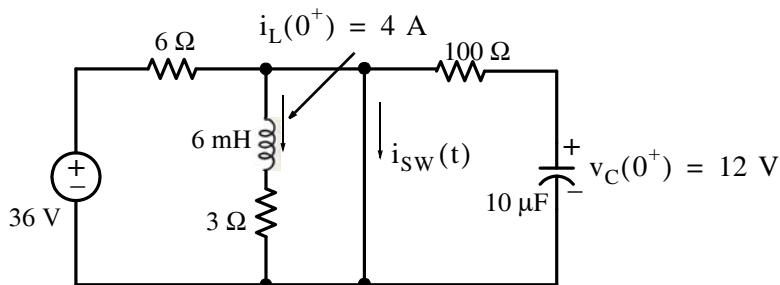
Then,

$$i_L(0^-) = \frac{36 \text{ V}}{(6 + 3) \Omega} = 4 \text{ A}$$

and

$$v_C(0^-) = 3 \times i_L(0^-) = 3 \times 4 = 12 \text{ V}$$

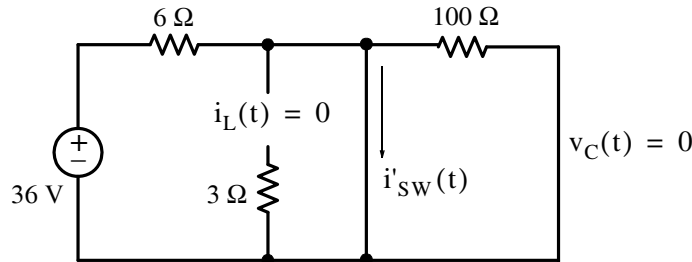
The circuit at  $t = 0^+$  is as shown below and the current  $i_{sw}(t)$  through the switch is the sum of the currents due to the 36 V voltage source, due to  $i_L(0^+) = 4 \text{ A}$ , and due to  $v_C(0^+) = 12 \text{ V}$ .



## Chapter 9 Natural Response

We will apply superposition three times. Thus for  $t > 0$ :

- I. With the 36 V voltage source acting alone where  $i_L = 0$  (open) and  $v_C = 0$  (shorted), the circuit is as shown below.

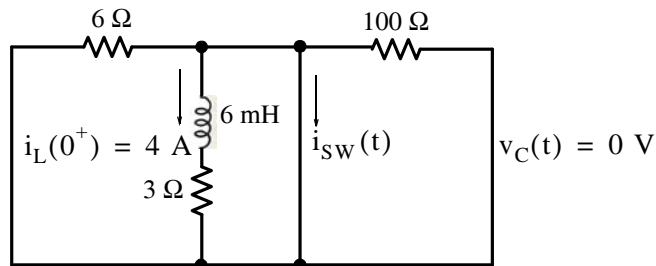


Since the 100 Ω is shorted out, we have

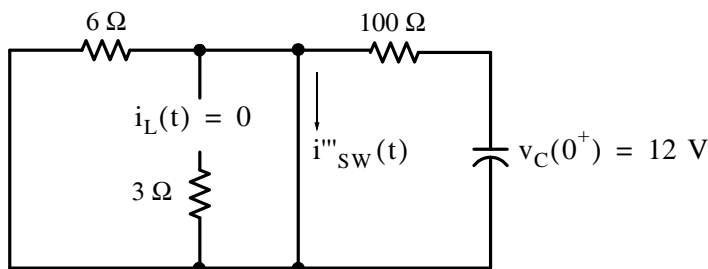
$$i'_{sw}(t) = 36/6 = 6 \text{ A}$$

- II. With the  $i_L(0^+) = 4 \text{ A}$  current source acting alone the circuit is as shown below where we observe that the 6 Ω and 100 Ω resistors are shorted out and thus  $i''_{sw}(t) = -i_L(t)$  where  $i_L(t) = I_0 e^{-(R/L)t}$ ,  $I_0 = 4 \text{ A}$ ,  $R = 3 \text{ Ω}$ ,  $L = 6 \text{ mH}$ ,  $R/L = 3/(6 \times 10^{-3}) = 500$  and thus

$$i''_{sw}(t) = -i_L(t) = -4e^{-500t}$$



- III. With the  $v_C(0^+) = V_0 = 12 \text{ V}$  voltage source acting alone the circuit is as shown below where we observe that the 6 Ω resistor is shorted out.



and thus

$$v_C(t) = V_0 e^{-(1/RC)t} = 12e^{-[1/(100 \times 10^{-5})]t} = 12e^{-1000t}.$$

Then,

$$i'''_{sw}(t) = v_C(t)/100 \Omega = 0.12e^{-1000t}$$

Therefore, the total current through the closed switch for  $t > 0$  is

$$i_{sw}(t) = i'_{sw}(t) + i''_{sw}(t) + i'''_{sw}(t) = 6 - 4e^{-500t} + 0.12e^{-1000t} \text{ A}$$

- 8.
- With the **Manual Switch** block in the upper position, all resistors are in parallel with the 72 V voltage source and thus the voltages across the 8, 2, and 4 Ohm resistors are 72 volts. Thus, the current through the 2 Ohm resistor is  $72/2 = 36$  amps, and the current through the 4 Ohm resistor is  $72/4 = 18$  amps. It is observed that immediately after the simulation command is issued, the current through the inductor resists any change, and finally stabilizes at 18 amps. The 6 and 3 Ohm resistors are shorted by the inductor.
  - With the **Manual Switch** block in the lower position, all resistors and the inductor to the right of the switch are grounded and thus all readings are zero. The 8 Ohm resistor is still in parallel with the 72 V voltage source and thus the voltages across it is 72 volts.

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# Chapter 10

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## Forced and Total Response in RL and RC Circuits

This chapter discusses the forced response of electric circuits. The term “forced” here implies that the circuit is excited by a voltage or current source, and its response to that excitation is analyzed. Then, the forced response is added to the natural response to form the total response.

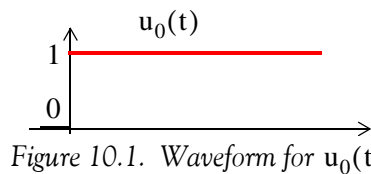
### 10.1 Unit Step Function $u_0(t)$

A function is said to be *discontinuous* if it exhibits points of discontinuity, that is, if the function jumps from one value to another without taking on any intermediate values.

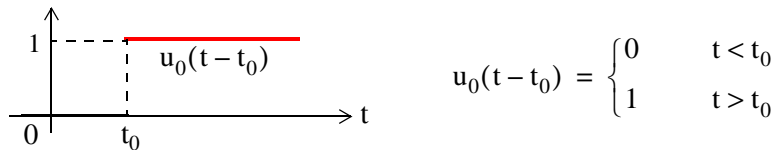
A well-known discontinuous function is the *unit step function*  $u_0(t)$ \* which is defined as

$$u_0(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases} \quad (10.1)$$

It is also represented by the waveform in Figure 10.1.



In the waveform of Figure 10.1, the unit step function  $u_0(t)$  changes abruptly from 0 to 1 at  $t = 0$ . But if it changes at  $t = t_0$  instead, its waveform and definition are as shown in Figure 10.2.



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\* In some books, the unit step function is denoted as  $u(t)$ , that is, without the subscript 0. In this text we will reserve this designation for any input.

## Chapter 10 Forced and Total Response in RL and RC Circuits

Likewise, if the unit step function changes from 0 to 1 at  $t = -t_0$  as shown in Figure 10.3, it is denoted as  $u_0(t + t_0)$

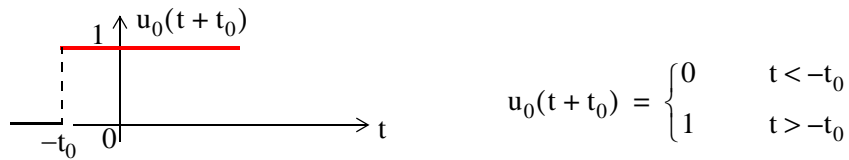


Figure 10.3. Waveform and definition of  $u_0(t + t_0)$

Other forms of the unit step function are shown in Figure 10.4.

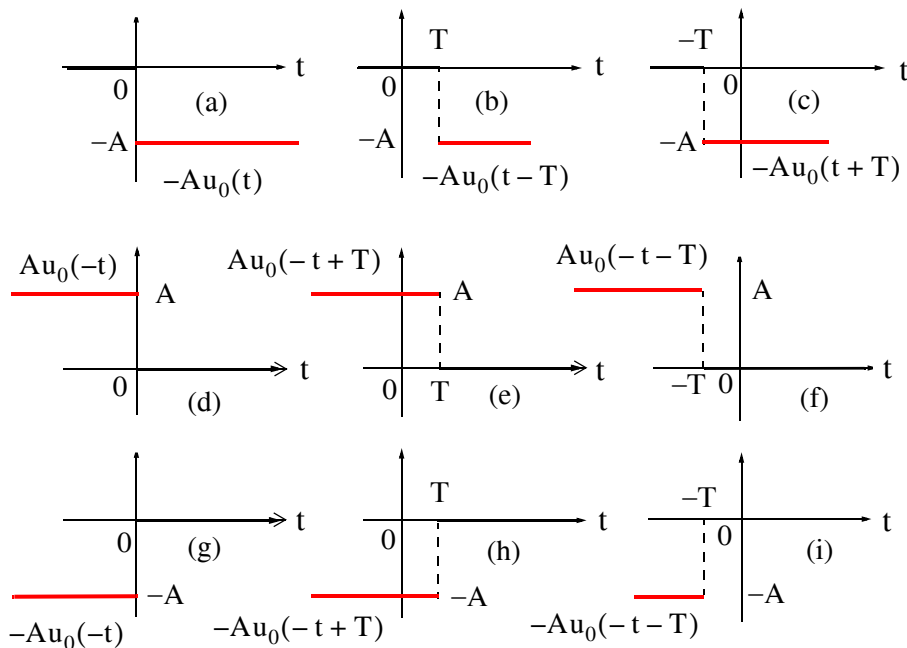


Figure 10.4. Other forms of the unit step function

Unit step functions can be used to represent other time-varying functions such as the rectangular pulse shown in Figure 10.5. This pulse is represented as  $u_0(t) - u_0(t - 1)$ .

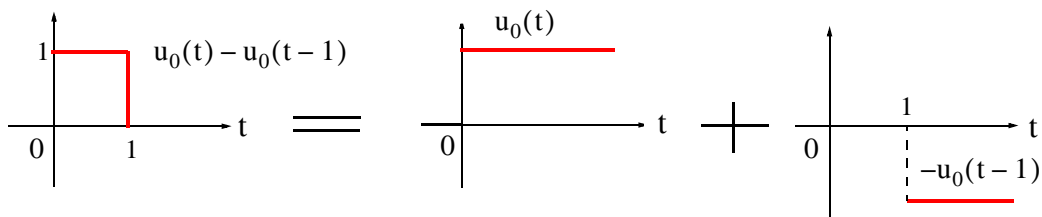


Figure 10.5. A rectangular pulse expressed as the sum of two unit step functions

The unit step function offers a convenient method of describing the sudden application of a voltage or current source. For example, a constant voltage source of 24 V applied at  $t = 0$ , can be denoted as  $24u_0(t)$  V. Likewise, a sinusoidal voltage source  $v(t) = V_m \cos \omega t$  V that is applied to a circuit at  $t = t_0$ , can be described as  $v(t) = (V_m \cos \omega t)u_0(t - t_0)$  V. Also, if the excitation in a circuit is a rectangular, or triangular, or sawtooth, or any other recurring pulse, it can be represented as a sum (difference) of unit step functions.

### Example 10.1

Express the square waveform of Figure 10.6 as a sum of unit step functions. The vertical dotted lines indicate the discontinuities at  $T, 2T, 3T$ , and so on.

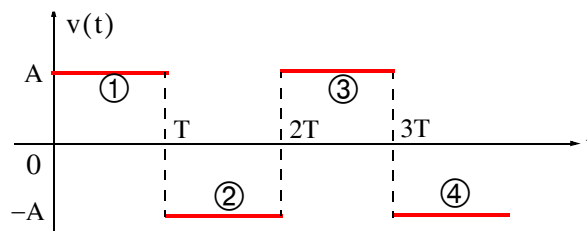


Figure 10.6. Square waveform for Example 10.1

#### Solution:

The line segment ① has height  $A$ , starts at  $t = 0$ , and terminates at  $t = T$  on the time axis. Then, as in Figure 10.5, this segment can be expressed as

$$v_1(t) = A[u_0(t) - u_0(t - T)] \quad (10.2)$$

The line segment ② has height  $-A$ , starts at  $t = T$ , on the time axis, and terminates at  $t = 2T$ . This segment can be expressed as

$$v_2(t) = -A[u_0(t - T) - u_0(t - 2T)] \quad (10.3)$$

Line segment ③ has height  $A$ , starts at  $t = 2T$ , and terminates at  $t = 3T$ . This segment can be expressed as

$$v_3(t) = A[u_0(t - 2T) - u_0(t - 3T)] \quad (10.4)$$

Line segment ④ has height  $-A$ , starts at  $t = 3T$ , and terminates at  $t = 4T$ . This segment can be expressed as

$$v_4(t) = -A[u_0(t - 3T) - u_0(t - 4T)] \quad (10.5)$$

Thus, the square waveform of Figure 10.6 can be expressed as the summation of (10.2) through (10.5), that is,

$$\begin{aligned}
 v(t) &= v_1(t) + v_2(t) + v_3(t) + v_4(t) \\
 &= A[u_0(t) - u_0(t - T)] - A[u_0(t - T) - u_0(t - 2T)] \\
 &\quad + A[u_0(t - 2T) - u_0(t - 3T)] - A[u_0(t - 3T) - u_0(t - 4T)]
 \end{aligned} \tag{10.6}$$

Combining like terms, we obtain

$$v(t) = A[u_0(t) - 2u_0(t - T) + 2u_0(t - 2T) - 2u_0(t - 3T) + \dots] \tag{10.7}$$

### Example 10.2

Express the symmetric rectangular pulse of Figure 10.7 as a sum of unit step functions.

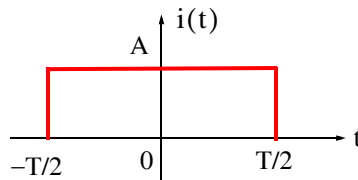


Figure 10.7. Symmetric rectangular pulse for Example 10.2

#### Solution:

This pulse has height  $A$ , it starts at  $t = -T/2$ , and terminates at  $t = T/2$ . Therefore, with reference to Figures 10.3 and 10.4 (b), we obtain

$$i(t) = Au_0\left(t + \frac{T}{2}\right) - Au_0\left(t - \frac{T}{2}\right) = A\left[u_0\left(t + \frac{T}{2}\right) - u_0\left(t - \frac{T}{2}\right)\right] \tag{10.8}$$

### Example 10.3

Express the symmetric triangular waveform shown in Figure 10.8 as a sum of unit step functions.

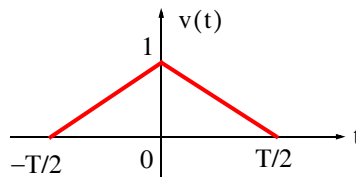


Figure 10.8. Symmetric triangular waveform for Example 10.3

**Solution:**

As a first step, we derive the equations of the linear segments ① and ② shown in Figure 10.9.

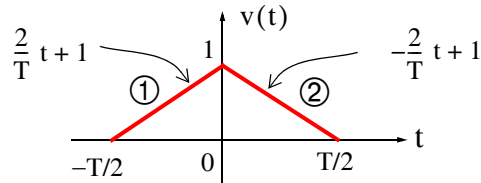


Figure 10.9. Equations for the linear segments of Figure 10.8

For line segment ①,

$$v_1(t) = \left(\frac{2}{T}t + 1\right) \left[ u_0\left(t + \frac{T}{2}\right) - u_0(t) \right] \quad (10.9)$$

and for line segment ②,

$$v_2(t) = \left(-\frac{2}{T}t + 1\right) \left[ u_0(t) - u_0\left(t - \frac{T}{2}\right) \right] \quad (10.10)$$

Combining (10.9) and (10.10), we obtain

$$\begin{aligned} v(t) &= v_1(t) + v_2(t) \\ &= \left(\frac{2}{T}t + 1\right) \left[ u_0\left(t + \frac{T}{2}\right) - u_0(t) \right] + \left(-\frac{2}{T}t + 1\right) \left[ u_0(t) - u_0\left(t - \frac{T}{2}\right) \right] \end{aligned} \quad (10.11)$$

### Example 10.4

Express the waveform shown in Figure 10.10 as a sum of unit step functions.

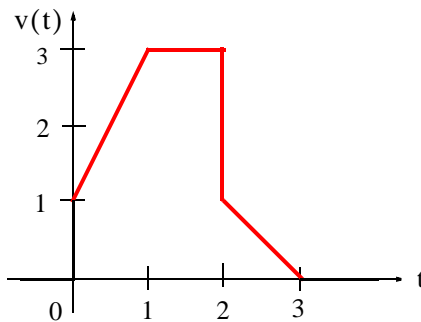


Figure 10.10. Waveform for Example 10.4



### Solution:

As in the previous example, we first find the equations of the linear segments ① and ② shown in Figure 10.11.

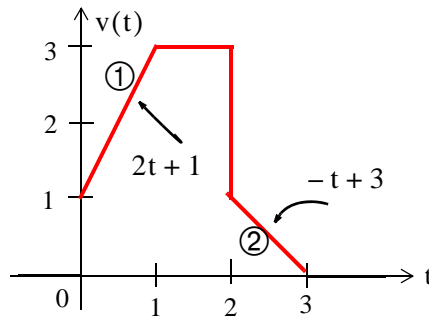


Figure 10.11. Equations for the linear segments of Figure 10.10

Following the same procedure as in the previous examples, we obtain

$$v(t) = (2t + 1)[u_0(t) - u_0(t - 1)] + 3[u_0(t - 1) - u_0(t - 2)] \\ + (-t + 3)[u_0(t - 2) - u_0(t - 3)]$$

Multiplying the values in parentheses by the values in the brackets, we obtain

$$v(t) = (2t + 1)u_0(t) - (2t + 1)u_0(t - 1) + 3u_0(t - 1) \\ - 3u_0(t - 2) + (-t + 3)u_0(t - 2) - (-t + 3)u_0(t - 3)$$

or

$$v(t) = (2t + 1)u_0(t) + [-(2t + 1) + 3]u_0(t - 1) \\ + [-3 + (-t + 3)]u_0(t - 2) - (-t + 3)u_0(t - 3)$$

and combining terms inside the brackets, we obtain

$$v(t) = (2t + 1)u_0(t) - 2(t - 1)u_0(t - 1) - tu_0(t - 2) + (t - 3)u_0(t - 3) \quad (10.12)$$

Two other functions of interest are the *unit ramp function* and the *unit impulse* or *delta function*. We will discuss the unit ramp function first.

### 10.2 Unit Ramp Function $u_1(t)$

The *unit ramp function*, denoted as  $u_1(t)$ , is defined as

$$u_1(t) = \int_{-\infty}^t u_0(\tau) d\tau \quad (10.13)$$

where  $\tau$  is a dummy variable.

We can evaluate the integral of (10.13) by considering the area under the unit step function  $u_0(t)$  from  $-\infty$  to  $t$  as shown in Figure 10.12.

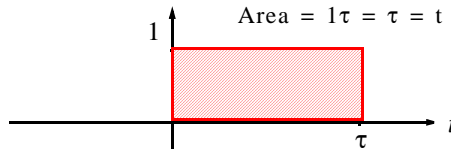


Figure 10.12. Area under the unit step function from  $-\infty$  to  $t$

Therefore,

$$\boxed{u_1(t) = \begin{cases} 0 & t < 0 \\ t & t \geq 0 \end{cases}} \quad (10.14)$$

and since  $u_1(t)$  is the integral of  $u_0(t)$ , then  $u_0(t)$  must be the derivative of  $u_1(t)$ , i.e.,

$$\boxed{\frac{d}{dt}u_1(t) = u_0(t)} \quad (10.15)$$

Higher order functions of  $t$  can be generated by repeated integration of the unit step function. For example, integrating  $u_0(t)$  twice and multiplying by 2, we define  $u_2(t)$  as

$$u_2(t) = \begin{cases} 0 & t < 0 \\ t^2 & t \geq 0 \end{cases} \quad \text{or} \quad u_2(t) = 2 \int_{-\infty}^t u_1(\tau) d\tau \quad (10.16)$$

Similarly,

$$u_3(t) = \begin{cases} 0 & t < 0 \\ t^3 & t \geq 0 \end{cases} \quad \text{or} \quad u_3(t) = 3 \int_{-\infty}^t u_2(\tau) d\tau \quad (10.17)$$

and in general,

$$u_n(t) = \begin{cases} 0 & t < 0 \\ t^n & t \geq 0 \end{cases} \quad \text{or} \quad u_n(t) = n \int_{-\infty}^t u_{n-1}(\tau) d\tau \quad (10.18)$$

Also,

$$u_{n-1}(t) = \frac{1}{n} \frac{d}{dt} u_n(t) \quad (10.19)$$

### 10.3 Delta Function $\delta(t)$

The *unit impulse* or *delta function*, denoted as  $\delta(t)$ , is the derivative of the unit step  $u_0(t)$ . It is generally defined as

$$\int_{-\infty}^t \delta(\tau) d\tau = u_0(t) \quad (10.20)$$

where

$$\delta(t) = 0 \text{ for all } t \neq 0 \quad (10.21)$$

To better understand the delta function  $\delta(t)$ , let us represent the unit step  $u_0(t)$  as shown in Figure 10.13 (a).

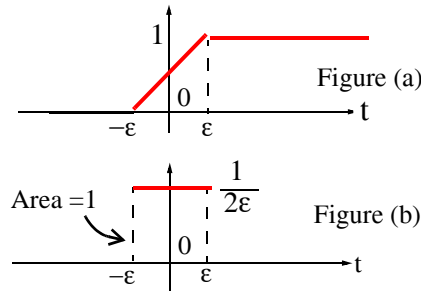


Figure 10.13. Representation of the unit step as a limit.

The function of Figure 10.13 (a) becomes the unit step as  $\epsilon \rightarrow 0$ . Figure 10.13 (b) is the derivative of Figure 10.13 (a), where we see that as  $\epsilon \rightarrow 0$ ,  $1/2\epsilon$  becomes unbounded, but the area of the rectangle remains 1. Therefore, in the limit, we can think of  $\delta(t)$  as approaching a very large spike or impulse at the origin, with unbounded amplitude, zero width, and area equal to 1.

Two useful properties of the delta function are the sampling property and the sifting property.

The *Sampling Property of the Delta Function* states that

$$f(t)\delta(t) = f(0)\delta(t) \quad (10.22)$$

or

$$f(t)\delta(t-a) = f(a)\delta(t) \quad (10.23)$$

that is, multiplication of any function  $f(t)$  by the delta function  $\delta(t)$  results in sampling the function at the time instants where the delta function is not zero. The study of discrete-time systems is based on this property.

The *Sifting Property of the Delta Function* states that

$$\int_{-\infty}^{\infty} f(t)\delta(t-\alpha)dt = f(\alpha) \quad (10.24)$$

that is, if we multiply any function  $f(t)$  by  $\delta(t-\alpha)$  and integrate from  $-\infty$  to  $+\infty$ , we will obtain the value of  $f(t)$  evaluated at  $t-\alpha$ .

The proofs of (10.22) through (10.24) and additional properties of the delta function are beyond the scope of this book. They are provided in *Signals and Systems with MATLAB Computing and Simulink Modeling*, ISBN 978-1-934404-11-9.

MATLAB has two built-in functions for the unit step and the delta functions. These are designated by the names of the mathematicians who used them in their work. The unit step  $u_0(t)$  is called **Heaviside(t)** and the delta function  $\delta(t)$  is called **Dirac(t)**. Shown below are examples of how they are being used.

```
syms k a t
u=k*sym('Heaviside(t-a)') % Create unit step function at t=a

u =
k*Heaviside(t-a)

d=diff(u) % Compute the derivative of the unit step function

d =
k*Dirac(t-a)

int(d) % Integrate the delta function

ans =
Heaviside(t-a)*k
```

### Example 10.5

For the circuit shown in Figure 10.14, the inputs are applied at different times as indicated.

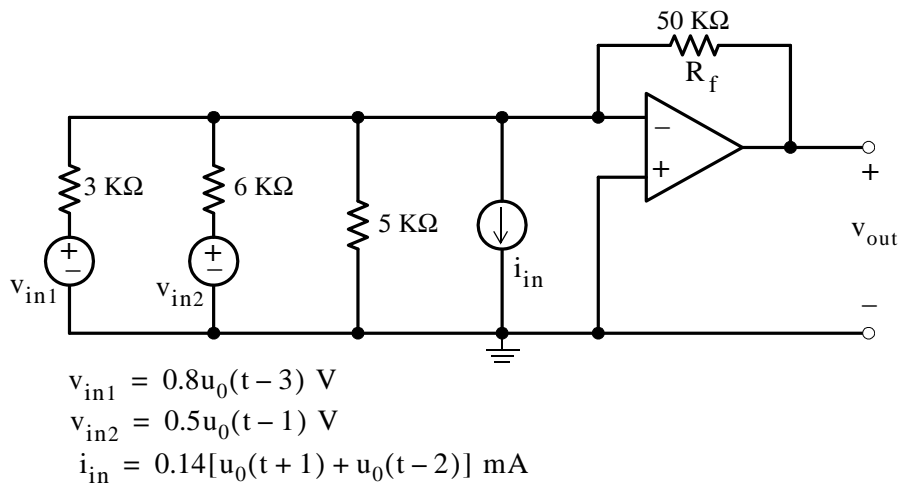


Figure 10.14. Circuit for Example 10.5

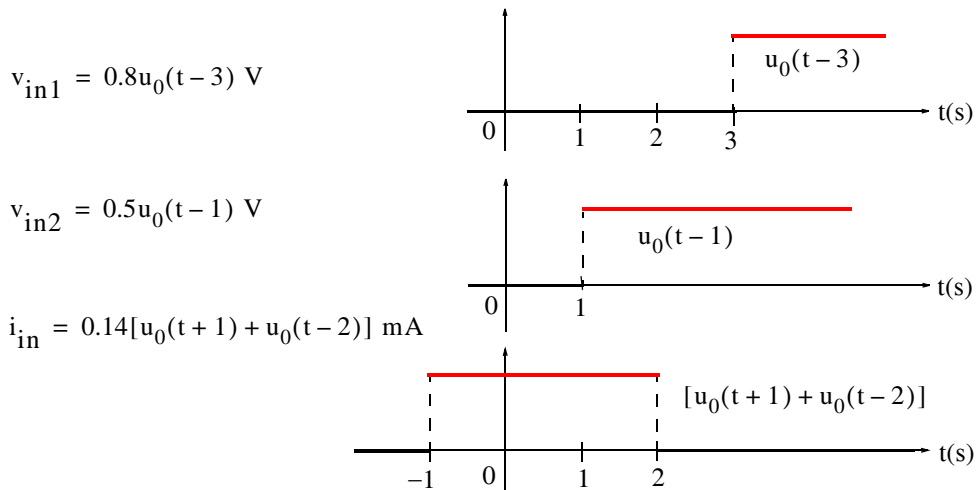
## Chapter 10 Forced and Total Response in RL and RC Circuits

Compute  $v_{\text{out}}$  at:

- a.  $t = -0.5$  s
- b.  $t = 1.5$  s
- c.  $t = 5$  s

**Solution:**

Let us first sketch the step functions for each of the inputs.



- a. At  $t = -0.5$  s only the signal due to  $i_{\text{in}}$  is active; therefore, exchanging the current source and its parallel resistance with an equivalent voltage source with a series resistance, the input circuit becomes as shown in Figure 10.15.

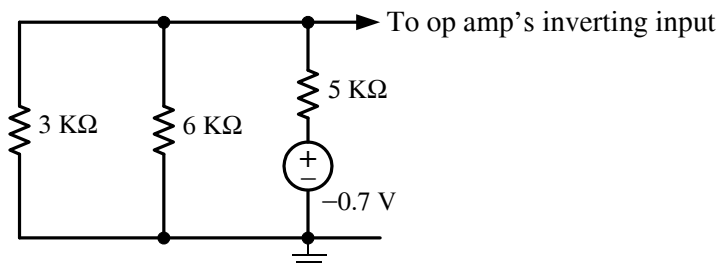


Figure 10.15. Input to the circuit of Example 10.5 when  $i_{\text{in}}$  is acting alone

Replacing the circuit of Figure 10.15 with its Thevenin equivalent, we obtain the network of Figure 10.16.

$$3 \text{ K}\Omega \parallel 6 \text{ K}\Omega = 2 \text{ K}\Omega$$

$$v_{\text{TH1}} = V_{2 \text{ K}\Omega} = \frac{2 \text{ K}\Omega}{2 \text{ K}\Omega + 5 \text{ K}\Omega}(-0.7) = -0.2 \text{ V}$$

$$R_{\text{TH1}} = \frac{2 \text{ K}\Omega \times 5 \text{ K}\Omega}{7 \text{ K}\Omega} = 10/7 \text{ K}\Omega$$

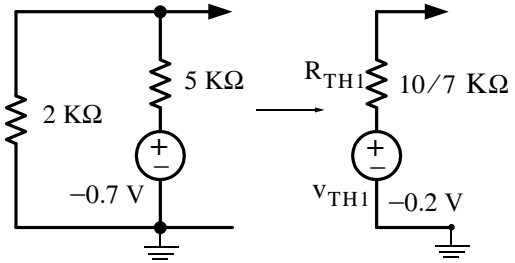


Figure 10.16. Simplified input to the circuit of Example 10.5 when  $i_{\text{in}}$  is acting alone

Now, we can compute  $v_{\text{out1}}$  with the circuit of Figure 10.17.

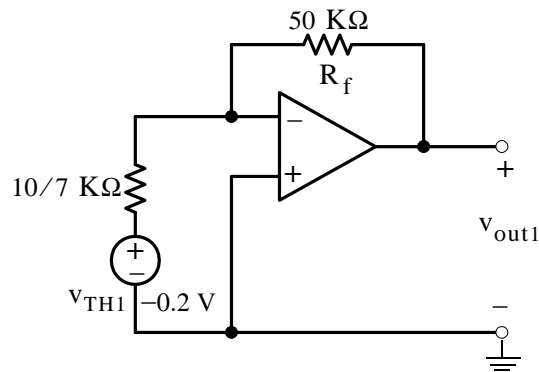


Figure 10.17. Circuit for computation of  $v_{\text{out1}}$

$$v_{\text{out1}} = -\left(\frac{50}{10/7}\right)v_{\text{TH1}} = -35 \times (-0.2 \text{ mV}) = 7 \text{ V} \quad (10.25)$$

b. At  $t = 1.5 \text{ s}$  the active inputs are

$$i_{\text{in}} = 0.14[u_0(t+1) + u_0(t-2)] \text{ mA}$$

and

$$v_{\text{in2}} = 0.5u_0(t-1) \text{ V}$$

Since we already know the output due to  $i_{\text{in}}$  acting alone, we will find the output due to  $v_{\text{in2}}$  acting alone and then apply superposition to find the output when both of these inputs are present. Thus, with the input  $v_{\text{in2}}$  acting alone, the input circuit is as shown in Figure 10.18.

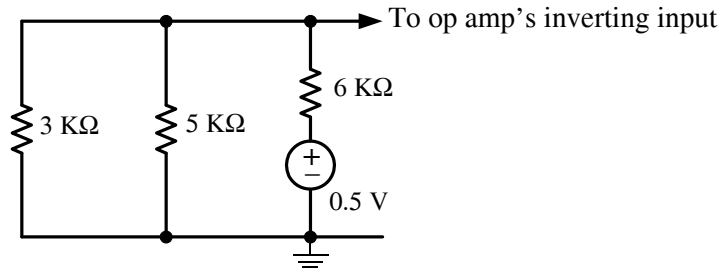


Figure 10.18. Input to the circuit of Example 10.5 when  $v_{in2}$  is acting alone

Replacing this circuit of Figure 10.18 with its Thevenin equivalent, we obtain the network of Figure 10.19.

$$3 \text{ K}\Omega \parallel 5 \text{ K}\Omega = 15/8 \text{ K}\Omega$$

$$v_{TH2} = v_{(15/8) \text{ K}\Omega} = \frac{15/8}{15/8 + 6}(0.5) = \frac{5}{42} \text{ V}$$

$$R_{TH2} = R_{TH1} = 10/7 \text{ K}\Omega$$

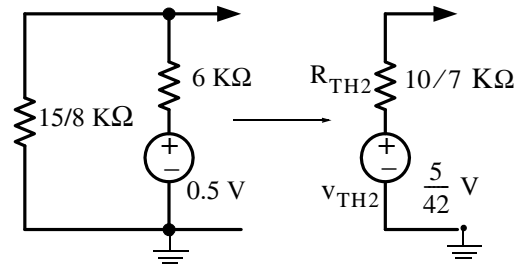


Figure 10.19. Simplified input to the circuit of Example 10.5 when  $v_{in2}$  is acting alone

Now, we can compute  $v_{out2}$  with the circuit of Figure 10.20.

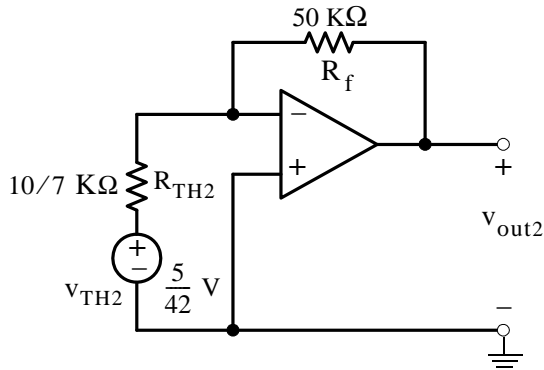


Figure 10.20. Circuit for computation of  $v_{out2}$

$$v_{out2} = -\left(\frac{50}{10/7}\right)v_{TH2} = -35 \times \left(\frac{5}{42}\right) = -\frac{25}{6} \text{ V} \quad (10.26)$$

Therefore, from (10.25) and (10.26) the op amp's output voltage at  $t = 1.5 \text{ s}$  is

$$v_{out1} + v_{out2} = 7 - \frac{25}{6} = \frac{17}{6} \text{ V} \quad (10.27)$$

c. At  $t = 5$  s the active inputs are

$$v_{in1} = 0.8u_0(t-3) \text{ V}$$

and

$$v_{in2} = 0.5u_0(t-1) \text{ V}$$

Since we already know the output due to  $v_{in2}$  acting alone, we will find the output due to  $v_{in1}$  acting alone and then apply superposition to find the output when both of these inputs are present. Thus, with the input  $v_{in1}$  acting alone, the input circuit is as shown in Figure 10.21.

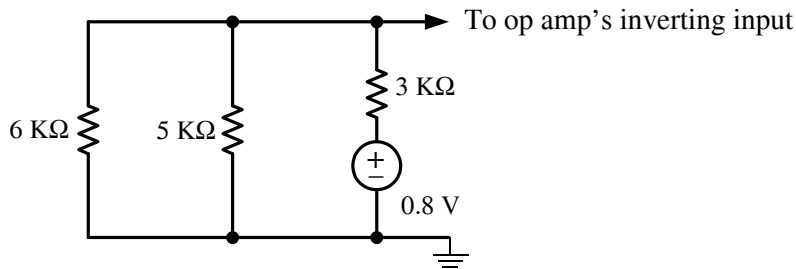


Figure 10.21. Input to the circuit of Example 10.5 when  $v_{in1}$  is acting alone

Replacing this circuit of Figure 10.21 with its Thevenin equivalent, we obtain the network of Figure 10.22.

$$6 \text{ k}\Omega \parallel 5 \text{ k}\Omega = 30/11 \text{ k}\Omega$$

$$V_{TH3} = v_{(30/11) \text{ k}\Omega} = \frac{30/11}{30/11 + 3}(0.8) = \frac{10}{21} \text{ V}$$

$$R_{TH3} = R_{TH2} = R_{TH1} = 10/7 \text{ k}\Omega$$

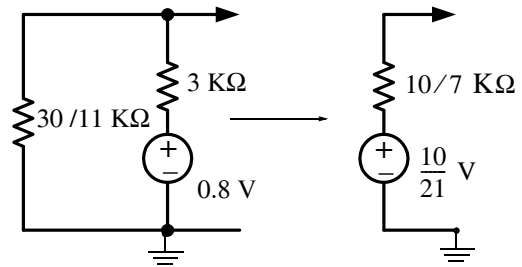


Figure 10.22. Simplified input to the circuit of Example 10.5 when  $v_{in1}$  is acting alone

Now, we can compute  $v_{out3}$  with the circuit of Figure 10.23.

$$v_{out3} = -\left(\frac{50}{10/7}\right)v_{TH3} = -35 \times \left(\frac{10}{21}\right) = -\frac{50}{3} \text{ V} \tag{10.28}$$

Therefore, from (10.26) and (10.28) the op amp's output voltage at  $t = 5$  s is

$$v_{out2} + v_{out3} = -\frac{25}{6} - \frac{50}{3} = -\frac{125}{6} \text{ V} \tag{10.29}$$



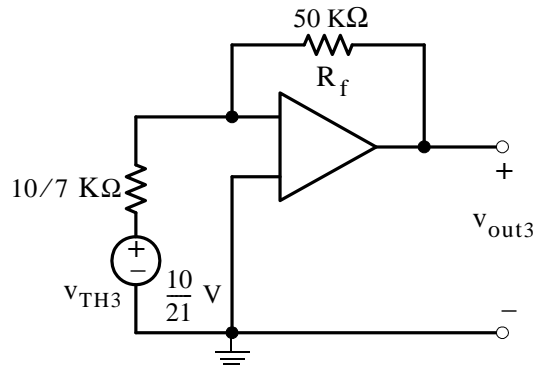


Figure 10.23. Circuit for computation of  $v_{out3}$

### 10.4 Forced and Total Response in an RL Circuit

For the circuit shown in Figure 10.24(a),  $V_S$  is constant. We will derive an expression for the inductor current  $i_L(t) = i(t)$  for  $t > 0$  given that the initial condition is  $i_L(0^-) = 0$ . Here, the inductor current  $i_L(t)$  will be referred to as the *total response*.

The switch in Figure 10.24 (a) can be omitted if we multiply the excitation  $V_S$  by the unit step function  $u_0(t)$  as shown in Figure 10.24 (b).

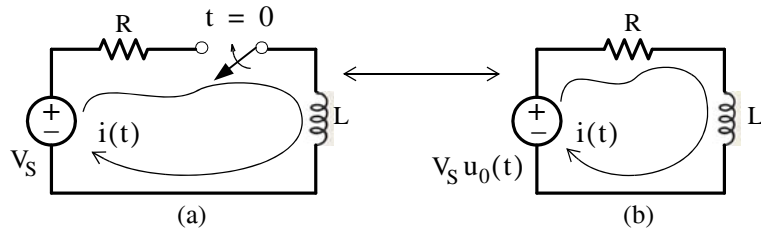


Figure 10.24. Circuits for derivation of the total response  $i_L(t) = i(t)$

We begin by applying KVL, that is,

$$L \frac{di}{dt} + Ri = V_S u_0(t) \quad (10.30)$$

The initial condition states that  $i_L(0^-) = 0$ ; thus for  $t < 0$ ,  $i(t) = 0$

For  $t > 0$ , we must solve the differential equation

$$L \frac{di}{dt} + Ri = V_S \quad (10.31)$$

It is shown in differential equations textbooks that a differential equation such as the above, can be solved by the method of separation of the variables. Thus, rearranging (10.31), separating the variables, and integrating we obtain:

$$L \frac{di}{dt} = V_S - Ri$$

or

$$\frac{Ldi}{V_S - Ri} = dt$$

or

$$\int \frac{Ldi}{V_S - Ri} = \int dt$$

and referring to a table of integrals, we obtain

$$-\frac{L}{R} \ln(V_S - Ri) = t + k \quad (10.32)$$

The constant  $k$  in (10.32) represents the constant of integration of both sides and it can be evaluated from the initial condition, and as we stated in the previous chapter

$$i_L(0^-) = i_L(0) = i_L(0^+) \quad (10.33)$$

Therefore, at  $t = 0^+$

$$-\frac{L}{R} \ln(V_S - 0) = 0 + k$$

$$k = -\frac{L}{R} \ln V_S$$

and by substitution into (10.32), we obtain

$$-\frac{L}{R} \ln(V_S - Ri) = t - \frac{L}{R} \ln V_S$$

$$-\frac{L}{R} [\ln(V_S - Ri) - \ln V_S] = t$$

$$-\frac{L}{R} \left[ \ln \frac{V_S - Ri}{V_S} \right] = t$$

$$\ln \frac{V_S - Ri}{V_S} = -\frac{R}{L} t$$

$$\frac{V_S - Ri}{V_S} = e^{-(R/L)t}$$

$$Ri = V_S - V_S e^{-(R/L)t}$$

$$i(t) = \frac{V_S}{R} - \frac{V_S}{R} e^{-(R/L)t}$$

The general expression for all  $t$  is

$$\boxed{i(t) = \left( \frac{V_S}{R} - \frac{V_S}{R} e^{-(R/L)t} \right) u_0(t)} \quad (10.34)$$

We observe that the right side of (10.34) consists of two terms,  $V_S/R$  which is constant called the *forced response*, and the exponential term  $-\frac{V_S}{R} e^{-(R/L)t}$  that has the same form as that of the previous chapter which we call the *natural response*.

The forced response  $V_S/R$  is a result of the application of the excitation (forcing) function  $V_S u_0(t)$  applied to the RL circuit. This value represents the steady-state condition reached as  $t \rightarrow \infty$  since the inductor  $L$  at this state behaves as a short circuit.

The amplitude of the natural response is  $-V_S/R$  and depends on the values of  $V_S$  and  $R$ .

The summation of the forced response and the natural response constitutes the *total response* or *complete response*, that is,

$$i(t)_{\text{total}} = i(t)_{\text{forced response}} + i(t)_{\text{natural response}}$$

or

$$\boxed{i_{\text{total}} = i_f + i_n} \quad (10.35)$$

Now, let us return to the RL circuit of Figure 10.24 to find the complete (total) response  $i_{\text{total}}$  by the summation of the forced and the natural responses as indicated in (10.35).

The forced response  $i_f$  is found from the circuit of Figure 10.25 where we let  $t \rightarrow \infty$

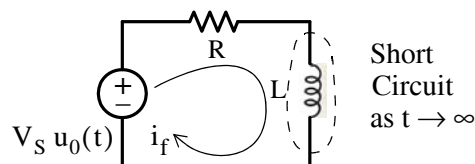


Figure 10.25. Circuit for derivation of the forced response  $i_f$

Then, from the circuit of Figure 10.25,

$$\boxed{i_f = \frac{V_S}{R}} \quad (10.36)$$

Next, we need to find the natural response. This is found by letting the excitation (forcing function)  $V_S u_0(t)$  go to zero as shown in the circuit of Figure 10.26.

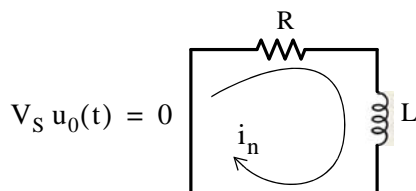


Figure 10.26. Circuit for derivation of the natural response  $i_n$

We found in Chapter 9 that the natural response  $i_n$  has the exponential form

$$i_n = Ae^{-(R/L)t} \quad (10.37)$$

Therefore, the total response is

$$i_{\text{total}} = i_f + i_n = \frac{V_S}{R} + Ae^{-(R/L)t} \quad (10.38)$$

where the constant  $A$  is evaluated from the initial condition  $i_L(0^-) = i_L(0) = i_L(0^+)$

Substitution of the initial condition into (10.38) yields

$$i(0) = 0 = \frac{V_S}{R} + Ae^0$$

or

$$A = -\frac{V_S}{R}$$

and with this substitution (10.38) is rewritten as

$$i_{\text{total}} = \left( \frac{V_S}{R} - \frac{V_S}{R} e^{-(R/L)t} \right) u_0(t) \quad (10.39)$$

and this is the same as (10.34).

We can sketch  $i_{\text{total}}$  easily if we sketch  $\frac{V_S}{R}$  and  $-\frac{V_S}{R}e^{-(R/L)t}$  separately and then add these. This is done with MATLAB and the plots are shown in Figure 10.27.

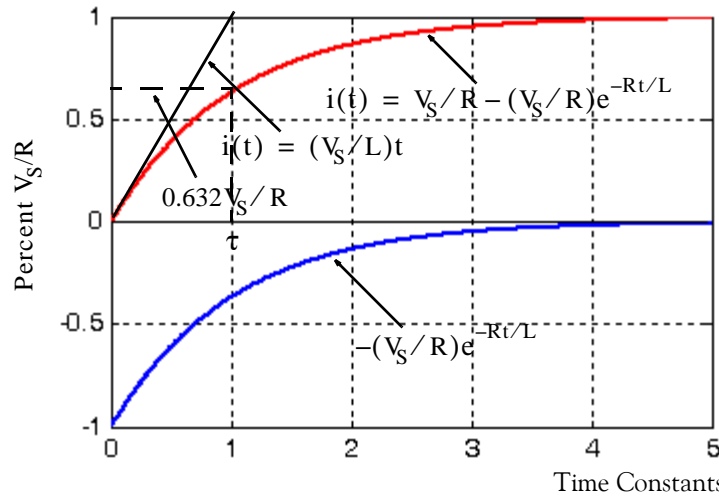


Figure 10.27. Curves for forced, natural, and total responses in a series RL circuit

The curves in Figure 10.27 were created with the following MATLAB script:

```
x=0:0.01:5; Vs=1; R=1; L=1; y=-(Vs./R).*exp(-R.*x./L); z=Vs./R+y; plot(x,y,x,z)
```

The time constant  $\tau$  is defined as before, and its numerical value can be found from the circuit constants  $R$  and  $L$  as follows:

The equation of the straight line with slope =  $V_S/L$  is found from

$$\left. \frac{d}{dt}(i_{\text{total}}) \right|_{t=0} = \left. \frac{R}{L} \cdot \frac{V_S}{R} e^{-(R/L)t} \right|_{t=0} = \frac{V_S}{L}$$

Assuming constant rate of change as shown in Figure 10.27, at  $t = \tau$ ,

$$i(t) = \frac{V_S}{R}$$

and thus

$$\frac{V_S}{R} = \frac{V_S}{L}\tau$$

or

$$\tau = \frac{L}{R}$$

as before. Also, from (10.39)

$$i(\tau) = \frac{V_S}{R} - \frac{V_S}{R} e^{-(R/L)(L/R)} = \frac{V_S}{R}(1 - e^{-1}) = \frac{V_S}{R}(1 - 0.368)$$

or

$$\boxed{i(\tau) = 0.632 \frac{V_S}{R}} \quad (10.40)$$

Therefore, the current in a series RL circuit which has been excited by a constant source, in one time constant has reached 63.2% of its final value.

### Example 10.6

For the circuit of Figure 10.28, compute the energy stored in the 10 mH inductor at  $t = 100$  ms.

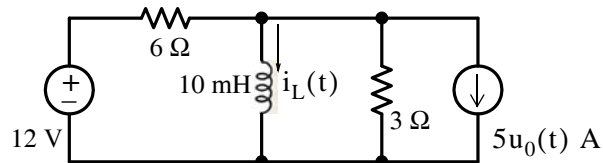


Figure 10.28. Circuit for Example 10.6

#### Solution:

For  $t < 0$ , the circuit is as shown in Figure 10.29 where the 3 Ω resistor is shorted out by the inductor.

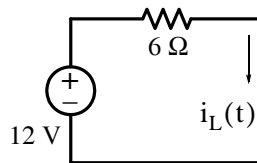


Figure 10.29. Circuit of Example 10.6 for  $t < 0$

From the circuit of Figure 10.29,

$$i_L(0^-) = \frac{12}{6} = 2 \text{ A}$$

and this value establishes our initial condition as

$$i_L(0^+) = 2 \text{ A} \tag{10.41}$$

For  $t > 0$ , the circuit is as shown in Figure 10.30.

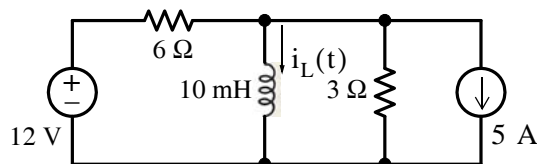


Figure 10.30. Circuit of Example 10.6 for  $t > 0$

We will compute  $i_L(t)$  from the relation

$$i_L(t) = i_f + i_n$$

The forced component  $i_f$  is found from the circuit at steady state conditions. It is shown in Figure 10.31 where the voltage source and its series resistance have been exchanged for an equivalent current source with a parallel resistor. The resistors have been shorted out by the inductor.

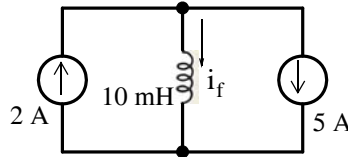


Figure 10.31. Circuit of Example 10.6 under steady-state conditions

By inspection,  $i_f = 2 - 5$  or

$$i_f = -3 \text{ A} \quad (10.42)$$

To find  $i_n$  we short the voltage source and open the current source. The circuit then reduces to that shown in Figure 10.32.

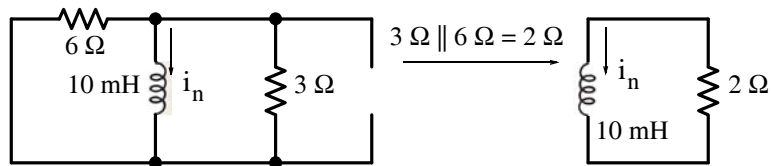


Figure 10.32. Circuit of Example 10.6 for determining the natural response

The natural response of RL circuit of Figure 10.32 is

$$i_n = Ae^{-(R/L)t} = Ae^{-(2/10 \times 10^{-3})t}$$

or

$$i_n = Ae^{-200t} \quad (10.43)$$

The total response is the summation of (10.42) and (10.43), that is,

$$i_{\text{total}} = i_f + i_n = -3 + Ae^{-200t} \quad (10.44)$$

Using the initial condition of (10.42), we obtain

$$i_L(0^+) = 2 = -3 + Ae^{-0}$$

or

$$A = 5$$

Finally, by substitution into (10.44) we obtain

$$i_{\text{total}} = (-3 + 5e^{-200t})u_0(t) \quad (10.45)$$

and the energy stored in the inductor at  $t = 100$  ms is

$$\begin{aligned} W_L \Big|_{t=100 \text{ ms}} &= \frac{1}{2} Li_L^2 \Big|_{t=100 \text{ ms}} = \frac{1}{2} 10 \times 10^{-3} (-3 + 5e^{-200 \times 100 \times 10^{-3}})^2 \\ &= 5 \times 10^{-3} (-3 + 5e^{-20})^2 = 45 \text{ mJ} \end{aligned} \quad (10.46)$$

## 10.5 Forced and Total Response in an RC Circuit

For the circuit shown in Figure 10.33 (a),  $V_S$  is constant. We will derive an expression for the capacitor voltage  $v_C(t)$  for  $t > 0$  given that the initial condition is  $v_C(0^-) = 0$ . Here, the capacitor voltage  $v_C(t)$  will be referred to as the *total response*.

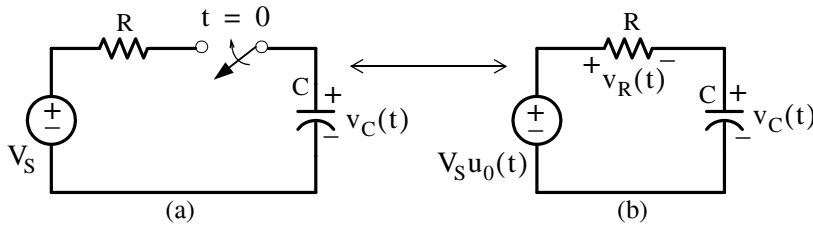


Figure 10.33. Circuits for derivation of the total response  $v_C(t)$

The switch in Figure 10.33 (a) can be omitted if we multiply the excitation  $V_S$  by the unit step function  $u_0(t)$  as shown in Figure 10.33 (b).

We begin by applying KVL, that is,

$$v_R + v_C = V_S u_0(t) \quad (10.47)$$

and since

$$i = i_C = C \frac{dv_C}{dt}$$

we can express  $v_R$  as

$$v_R = Ri = RC \frac{dv_C}{dt}$$

By substitution into (10.47), we obtain

$$RC \frac{dv_C}{dt} + v_C = V_S u_0(t) \quad (10.48)$$



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## Chapter 10 Forced and Total Response in RL and RC Circuits

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The initial condition states that  $v_C(0^-) = 0$ ; thus for  $t < 0$ ,  $v_C(t) = 0$

For  $t > 0$ , we must solve the differential equation

$$RC \frac{dv_C}{dt} + v_C = V_S \quad (10.49)$$

Rearranging, separating variables and integrating, we obtain:

$$RC dv_C = (V_S - v_C) dt$$

$$\frac{dv_C}{v_C - V_S} = -\frac{1}{RC} dt$$

$$\int \frac{dv_C}{v_C - V_S} dt = -\frac{1}{RC} \int dt \quad (10.50)$$

or

$$\ln(v_C - V_S) = -\frac{1}{RC} t + k$$

where  $k$  represents the constant of integration of both sides of (10.51). Then,

$$v_C - V_S = e^{-(1/RC)t + k} = e^k e^{-(1/RC)t} = k_1 e^{-(1/RC)t}$$

or

$$v_C = V_S - k_1 e^{-(1/RC)t} \quad (10.51)$$

The constant  $k_1$  can be evaluated from the initial condition  $v_C(0^+) = v_C(0^-) = 0$  where by substitution into (10.51) we obtain

$$v_C(0^+) = 0 = V_S - k_1 e^0$$

or

$$k_1 = V_S$$

Therefore, the solution of (10.49) is

$$\boxed{v_C(t) = (V_S - V_S e^{-(1/RC)t}) u_0(t)} \quad (10.52)$$

As with the RL circuit of the previous section, we observe that the solution consists of a *forced response* and a *natural response*. The constant term  $V_S$  is the voltage attained across the capacitor as  $t \rightarrow \infty$  and represents the steady-state condition since the capacitor  $C$  at this state behaves as an open circuit.

The amplitude of the exponential term natural response is  $-V_S$ .

The summation of the forced response and the natural response constitutes the total response, i.e.,

$$v_C(t)_{\text{complete response}} = v_C(t)_{\text{forced response}} + v_C(t)_{\text{natural response}}$$

or

$$v_{C_{\text{total}}} = v_{C_f} + v_{C_n} \quad (10.53)$$

Now, let us return to the RC circuit of Figure 10.33(b) to find the complete (total) response by summing the forced and the natural responses indicated in (10.53).

The forced response  $v_{C_f}$  is found from the circuit of Figure 10.34 where we let  $t \rightarrow \infty$ .

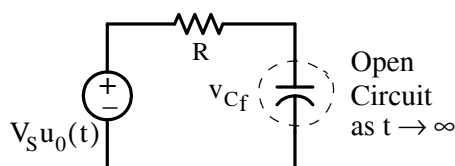


Figure 10.34. Circuit for derivation of the forced response  $v_{C_f}$

Then, from the circuit of Figure 10.34,

$$\boxed{v_{C_f} = V_S} \quad (10.54)$$

Next, we need to find the natural response and this is found by letting the excitation (forcing function)  $V_S u_0(t)$  go to zero as shown in Figure 10.35.

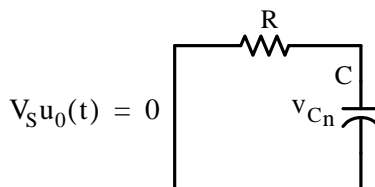


Figure 10.35. Circuit for derivation of the natural response  $v_{C_n}$

We found in Chapter 9 that the natural response  $v_{C_n}$  has the exponential form

$$v_{C_n} = A e^{-(1/RC)t}$$

and thus the total response is

$$v_C(t) = v_{C_f} + v_{C_n} = V_S + A e^{-(1/RC)t} \quad (10.55)$$

where the constant  $A$  is evaluated from the initial condition  $v_C(0^-) = v_C(0) = v_C(0^+) = 0$

Substitution of the initial condition into (10.55) yields

$$v_C(0^+) = 0 = V_S - Ae^0$$

or

$$A = -V_S$$

With this substitution (10.55) is rewritten as

$$v_C(t) = (V_S - V_S e^{-(1/RC)t}) u_0(t) \quad (10.56)$$

and this is the same as (10.52).

We can sketch  $v_{C\text{total}}$  easily if we sketch  $V_S$  and  $-V_S e^{-(1/RC)t}$  separately and then add these. This is done with MATLAB and the plots are shown in Figure 10.36.

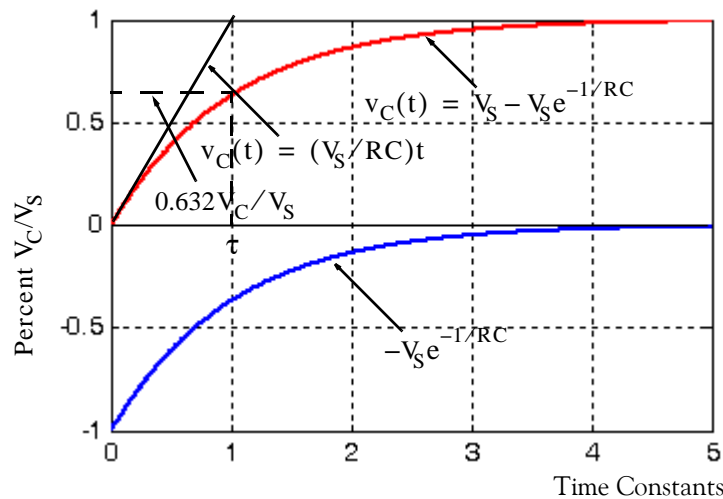


Figure 10.36. Curves for forced, natural, and total responses in a series RC circuit

The time constant  $\tau$  is defined as before, and its numerical value can be found from the circuit constants  $R$  and  $C$  as follows:

The equation of the straight line with slope =  $V_S/RC$  is found from

$$\left. \frac{d}{dt} v_C(t) \right|_{t=0} = \frac{1}{RC} \cdot V_S e^{-(1/RC)t} \Big|_{t=0} = \frac{V_S}{RC}$$

Assuming constant rate of change as shown in Figure 10.36, at  $t = \tau$ ,

$$v_C(t) = V_S$$

and thus

$$V_S = \frac{V_S}{RC} \tau$$

or

$$\tau = RC$$

as before. Also, from (10.56)

$$v_C(\tau) = V_S - V_S e^{-(1/RC)RC} = V_S(1 - e^{-1}) = V_S(1 - 0.368)$$

or

$$\boxed{v_C(\tau) = 0.632V_S} \quad (10.57)$$

Therefore, the voltage across a capacitor in a series RC circuit which has been excited by a constant source, in one time constant has reached 63.2% of its final value.

### Example 10.7

For the circuit shown in Figure 10.37 find:

- $v_C(1^-)$  and  $i_C(1^-)$
- $v_C(1^+)$  and  $i_C(1^+)$
- $v_C(t = 10 \text{ min.})$  and  $i_C(t = 10 \text{ min.})$
- $i_C(t)$  for  $t > 1$

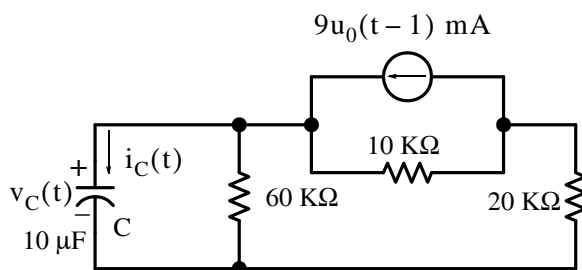


Figure 10.37. Circuit for Example 10.7

### Solution:

- No initial condition is given so we must assume that sufficient time has elapsed for steady-state conditions to exist for all  $t < 1$  s. We assume time is in seconds since we are not told otherwise. Then, since there is no voltage or current source present to cause current to flow, we obtain

$$v_C(1^-) = 0$$

and

$$i_C(1^-) = 0$$

- b. Exchanging the current source and the  $10\text{ K}\Omega$  resistor with a voltage source with a  $10\text{ K}\Omega$  series resistor, the circuit at  $t = 1^+$  is as shown in Figure 10.38.

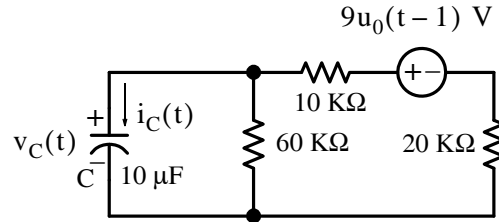


Figure 10.38. Circuit for Example 10.7 at  $t = 1^+$

Now, since  $v_C(1^+) = v_C(1^-)$ , no current flows through the  $60\text{ k}\Omega$  resistor at  $t = 1^+$ ; if it did, the voltage across the capacitor would change instantaneously, and as we know, this is a physical impossibility. Instead, the current path is through the capacitor which at exactly  $t = 1^+$  acts as a short circuit since  $v_C(1^+) = v_C(1^-) = 0$ . Therefore,

$$i_C(1^+) = \frac{90\text{ V}}{(20 + 10)\text{ K}\Omega} = 3\text{ mA} \quad (10.58)$$

- c. The time  $t = 10\text{ min}$  is essentially the same as  $t = \infty$ , and at this time the capacitor voltage  $v_C(t = 10\text{ min})$  is constant and equal to the voltage across the  $60\text{ K}\Omega$  resistor, i.e.,

$$v_C(t = 10\text{ min}) = v_C(\infty) = v_{60\text{ K}\Omega} = \frac{90\text{ V}}{(20 + 10 + 60)\text{ K}\Omega} \cdot 60\text{ K}\Omega = 60\text{ V}$$

Also,

$$i_C(t)|_{t=\infty} = C \frac{dv_C}{dt} = 0$$

- d. For  $t > 1$

$$i_C(t)|_{t>1} = i_{Cf} + i_{Cn}$$

where from part (c)

$$i_{Cf}(\infty) = 0$$

and

$$i_{Cn} = Ae^{-(1/R_{eq}C)t}$$

With the voltage source shorted in the circuit of Figure 10.38, the equivalent resistance is

$$R_{eq} = (10\text{ K}\Omega + 20\text{ K}\Omega) \parallel 60\text{ K}\Omega = 20\text{ K}\Omega$$

or

$$R_{eq}C = 20 \times 10^3 \times 10 \times 10^{-6} = 0.2 \text{ s}$$

Therefore,

$$i_{Cn} = Ae^{-(1/0.2)t} = Ae^{-5t} \quad (10.59)$$

We can evaluate the constant A using (10.59) where

$$i_C(1^+) = 3 \text{ mA} = Ae^{-5}$$

or

$$A = \frac{3 \times 10^{-3}}{e^{-5}} = 0.445$$

and by substitution into (10.59),

$$i_C(t)|_{t>1} = i_{cf} + i_{cn} = i_{cn} = 0.445e^{-5t}u_0(t-1) \text{ mA} \quad (10.60)$$

### Example 10.8

In the circuit shown in Figure 10.39, the switch is actually an electronic switch and it is open for 15  $\mu\text{s}$  and closed for 15  $\mu\text{s}$ . Initially, the capacitor is discharged, i.e.,  $v_C(0) = 0$ . Compute and sketch the voltage across the capacitor for two repetitive cycles.

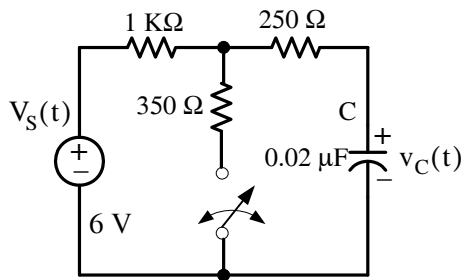


Figure 10.39. Circuit for Example 10.8

**Solution:**

With the switch in the open position the circuit is as shown in Figure 10.40.

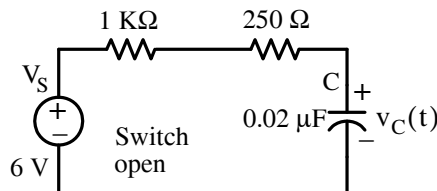


Figure 10.40. Circuit for Example 10.8 with the switch in the open position

## Chapter 10 Forced and Total Response in RL and RC Circuits

For the time period  $0 < t_{\text{open}} < 15 \mu\text{s}$  the time constant for the circuit of Figure 10.40 is

$$\tau_{\text{open}} = R_{\text{eq}}C = (1 \text{ K}\Omega + 0.25 \text{ K}\Omega) \times 0.02 \times 10^{-6} = 25 \mu\text{s}$$

Thus, at the end of the first period when the switch is open, the voltage across the capacitor is

$$v_C(t)|_{t=15 \mu\text{s}} = v_{Cf} + v_{Cn} = V_S - V_S e^{-t/RC} = 6 - 6e^{-4 \times 10^4 t} = 6 - 6e^{-0.6} = 2.71 \text{ V} \quad (10.61)$$

Next, with the switch closed for  $15 < t_{\text{closed}} < 30 \mu\text{s}$  the circuit is as shown in Figure 10.41.

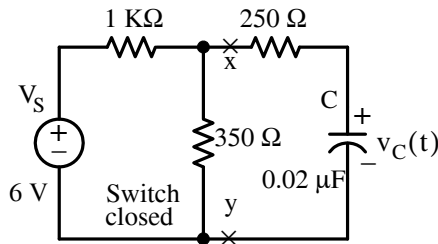


Figure 10.41. Circuit for Example 10.8 with the switch in the closed position

Replacing the circuit to the left of points  $x$  and  $y$  by its Thevenin equivalent, we obtain the circuit shown in Figure 10.42.

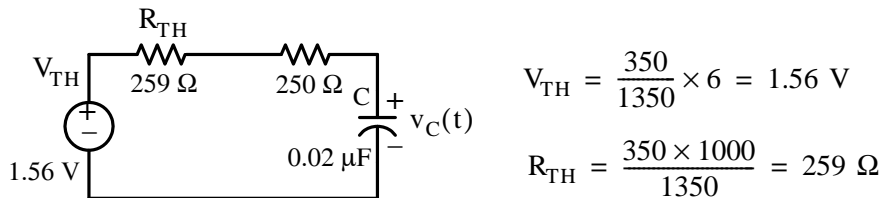


Figure 10.42. Thevenin equivalent circuit for the circuit of Figure 10.41

The time constant for the circuit of Figure 10.42 where the switch is closed, is

$$\tau_{\text{closed}} = R_{\text{eq}}C = (259 \Omega + 250 \Omega) \times 0.02 \times 10^{-6} = 10.2 \mu\text{s}$$

The capacitor voltage  $v_C(t)$  for the circuit of Figure 10.42 is

$$v_C(t) = v_{Cf} + v_{Cn} = V_{\text{TH}} + A_1 e^{-(1/RC)(t-15\mu\text{s})} = 1.56 + A_1 e^{-(1/10.2\mu\text{s})(t-15\mu\text{s})} \quad (10.62)$$

and the constant  $A_1$  is evaluated from initial condition at  $t = 15 \mu\text{s}$  which by (10.62) is

$$v_C(t)|_{t=15 \mu\text{s}} = 2.71 \text{ V}$$

Then,

$$v_C(t)|_{15 < t < 30 \mu\text{s}} = 2.71 = 1.56 + A_1 e^{-(1/10.2\mu\text{s})(15-15)\mu\text{s}}$$

or

$$A_1 = 1.15$$

and by substitution into (10.62)

$$v_C(t)|_{15 < t < 30 \mu s} = 1.56 + 1.15e^{-(1/10.2\mu s)(t-15\mu s)} \quad (10.63)$$

At the end of the first period when the switch is closed, the voltage across the capacitor is

$$v_C(t)|_{t=30 \mu s} = 1.56 + 1.15e^{-(1/10.2\mu s)(30-15)\mu s} = 1.82 \text{ V} \quad (10.64)$$

For the next cycle, that is, for  $30 < t_{\text{open}} < 45 \mu s$  when the switch is open, the time constant  $\tau_{\text{open}}$  is the same as before, i.e.,  $\tau_{\text{open}} = 25 \mu s$  and the capacitor voltage is

$$v_C(t) = v_{Cf} + v_{Cn} = 6 + A_2e^{-(1/25\mu s)(t-30\mu s)} \quad (10.65)$$

The constant  $A_2$  is computed with (10.65) as

$$v_C(t)|_{t=30 \mu s} = 1.82 = 6 + A_2e^{-(1/25\mu s)(30-30)\mu s}$$

or

$$A_2 = -4.18$$

and by substitution into (10.65)

$$v_C(t)|_{30 < t < 45 \mu s} = 6 - 4.18e^{-(1/25\mu s)(t-30\mu s)} \quad (10.66)$$

At the end of the second period when the switch is open, the voltage across the capacitor is

$$v_C(t)|_{t=45 \mu s} = 6 - 4.18e^{-(1/25\mu s)(45-30)\mu s} = 3.71 \text{ V} \quad (10.67)$$

The second period when the switch is closed is  $45 < t_{\text{closed}} < 60 \mu s$ . Then,

$$v_C(t)|_{45 < t < 60 \mu s} = v_{Cf} + v_{Cn} = V_{TH} + A_3e^{-(1/RC)(t-45)} = 1.56 + A_3e^{-(1/10.2\mu s)(t-45\mu s)} \quad (10.68)$$

and with (10.67) we obtain

$$A_3 = 2.15$$

Therefore,

$$v_C(t)|_{45 < t < 60 \mu s} = 1.56 + 2.15e^{-(1/10.2\mu s)(t-45\mu s)} \quad (10.69)$$

and

$$v_C(t)|_{t=60 \mu s} = 1.56 + 2.15e^{-(1/10.2\mu s)(60-45)\mu s} = 2.05 \text{ V} \quad (10.70)$$



Repeating the above steps for the third open and closed switch periods, we obtain

$$v_C(t)\Big|_{60 < t < 75 \mu\text{s}} = 6 - 3.95e^{-(1/25\mu\text{s})(t-60\mu\text{s})} \quad (10.71)$$

and

$$v_C(t)\Big|_{t=75 \mu\text{s}} = 3.83 \text{ V} \quad (10.72)$$

Likewise,

$$v_C(t)\Big|_{75 < t < 90 \mu\text{s}} = 1.56 + 2.27e^{-(1/10.2\mu\text{s})(t-75\mu\text{s})} \quad (10.73)$$

and

$$v_C(t)\Big|_{t=90 \mu\text{s}} = 2.08 \text{ V} \quad (10.74)$$

and using the MATLAB script below we obtain the waveform shown in Figure 10.43.

```
t0=(0:0.01:15)*10^-6;
v0=6-6.*exp(-4.*10.^4.*t0);
t1=(15:0.01:30)*10^-6;
v1=1.56+1.15.*exp((-1./((10.2.*10.^(-6))*(t1-15.*10.^(-6)))));
t2=(30:0.01:45)*10^-6;
v2=6-4.18.*exp((-1./((25.*10.^(-6))*(t2-30.*10.^(-6)))));
t3=(45:0.01:60)*10^-6;
v3=1.56+2.15.*exp((-1./((10.2.*10.^(-6))*(t3-45.*10.^(-6)))));
t4=(60:0.01:75)*10^-6;
v4=6-3.95.*exp((-1./((25.*10.^(-6))*(t4-60.*10.^(-6)))));
t5=(75:0.01:90)*10^-6;
v5=1.56+2.27.*exp((-1./((10.2.*10.^(-6))*(t5-75.*10.^(-6)))));
plot(t0,v0,t1,v1,t2,v2,t3,v3,t4,v4,t5,v5)
```

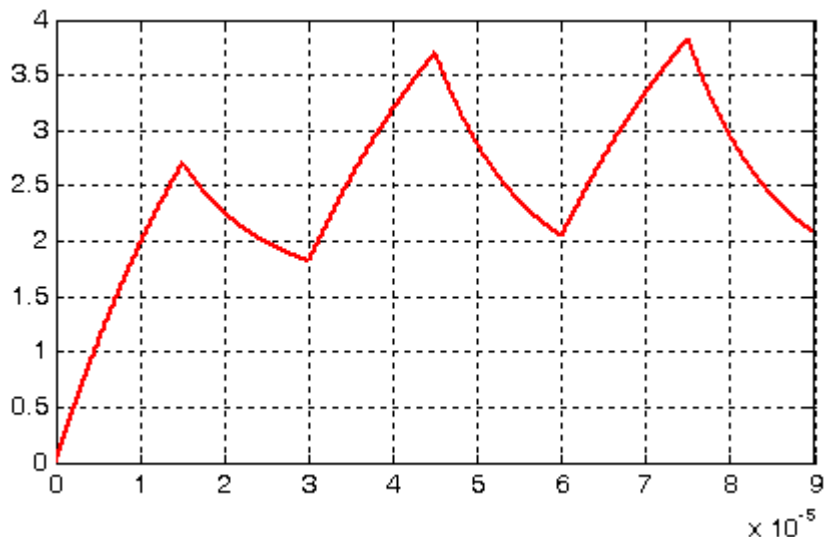


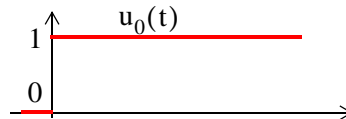
Figure 10.43. Voltage across the capacitor for the circuit of Example 10.8

## 10.6 Summary

- The unit step function  $u_0(t)$  is defined as

$$u_0(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

and it is represented by the waveform below.



- Unit step functions can be used to represent other time-varying functions.
- The unit step function offers a convenient method of describing the sudden application of a voltage or current source.
- The unit ramp function  $u_1(t)$ , is defined as the integral of the unit step function, that is,

$$u_1(t) = \int_{-\infty}^t u_0(\tau) d\tau$$

where  $\tau$  is a dummy variable. It is also expressed as

$$u_1(t) = \begin{cases} 0 & t < 0 \\ t & t \geq 0 \end{cases}$$

- The unit impulse or delta function, denoted as  $\delta(t)$ , is the derivative of the unit step  $u_0(t)$ . It is defined as

$$\delta(\tau) = \frac{d}{d\tau} u_0(\tau)$$

or

$$\int_{-\infty}^t \delta(\tau) d\tau = u_0(t)$$

and

$$\delta(t) = 0 \text{ for all } t \neq 0$$

- In a simple RL circuit that is excited by a voltage source  $V_S u_0(t)$  the current is

$$i(t) = i_f + i_n = \left( \frac{V_S}{R} - \frac{V_S}{R} e^{-(R/L)t} \right) u_0(t)$$

where the forced response  $i_f$  represents the steady-state condition reached as  $t \rightarrow \infty$ . Since the inductor  $L$  at this state behaves as a short circuit,  $i_f = V_S / R$ . The natural response  $i_n$  is the second term in the parenthesis of the above expression, that is,  $i_n = (-V_S / R)e^{-(R/L)t}$

- In a simple RC circuit that is excited by a voltage source  $V_S u_0(t)$  the voltage across the capacitor is

$$v_C(t) = v_{Cf} + v_{Cn} = (V_S + Ae^{-(1/RC)t})u_0(t)$$

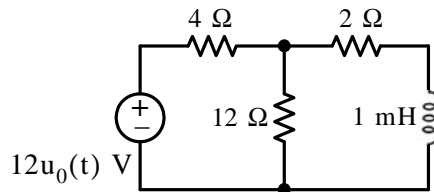
where the forced response  $v_{Cf}$  represents the steady-state condition reached as  $t \rightarrow \infty$ . Since the capacitor  $C$  at this state behaves as an open circuit,  $v_{Cf} = V_S$ . The natural response  $v_{Cn}$  is the second term in the parenthesis of the above expression, that is,  $v_{Cn} = Ae^{-(1/RC)t}$ . The constant  $A$  must be evaluated from the total response.

## 10.7 Exercises

### Multiple Choice

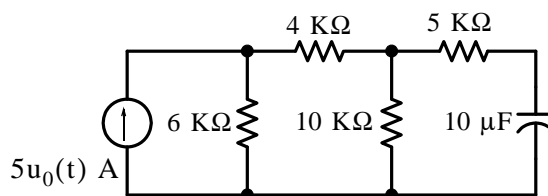
1. For the circuit below the time constant is

- A. 0.5 ms
- B. 71.43  $\mu$ s
- C. 2,000 s
- D. 0.2 ms
- E. none of the above



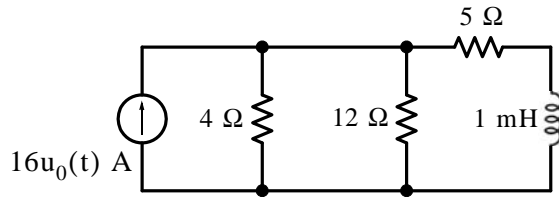
2. For the circuit below the time constant is

- A. 50 ms
- B. 100 ms
- C. 190 ms
- D. 78.6 ms
- E. none of the above

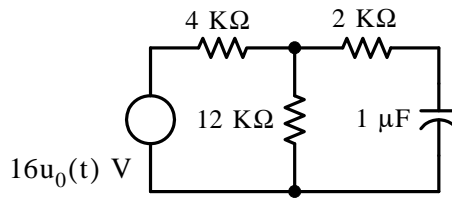


3. The forced response component  $i_{L_f}$  of the inductor current for the circuit below is

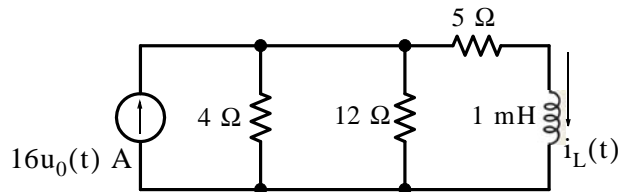
- A. 16 A
- B. 10 A
- C. 6 A
- D. 2 A
- E. none of the above



4. The forced response component  $v_{Cf}$  of the capacitor voltage for the circuit below is
- A. 10 V
  - B. 2 V
  - C.  $32/3$  V
  - D. 8 V
  - E. none of the above

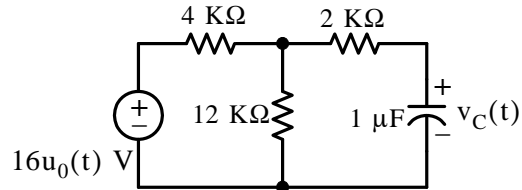


5. For the circuit below  $i_L(0^-) = 2$  A. For  $t > 0$  the total response of  $i_L(t)$  is
- A. 6 A
  - B.  $6e^{-5000t}$  A
  - C.  $6 + 6e^{-5000t}$  A
  - D.  $6 - 4e^{-5000t}$  A
  - E. none of the above



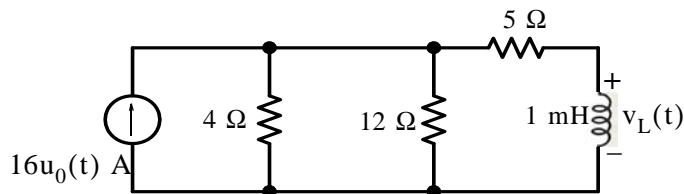
6. For the circuit below  $v_C(0^-) = 5$  V. For  $t > 0$  the total response of  $v_C(t)$  is

- A. 12 V
- B.  $10 - 5e^{-500t}$  V
- C.  $12 - 7e^{-200t}$  V
- D.  $12 + 7e^{-200t}$  V
- E. none of the above



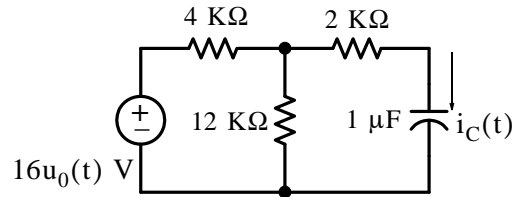
7. For the circuit below  $i_L(0^-) = 2$  A. For  $t > 0$  the total response of  $v_L(t)$  is

- A.  $20e^{-5000t}$  V
- B.  $20e^{-5000t}$  V
- C.  $-32e^{-8000t}$  V
- D.  $32e^{-8000t}$  V
- E. none of the above



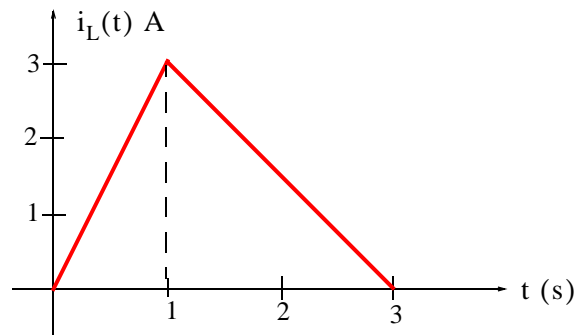
8. For the circuit below  $v_C(0^-) = 5$  V. For  $t > 0$  the total response  $i_C(t)$  is

- A.  $1400e^{-200t}$  A
- B.  $1.4e^{-200t}$  A
- C.  $3500e^{-500t}$  A
- D.  $3.5e^{-500t}$  A
- E. none of the above



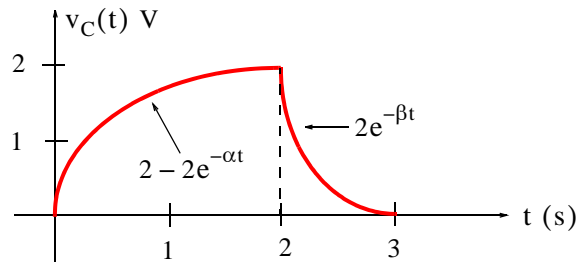
9. The waveform below can be expressed as

- A.  $3tu_0(t)$  A
- B.  $3[u_0(t)] - 3[u_0(t-3)]$  A
- C.  $3t[u_0(t) - u_0(t-1)] + (-1.5t + 4.5)[u_0(t-1) - u_0(t-3)]$  A
- D.  $3t[u_0(t) - u_0(t-3)] + (-1.5t + 4.5)[u_0(t) - u_0(t-3)]$  A
- E. none of the above



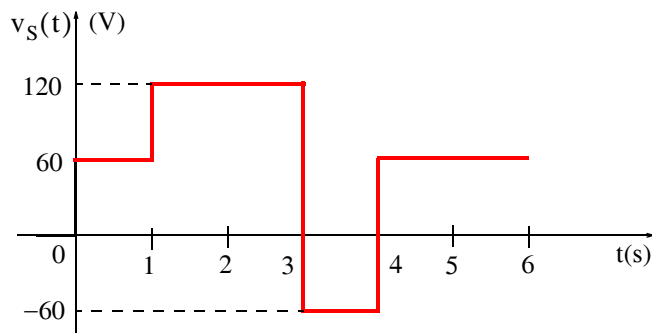
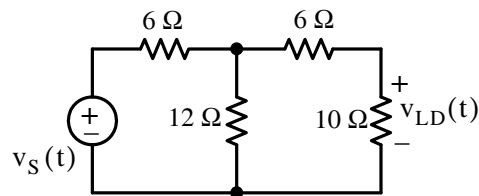
10. The waveform below can be expressed as

- A.  $2(1 - e^{-\alpha t} - e^{-\beta t})u_0(t)$  V
- B.  $(2 - 2e^{-\alpha t})[u_0(t) - u_0(t-2)] + (2e^{-\beta t})[u_0(t-2) - u_0(t-3)]$  V
- C.  $(2 - 2e^{-\alpha t})[u_0(t) - u_0(t-2)] - (2e^{-\beta t})[u_0(t-2) - u_0(t-3)]$  V
- D.  $(2 - 2e^{-\alpha t})[u_0(t)] - (2e^{-\beta t})[u_0(t-3)]$  V
- E. none of the above

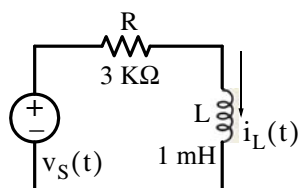


### Problems

1. In the circuit below, the voltage source  $v_S(t)$  varies with time as shown by the waveform below it. Compute, sketch, and express  $v_{LD}(t)$  as a sum of unit step functions for  $0 \leq t \leq 5$  s.



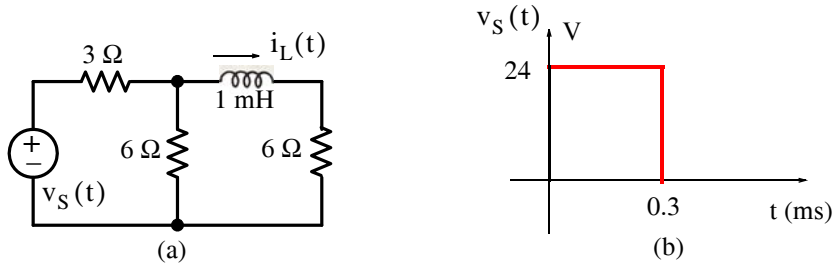
2. In the circuit below  $v_S(t) = 15u_0(t) - 30u_0(t - 2)$  V. Compute  $i_L(t)$  for  $t > 0$ .



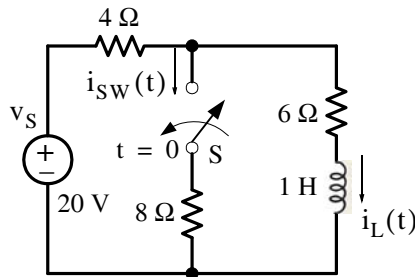


## Chapter 10 Forced and Total Response in RL and RC Circuits

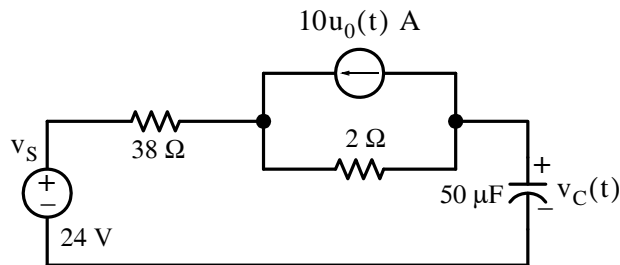
3. In the circuit below the excitation  $v_S(t)$  is a pulse shown next to it.
- Compute  $i_L(t)$  for  $0 < t < 0.3$  ms
  - Compute and sketch  $i_L(t)$  for all  $t > 0$



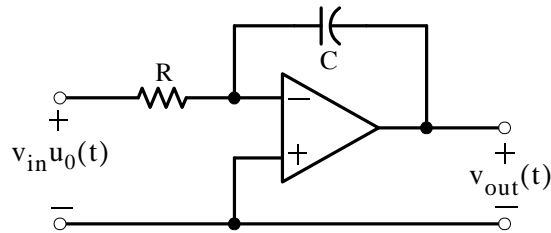
4. In the circuit below switch  $S$  has been open for a very long time and closes at  $t = 0$ . Compute and sketch  $i_L(t)$  and  $i_{SW}(t)$  for  $t > 0$ .



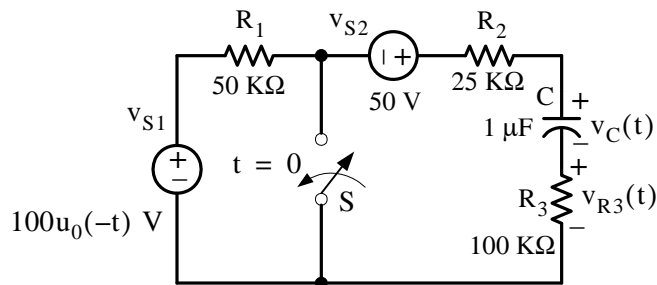
5. For the circuit below compute  $v_C(t)$  for  $t > 0$ .



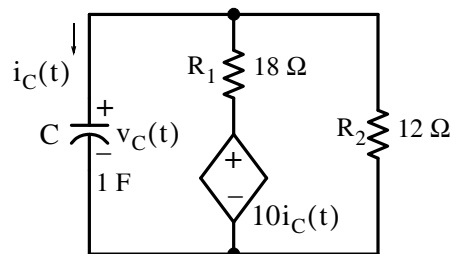
6. For the op amp circuit below compute  $v_{\text{out}}(t)$  for  $t > 0$  in terms of  $R$ ,  $C$ , and  $v_{\text{in}}u_0(t)$  given that  $v_C(0^-) = 0$



7. In the circuit of Figure 10.61, switch  $S$  has been open for a very long time and closes at  $t = 0$ . Compute and sketch  $v_C(t)$  and  $v_{R3}(t)$  for  $t > 0$ .



8. For the circuit below it is given that  $v_C(0^-) = 5 \text{ V}$ . Compute  $i_C(t)$  for  $t > 0$ . Hint: Be careful in deriving the time constant for this circuit.



9. A 12V DC, a 1 MΩ, and a 1 μF are connected in series. Create a Simulink / SimPower Systems model to display the waveform of the voltage across the capacitor as a function of time.

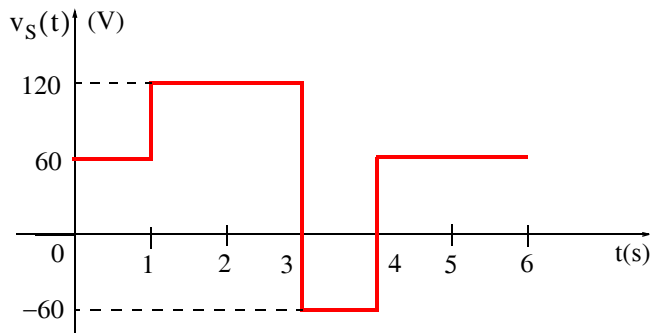
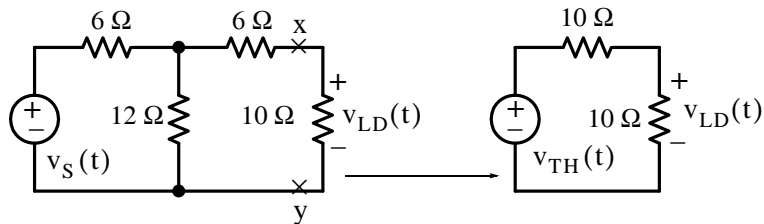
10.8 Answers / Solutions to End-of-Chapter Exercises

Multiple Choice

1. D
2. B
3. C
4. E 12 V
5. E  $6 - 4e^{-8000t}$  A
6. C
7. D
8. A
9. C
10. B

Problems

1. We replace the given circuit shown below with its Thevenin equivalent.



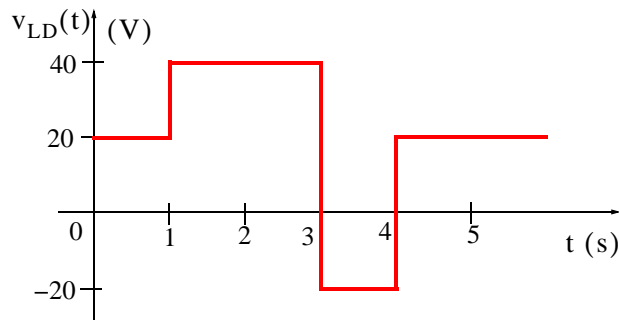
For the Thevenin equivalent voltage at different time intervals is as shown below.

$$v_{TH}(t) = \begin{cases} \frac{12}{18}v_s(t) = \frac{2}{3} \times 60 = 40 \text{ V} & 0 < t < 1 \text{ s} \\ \frac{12}{18}v_s(t) = \frac{2}{3} \times 120 = 80 \text{ V} & 1 < t < 3 \text{ s} \\ \frac{12}{18}v_s(t) = \frac{2}{3} \times (-60) = -40 \text{ V} & 3 < t < 4 \text{ s} \\ \frac{12}{18}v_s(t) = \frac{2}{3} \times 60 = 40 \text{ V} & t > 4 \text{ s} \end{cases}$$

and

$$v_{LD}(t) = \frac{10}{20}v_{TH}(t) = 0.5v_{TH}(t) = \begin{cases} 0.5 \times 40 = 20 \text{ V} & 0 < t < 1 \text{ s} \\ 0.5 \times 80 = 40 \text{ V} & 1 < t < 3 \text{ s} \\ 0.5 \times (-40) = -20 \text{ V} & 3 < t < 4 \text{ s} \\ 0.5 \times 40 = 20 \text{ V} & t > 4 \text{ s} \end{cases}$$

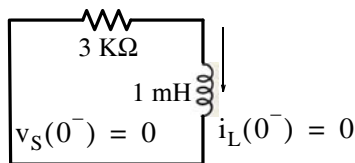
The waveform of the voltage across the load is as shown below.



The waveform above can now be expressed as a sum of unit step functions as follows:

$$\begin{aligned} v_{LD}(t) &= 20u_0(t) - 20u_0(t-1) + 40u_0(t-1) - 40u_0(t-3) + 20u_0(t-4) \\ &\quad - 20u_0(t-3) + 20u_0(t-4) + 20u_0(t-4) \\ &= 20u_0(t) + 20u_0(t-1) - 60u_0(t-3) + 40u_0(t-4) \end{aligned}$$

2. The circuit at  $t = 0^-$  is as shown below and since we are not told otherwise, we will assume that  $i_L(0^-) = 0$



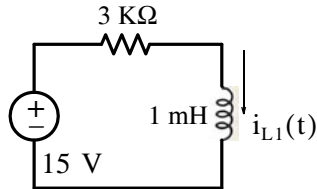
---

## Chapter 10 Forced and Total Response in RL and RC Circuits

---

For  $t > 0$  we let  $i_{L1}(t)$  be the inductor current when the  $15u_0(t)$  voltage source acts alone and  $i_{L2}(t)$  when the  $30u_0(t-2)$  voltage source acts alone. Then,  $i_{L\text{ TOTAL}}(t) = i_{L1}(t) + i_{L2}(t)$

For  $0 < t < 2$  s the circuit is as shown below.

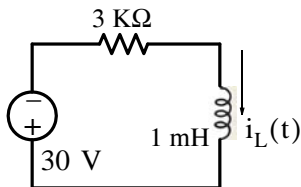


Then,  $i_{L1}(t) = i_{L1f} + i_{L1n}$  where  $i_{L1f} = \frac{15}{3\text{ k}\Omega} = 5\text{ mA}$  and  $i_{L1n} = A_1 e^{-(R/L)t} = A e^{-3 \times 10^6 t}$

Thus,  $i_{L1}(t) = 5 + A_1 e^{-3 \times 10^6 t}$  mA and using the initial condition  $i_L(0^-) = i_L(0^+) = 0$ , we obtain  $i_{L1}(0) = 5 + A_1 e^0$  mA or  $A_1 = -5$ . Therefore,

$$i_{L1}(t) = 5 - 5e^{-3 \times 10^6 t} \quad (1)$$

Next, with the  $30u_0(t-2)$  voltage source acts alone the circuit is as shown below.



Then,

$$i_{L2}(t) = i_{L2f} + i_{L2n}$$

$$i_{L2f} = \frac{-30}{3\text{ k}\Omega} = -10\text{ mA}$$

and

$$i_{L2n} = A_2 e^{-(R/L)(t-2)} = B e^{-3 \times 10^6 (t-2)}$$

Thus,

$$i_{L2}(t) = -10 + A_2 e^{-3 \times 10^6 (t-2)} \text{ mA}$$

and the initial condition at  $t = 2$  is found from (1) above as

$$i_{L1}\Big|_{t=2\text{ s}} = 5 - 5e^{-6 \times 10^6 t} \approx 5 \text{ mA}$$

Therefore,

$$i_{L2}\Big|_{t=2\text{ s}} = i_{L1}\Big|_{t=2\text{ s}} = 5 = -10 + A_2 e^{-3 \times 10^6 (2-2)} \text{ mA}$$

or

$$A_2 = 15$$

and

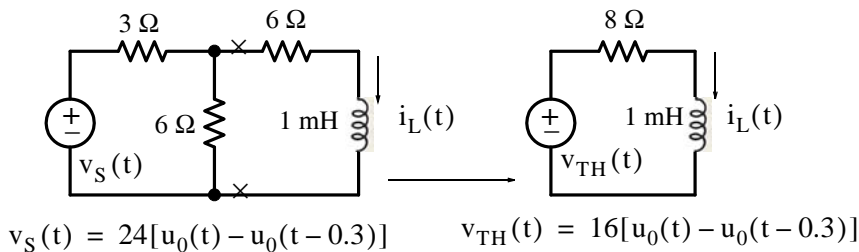
$$i_{L2}(t) = -10 + 15e^{-3 \times 10^6 (t-2)} \text{ mA} \quad (2)$$

Thus, the total current when both voltage sources are present is the summation of (1) and (2), that is,

$$\begin{aligned} i_{L\text{ TOTAL}}(t) &= i_{L1}(t) + i_{L2}(t) = 5 - 5e^{-3 \times 10^6 t} - 10 + 15e^{-3 \times 10^6 (t-2)} \text{ mA} \\ &= -5 - 5e^{-3 \times 10^6 t} + 15e^{-3 \times 10^6 (t-2)} \text{ mA} \end{aligned}$$

3.

- a. For this circuit  $v_S(t) = 24[u_0(t) - u_0(t - 0.3)]$  and since we are not told otherwise, we will assume that  $i_L(0^-) = 0$ . For  $0 < t < 0.3$  ms the circuit and its Thevenin equivalent are as shown below.



Then,

$$i_L(t) = i_{Lf} + i_{Ln} = 16/8 + A_1 e^{-(R/L)t} = 2 + A_1 e^{-8000t}$$

and at  $t = 0$

$$i_L(0) = i_L(0^-) = 0 = 2 + A_1 e^0$$

or

$$A_1 = -2$$

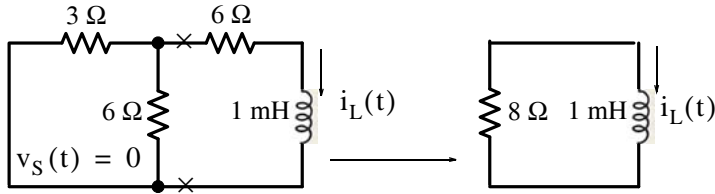
and thus for  $0 < t < 0.3$  ms

$$i_L(t) = 2 - 2e^{-8000t} \quad (1)$$

- b. For  $t > 0.3$  ms the circuit is as shown below. For this circuit

$$i_L(t) = A_2 e^{-(R/L)(t-0.3)} = A_2 e^{-8000(t-0.3)} \quad (2)$$

and  $A_2$  is found from the initial condition at  $t = 0.3$  ms, that is, with (1) above we obtain



$$i_L|_{t=0.3 \text{ ms}} = 2 - 2e^{-8 \times 10^3 \times 0.3 \times 10^{-3}} = 2 - 2e^{2.4} = 1.82 \text{ A}$$

and by substitution into (2) above

$$i_L|_{t=0.3 \text{ ms}} = 1.82 = A_2 e^{-8000(0.3-0.3)}$$

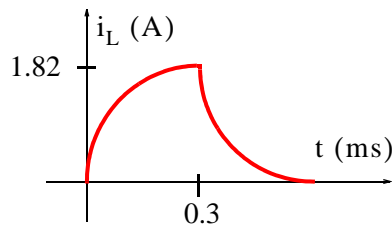
or

$$A_2 = 1.82$$

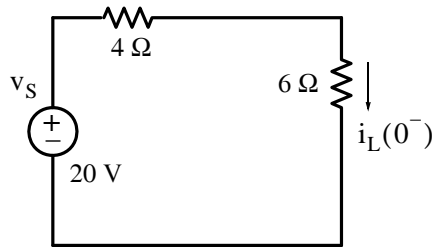
Therefore for  $t > 0.3$  ms

$$i_L(t) = 1.82 e^{-(t-0.3 \text{ ms})}$$

The waveform for the inductor current  $i_L(t)$  for all  $t > 0$  is shown below.



4. At  $t = 0^-$  the circuit is as shown below where  $i_L(0^-) = 20/(4 + 6) = 2$  A and thus the initial condition has been established.

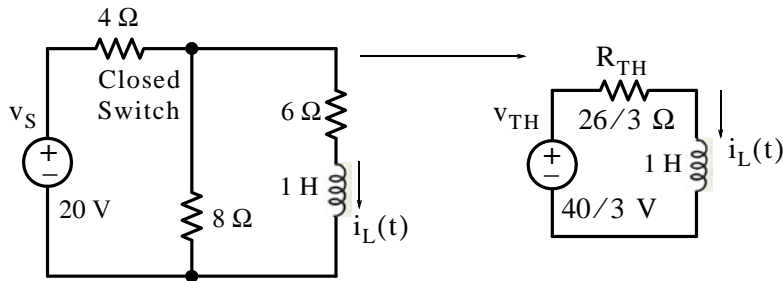


For  $t > 0$  the circuit and its Thevenin equivalent are as shown below where

$$v_{TH} = \frac{8}{4+8} \times 20 = 40/3 \text{ V}$$

and

$$R_{TH} = \frac{8 \times 4}{8+4} + 6 = 26/3 \text{ } \Omega$$



Then,

$$i_L(t) = i_{Lf} + i_{Ln} = \frac{40/3}{26/3} + Ae^{-(R/L)t} = 20/13 + Ae^{-(26/3)t}$$

and  $A$  is evaluated from the initial condition, i.e.,

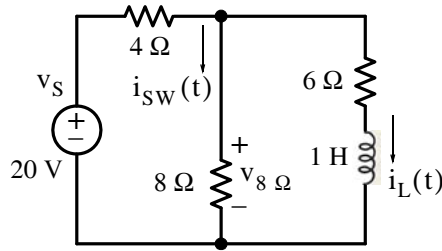
$$i_L(0^-) = i_L(0^+) = 2 = 20/13 + Ae^0$$

from which  $A = 6/13$  and thus for  $t > 0$

$$i_L(t) = \frac{20}{13} + \frac{6}{13}e^{-(26/3)t} = 1.54 + 0.46e^{-8.67t} \text{ A} \quad (1)$$

Next, to find  $i_{S_W}(t)$  we observe that this current flows also through the  $8 \text{ } \Omega$  resistor and this can be found from  $v_{8 \text{ } \Omega}$  shown on the circuit below.





Now,

$$\begin{aligned}
 v_{8\ \Omega} &= v_{6\ \Omega} + v_L(t) = 6i_L(t) + L \frac{di_L}{dt} \\
 &= 6(1.54 + 0.46e^{-8.67t}) + 1 \times \frac{d}{dt}(1.54 + 0.46e^{-8.67t}) \\
 &= 9.24 + 2.76e^{-8.67t} - 8.67 \times 0.46e^{-8.67t} \\
 &= 9.24 - 1.23e^{-8.67t}
 \end{aligned}$$

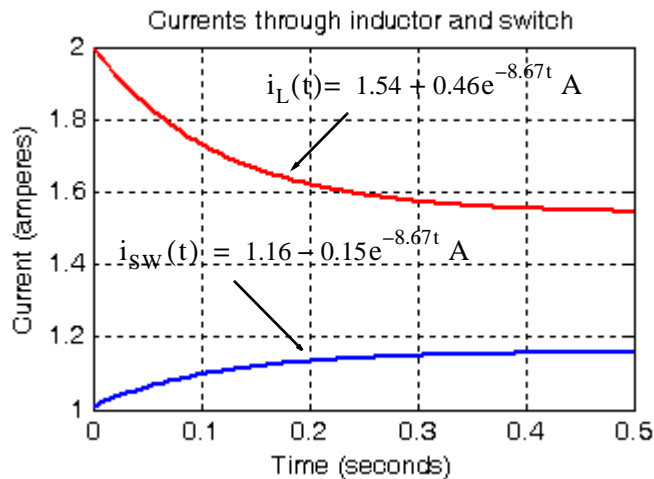
and

$$i_{sw}(t) = i_{8\ \Omega} = \frac{v_{8\ \Omega}}{8} = \frac{9.24 - 1.23e^{-8.67t}}{8} = 1.16 - 0.15e^{-8.67t} \text{ A} \quad (2)$$

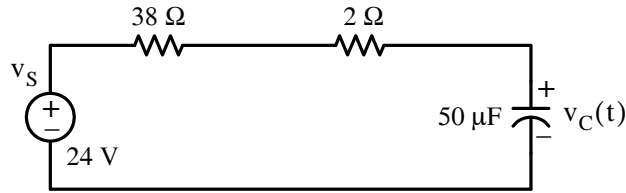
Therefore, from the initial condition, (1) and (2) above we have

$$i_L(0^+) = 2 \quad i_L(\infty) = 1.54 \quad i_{sw}(0^+) = 1.16 - 0.15 = 1.01 \quad i_{sw}(\infty) = 1.16$$

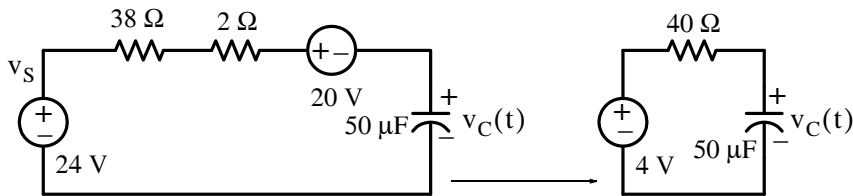
and with these values we sketch  $i_L(t)$  and  $i_{sw}(t)$  as shown below.



5. At  $t = 0^-$  the circuit is as shown below where  $v_C(0^-) = 24 \text{ V}$  and thus the initial condition has been established.



The circuit for  $t > 0$  is shown below where the current source has been replaced with a voltage source.



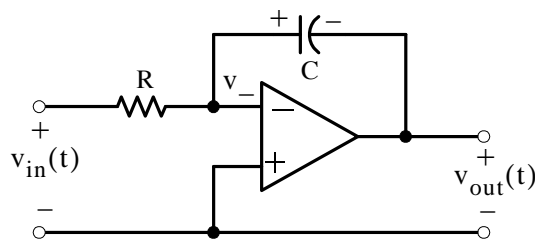
Now,

$$v_C(t) = v_{Cf} + v_{Cn} = 4 + Ae^{-1/(RC)t} = 4 + Ae^{-500t}$$

and with the initial condition  $v_C(0^-) = v_C(0^+) = 24 \text{ V} = 4 + Ae^0$  from which  $A = 20$  we obtain

$$v_C(t) = 4 + 20e^{-500t}$$

6. For  $t > 0$  the op amp circuit is as shown below.



Application of KCL at the minus (-) input yields

$$\frac{v_- - v_{in}}{R} + C \frac{dv_C}{dt} = 0$$

and since  $v_- = 0$

$$C \frac{dv_C}{dt} = \frac{v_{in}}{R}$$

or

$$\frac{dv_C}{dt} = \frac{v_{in}}{RC}$$

Integrating both sides and observing that  $v_{out}(t) = -v_C(t)$  we obtain

$$v_{out}(t) = -\frac{v_{in}}{RC}t + k$$

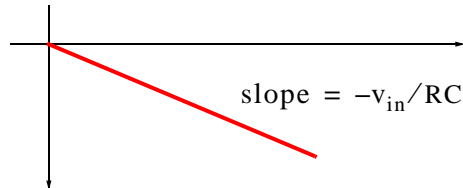
where  $k$  is the constant of integration of both sides and it is evaluated from the given initial condition. Then,

$$v_C(0^-) = v_C(0^+) = 0 = 0 + k$$

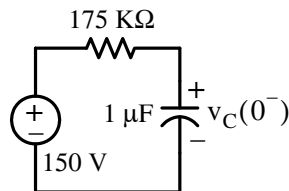
or  $k = 0$ . Therefore,

$$v_{out}(t) = -\left(\frac{v_{in}}{RC} t\right)u_0(t)$$

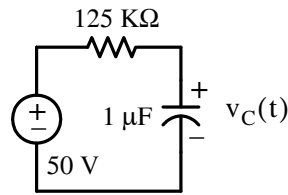
and  $v_{in}/RC$  is the slope as shown below.



7. At  $t = 0^-$  the circuit is as shown below where  $v_C(0^-) = 150$  V and thus the initial condition has been established.



The circuit for  $t > 0$  is shown below where the voltage source  $v_{s1}$  is absent for all positive time and the  $50$  KΩ is shorted out by the closed switch.



For the circuit above

$$v_C(t) = v_{Cf} + v_{Cn} = 50 + Ae^{-t/(RC)} = 50 + Ae^{-8t}$$

and with the initial condition

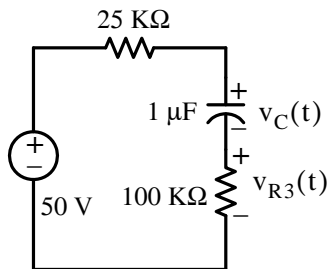
$$v_C(0^-) = v_C(0^+) = 150 = 50 + Ae^0$$

from which  $A = 100$  and thus for  $t > 0$

$$v_C(t) = 50 + 100e^{-8t} \text{ V}$$

To find  $v_{R3}(t)$  we will first find  $i_C(t)$  from the circuit below where

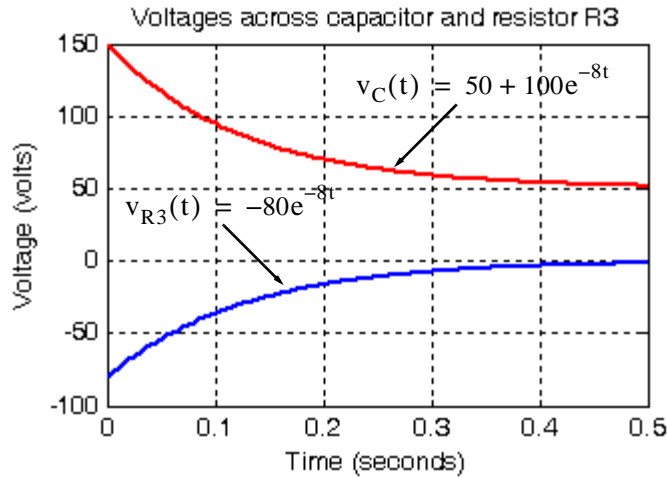
$$i_C(t) = C \frac{dv_C}{dt} = 10^{-6} \times (-8 \times 10^{-4} e^{-8t})$$



Then,

$$v_{R3}(t) = (100 \text{ K}\Omega)i_C = 10^5(-8 \times 10^{-4} e^{-8t}) = -80e^{-8t} \text{ V}$$

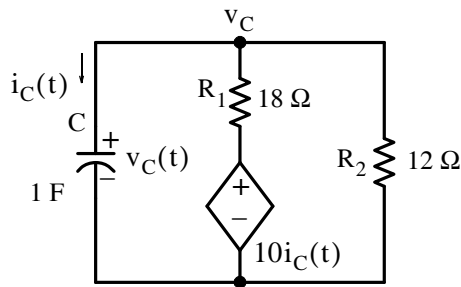
The sketches below show  $v_C(t)$  and  $v_{R3}(t)$  as they approach their final values.



8. For this circuit we cannot short the dependent source and therefore we cannot find  $R_{eq}$  by combining the resistances  $R_1$  and  $R_2$  in parallel combination in order to find the time constant  $\tau = RC$ . Instead, we will derive the time constant from the differential equation of (9.9) of the previous chapter, that is,

$$\frac{dv_C}{dt} + \frac{v_C}{RC} = 0$$

From the given circuit shown below,



$$i_C + \frac{v_C - 10i_C}{R_1} + \frac{v_C}{R_2}$$

or

$$C \frac{dv_C}{dt} + \left( \frac{1}{R_1} + \frac{1}{R_2} \right) v_C - \frac{10}{R_1} C \frac{dv_C}{dt} = 0$$

or

$$\left( 1 - \frac{10}{R_1} \right) \left( C \frac{dv_C}{dt} \right) + \left( \frac{R_1 + R_2}{R_1 \cdot R_2} \right) v_C = 0$$

or

$$\frac{dv_C}{dt} + \frac{\left(\frac{R_1 + R_2}{R_1 \cdot R_2}\right)}{\left(1 - \frac{10}{R_1}\right) \cdot C} v_C = 0$$

and from this differential equation we see that the coefficient of  $v_C$  is

$$\frac{1}{R_{eq}C} = \frac{1}{\tau} = \frac{30/216}{\left(1 - \frac{10}{18}\right) \cdot 1} = \frac{30/216}{8/18} = \frac{15 \times 9}{4 \times 108} = \frac{135}{432} = \frac{5}{16} = 0.3125$$

and thus

$$v_C(t) = Ae^{-0.3125t}$$

and with the given initial condition  $v_C(0^-) = V_0 = A = 5 \text{ V}$  we obtain

$$v_C(t) = 5e^{-0.3125t}$$

Then, using the relation

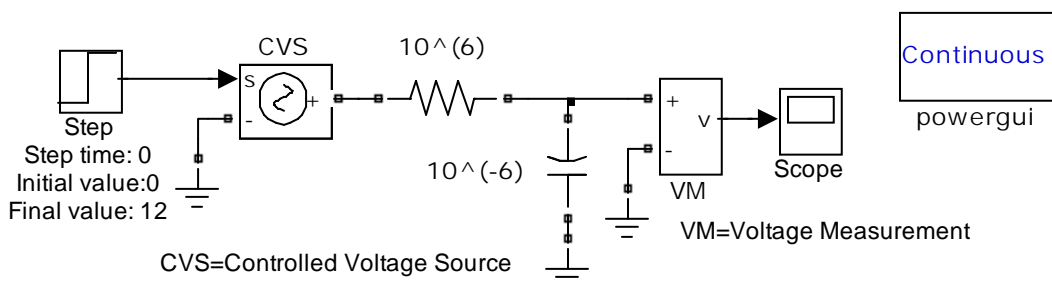
$$i_C = C \frac{dv_C}{dt}$$

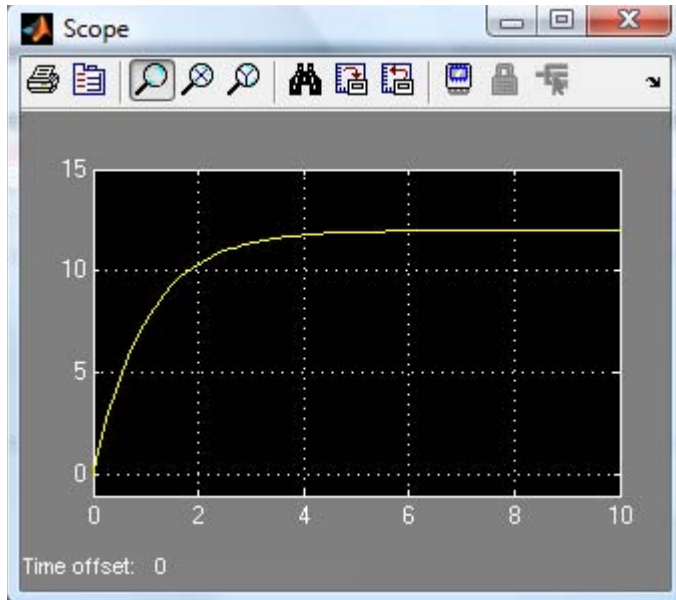
we find that for  $t > 0$

$$i_C(t) = (1)(-0.3125 \times 5e^{-0.3125t}) = -1.5625e^{-0.3125t}$$

and the minus (-) sign indicates that the  $i_C(t)$  direction is opposite to that shown.

9.





This appendix serves as an introduction to the basic MATLAB commands and functions, procedures for naming and saving the user generated files, comment lines, access to MATLAB's Editor / Debugger, finding the roots of a polynomial, and making plots. Several examples are provided with detailed explanations.

### A.1 MATLAB® and Simulink®

MATLAB and Simulink are products of The MathWorks,™ Inc. These are two outstanding software packages for scientific and engineering computations and are used in educational institutions and in industries including automotive, aerospace, electronics, telecommunications, and environmental applications. MATLAB enables us to solve many advanced numerical problems rapidly and efficiently.

### A.2 Command Window

To distinguish the screen displays from the user commands, important terms, and MATLAB functions, we will use the following conventions:

*Click*: Click the left button of the mouse

*Courier Font*: Screen displays

*Helvetica Font*: User inputs at MATLAB's command window prompt >> or EDU>>\*

**Helvetica Bold**: MATLAB functions

**Times Bold Italic**: Important terms and facts, notes and file names

When we first start MATLAB, we see various help topics and other information. Initially, we are interested in the *command screen* which can be selected from the Window drop menu. When the command screen, we see the prompt >> or EDU>>. This prompt is displayed also after execution of a command; MATLAB now waits for a new command from the user. It is highly recommended that we use the *Editor/Debugger* to write our program, save it, and return to the command screen to execute the program as explained below.

To use the Editor/Debugger:

1. From the *File* menu on the toolbar, we choose *New* and click on *M-File*. This takes us to the *Editor Window* where we can type our *script* (list of statements) for a new file, or open a previously saved file. We must save our program with a file name which starts with a letter.

---

\* EDU>> is the MATLAB prompt in the Student Version



**Important!** MATLAB is *case sensitive*, that is, it distinguishes between upper- and lower-case letters. Thus, *t* and *T* are two different letters in MATLAB language. The files that we create are saved with the file name we use and the extension *.m*; for example, *myfile01.m*. It is a good practice to save the script in a file name that is descriptive of our script content. For instance, if the script performs some matrix operations, we ought to name and save that file as *matrices01.m* or any other similar name. We should also use a floppy disk or an external drive to backup our files.

2. Once the script is written and saved as an *m-file*, we may exit the *Editor/Debugger* window by clicking on *Exit Editor/Debugger* of the *File* menu. MATLAB then returns to the command window.
3. To execute a program, we type the file name **without** the *.m* extension at the `>>` prompt; then, we press `<enter>` and observe the execution and the values obtained from it. If we have saved our file in drive *a* or any other drive, we must make sure that it is added to the desired directory in MATLAB's search path. The MATLAB User's Guide provides more information on this topic.

Henceforth, it will be understood that each input command is typed after the `>>` prompt and followed by the `<enter>` key.

The command `help matlab\iofun` will display input/output information. To get help with other MATLAB topics, we can type `help` followed by any topic from the displayed menu. For example, to get information on graphics, we type `help matlab\graphics`. The MATLAB User's Guide contains numerous help topics.

To appreciate MATLAB's capabilities, we type `demo` and we see the MATLAB Demos menu. We can do this periodically to become familiar with them. Whenever we want to return to the command window, we click on the `Close` button.

When we are done and want to leave MATLAB, we type `quit` or `exit`. But if we want to clear all previous values, variables, and equations without exiting, we should use the command `clear`. This command erases everything; it is like exiting MATLAB and starting it again. The command `clc` clears the screen but MATLAB still remembers all values, variables and equations that we have already used. In other words, if we want to clear all previously entered commands, leaving only the `>>` prompt on the upper left of the screen, we use the `clc` command.

All text after the `%` (percent) symbol is interpreted as a *comment line* by MATLAB, and thus it is ignored during the execution of a program. A comment can be typed on the same line as the function or command or as a separate line. For instance,

```
conv(p,q) % performs multiplication of polynomials p and q
```

```
% The next statement performs partial fraction expansion of p(x) / q(x)
```

are both correct.

One of the most powerful features of MATLAB is the ability to do computations involving *complex numbers*. We can use either `i`, or `j` to denote the imaginary part of a complex number, such as `3-4i` or `3-4j`. For example, the statement

```
z=3-4j
```

displays

```
z = 3.0000-4.0000i
```

In the above example, a multiplication (\*) sign between 4 and j was not necessary because the complex number consists of numerical constants. However, if the imaginary part is a function, or variable such as `cos(x)`, we must use the multiplication sign, that is, we must type `cos(x)*j` or `j*cos(x)` for the imaginary part of the complex number.

### A.3 Roots of Polynomials

In MATLAB, a polynomial is expressed as a *row vector* of the form  $[a_n \ a_{n-1} \ \dots \ a_2 \ a_1 \ a_0]$ . These are the coefficients of the polynomial in descending order. **We must include terms whose coefficients are zero.**

We find the roots of any polynomial with the **roots(p)** function; **p** is a row vector containing the polynomial coefficients in descending order.

---

#### Example A.1

Find the roots of the polynomial

$$p_1(x) = x^4 - 10x^3 + 35x^2 - 50x + 24$$

**Solution:**

The roots are found with the following two statements where we have denoted the polynomial as `p1`, and the roots as `roots_p1`.

```
p1=[1 -10 35 -50 24]    % Specify and display the coefficients of p1(x)
```

```
p1 =
     1    -10     35    -50     24
```

```
roots_p1=roots(p1)      % Find the roots of p1(x)
```

```
roots_p1 =
     4.0000
     3.0000
     2.0000
     1.0000
```

We observe that MATLAB displays the polynomial coefficients as a row vector, and the roots as a column vector.

---

### Example A.2

Find the roots of the polynomial

$$p_2(x) = x^5 - 7x^4 + 16x^2 + 25x + 52$$

#### Solution:

There is no cube term; therefore, we must enter zero as its coefficient. The roots are found with the statements below, where we have defined the polynomial as **p2**, and the roots of this polynomial as **roots\_p2**. The result indicates that this polynomial has three real roots, and two complex roots. Of course, complex roots always occur in *complex conjugate*\* pairs.

```
p2=[1 -7 0 16 25 52]
```

```
p2 =  
    1    -7     0    16    25    52
```

```
roots_p2=roots(p2)
```

```
roots_p2 =  
    6.5014  
    2.7428  
   -1.5711  
   -0.3366 + 1.3202i  
   -0.3366 - 1.3202i
```

---

## A.4 Polynomial Construction from Known Roots

We can compute the coefficients of a polynomial, from a given set of roots, with the **poly(r)** function where **r** is a row vector containing the roots.

---

### Example A.3

It is known that the roots of a polynomial are 1, 2, 3, and 4. Compute the coefficients of this polynomial.

---

\* By definition, the conjugate of a complex number  $A = a + jb$  is  $A^* = a - jb$

### Solution:

We first define a row vector, say  $r3$ , with the given roots as elements of this vector; then, we find the coefficients with the **poly(r)** function as shown below.

```
r3=[1 2 3 4]      % Specify the roots of the polynomial
```

```
r3 =
```

```
     1     2     3     4
```

```
poly_r3=poly(r3)  % Find the polynomial coefficients
```

```
poly_r3 =
```

```
     1    -10     35    -50     24
```

We observe that these are the coefficients of the polynomial  $p_1(x)$  of Example A.1.

---

### Example A.4

It is known that the roots of a polynomial are  $-1$ ,  $-2$ ,  $-3$ ,  $4 + j5$ , and  $4 - j5$ . Find the coefficients of this polynomial.

### Solution:

We form a row vector, say  $r4$ , with the given roots, and we find the polynomial coefficients with the **poly(r)** function as shown below.

```
r4=[-1 -2 -3 4+5j 4-5j]
```

```
r4 =
```

```
Columns 1 through 4
```

```
-1.0000    -2.0000   -3.0000   -4.0000+ 5.0000i
```

```
Column 5
```

```
-4.0000- 5.0000i
```

```
poly_r4=poly(r4)
```

```
poly_r4 =
```

```
     1     14     100     340     499     246
```

Therefore, the polynomial is

$$p_4(x) = x^5 + 14x^4 + 100x^3 + 340x^2 + 499x + 246$$

---

### A.5 Evaluation of a Polynomial at Specified Values

The **polyval(p,x)** function evaluates a polynomial  $p(x)$  at some specified value of the independent variable  $x$ .

---

#### Example A.5

Evaluate the polynomial

$$p_5(x) = x^6 - 3x^5 + 5x^3 - 4x^2 + 3x + 2 \quad (\text{A.1})$$

at  $x = -3$ .

**Solution:**

```
p5=[1 -3 0 5 -4 3 2]; % These are the coefficients of the given polynomial
% The semicolon (;) after the right bracket suppresses the
% display of the row vector that contains the coefficients of p5.
%
val_minus3=polyval(p5, -3) % Evaluate p5 at x=-3; no semicolon is used here
% because we want the answer to be displayed

val_minus3 =
    1280
```

---

Other MATLAB functions used with polynomials are the following:

**conv(a,b)** – multiplies two polynomials **a** and **b**

**[q,r]=deconv(c,d)** –divides polynomial **c** by polynomial **d** and displays the quotient **q** and remainder **r**.

**polyder(p)** – produces the coefficients of the derivative of a polynomial **p**.

---

#### Example A.6

Let

$$p_1 = x^5 - 3x^4 + 5x^2 + 7x + 9$$

and

$$p_2 = 2x^6 - 8x^4 + 4x^2 + 10x + 12$$

Compute the product  $p_1 \cdot p_2$  using the **conv(a,b)** function.

**Solution:**

```
p1=[1 -3 0 5 7 9];      % The coefficients of p1
p2=[2 0 -8 0 4 10 12]; % The coefficients of p2
p1p2=conv(p1,p2)       % Multiply p1 by p2 to compute coefficients of the product p1p2
```

```
p1p2 =
     2    -6    -8    34    18   -24   -74   -88    78   166   174   108
```

Therefore,

$$p_1 \cdot p_2 = 2x^{11} - 6x^{10} - 8x^9 + 34x^8 + 18x^7 - 24x^6 - 74x^5 - 88x^4 + 78x^3 + 166x^2 + 174x + 108$$

---

### Example A.7

Let

$$p_3 = x^7 - 3x^5 + 5x^3 + 7x + 9$$

and

$$p_4 = 2x^6 - 8x^5 + 4x^2 + 10x + 12$$

Compute the quotient  $p_3/p_4$  using the **[q,r]=deconv(c,d)** function.

**Solution:**

```
% It is permissible to write two or more statements in one line separated by semicolons
p3=[1 0 -3 0 5 7 9]; p4=[2 -8 0 0 4 10 12]; [q,r]=deconv(p3,p4)
```

```
q =
    0.5000
r =
     0     4    -3     0     3     2     3
```

Therefore,

$$q = 0.5 \quad r = 4x^5 - 3x^4 + 3x^2 + 2x + 3$$

---

### Example A.8

Let

$$p_5 = 2x^6 - 8x^4 + 4x^2 + 10x + 12$$

Compute the derivative  $\frac{d}{dx}p_5$  using the **polyder(p)** function.

### Solution:

```
p5=[2 0 -8 0 4 10 12]; % The coefficients of p5
der_p5=polyder(p5) % Compute the coefficients of the derivative of p5
```

```
der_p5 =
    12     0   -32     0     8    10
```

Therefore,

$$\frac{d}{dx}p_5 = 12x^5 - 32x^3 + 4x^2 + 8x + 10$$

---

## A.6 Rational Polynomials

*Rational Polynomials* are those which can be expressed in ratio form, that is, as

$$R(x) = \frac{\text{Num}(x)}{\text{Den}(x)} = \frac{b_n x^n + b_{n-1} x^{n-1} + b_{n-2} x^{n-2} + \dots + b_1 x + b_0}{a_m x^m + a_{m-1} x^{m-1} + a_{m-2} x^{m-2} + \dots + a_1 x + a_0} \quad (\text{A.2})$$

where some of the terms in the numerator and/or denominator may be zero. We can find the roots of the numerator and denominator with the **roots(p)** function as before.

As noted in the comment line of Example A.7, we can write MATLAB statements in one line, if we separate them by commas or semicolons. **Commas will display the results whereas semicolons will suppress the display.**

---

### Example A.9

Let

$$R(x) = \frac{p_{\text{num}}}{p_{\text{den}}} = \frac{x^5 - 3x^4 + 5x^2 + 7x + 9}{x^6 - 4x^4 + 2x^2 + 5x + 6}$$

Express the numerator and denominator in factored form, using the **roots(p)** function.

### Solution:

```
num=[1 -3 0 5 7 9]; den=[1 0 -4 0 2 5 6]; % Do not display num and den coefficients
roots_num=roots(num), roots_den=roots(den) % Display num and den roots
```

```
roots_num =
    2.4186 + 1.0712i    2.4186 - 1.0712i   -1.1633
   -0.3370 + 0.9961i   -0.3370 - 0.9961i
```

```
roots_den =
    1.6760 + 0.4922i    1.6760 - 0.4922i    -1.9304
   -0.2108 + 0.9870i   -0.2108 - 0.9870i    -1.0000
```

As expected, the complex roots occur in complex conjugate pairs.

For the numerator, we have the factored form

$$P_{\text{num}} = (x-2.4186 - j1.0712)(x-2.4186 + j1.0712)(x + 1.1633) \\ (x + 0.3370 - j0.9961)(x + 0.3370 + j0.9961)$$

and for the denominator, we have

$$P_{\text{den}} = (x-1.6760 - j0.4922)(x-1.6760 + j0.4922)(x + 1.9304) \\ (x + 0.2108 - j0.9870)(x + 0.2108 + j0.9870)(x + 1.0000)$$

We can also express the numerator and denominator of this rational function as a combination of *linear* and *quadratic* factors. We recall that, in a quadratic equation of the form  $x^2 + bx + c = 0$  whose roots are  $x_1$  and  $x_2$ , the negative sum of the roots is equal to the coefficient  $b$  of the  $x$  term, that is,  $-(x_1 + x_2) = b$ , while the product of the roots is equal to the constant term  $c$ , that is,  $x_1 \cdot x_2 = c$ . Accordingly, we form the coefficient  $b$  by addition of the complex conjugate roots and this is done by inspection; then we multiply the complex conjugate roots to obtain the constant term  $c$  using MATLAB as follows:

```
(2.4186 + 1.0712i)*(2.4186 -1.0712i)
ans = 6.9971
(-0.3370+ 0.9961i)*(-0.3370-0.9961i)
ans = 1.1058
(1.6760+ 0.4922i)*(1.6760-0.4922i)
ans = 3.0512
(-0.2108+ 0.9870i)*(-0.2108-0.9870i)
ans = 1.0186
```

Thus,

$$R(x) = \frac{P_{\text{num}}}{P_{\text{den}}} = \frac{(x^2 - 4.8372x + 6.9971)(x^2 + 0.6740x + 1.1058)(x + 1.1633)}{(x^2 - 3.3520x + 3.0512)(x^2 + 0.4216x + 1.0186)(x + 1.0000)(x + 1.9304)}$$



We can check this result of Example A.9 above with MATLAB's *Symbolic Math Toolbox* which is a collection of tools (functions) used in solving symbolic expressions. They are discussed in detail in MATLAB's Users Manual. For the present, our interest is in using the **collect(s)** function that is used to multiply two or more symbolic expressions to obtain the result in polynomial form. We must remember that the **conv(p,q)** function is used with numeric expressions only, that is, polynomial coefficients.

Before using a symbolic expression, we must create one or more symbolic variables such as  $x$ ,  $y$ ,  $t$ , and so on. For our example, we use the following script:

```
syms x % Define a symbolic variable and use collect(s) to express numerator in polynomial form
collect((x^2-4.8372*x+6.9971)*(x^2+0.6740*x+1.1058)*(x+1.1633))
```

```
ans =
x^5-29999/10000*x^4-1323/3125000*x^3+7813277909/
1562500000*x^2+1750276323053/250000000000*x+4500454743147/
500000000000
```

and if we simplify this, we find that is the same as the numerator of the given rational expression in polynomial form. We can use the same procedure to verify the denominator.

### A.7 Using MATLAB to Make Plots

Quite often, we want to plot a set of ordered pairs. This is a very easy task with the MATLAB **plot(x,y)** command that plots  $y$  versus  $x$ , where  $x$  is the horizontal axis (abscissa) and  $y$  is the vertical axis (ordinate).

---

#### Example A.10

Consider the electric circuit of Figure A.1, where the radian frequency  $\omega$  (radians/second) of the applied voltage was varied from 300 to 3000 in steps of 100 radians/second, while the amplitude was held constant.

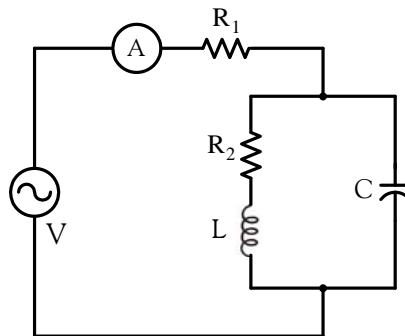


Figure A.1. Electric circuit for Example A.10

The ammeter readings were then recorded for each frequency. The magnitude of the impedance  $|Z|$  was computed as  $|Z| = |V/A|$  and the data were tabulated on Table A.1.

TABLE A.1 Table for Example A.10

$\omega$ (rads/s)	$ Z $ Ohms	$\omega$ (rads/s)	$ Z $ Ohms
300	39.339	1700	90.603
400	52.589	1800	81.088
500	71.184	1900	73.588
600	97.665	2000	67.513
700	140.437	2100	62.481
800	222.182	2200	58.240
900	436.056	2300	54.611
1000	1014.938	2400	51.428
1100	469.83	2500	48.717
1200	266.032	2600	46.286
1300	187.052	2700	44.122
1400	145.751	2800	42.182
1500	120.353	2900	40.432
1600	103.111	3000	38.845

Plot the magnitude of the impedance, that is,  $|Z|$  versus radian frequency  $\omega$ .

### Solution:

We cannot type  $\omega$  (omega) in the MATLAB Command prompt, so we will use the English letter  $w$  instead.

If a statement, or a row vector is too long to fit in one line, it can be continued to the next line by typing three or more periods, then pressing `<enter>` to start a new line, and continue to enter data. This is illustrated below for the data of  $w$  and  $z$ . Also, as mentioned before, we use the semi-colon (;) to suppress the display of numbers that we do not care to see on the screen.

The data are entered as follows:

```
w=[300 400 500 600 700 800 900 1000 1100 1200 1300 1400 1500 1600 1700 1800 1900...
2000 2100 2200 2300 2400 2500 2600 2700 2800 2900 3000];
%
z=[39.339 52.789 71.104 97.665 140.437 222.182 436.056...
1014.938 469.830 266.032 187.052 145.751 120.353 103.111...
90.603 81.088 73.588 67.513 62.481 58.240 54.611 51.468...
48.717 46.286 44.122 42.182 40.432 38.845];
```

Of course, if we want to see the values of  $w$  or  $z$  or both, we simply type  $w$  or  $z$ , and we press

`<enter>`. To plot  $z$  (y-axis) versus  $w$  (x-axis), we use the **plot(x,y)** command. For this example, we use **plot(w,z)**. When this command is executed, MATLAB displays the plot on MATLAB's *graph screen* and MATLAB denotes this plot as Figure 1. This plot is shown in Figure A.2.

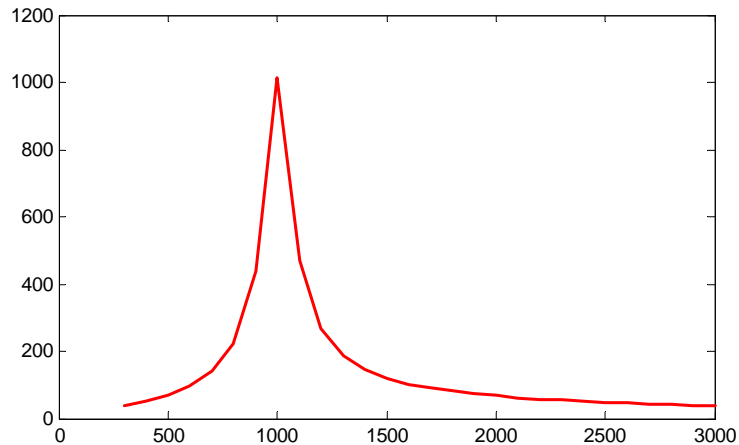


Figure A.2. Plot of impedance  $|z|$  versus frequency  $\omega$  for Example A.10

This plot is referred to as the *magnitude frequency response* of the circuit.

To return to the command window, we press any key, or from the *Window* pull-down menu, we select *MATLAB Command Window*. To see the graph again, we click on the *Window* pull-down menu, and we choose *Figure 1*.

We can make the above, or any plot, more presentable with the following commands:

**grid on:** This command adds grid lines to the plot. The **grid off** command removes the grid. The command **grid** toggles them, that is, changes from off to on or vice versa. The default\* is off.

**box off:** This command removes the box (the solid lines which enclose the plot), and **box on** restores the box. The command **box** toggles them. The default is on.

**title('string')**: This command adds a line of the text **string** (label) at the top of the plot.

**xlabel('string')** and **ylabel('string')** are used to label the x- and y-axis respectively.

The magnitude frequency response is usually represented with the x-axis in a logarithmic scale. We can use the **semilogx(x,y)** command which is similar to the **plot(x,y)** command, except that the x-axis is represented as a log scale, and the y-axis as a linear scale. Likewise, the **semilogy(x,y)** command is similar to the **plot(x,y)** command, except that the y-axis is represented as a

---

\* A default is a particular value for a variable that is assigned automatically by an operating system and remains in effect unless canceled or overridden by the operator.

log scale, and the x-axis as a linear scale. The **loglog(x,y)** command uses logarithmic scales for both axes.

Throughout this text it will be understood that **log** is the common (base 10) logarithm, and **ln** is the natural (base e) logarithm. We must remember, however, the function **log(x)** in MATLAB is the natural logarithm, whereas the common logarithm is expressed as **log10(x)**, and the logarithm to the base 2 as **log2(x)**.

Let us now redraw the plot with the above options by adding the following statements:

```
semilogx(w,z); grid;           % Replaces the plot(w,z) command
title('Magnitude of Impedance vs. Radian Frequency');
xlabel('w in rads/sec'); ylabel('|Z| in Ohms')
```

After execution of these commands, the plot is as shown in Figure A.3.

If the y-axis represents power, voltage or current, the x-axis of the frequency response is more often shown in a logarithmic scale, and the y-axis in dB (decibels).

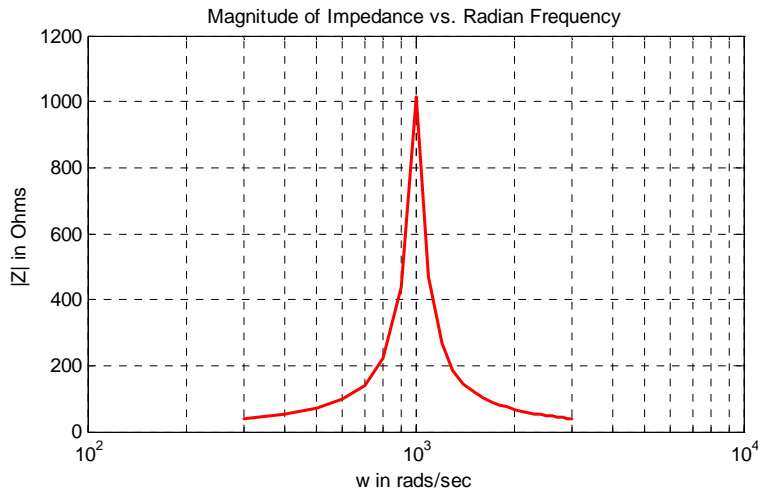


Figure A.3. Modified frequency response plot of Figure A.2.

To display the voltage  $v$  in a dB scale on the y-axis, we add the relation  $\text{dB}=20*\log_{10}(v)$ , and we replace the **semilogx(w,z)** command with **semilogx(w,dB)**.

The command **gtext('string')**<sup>\*</sup> switches to the current *Figure Window*, and displays a cross-hair that can be moved around with the mouse. For instance, we can use the command **gtext('Impedance |Z| versus Frequency')**, and this will place a cross-hair in the *Figure window*. Then, using

\* With the latest MATLAB Versions 6 and 7 (Student Editions 13 and 14), we can add text, lines and arrows directly into the graph using the tools provided on the *Figure Window*. For advanced MATLAB graphics, please refer to *The Math-Works Using MATLAB Graphics* documentation.

the mouse, we can move the cross-hair to the position where we want our label to begin, and we press <enter>.

The command **text(x,y,'string')** is similar to **gtext('string')**. It places a label on a plot in some specific location specified by **x** and **y**, and **string** is the label which we want to place at that location. We will illustrate its use with the following example which plots a 3-phase sinusoidal waveform.

The first line of the script below has the form

**linspace(first\_value, last\_value, number\_of\_values)**

This function specifies *the number of data points* but not the increments between data points. An alternate function is

**x=first: increment: last**

and this specifies *the increments between points* but not the number of data points.

The script for the 3-phase plot is as follows:

```
x=linspace(0, 2*pi, 60);      % pi is a built-in function in MATLAB;
% we could have used x=0:0.02*pi:2*pi or x = (0:0.02:2)*pi instead;
y=sin(x); u=sin(x+2*pi/3); v=sin(x+4*pi/3);
plot(x,y,x,u,x,v);          % The x-axis must be specified for each function
grid on, box on,            % turn grid and axes box on
text(0.75, 0.65, 'sin(x)'); text(2.85, 0.65, 'sin(x+2*pi/3)'); text(4.95, 0.65, 'sin(x+4*pi/3)')
```

These three waveforms are shown on the same plot of Figure A.4.

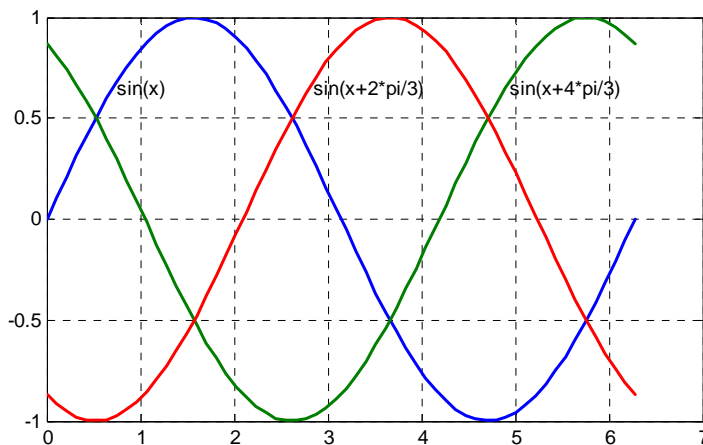


Figure A.4. Three-phase waveforms

In our previous examples, we did not specify line styles, markers, and colors for our plots. However, MATLAB allows us to specify various line types, plot symbols, and colors. These, or a combination of these, can be added with the **plot(x,y,s)** command, where **s** is a character string containing one or more characters shown on the three columns of Table A.2. MATLAB has no default color; it starts with blue and cycles through the first seven colors listed in Table A.2 for each additional line in the plot. Also, there is no default marker; no markers are drawn unless they are selected. The default line is the solid line. But with the latest MATLAB versions, we can select the line color, line width, and other options directly from the *Figure Window*.

TABLE A.2 Styles, colors, and markets used in MATLAB

Symbol	Color	Symbol	Marker	Symbol	Line Style
b	blue	.	point	—	solid line
g	green	o	circle	:	dotted line
r	red	x	x-mark	-.	dash-dot line
c	cyan	+	plus	---	dashed line
m	magenta	*	star		
y	yellow	s	square		
k	black	d	diamond		
w	white	v	triangle down		
		^	triangle up		
		<	triangle left		
		>	triangle right		
		p	pentagram		
		h	hexagram		

For example, **plot(x,y,'m\*:')** plots a magenta dotted line with a star at each data point, and **plot(x,y,'rs')** plots a red square at each data point, but does not draw any line because no line was selected. If we want to connect the data points with a solid line, we must type **plot(x,y,'rs-')**. For additional information we can type **help plot** in MATLAB's command screen.

The plots we have discussed thus far are two-dimensional, that is, they are drawn on two axes. MATLAB has also a three-dimensional (three-axes) capability and this is discussed next.

The **plot3(x,y,z)** command plots a line in 3-space through the points whose coordinates are the elements of *x*, *y* and *z*, where *x*, *y* and *z* are three vectors of the same length.

The general format is **plot3(x<sub>1</sub>,y<sub>1</sub>,z<sub>1</sub>,s<sub>1</sub>,x<sub>2</sub>,y<sub>2</sub>,z<sub>2</sub>,s<sub>2</sub>,x<sub>3</sub>,y<sub>3</sub>,z<sub>3</sub>,s<sub>3</sub>,...)** where **x<sub>n</sub>**, **y<sub>n</sub>** and **z<sub>n</sub>** are vectors or matrices, and **s<sub>n</sub>** are strings specifying color, marker symbol, or line style. These strings are the same as those of the two-dimensional plots.

### Example A.11

Plot the function

$$z = -2x^3 + x + 3y^2 - 1 \quad (\text{A.3})$$

**Solution:**

We arbitrarily choose the interval (length) shown on the script below.

```
x = -10: 0.5: 10;      % Length of vector x
y = x;                % Length of vector y must be same as x
z = -2.*x.^3+x+3.*y.^2-1; % Vector z is function of both x and y*
plot3(x,y,z); grid
```

The three-dimensional plot is shown in Figure A.5.

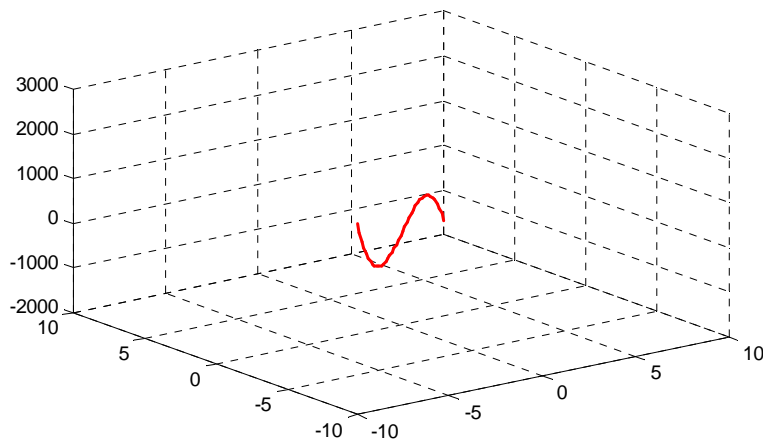


Figure A.5. Three dimensional plot for Example A.11

---

In a two-dimensional plot, we can set the limits of the *x*- and *y*-axes with the **axis([xmin xmax ymin ymax])** command. Likewise, in a three-dimensional plot we can set the limits of all three axes with the **axis([xmin xmax ymin ymax zmin zmax])** command. It must be placed after the **plot(x,y)** or **plot3(x,y,z)** commands, or on the same line without first executing the **plot** command. This must be done for each plot. The three-dimensional **text(x,y,z,'string')** command will place **string** beginning at the co-ordinate (x,y,z) on the plot.

For three-dimensional plots, **grid on** and **box off** are the default states.

---

\* This statement uses the so called dot multiplication, dot division, and dot exponentiation where the multiplication, division, and exponential operators are preceded by a dot. These important operations will be explained in Section A.9.

We can also use the **mesh(x,y,z)** command with two vector arguments. These must be defined as  $\text{length}(x) = n$  and  $\text{length}(y) = m$  where  $[m, n] = \text{size}(Z)$ . In this case, the vertices of the mesh lines are the triples  $\{x(j), y(i), Z(i, j)\}$ . We observe that **x** corresponds to the columns of *Z*, and **y** corresponds to the rows.

To produce a mesh plot of a function of two variables, say  $z = f(x, y)$ , we must first generate the *X* and *Y* matrices that consist of repeated rows and columns over the range of the variables *x* and *y*. We can generate the matrices *X* and *Y* with the **[X,Y]=meshgrid(x,y)** function that creates the matrix *X* whose rows are copies of the vector **x**, and the matrix *Y* whose columns are copies of the vector **y**.

### Example A.12

The volume *V* of a right circular cone of radius *r* and height *h* is given by

$$V = \frac{1}{3}\pi r^2 h \quad (\text{A.4})$$

Plot the volume of the cone as *r* and *h* vary on the intervals  $0 \leq r \leq 4$  and  $0 \leq h \leq 6$  meters.

#### Solution:

The volume of the cone is a function of both the radius *r* and the height *h*, that is,

$$V = f(r, h)$$

The three-dimensional plot is created with the following MATLAB script where, as in the previous example, in the second line we have used the dot multiplication, dot division, and dot exponentiation. This will be explained in Section A.9.

```
[R,H]=meshgrid(0:4,0:6); % Creates R and H matrices from vectors r and h;...
V=(pi .* R.^2 .* H) ./3; mesh(R, H, V);...
xlabel('x-axis, radius r (meters)'); ylabel('y-axis, altitude h (meters)');...
zlabel('z-axis, volume (cubic meters)'); title('Volume of Right Circular Cone'); box on
```

The three-dimensional plot of Figure A.6 shows how the volume of the cone increases as the radius and height are increased.

The plots of Figure A.5 and A.6 are rudimentary; MATLAB can generate very sophisticated three-dimensional plots. The MATLAB User's Manual and the Using MATLAB Graphics Manual contain numerous examples.



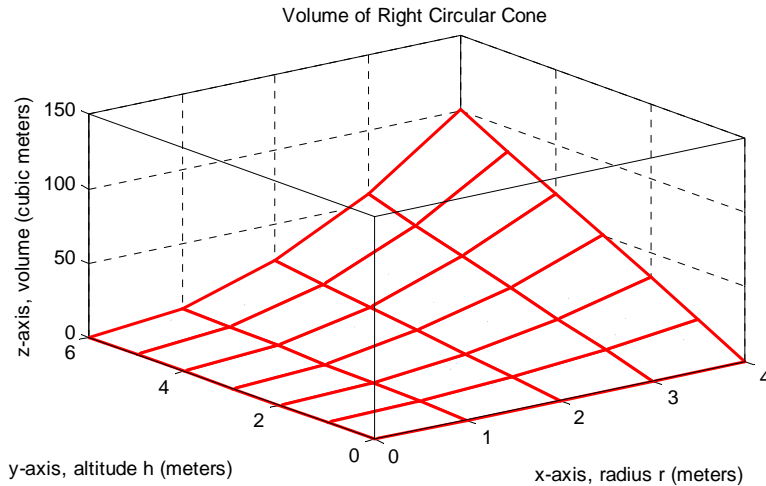


Figure A.6. Volume of a right circular cone.

### A.8 Subplots

MATLAB can display up to four windows of different plots on the *Figure* window using the command **subplot(m,n,p)**. This command divides the window into an  $m \times n$  matrix of plotting areas and chooses the  $p$ th area to be active. No spaces or commas are required between the three integers  $m$ ,  $n$  and  $p$ . The possible combinations are shown in Figure A.7.

We will illustrate the use of the **subplot(m,n,p)** command following the discussion on multiplication, division and exponentiation that follows.

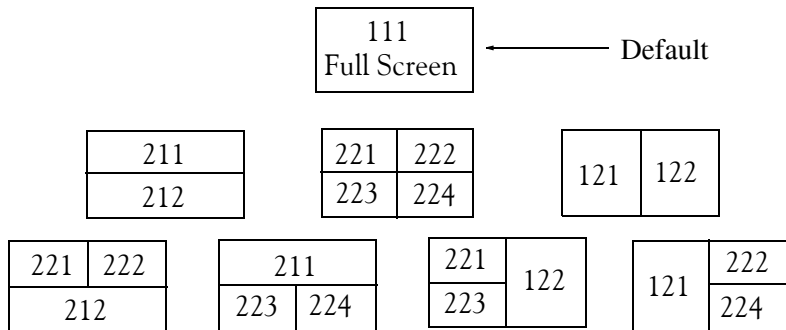


Figure A.7. Possible subplot arrangements in MATLAB

### A.9 Multiplication, Division, and Exponentiation

MATLAB recognizes two types of multiplication, division, and exponentiation. These are the **matrix** multiplication, division, and exponentiation, and the **element-by-element** multiplication, division, and exponentiation. They are explained in the following paragraphs.

In Section A.2, the arrays [a b c ...], such as those that contained the coefficients of polynomials, consisted of one row and multiple columns, and thus are called **row vectors**. If an array has one column and multiple rows, it is called a **column vector**. We recall that the elements of a row vector are separated by spaces. To distinguish between row and column vectors, the elements of a column vector must be separated by semicolons. An easier way to construct a column vector, is to write it first as a row vector, and then transpose it into a column vector. MATLAB uses the single quotation character (') to transpose a vector. Thus, a column vector can be written either as

```
b=[-1; 3; 6; 11]
```

or as

```
b=[-1 3 6 11]'
```

As shown below, MATLAB produces the same display with either format.

```
b=[-1; 3; 6; 11]
```

```
b =  
    -1  
     3  
     6  
    11
```

```
b=[-1 3 6 11]'      % Observe the single quotation character (')
```

```
b =  
    -1  
     3  
     6  
    11
```

We will now define Matrix Multiplication and Element-by-Element multiplication.

### 1. Matrix Multiplication (multiplication of row by column vectors)

Let

$$\mathbf{A} = [a_1 \ a_2 \ a_3 \ \dots \ a_n]$$

and

$$\mathbf{B} = [b_1 \ b_2 \ b_3 \ \dots \ b_n]'$$

be two vectors. We observe that  $\mathbf{A}$  is defined as a row vector whereas  $\mathbf{B}$  is defined as a column vector, as indicated by the transpose operator ('). Here, multiplication of the row vector  $\mathbf{A}$  by the column vector  $\mathbf{B}$ , is performed with the matrix multiplication operator (\*). Then,

$$\mathbf{A} * \mathbf{B} = [a_1 b_1 + a_2 b_2 + a_3 b_3 + \dots + a_n b_n] = \text{single value} \quad (\text{A.5})$$

For example, if

$$\mathbf{A} = [1 \ 2 \ 3 \ 4 \ 5]$$

and

$$\mathbf{B} = [-2 \ 6 \ -3 \ 8 \ 7]'$$

the matrix multiplication  $\mathbf{A}*\mathbf{B}$  produces the single value 68, that is,

$$\mathbf{A}*\mathbf{B} = 1 \times (-2) + 2 \times 6 + 3 \times (-3) + 4 \times 8 + 5 \times 7 = 68$$

and this is verified with the MATLAB script

```
A=[1 2 3 4 5]; B=[-2 6 -3 8 7]'; A*B % Observe transpose operator (') in B
ans =
    68
```

Now, let us suppose that both  $\mathbf{A}$  and  $\mathbf{B}$  are row vectors, and we attempt to perform a row-by-row multiplication with the following MATLAB statements.

```
A=[1 2 3 4 5]; B=[-2 6 -3 8 7]; A*B % No transpose operator (') here
```

When these statements are executed, MATLAB displays the following message:

```
??? Error using ==> *

```

```
Inner matrix dimensions must agree.
```

Here, because we have used the matrix multiplication operator (\*) in  $\mathbf{A}*\mathbf{B}$ , MATLAB expects vector  $\mathbf{B}$  to be a column vector, not a row vector. It recognizes that  $\mathbf{B}$  is a row vector, and warns us that we cannot perform this multiplication using the matrix multiplication operator (\*). Accordingly, we must perform this type of multiplication with a different operator. This operator is defined below.

### 2. Element-by-Element Multiplication (multiplication of a row vector by another row vector)

Let

$$\mathbf{C} = [c_1 \ c_2 \ c_3 \ \dots \ c_n]$$

and

$$\mathbf{D} = [d_1 \ d_2 \ d_3 \ \dots \ d_n]$$

be two row vectors. Here, multiplication of the row vector  $\mathbf{C}$  by the row vector  $\mathbf{D}$  is performed with the *dot multiplication operator* (.\*). There is no space between the dot and the multiplication symbol. Thus,

$$\mathbf{C}.*\mathbf{D} = [c_1d_1 \ c_2d_2 \ c_3d_3 \ \dots \ c_nd_n] \quad (\text{A.6})$$

This product is another row vector with the same number of elements, as the elements of **C** and **D**.

As an example, let

$$\mathbf{C} = [1 \ 2 \ 3 \ 4 \ 5]$$

and

$$\mathbf{D} = [-2 \ 6 \ -3 \ 8 \ 7]$$

Dot multiplication of these two row vectors produce the following result.

$$\mathbf{C}.*\mathbf{D} = 1 \times (-2) \quad 2 \times 6 \quad 3 \times (-3) \quad 4 \times 8 \quad 5 \times 7 = -2 \quad 12 \quad -9 \quad 32 \quad 35$$

Check with MATLAB:

```
C=[1 2 3 4 5];    % Vectors C and D must have
D=[-2 6 -3 8 7]; % same number of elements
C.*D              % We observe that this is a dot multiplication
ans =
    -2     12     -9     32     35
```

Similarly, the division (/) and exponentiation (^) operators, are used for matrix division and exponentiation, whereas dot division (./) and dot exponentiation (.^) are used for element-by-element division and exponentiation, as illustrated in Examples A.11 and A.12 above.

We must remember that *no space is allowed between the dot (.) and the multiplication, division, and exponentiation operators.*

**Note:** A dot (.) is never required with the plus (+) and minus (-) operators.

---

### Example A.13

Write the MATLAB script that produces a simple plot for the waveform defined as

$$y = f(t) = 3e^{-4t} \cos 5t - 2e^{-3t} \sin 2t + \frac{t^2}{t+1} \quad (\text{A.7})$$

in the  $0 \leq t \leq 5$  seconds interval.

#### Solution:

The MATLAB script for this example is as follows:

```
t=0: 0.01: 5;          % Define t-axis in 0.01 increments
y=3 .* exp(-4 .* t) .* cos(5 .* t)-2 .* exp(-3 .* t) .* sin(2 .* t) + t.^2 ./ (t+1);
plot(t,y); grid; xlabel('t'); ylabel('y=f(t)'); title('Plot for Example A.13')
```

The plot for this example is shown in Figure A.8.

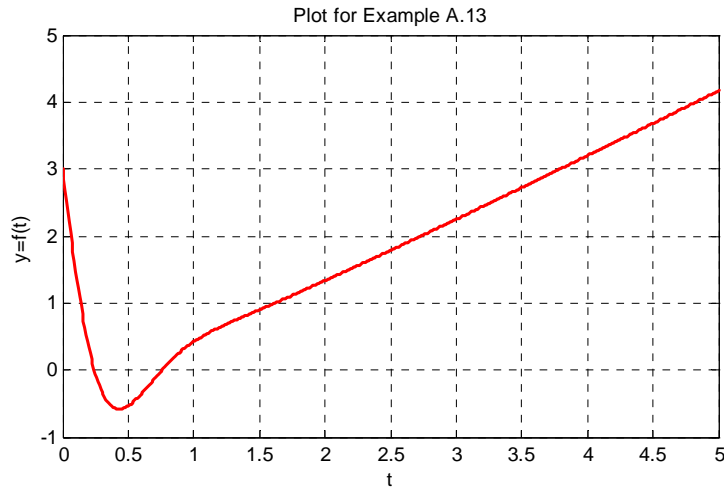


Figure A.8. Plot for Example A.13

Had we, in this example, defined the time interval starting with a negative value equal to or less than  $-1$ , say as  $-3 \leq t \leq 3$ , MATLAB would have displayed the following message:

Warning: Divide by zero.

This is because the last term (the rational fraction) of the given expression, is divided by zero when  $t = -1$ . To avoid division by zero, we use the special MATLAB function **eps**, which is a number approximately equal to  $2.2 \times 10^{-16}$ . It will be used with the next example.

---

The command **axis([xmin xmax ymin ymax])** scales the current plot to the values specified by the arguments **xmin**, **xmax**, **ymin** and **ymax**. There are no commas between these four arguments. This command must be placed *after* the plot command and must be repeated for each plot. The following example illustrates the use of the dot multiplication, division, and exponentiation, the **eps** number, the **axis([xmin xmax ymin ymax])** command, and also MATLAB's capability of displaying up to four windows of different plots.

---

### Example A.14

Plot the functions

$$y = \sin^2x, \quad z = \cos^2x, \quad w = \sin^2x \cdot \cos^2x, \quad v = \sin^2x / \cos^2x$$

in the interval  $0 \leq x \leq 2\pi$  using 100 data points. Use the **subplot** command to display these functions on four windows on the same graph.

**Solution:**

The MATLAB script to produce the four subplots is as follows:

```
x=linspace(0,2*pi,100);           % Interval with 100 data points
y=(sin(x).^ 2); z=(cos(x).^ 2);
w=y.* z;
v=y./ (z+eps);% add eps to avoid division by zero
subplot(221);% upper left of four subplots
plot(x,y); axis([0 2*pi 0 1]);
title('y = (sinx) ^ 2');
subplot(222);                       % upper right of four subplots
plot(x,z); axis([0 2*pi 0 1]);
title('z = (cosx) ^ 2');
subplot(223);                       % lower left of four subplots
plot(x,w); axis([0 2*pi 0 0.3]);
title('w = (sinx) ^ 2*(cosx) ^ 2');
subplot(224);                       % lower right of four subplots
plot(x,v); axis([0 2*pi 0 400]);
title('v = (sinx) ^ 2/(cosx) ^ 2');
```

These subplots are shown in Figure A.9.

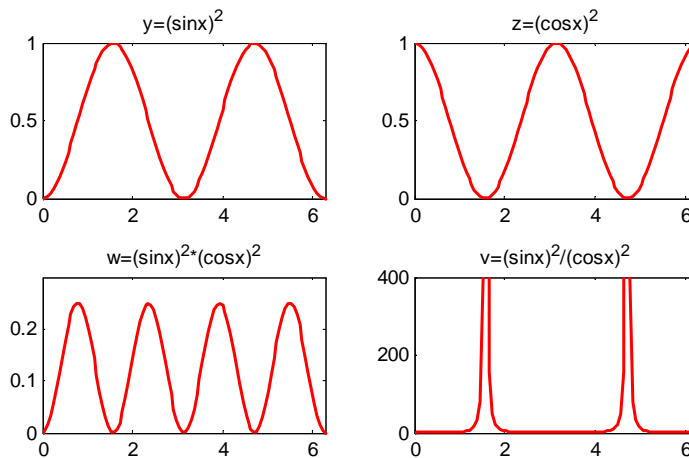


Figure A.9. Subplots for the functions of Example A.14

---

The next example illustrates MATLAB's capabilities with imaginary numbers. We will introduce the **real(z)** and **imag(z)** functions that display the real and imaginary parts of the complex quantity  $z = x + iy$ , the **abs(z)**, and the **angle(z)** functions that compute the absolute value (magnitude) and phase angle of the complex quantity  $z = x + iy = r\angle\theta$ . We will also use the **polar(theta,r)** function that produces a plot in polar coordinates, where **r** is the magnitude, **theta**

is the angle in radians, and the **round(n)** function that rounds a number to its nearest integer.

**Example A.15**

Consider the electric circuit of Figure A.10.

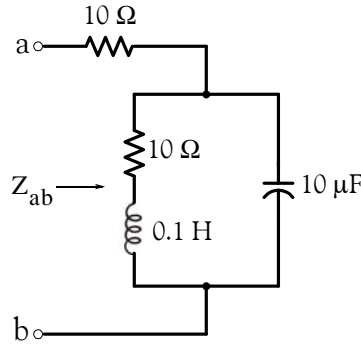


Figure A.10. Electric circuit for Example A.15

With the given values of resistance, inductance, and capacitance, the impedance  $Z_{ab}$  as a function of the radian frequency  $\omega$  can be computed from the following expression:

$$Z_{ab} = Z = 10 + \frac{10^4 - j(10^6/\omega)}{10 + j(0.1\omega - 10^5/\omega)} \tag{A.8}$$

- a. Plot  $\text{Re}\{Z\}$  (the real part of the impedance  $Z$ ) versus frequency  $\omega$ .
- b. Plot  $\text{Im}\{Z\}$  (the imaginary part of the impedance  $Z$ ) versus frequency  $\omega$ .
- c. Plot the impedance  $Z$  versus frequency  $\omega$  in polar coordinates.

**Solution:**

The MATLAB script below computes the real and imaginary parts of  $Z_{ab}$  which, for simplicity, are denoted as  $z$ , and plots these as two separate graphs (parts a & b). It also produces a polar plot (part c).

```
w=0: 1: 2000; % Define interval with one radian interval;...
z=(10+(10.^4 -j.* 10.^6 ./ (w+eps)) ./ (10 + j.* (0.1.* w -10.^5./ (w+eps))));...
%
% The first five statements (next two lines) compute and plot Re{z}
real_part=real(z); plot(w,real_part);...
xlabel('radian frequency w'); ylabel('Real part of Z'); grid
```

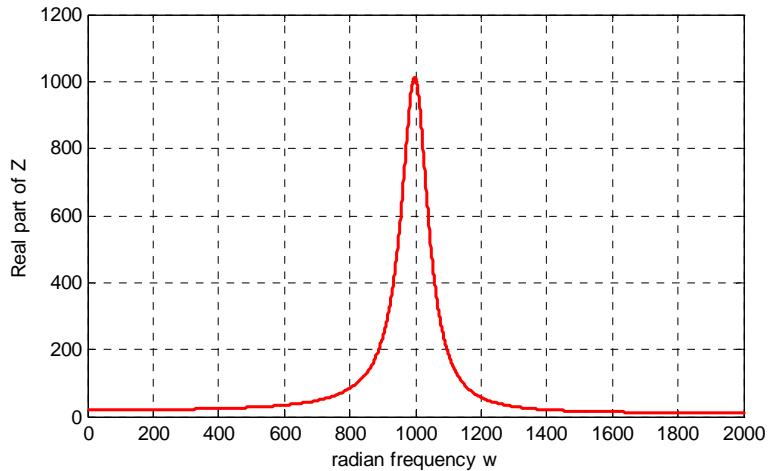


Figure A.11. Plot for the real part of the impedance in Example A.15

```
% The next five statements (next two lines) compute and plot Im{z}
imag_part=imag(z); plot(w,imag_part);...
xlabel('radian frequency w'); ylabel('Imaginary part of Z'); grid
```

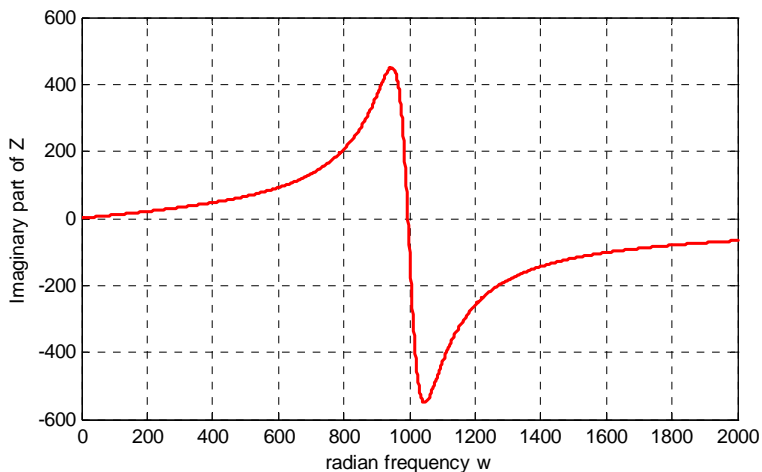


Figure A.12. Plot for the imaginary part of the impedance in Example A.15

```
% The last six statements (next five lines) below produce the polar plot of z
mag=abs(z);           % Computes |Z|;...
rndz=round(abs(z));  % Rounds |Z| to read polar plot easier;...
theta=angle(z);     % Computes the phase angle of impedance Z;...
polar(theta,rndz);   % Angle is the first argument
ylabel('Polar Plot of Z'); grid
```



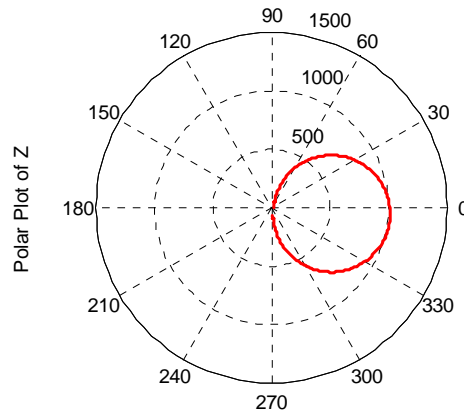


Figure A.13. Polar plot of the impedance in Example A.15

Example A.15 clearly illustrates how powerful, fast, accurate, and flexible MATLAB is.

---

### A.10 Script and Function Files

MATLAB recognizes two types of files: **script files** and **function files**. Both types are referred to as **m-files** since both require the **.m** extension.

A **script file** consists of two or more built-in functions such as those we have discussed thus far. Thus, the script for each of the examples we discussed earlier, make up a script file. Generally, a script file is one which was generated and saved as an m-file with an editor such as the MATLAB's Editor/Debugger.

A **function file** is a user-defined function using MATLAB. We use function files for repetitive tasks. The first line of a function file must contain the word **function**, followed by the output argument, the equal sign (**=**), and the input argument enclosed in parentheses. The function name and file name must be the same, but the file name must have the extension **.m**. For example, the function file consisting of the two lines below

```
function y = myfunction(x)
y=x.^ 3 + cos(3.* x)
```

is a function file and must be saved as **myfunction.m**

For the next example, we will use the following MATLAB functions:

**fzero(f,x)** – attempts to find a zero of a function of one variable, where **f** is a string containing the name of a real-valued function of a single real variable. MATLAB searches for a value near a point where the function **f** changes sign, and returns that value, or returns NaN if the search fails.

**Important:** We must remember that we use **roots(p)** to find the roots of polynomials only, such as those in Examples A.1 and A.2.

**fplot(fcn,lims)** – plots the function specified by the string **fcn** between the x-axis limits specified by **lims = [xmin xmax]**. Using **lims = [xmin xmax ymin ymax]** also controls the y-axis limits. The string **fcn** must be the name of an *m-file* function or a string with variable *x*.

**NaN** (Not-a-Number) is not a function; it is MATLAB's response to an undefined expression such as  $0/0$ ,  $\infty/\infty$ , or inability to produce a result as described on the next paragraph. We can avoid division by zero using the **eps** number, which we mentioned earlier.

### Example A.16

Find the zeros, the minimum, and the maximum values of the function

$$f(x) = \frac{1}{(x - 0.1)^2 + 0.01} - \frac{1}{(x - 1.2)^2 + 0.04} - 10 \quad (\text{A.9})$$

in the interval  $-1.5 \leq x \leq 1.5$

#### Solution:

We first plot this function to observe the approximate zeros, maxima, and minima using the following script.

```
x=-1.5:0.01:1.5;
y=1./((x-0.1).^2+0.01)-1./((x-1.2).^2+0.04)-10;
plot(x,y); grid
```

The plot is shown in Figure A.14.

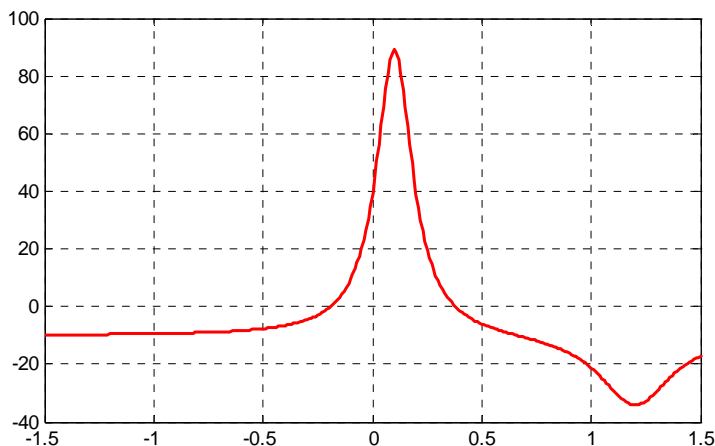


Figure A.14. Plot for Example A.16 using the `plot` command

The roots (zeros) of this function appear to be in the neighborhood of  $x = -0.2$  and  $x = 0.3$ . The maximum occurs at approximately  $x = 0.1$  where, approximately,  $y_{\max} = 90$ , and the minimum occurs at approximately  $x = 1.2$  where, approximately,  $y_{\min} = -34$ .

Next, we define and save  $f(x)$  as the **funczero01.m** function m-file with the following script:

```
function y=funczero01(x)
% Finding the zeros of the function shown below
y=1/((x-0.1)^2+0.01)-1/((x-1.2)^2+0.04)-10;
```

To save this file, from the File drop menu on the Command Window, we choose New, and when the Editor Window appears, we type the script above and we save it as **funczero01**. MATLAB appends the extension **.m** to it.

Now, we can use the **fplot(fcn,lims)** command to plot  $f(x)$  as follows:

```
fplot('funczero01', [-1.5 1.5]); grid
```

This plot is shown in Figure A.15. As expected, this plot is identical to the plot of Figure A.14 which was obtained with the **plot(x,y)** command as shown in Figure A.14.

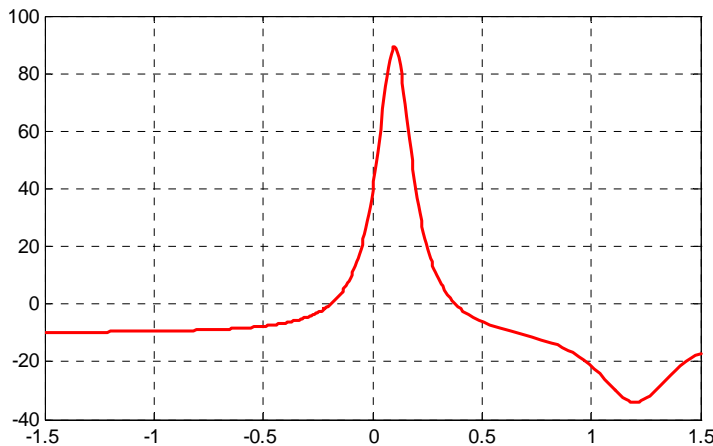


Figure A.15. Plot for Example A.16 using the *fplot* command

We will use the **fzero(f,x)** function to compute the roots of  $f(x)$  in Equation (A.9) more precisely. The MATLAB script below will accomplish this.

```
x1= fzero('funczero01', -0.2);
x2= fzero('funczero01', 0.3);
fprintf('The roots (zeros) of this function are r1= %3.4f', x1);
fprintf(' and r2= %3.4f \n', x2)
```

MATLAB displays the following:

The roots (zeros) of this function are  $r_1 = -0.1919$  and  $r_2 = 0.3788$

The earlier MATLAB versions included the function **fmin(f,x1,x2)** and with this function we could compute both a minimum of some function  $f(x)$  or a maximum of  $f(x)$  since a maximum of  $f(x)$  is equal to a minimum of  $-f(x)$ . This can be visualized by flipping the plot of a function  $f(x)$  upside-down. This function is no longer used in MATLAB and thus we will compute the maxima and minima from the derivative of the given function.

From elementary calculus, we recall that the maxima or minima of a function  $y = f(x)$  can be found by setting the first derivative of a function equal to zero and solving for the independent variable  $x$ . For this example we use the **diff(x)** function which produces the approximate derivative of a function. Thus, we use the following MATLAB script:

```
syms x ymin zmin; ymin=1/((x-0.1)^2+0.01)-1/((x-1.2)^2+0.04)-10;...
zmin=diff(ymin)
```

```
zmin =
-1/((x-1/10)^2+1/100)^2*(2*x-1/5)+1/((x-6/5)^2+1/25)^2*(2*x-12/5)
```

When the command

```
solve(zmin)
```

is executed, MATLAB displays a very long expression which when copied at the command prompt and executed, produces the following:

```
ans =
    0.6585 + 0.3437i
ans =
    0.6585 - 0.3437i
ans =
    1.2012
```

The real value 1.2012 above is the value of  $x$  at which the function  $y$  has its minimum value as we observe also in the plot of Figure A.15.

To find the value of  $y$  corresponding to this value of  $x$ , we substitute it into  $f(x)$ , that is,

```
x=1.2012; ymin=1/((x-0.1)^2+0.01)-1/((x-1.2)^2+0.04)-10
ymin = -34.1812
```

We can find the maximum value from  $-f(x)$  whose plot is produced with the script

```
x=-1.5:0.01:1.5; ymax=-1./((x-0.1).^2+0.01)+1./((x-1.2).^2+0.04)+10; plot(x,ymax); grid
```

and the plot is shown in Figure A.16.

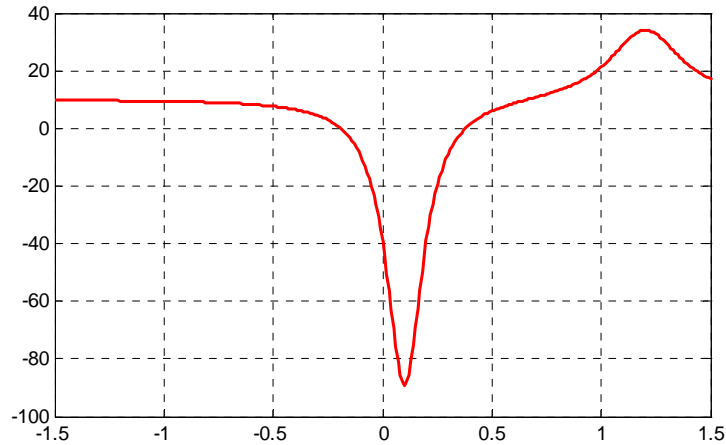


Figure A.16. Plot of  $-f(x)$  for Example A.16

Next we compute the first derivative of  $-f(x)$  and we solve for  $x$  to find the value where the maximum of  $y_{max}$  occurs. This is accomplished with the MATLAB script below.

```
syms x ymax zmax; ymax=-(1/((x-0.1)^2+0.01))-1/((x-1.2)^2+0.04)-10; zmax=diff(ymax)
zmax =
 1/((x-1/10)^2+1/100)^2*(2*x-1/5)-1/((x-6/5)^2+1/25)^2*(2*x-12/5)
solve(zmax)
```

When the command

```
solve(zmax)
```

is executed, MATLAB displays a very long expression which when copied at the command prompt and executed, produces the following:

```
ans =
 0.6585 + 0.3437i
ans =
 0.6585 - 0.3437i
ans =
 1.2012
ans =
 0.0999
```

From the values above we choose  $x = 0.0999$  which is consistent with the plots of Figures A.15 and A.16. Accordingly, we execute the following script to obtain the value of  $y_{min}$ .

```
x=0.0999;           % Using this value find the corresponding value of ymax
ymax=1 / ((x-0.1) ^ 2 + 0.01) -1 / ((x-1.2) ^ 2 + 0.04) -10

ymax = 89.2000
```

## A.11 Display Formats

MATLAB displays the results on the screen in integer format without decimals if the result is an integer number, or in short floating point format with four decimals if it a fractional number. The format displayed has nothing to do with the accuracy in the computations. MATLAB performs all computations with accuracy up to 16 decimal places.

The output format can changed with the **format** command. The available MATLAB formats can be displayed with the **help format** command as follows:

```
help format
```

```
FORMAT Set output format.
All computations in MATLAB are done in double precision.
FORMAT may be used to switch between different output display formats
as follows:
```

```
FORMAT Default. Same as SHORT.
FORMAT SHORT Scaled fixed point format with 5 digits.
FORMAT LONG Scaled fixed point format with 15 digits.
FORMAT SHORT E Floating point format with 5 digits.
FORMAT LONG E Floating point format with 15 digits.
FORMAT SHORT G Best of fixed or floating point format with 5 digits.
FORMAT LONG G Best of fixed or floating point format with 15 digits.
FORMAT HEX Hexadecimal format.
FORMAT + The symbols +, - and blank are printed for positive, negative,
and zero elements. Imaginary parts are ignored.
FORMAT BANK Fixed format for dollars and cents.
FORMAT RAT Approximation by ratio of small integers.
```

```
Spacing:
```

```
FORMAT COMPACT Suppress extra line-feeds.
FORMAT LOOSE Puts the extra line-feeds back in.
```

Some examples with different format displays age given below.

```
format short 33.3335 Four decimal digits (default)
format long 33.33333333333334 16 digits
format short e 3.3333e+01 Four decimal digits plus exponent
format short g 33.333 Better of format short or format short e
format bank 33.33 two decimal digits
format + only + or - or zero are printed
```

---

## Appendix A Introduction to MATLAB®

---

```
format rat 100/3 rational approximation
```

The **disp(X)** command displays the array **X** without printing the array name. If **X** is a string, the text is displayed.

The **fprintf(format,array)** command displays and prints both text and arrays. It uses specifiers to indicate where and in which format the values would be displayed and printed. Thus, if **%f** is used, the values will be displayed and printed in fixed decimal format, and if **%e** is used, the values will be displayed and printed in scientific notation format. With this command only the real part of each parameter is processed.

This appendix is just an introduction to MATLAB.\* This outstanding software package consists of many applications known as *Toolboxes*. The MATLAB Student Version contains just a few of these Toolboxes. Others can be bought directly from The MathWorks,™ Inc., as add-ons.

---

\* For more MATLAB applications, please refer to *Numerical Analysis Using MATLAB and Excel*, ISBN 978-1-934404-03-4.

This appendix is a brief introduction to Simulink. This author feels that we can best introduce Simulink with a few examples. Some familiarity with MATLAB is essential in understanding Simulink, and for this purpose, Appendix A is included as an introduction to MATLAB.

### B.1 Simulink and its Relation to MATLAB

The MATLAB® and Simulink® environments are integrated into one entity, and thus we can analyze, simulate, and revise our models in either environment at any point. We invoke Simulink from within MATLAB. We will introduce Simulink with a few illustrated examples.

---

#### Example B.1

For the circuit of Figure B.1, the initial conditions are  $i_L(0^-) = 0$ , and  $v_C(0^-) = 0.5$  V. We will compute  $v_C(t)$ .

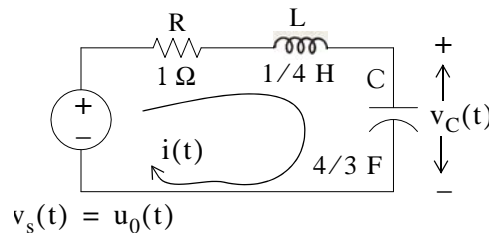


Figure B.1. Circuit for Example B.1

For this example,

$$i = i_L = i_C = C \frac{dv_C}{dt} \quad (\text{B.1})$$

and by Kirchoff's voltage law (KVL),

$$Ri_L + L \frac{di_L}{dt} + v_C = u_0(t) \quad (\text{B.2})$$

Substitution of (B.1) into (B.2) yields



$$RC \frac{dv_C}{dt} + LC \frac{d^2v_C}{dt^2} + v_C = u_0(t) \quad (\text{B.3})$$

Substituting the values of the circuit constants and rearranging we obtain:

$$\frac{1}{3} \frac{d^2v_C}{dt^2} + \frac{4}{3} \frac{dv_C}{dt} + v_C = u_0(t)$$

$$\frac{d^2v_C}{dt^2} + 4 \frac{dv_C}{dt} + 3v_C = 3u_0(t) \quad (\text{B.4})$$

$$\frac{d^2v_C}{dt^2} + 4 \frac{dv_C}{dt} + 3v_C = 3 \quad t > 0 \quad (\text{B.5})$$

To appreciate Simulink's capabilities, for comparison, three different methods of obtaining the solution are presented, and the solution using Simulink follows.

### First Method – Assumed Solution

Equation (B.5) is a second-order, non-homogeneous differential equation with constant coefficients, and thus the complete solution will consist of the sum of the forced response and the natural response. It is obvious that the solution of this equation cannot be a constant since the derivatives of a constant are zero and thus the equation is not satisfied. Also, the solution cannot contain sinusoidal functions (sine and cosine) since the derivatives of these are also sinusoids.

However, decaying exponentials of the form  $ke^{-at}$  where  $k$  and  $a$  are constants, are possible candidates since their derivatives have the same form but alternate in sign.

It can be shown\* that if  $k_1e^{-s_1t}$  and  $k_2e^{-s_2t}$  where  $k_1$  and  $k_2$  are constants and  $s_1$  and  $s_2$  are the roots of the characteristic equation of the homogeneous part of the given differential equation, the natural response is the sum of the terms  $k_1e^{-s_1t}$  and  $k_2e^{-s_2t}$ . Therefore, the total solution will be

$$v_C(t) = \text{natural response} + \text{forced response} = v_{cn}(t) + v_{cf}(t) = k_1e^{-s_1t} + k_2e^{-s_2t} + v_{cf}(t) \quad (\text{B.6})$$

The values of  $s_1$  and  $s_2$  are the roots of the characteristic equation

---

\* Please refer to *Circuit Analysis II with MATLAB Applications*, ISBN 0-9709511-5-9, Appendix B for a thorough discussion.

$$s^2 + 4s + 3 = 0 \quad (\text{B.7})$$

Solution of (B.7) yields of  $s_1 = -1$  and  $s_2 = -3$  and with these values (B.6) is written as

$$v_c(t) = k_1 e^{-t} + k_2 e^{-3t} + v_{cf}(t) \quad (\text{B.8})$$

The forced component  $v_{cf}(t)$  is found from (B.5), i.e.,

$$\frac{d^2 v_C}{dt^2} + 4 \frac{dv_C}{dt} + 3v_C = 3 \quad t > 0 \quad (\text{B.9})$$

Since the right side of (B.9) is a constant, the forced response will also be a constant and we denote it as  $v_{Cf} = k_3$ . By substitution into (B.9) we obtain

$$0 + 0 + 3k_3 = 3$$

or

$$v_{Cf} = k_3 = 1 \quad (\text{B.10})$$

Substitution of this value into (B.8), yields the total solution as

$$v_C(t) = v_{Cn}(t) + v_{Cf} = k_1 e^{-t} + k_2 e^{-3t} + 1 \quad (\text{B.11})$$

The constants  $k_1$  and  $k_2$  will be evaluated from the initial conditions. First, using  $v_C(0) = 0.5$  V and evaluating (B.11) at  $t = 0$ , we obtain

$$v_C(0) = k_1 e^0 + k_2 e^0 + 1 = 0.5$$

$$k_1 + k_2 = -0.5 \quad (\text{B.12})$$

Also,

$$i_L = i_C = C \frac{dv_C}{dt}, \quad \frac{dv_C}{dt} = \frac{i_L}{C}$$

and

$$\left. \frac{dv_C}{dt} \right|_{t=0} = \frac{i_L(0)}{C} = \frac{0}{C} = 0 \quad (\text{B.13})$$

Next, we differentiate (B.11), we evaluate it at  $t = 0$ , and equate it with (B.13). Thus,

$$\left. \frac{dv_C}{dt} \right|_{t=0} = -k_1 - 3k_2 \quad (\text{B.14})$$

By equating the right sides of (B.13) and (B.14) we obtain

$$-k_1 - 3k_2 = 0 \tag{B.15}$$

Simultaneous solution of (B.12) and (B.15), gives  $k_1 = -0.75$  and  $k_2 = 0.25$ . By substitution into (B.8), we obtain the total solution as

$$v_C(t) = (-0.75e^{-t} + 0.25e^{-3t} + 1)u_0(t) \tag{B.16}$$

Check with MATLAB:

```

syms t                               % Define symbolic variable t
y0=-0.75*exp(-t)+0.25*exp(-3*t)+1;  % The total solution y(t), for our example, vc(t)
y1=diff(y0)                           % The first derivative of y(t)

y1 =
3/4*exp(-t) - 3/4*exp(-3*t)

y2=diff(y0,2)                          % The second derivative of y(t)

y2 =
-3/4*exp(-t) + 9/4*exp(-3*t)

y=y2+4*y1+3*y0                          % Summation of y and its derivatives

Y =
3
    
```

Thus, the solution has been verified by MATLAB. Using the expression for  $v_C(t)$  in (B.16), we find the expression for the current as

$$i = i_L = i_C = C \frac{dv_C}{dt} = \frac{4}{3} \left( \frac{3}{4}e^{-t} - \frac{3}{4}e^{-3t} \right) = e^{-t} - e^{-3t} \text{ A} \tag{B.17}$$

### Second Method – Using the Laplace Transformation

The transformed circuit is shown in Figure B.2.

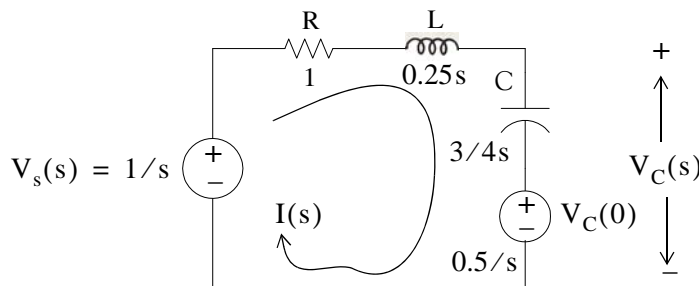


Figure B.2. Transformed Circuit for Example B.1

By the voltage division\* expression,

$$V_C(s) = \frac{3/4s}{(1 + 0.25s + 3/4s)} \cdot \left( \frac{1}{s} - \frac{0.5}{s} \right) + \frac{0.5}{s} = \frac{1.5}{s(s^2 + 4s + 3)} + \frac{0.5}{s} = \frac{0.5s^2 + 2s + 3}{s(s+1)(s+3)}$$

Using partial fraction expansion,† we let

$$\frac{0.5s^2 + 2s + 3}{s(s+1)(s+3)} = \frac{r_1}{s} + \frac{r_2}{(s+1)} + \frac{r_3}{(s+3)} \quad (\text{B.18})$$

$$r_1 = \left. \frac{0.5s^2 + 2s + 3}{(s+1)(s+3)} \right|_{s=0} = 1$$

$$r_2 = \left. \frac{0.5s^2 + 2s + 3}{s(s+3)} \right|_{s=-1} = -0.75$$

$$r_3 = \left. \frac{0.5s^2 + 2s + 3}{s(s+1)} \right|_{s=-3} = 0.25$$

and by substitution into (B.18)

$$V_C(s) = \frac{0.5s^2 + 2s + 3}{s(s+1)(s+3)} = \frac{1}{s} + \frac{-0.75}{(s+1)} + \frac{0.25}{(s+3)}$$

Taking the Inverse Laplace transform‡ we find that

$$\mathbf{v_C(t) = 1 - 0.75e^{-t} + 0.25e^{-3t}}$$

### Third Method – Using State Variables

$$Ri_L + L \frac{di_L}{dt} + v_C = u_0(t) \quad **$$

---

\* For derivation of the voltage division and current division expressions, please refer to *Circuit Analysis I with MATLAB Applications*, ISBN 0-9709511-2-4.

† Partial fraction expansion is discussed in Chapter 3, this text.

‡ For an introduction to Laplace Transform and Inverse Laplace Transform, please refer Chapters 2 and 3, this text.

\*\* Usually, in *State-Space and State Variables Analysis*,  $u(t)$  denotes any input. For distinction, we will denote the Unit Step Function as  $u_0(t)$ . For a detailed discussion on *State-Space and State Variables Analysis*, please refer to Chapter 5, this text.

By substitution of given values and rearranging, we obtain

$$\frac{1}{4} \frac{di_L}{dt} = (-1)i_L - v_C + 1$$

or

$$\frac{di_L}{dt} = -4i_L - 4v_C + 4 \quad (\text{B.19})$$

Next, we define the state variables  $x_1 = i_L$  and  $x_2 = v_C$ . Then,

$$\dot{x}_1 = \frac{di_L}{dt} \quad (\text{B.20})$$

and

$$\dot{x}_2 = \frac{dv_C}{dt} \quad (\text{B.21})$$

Also,

$$i_L = C \frac{dv_C}{dt}$$

and thus,

$$x_1 = i_L = C \frac{dv_C}{dt} = C \dot{x}_2 = \frac{4}{3} \dot{x}_2$$

or

$$\dot{x}_2 = \frac{3}{4} \dot{x}_1 \quad (\text{B.22})$$

Therefore, from (B.19), (B.20), and (B.22), we obtain the state equations

$$\dot{x}_1 = -4x_1 - 4x_2 + 4$$

$$\dot{x}_2 = \frac{3}{4} x_1$$

and in matrix form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -4 & -4 \\ 3/4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \end{bmatrix} u_0(t) \quad (\text{B.23})$$

Solution<sup>†</sup> of (B.23) yields

---

\* The notation  $\dot{x}$  ( $x$  dot) is often used to denote the first derivative of the function  $x$ , that is,  $\dot{x} = dx/dt$ .

† The detailed solution of (B.23) is given in Chapter 5, Example 5.10, Page 5–23, this text.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} e^{-t} - e^{-3t} \\ 1 - 0.75e^{-t} + 0.25e^{-3t} \end{bmatrix}$$

Then,

$$x_1 = i_L = e^{-t} - e^{-3t} \quad (\text{B.24})$$

and

$$x_2 = v_C = 1 - 0.75e^{-t} + 0.25e^{-3t} \quad (\text{B.25})$$

### Modeling the Differential Equation of Example B.1 with Simulink

To run Simulink, we must first invoke MATLAB. Make sure that Simulink is installed in your system. In the MATLAB Command prompt, we type:

`simulink`

Alternately, we can click on the Simulink icon shown in Figure B.3. It appears on the top bar on MATLAB's Command prompt.



Figure B.3. The Simulink icon

Upon execution of the Simulink command, the **Commonly Used Blocks** appear as shown in Figure B.4.

In Figure B.4, the left side is referred to as the **Tree Pane** and displays all Simulink libraries installed. The right side is referred to as the **Contents Pane** and displays the blocks that reside in the library currently selected in the Tree Pane.

Let us express the differential equation of Example B.1 as

$$\frac{d^2 v_C}{dt^2} = -4 \frac{dv_C}{dt} - 3v_C + 3u_0(t) \quad (\text{B.26})$$

A block diagram representing relation (B.26) above is shown in Figure B.5. We will use Simulink to draw a similar block diagram.\*

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\* Henceforth, all Simulink block diagrams will be referred to as models.

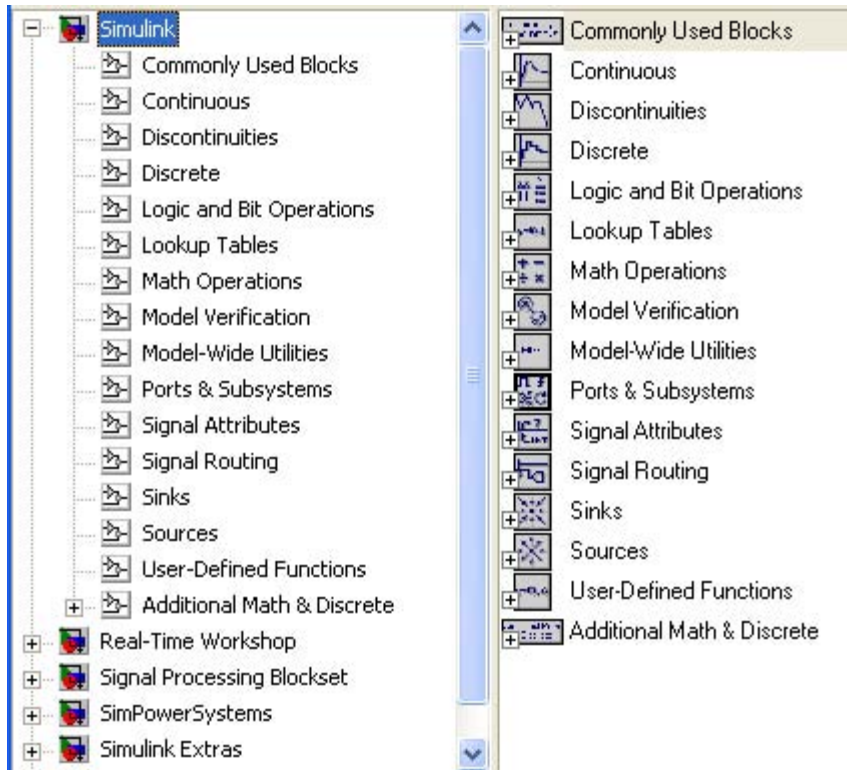


Figure B.4. The Simulink Library Browser

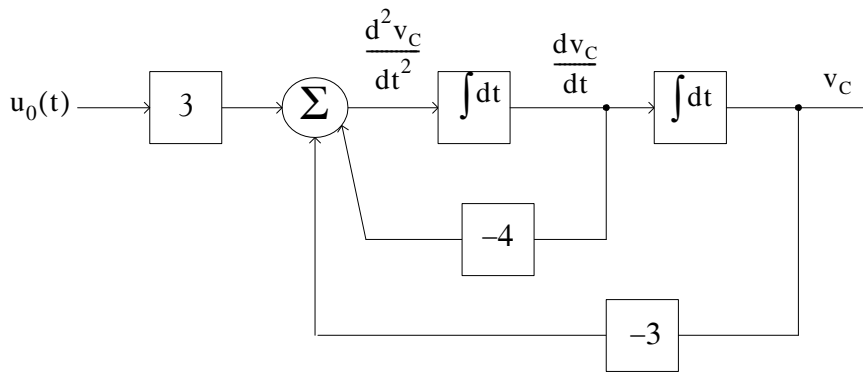


Figure B.5. Block diagram for equation (B.26)

To model the differential equation (B.26) using Simulink, we perform the following steps:

1. On the **Simulink Library Browser**, we click on the leftmost icon shown as a blank page on the top title bar. A new model window named **untitled** will appear as shown in Figure B.6.

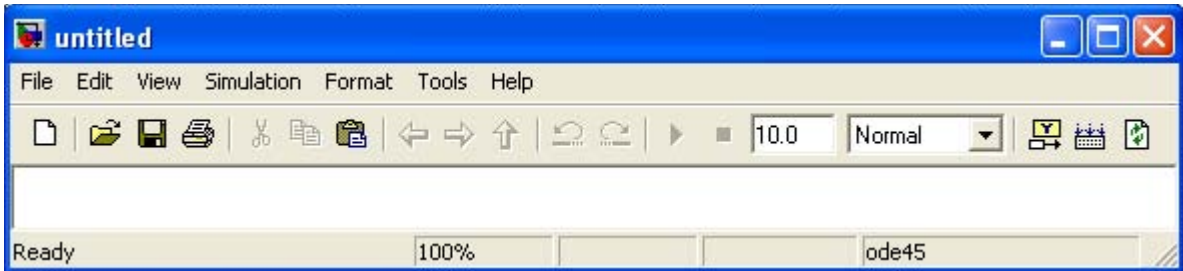


Figure B.6. The Untitled model window in Simulink.

The window of Figure B.6 is the model window where we enter our blocks to form a block diagram. We save this as model file name **Equation\_1\_26**. This is done from the File drop menu of Figure B.6 where we choose **Save as** and name the file as **Equation\_1\_26**. Simulink will add the extension **.mdl**. The new model window will now be shown as **Equation\_1\_26**, and all saved files will have this appearance. See Figure B.7.

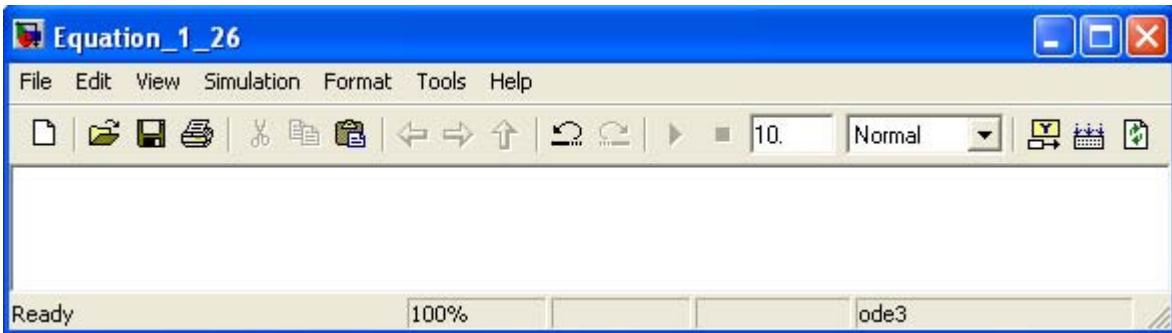


Figure B.7. Model window for Equation\_1\_26.mdl file

2. With the **Equation\_1\_26** model window and the **Simulink Library Browser** both visible, we click on the **Sources** appearing on the left side list, and on the right side we scroll down until we see the unit step function shown as **Step**. See Figure B.8. We select it, and we drag it into the **Equation\_1\_26** model window which now appears as shown in Figure B.8. We save file **Equation\_1\_26** using the File drop menu on the **Equation\_1\_26** model window (right side of Figure B.8).
3. With reference to block diagram of Figure B.5, we observe that we need to connect an amplifier with Gain 3 to the unit step function block. The gain block in Simulink is under **Commonly Used Blocks** (first item under Simulink on the **Simulink Library Browser**). See Figure B.8. If the **Equation\_1\_26** model window is no longer visible, it can be recalled by clicking on the white page icon on the top bar of the **Simulink Library Browser**.
4. We choose the gain block and we drag it to the right of the unit step function. The triangle on the right side of the unit step function block and the **>** symbols on the left and right sides of the gain block are connection points. We point the mouse close to the connection point of the unit step function until it shows as a cross hair, and draw a straight line to connect the two



blocks.\* We double-click on the gain block and on the **Function Block Parameters**, we change the gain from 1 to 3. See Figure B.9.

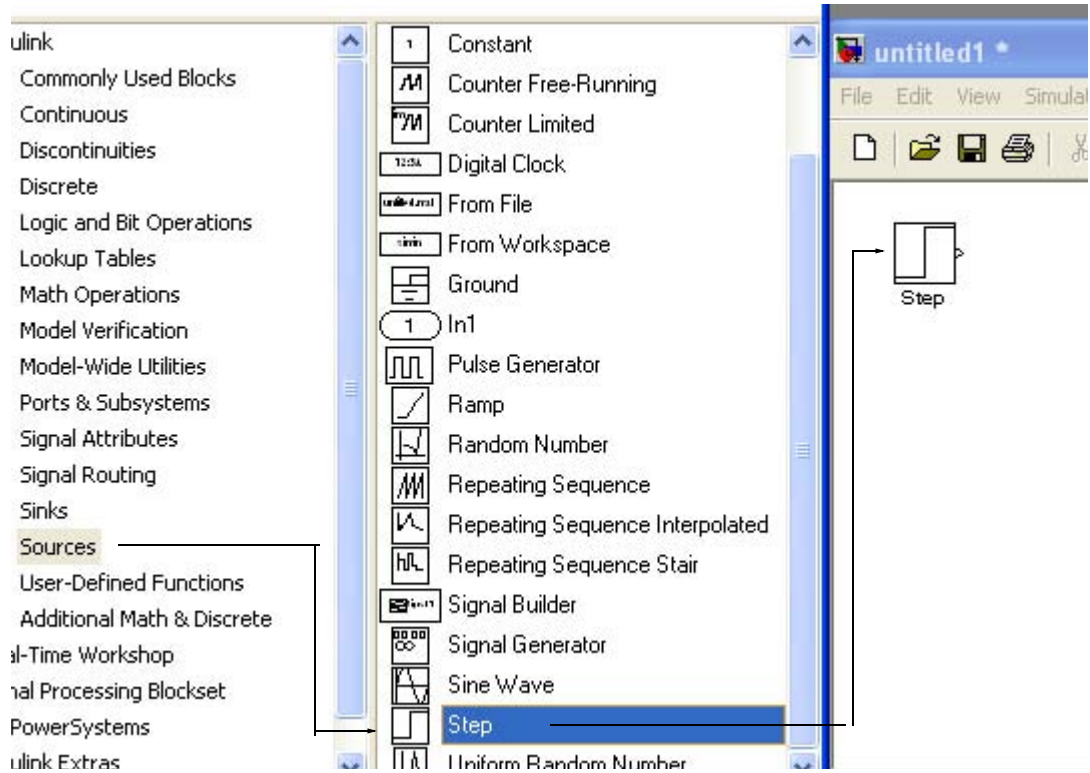


Figure B.8. Dragging the unit step function into File Equation\_1\_26

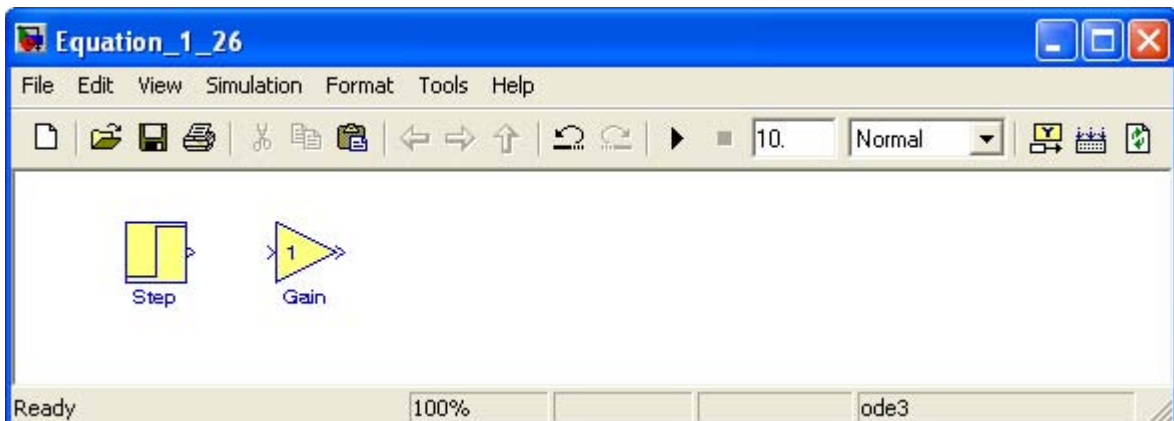


Figure B.9. File Equation\_1\_26 with added Step and Gain blocks

\* An easy method to interconnect two Simulink blocks by clicking on the source block to select it, then hold down the **Ctrl** key and left-click on the destination block.

- Next, we need to add a three-input adder. The adder block appears on the right side of the **Simulink Library Browser** under **Math Operations**. We select it, and we drag it into the `Equation_1_26` model window. We double click it, and on the **Function Block Parameters** window which appears, we specify 3 inputs. We then connect the output of the of the gain block to the first input of the adder block as shown in Figure B.10.

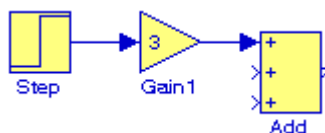


Figure B.10. File `Equation_1_26` with added gain block

- From the **Commonly Used Blocks** of the **Simulink Library Browser**, we choose the **Integrator** block, we drag it into the `Equation_1_26` model window, and we connect it to the output of the **Add** block. We repeat this step and to add a second **Integrator** block. We click on the text “Integrator” under the first integrator block, and we change it to **Integrator 1**. Then, we change the text “Integrator 1” under the second Integrator to “Integrator 2” as shown in Figure B.11.

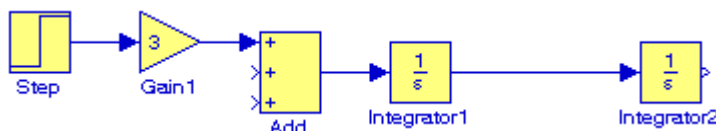


Figure B.11. File `Equation_1_26` with the addition of two integrators

- To complete the block diagram, we add the **Scope** block which is found in the **Commonly Used Blocks** on the **Simulink Library Browser**, we click on the Gain block, and we copy and paste it twice. We flip the pasted Gain blocks by using the **Flip Block** command from the **Format** drop menu, and we label these as **Gain 2** and **Gain 3**. Finally, we double-click on these gain blocks and on the **Function Block Parameters** window, we change the gains from to  $-4$  and  $-3$  as shown in Figure B.12.

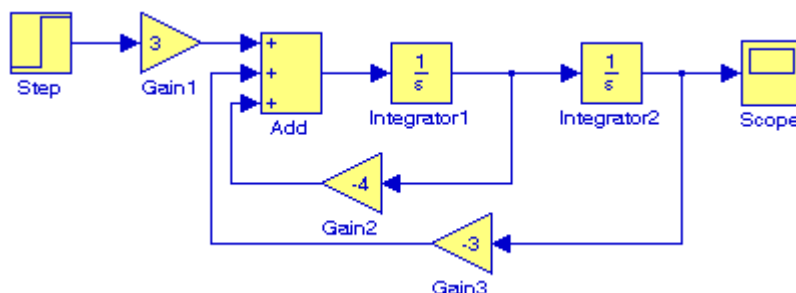




Figure B.12. File `Equation_1_26` complete block diagram

8. The initial conditions  $i_L(0^-) = C \frac{dv_C}{dt} \Big|_{t=0} = 0$ , and  $v_C(0^-) = 0.5 \text{ V}$  are entered by double clicking the Integrator blocks and entering the values 0 for the first integrator, and 0.5 for the second integrator. We also need to specify the simulation time. This is done by specifying the simulation time to be 10 seconds on the **Configuration Parameters** from the **Simulation** drop menu. We can start the simulation on **Start** from the **Simulation** drop menu or by clicking on the  icon.
9. To see the output waveform, we double click on the **Scope** block, and then clicking on the Autoscale  icon, we obtain the waveform shown in Figure B.13.

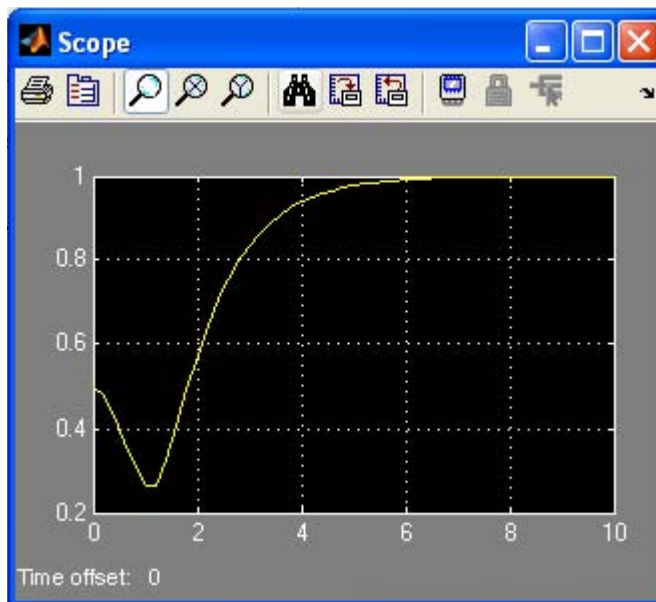


Figure B.13. The waveform for the function  $v_C(t)$  for Example B.1

Another easier method to obtain and display the output  $v_C(t)$  for Example B.1, is to use **State-Space** block from **Continuous** in the Simulink Library Browser, as shown in Figure B.14.

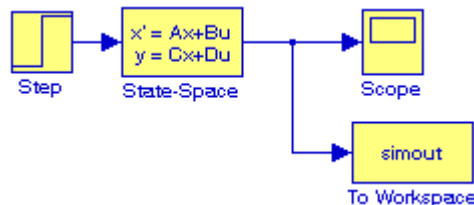


Figure B.14. Obtaining the function  $v_C(t)$  for Example B.1 with the State-Space block.

The **simout To Workspace** block shown in Figure B.14 writes its input to the workspace. The data and variables created in the MATLAB Command window, reside in the MATLAB Workspace. This block writes its output to an array or structure that has the name specified by the block's Variable name parameter. This gives us the ability to delete or modify selected variables. We issue the command **who** to see those variables. From Equation B.23, Page B–6,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -4 & -4 \\ 3/4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \end{bmatrix} u_0(t)$$

The output equation is

$$y = Cx + du$$

or

$$y = [0 \ 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0]u$$

We double-click on the **State-Space** block, and in the **Functions Block Parameters** window we enter the constants shown in Figure B.15.

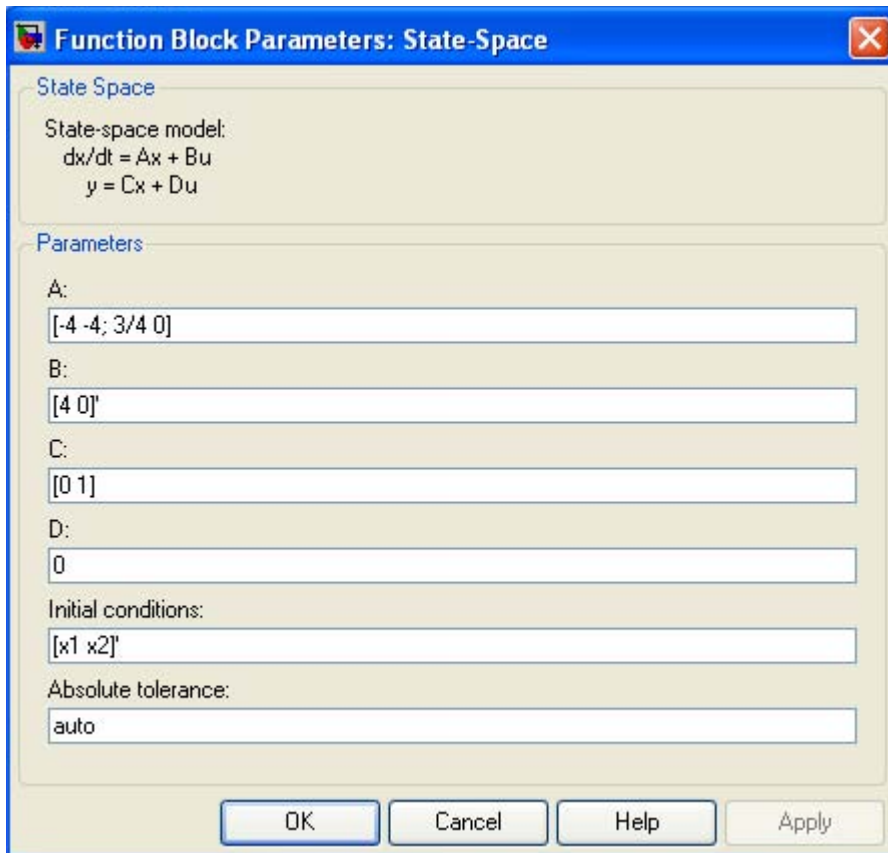




Figure B.15. The Function block parameters for the State-Space block.

The initial conditions  $[x_1 \ x_2]'$  are specified in MATLAB's Command prompt as  
 $x_1=0; x_2=0.5;$

As before, to start the simulation we click on the  icon, and to see the output waveform, we double click on the **Scope** block, and then clicking on the Autoscale  icon, we obtain the waveform shown in Figure B.16.

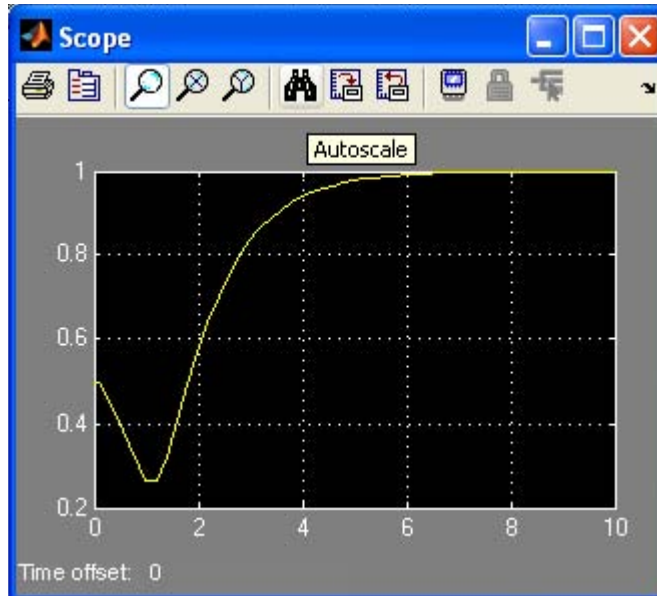


Figure B.16. The waveform for the function  $v_C(t)$  for Example B.1 with the State–Space block.

The state–space block is the best choice when we need to display the output waveform of three or more variables as illustrated by the following example.

### Example B.2

A fourth–order network is described by the differential equation

$$\frac{d^4 y}{dt^4} + a_3 \frac{d^3 y}{dt^3} + a_2 \frac{d^2 y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y(t) = u(t) \quad (\text{B.27})$$

where  $y(t)$  is the output representing the voltage or current of the network, and  $u(t)$  is any input, and the initial conditions are  $y(0) = y'(0) = y''(0) = y'''(0) = 0$ .

a. We will express (B.27) as a set of state equations

b. It is known that the solution of the differential equation

$$\frac{d^4 y}{dt^4} + 2 \frac{d^2 y}{dt^2} + y(t) = \sin t \quad (\text{B.28})$$

subject to the initial conditions  $y(0) = y'(0) = y''(0) = y'''(0) = 0$ , has the solution

$$y(t) = 0.125[(3 - t^2) - 3t \cos t] \quad (\text{B.29})$$

In our set of state equations, we will select appropriate values for the coefficients  $a_3, a_2, a_1$ , and  $a_0$  so that the new set of the state equations will represent the differential equation of (B.28), and using Simulink, we will display the waveform of the output  $y(t)$ .

1. The differential equation of (B.28) is of fourth-order; therefore, we must define four state variables that will be used with the four first-order state equations.

We denote the state variables as  $x_1, x_2, x_3$ , and  $x_4$ , and we relate them to the terms of the given differential equation as

$$x_1 = y(t) \quad x_2 = \frac{dy}{dt} \quad x_3 = \frac{d^2 y}{dt^2} \quad x_4 = \frac{d^3 y}{dt^3} \quad (\text{B.30})$$

We observe that

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= x_4 \\ \frac{d^4 y}{dt^4} &= \dot{x}_4 = -a_0 x_1 - a_1 x_2 - a_2 x_3 - a_3 x_4 + u(t) \end{aligned} \quad (\text{B.31})$$

and in matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t) \quad (\text{B.32})$$

In compact form, (B.32) is written as

$$\dot{x} = Ax + bu \quad (\text{B.33})$$

Also, the output is

$$y = Cx + du \quad (\text{B.34})$$

where

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \text{and } u = u(t) \quad (\text{B.35})$$

and since the output is defined as

$$y(t) = x_1$$

relation (B.34) is expressed as

$$y = [1 \ 0 \ 0 \ 0] \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + [0]u(t) \quad (\text{B.36})$$

2. By inspection, the differential equation of (B.27) will be reduced to the differential equation of (B.28) if we let

$$a_3 = 0 \quad a_2 = 2 \quad a_1 = 0 \quad a_0 = 1 \quad u(t) = \text{sint}$$

and thus the differential equation of (B.28) can be expressed in state–space form as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \text{sint} \quad (\text{B.37})$$

where

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & 0 & -2 & 0 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \text{and } u = \text{sint} \quad (\text{B.38})$$

Since the output is defined as

$$y(t) = x_1$$

in matrix form it is expressed as

$$y = [1 \ 0 \ 0 \ 0] \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + [0] \sin t \quad (\text{B.39})$$

We invoke MATLAB, we start Simulink by clicking on the Simulink icon, on the **Simulink Library Browser** we click on the **Create a new model** (blank page icon on the left of the top bar), and we save this model as **Example\_1\_2**. On the **Simulink Library Browser** we select **Sources**, we drag the **Signal Generator** block on the **Example\_1\_2** model window, we click and drag the **State-Space** block from the **Continuous** on **Simulink Library Browser**, and we click and drag the **Scope** block from the **Commonly Used Blocks** on the **Simulink Library Browser**. We also add the **Display** block found under **Sinks** on the **Simulink Library Browser**. We connect these four blocks and the complete block diagram is as shown in Figure B.17.

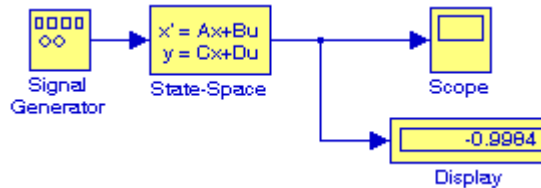


Figure B.17. Block diagram for Example B.2

We now double-click on the **Signal Generator** block and we enter the following in the **Function Block Parameters**:

Wave form: sine

Time (t): Use simulation time

Amplitude: 1

Frequency: 2

Units: Hertz

Next, we double-click on the **state-space** block and we enter the following parameter values in the **Function Block Parameters**:

A: [0 1 0 0; 0 0 1 0; 0 0 0 1; -a0 -a1 -a2 -a3]

B: [0 0 0 1]'

C: [1 0 0 0]

D: [0]



Initial conditions: x0



Absolute tolerance: auto

Now, we switch to the MATLAB Command prompt and we type the following:

```
>> a0=1; a1=0; a2=2; a3=0; x0=[0 0 0 0]';
```

We change the **Simulation Stop time** to 25, and we start the simulation by clicking on the  icon. To see the output waveform, we double click on the **Scope** block, then clicking on the Autoscale  icon, we obtain the waveform shown in Figure B.18.

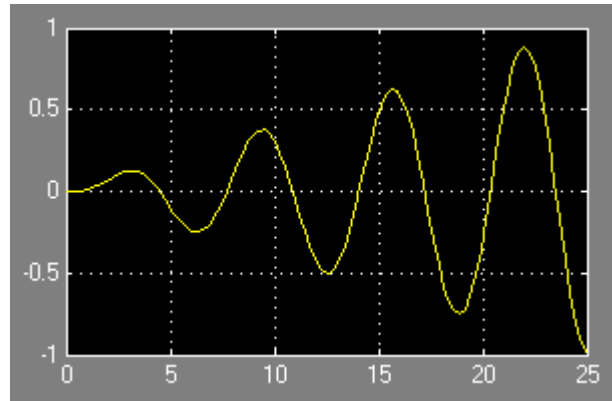


Figure B.18. Waveform for Example B.2

The **Display** block in Figure B.17 shows the value at the end of the simulation stop time.

Examples B.1 and B.2 have clearly illustrated that the State–Space is indeed a powerful block. We could have obtained the solution of Example B.2 using four Integrator blocks by this approach would have been more time consuming.

### Example B.3

Using **Algebraic Constraint** blocks found in the **Math Operations** library, **Display** blocks found in the **Sinks** library, and **Gain** blocks found in the **Commonly Used Blocks** library, we will create a model that will produce the simultaneous solution of three equations with three unknowns.

The model will display the values for the unknowns  $z_1$ ,  $z_2$ , and  $z_3$  in the system of the equations

$$\begin{aligned} a_1 z_1 + a_2 z_2 + a_3 z_3 + k_1 &= 0 \\ a_4 z_1 + a_5 z_2 + a_6 z_3 + k_2 &= 0 \\ a_7 z_1 + a_8 z_2 + a_9 z_3 + k_3 &= 0 \end{aligned} \tag{B.40}$$

The model is shown in Figure B.19.

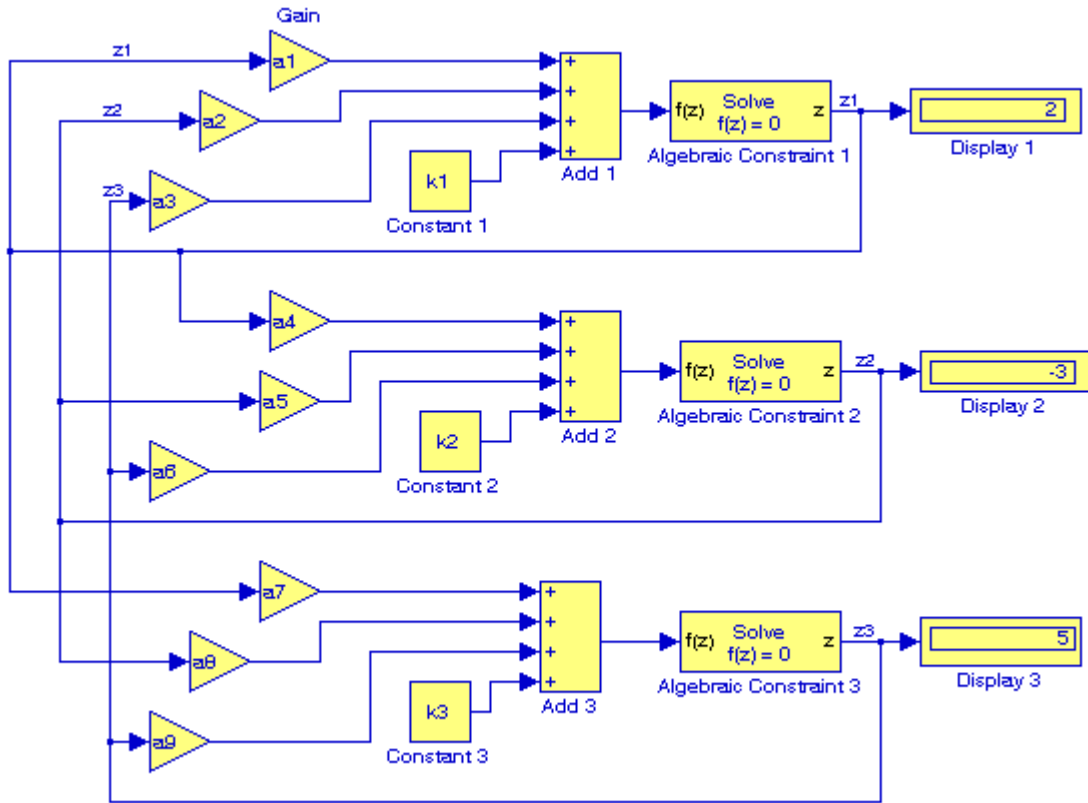


Figure B.19. Model for Example B.3

Next, we go to MATLAB's Command prompt and we enter the following values:

```
a1=2; a2=-3; a3=-1; a4=1; a5=5; a6=4; a7=-6; a8=1; a9=2;...
k1=-8; k2=-7; k3=5;
```

After clicking on the simulation icon, we observe the values of the unknowns as  $z_1 = 2$ ,  $z_2 = -3$ , and  $z_3 = 5$ . These values are shown in the Display blocks of Figure B.19.

The **Algebraic Constraint** block constrains the input signal  $f(z)$  to zero and outputs an algebraic state  $z$ . The block outputs the value necessary to produce a zero at the input. The output must affect the input through some feedback path. This enables us to specify algebraic equations for index 1 differential/algebraic systems (DAEs). By default, the Initial guess parameter is zero. We can improve the efficiency of the algebraic loop solver by providing an Initial guess for the algebraic state  $z$  that is close to the solution value.

An outstanding feature in Simulink is the representation of a large model consisting of many blocks and lines, to be shown as a single Subsystem block.\* For instance, we can group all blocks and lines in the model of Figure B.19 except the display blocks, we choose **Create Subsystem** from the **Edit** menu, and this model will be shown as in Figure B.20† where in MATLAB's Command prompt we have entered:

```
a1=5; a2=-1; a3=4; a4=11; a5=6; a6=9; a7=-8; a8=4; a9=15;...  
k1=14; k2=-6; k3=9;
```

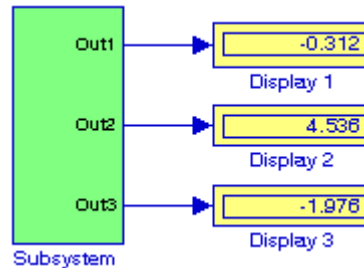


Figure B.20. The model of Figure B.19 represented as a subsystem

The Display blocks in Figure B.20 show the values of  $z_1$ ,  $z_2$ , and  $z_3$  for the values specified in MATLAB's Command prompt.

## B.2 Simulink Demos

At this time, the reader with no prior knowledge of Simulink, should be ready to learn Simulink's additional capabilities. It is highly recommended that the reader becomes familiar with the block libraries found in the Simulink Library Browser. Then, the reader can follow the steps delineated in The MathWorks Simulink User's Manual to run the Demo Models beginning with the **thermo** model. This model can be seen by typing

```
thermo
```

at the MATLAB Command prompt.

---

\* The Subsystem block is described in detail in Chapter 2, Section 2.1, Page 2-2, Introduction to Simulink with Engineering Applications, 978-1-934404-09-6.

† The contents of the Subsystem block are not lost. We can double-click on the Subsystem block to see its contents. The Subsystem block replaces the inputs and outputs of the model with Inport and Outport blocks. These blocks are described in Section 2.1, Chapter 2, Page 2-2, Introduction to Simulink with Engineering Applications, ISBN 978-1-934404-09-6.

This appendix is a brief introduction to **SimPowerSystems**® blockset that operates in the **Simulink**® environment. An introduction to **Simulink** is presented in Appendix B. For additional help with Simulink, please refer to the Simulink documentation.

### C.1 Simulation of Electric Circuits with SimPowerSystems

As stated in Appendix B, the MATLAB® and Simulink® environments are integrated into one entity, and thus we can analyze, simulate, and revise our models in either environment at any point. We can invoke **Simulink** from within MATLAB or by typing `simulink` at the MATLAB command prompt, and we can invoke **SimPowerSystems** from within Simulink or by typing `powerlib` at the MATLAB command prompt. We will introduce **SimPowerSystems** with two illustrated examples, a DC electric circuit, and an AC electric circuit

---

#### Example C.1

For the simple resistive circuit in Figure C.1,  $v_S = 12\text{v}$ ,  $R_1 = 7\Omega$ , and  $R_2 = 5\Omega$ . From the voltage division expression,  $v_{R_2} = R_2 \times v_S / (R_1 + R_2) = 5 \times 12 / 12 = 5\text{v}$  and from Ohm's law,  $i = v_S / (R_1 + R_2) = 1\text{A}$ .

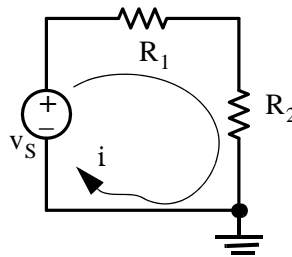


Figure C.1. Circuit for Example C.1

To model the circuit in Figure C.1, we enter the following command at the MATLAB prompt.

```
powerlib
```

and upon execution of this command, the `powerlib` window shown in Figure C.2 is displayed.

From the **File** menu in Figure C.2, we open a new window and we name it `Sim_Fig_C3` as shown in Figure C.3.

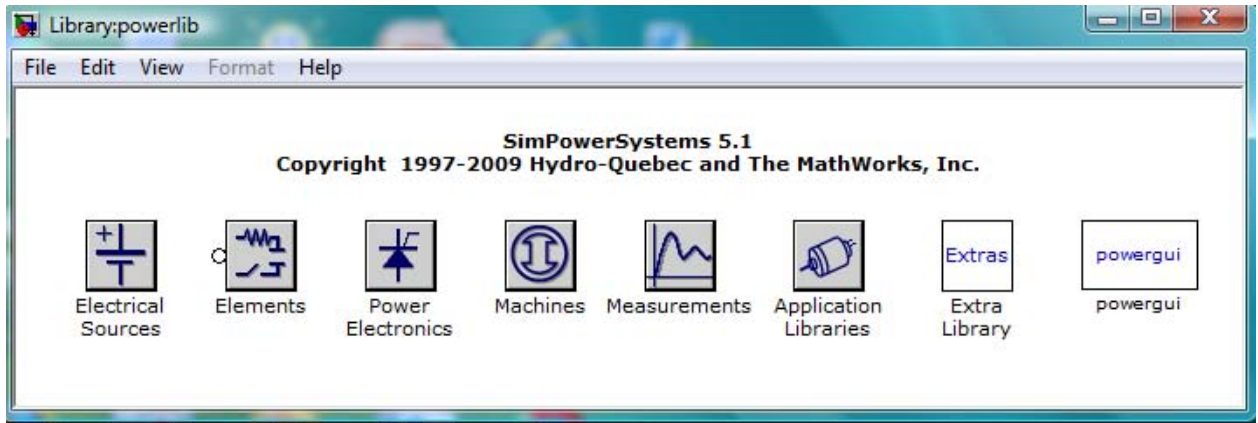


Figure C.2. Library blocks for SimPowerSystems

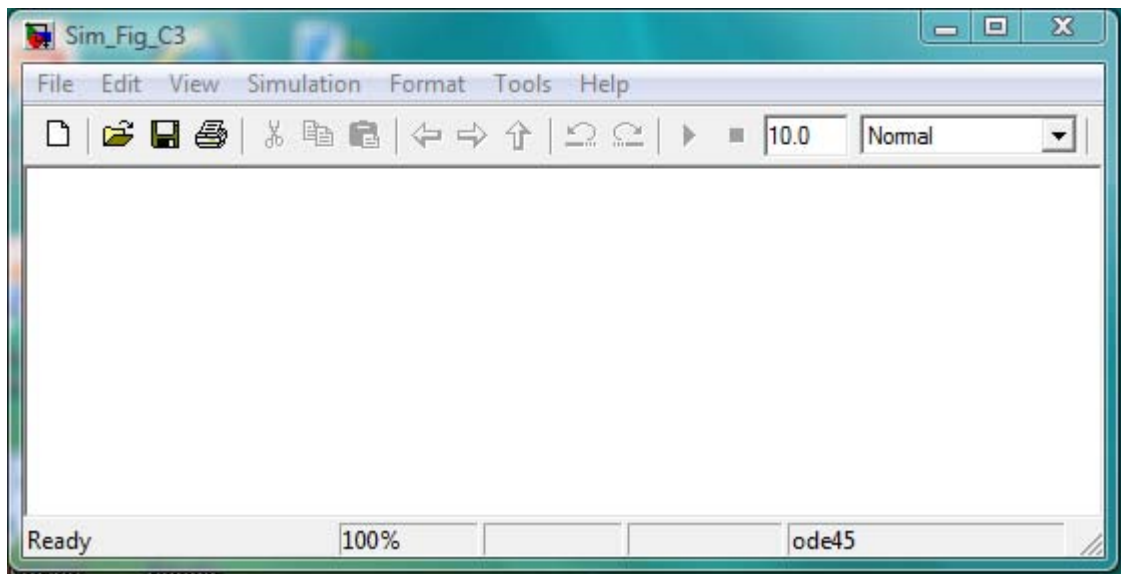


Figure C.3. New window for modeling the circuit shown in Figure C.1

The **powergui** block in Figure C.2 is referred to as the **Environmental** block for **SimPowerSystems** models and it must be included in every model containing **SimPowerSystems** blocks. Accordingly, we begin our model by adding this block as shown in Figure C.4.

We observe that in Figure C.4, the **powergui** block is named **Continuous**. This is the default method of solving an electric circuit and uses a variable step Simulink solver. Other methods are the **Discrete** method used when the discretization of the system at fixed time steps is desired, and the **Phasors** method which performs phasor simulation at the frequency specified by the Phasor frequency parameter. These methods are described in detail in the **SimPowerSystems** documentation.

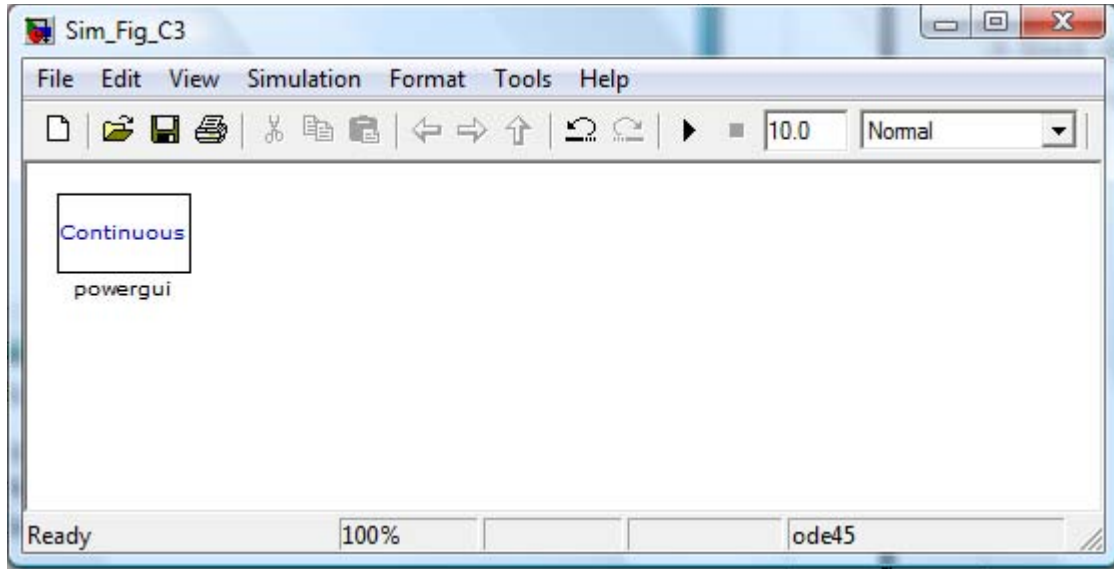


Figure C.4. Window with the addition of the powergui block

Next, we need to the components of the electric circuit shown in Figure C.1. From the **Electrical Sources** library in Figure C.2 we select the **DC Voltage Source** block and drag it into the model, from the **Elements** library we select and drag the **Series RLC Branch** block and the **Ground** block, from the **Measurements** library we select the **Current Measurement** and the **Voltage Measurement** blocks, and from the **Simulink Sinks** library we select and drag the **Display** block. The model now appears as shown in Figure C.5.

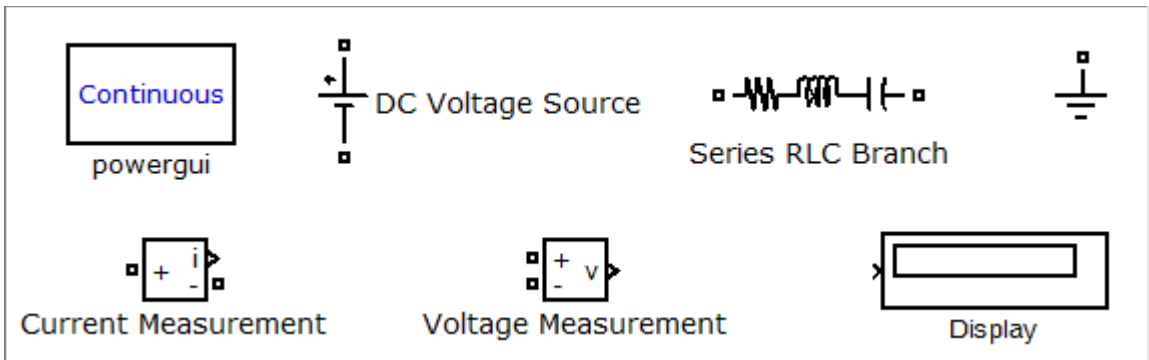


Figure C.5. The circuit components for our model

From the **Series RLC Branch** block we only need the resistor, and to eliminate the inductor and the capacitor, we double click it and from the **Block Parameters** window we select the **R** component with value set at  $7 \Omega$  as shown in Figure C.6.

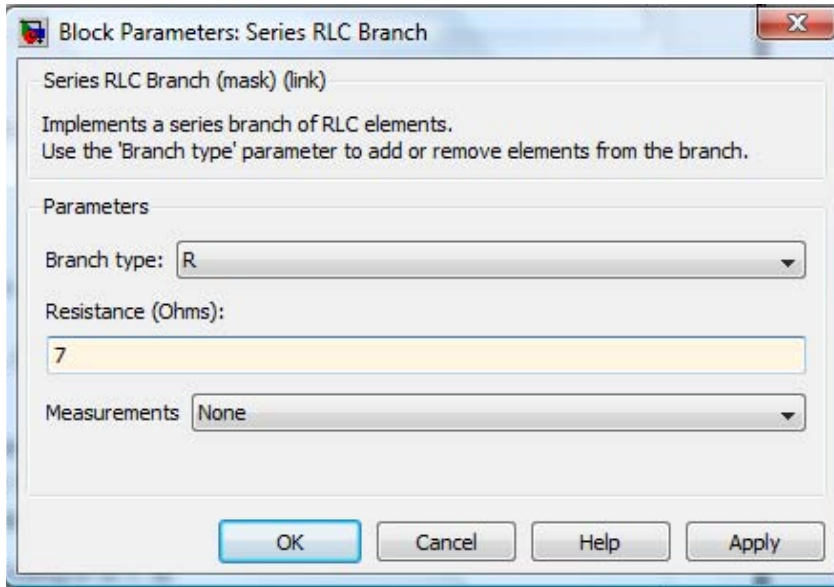


Figure C.6. The Block Parameters window for the Series RLC Branch

We need two resistors for our model and thus we copy and paste the resistor into the model, using the **Block Parameters** window we change its value to  $5 \Omega$ , and from the **Format** drop window we click the **Rotate** block option and we rotate it **clockwise**. We also need two **Display** blocks, one for the current measurement and the second for the voltage measurement and thus we copy and paste the **Display** block into the model. We also copy and paste twice the **Ground** block and the model is now as shown in Figure C.7 where we also have renamed the blocks to shorter names.

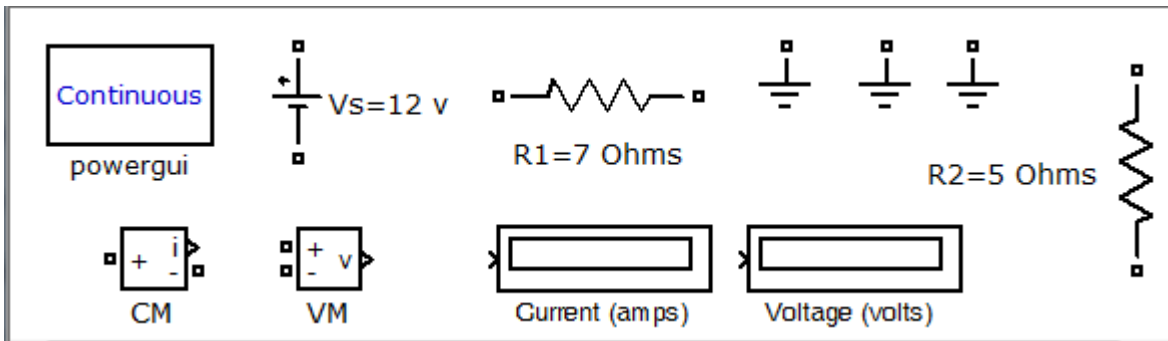


Figure C.7. Model with blocks renamed

From Figure C.7 above, we observe that both terminals of the voltage source and the resistors are shown with small square ( ) ports, the left ports of the CM (Current Measurement), and VM (Voltage Measurement) are also shown with ports, but the terminals on the right are shown with the **Simulink** output ports as >. The rules for the **SimPowerSystems** electrical terminal ports and connection lines are as follows:

1. We can connect **Simulink** ports (>) only to other **Simulink** ports.
2. We can connect **SimPowerSystems** ports ( ) only to other **SimPowerSystems** ports.\*
3. If it is necessary to connect Simulink ports (>) to SimPowerSystems ports ( ), we can use SimPowerSystems blocks that contain both Simulink and SimPowerSystems ports such as the Current Measurement (CM) block and the Voltage Measurement (VM) block shown in Figure C.7.

The model for the electric circuit in Figure C.1 is shown in Figure C.8.

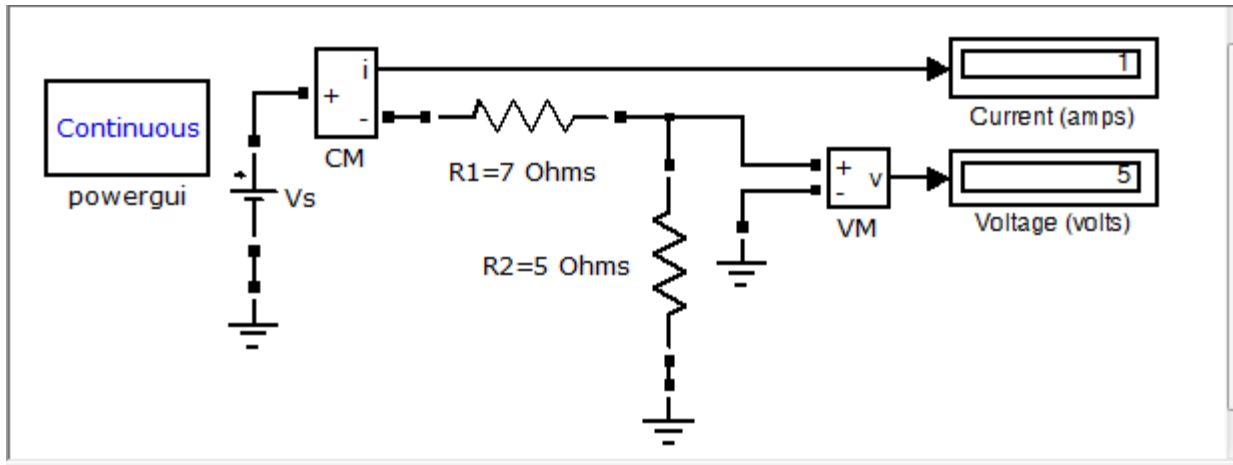


Figure C.8. The final form of the SimPowerSystems model for the electric circuit in Figure C.1

For the model in Figure C.8 we used the **DC Voltage Source** block. The **SimPowerSystems** documentation states that we can also use the **AC Voltage Source** block as a **DC Voltage Source** block provided that we set the frequency at 0 Hz and the phase at 90 degrees in the **Block Parameters** window as shown in Figure C.9.

\* As in Simulink, we can autoconnect two **SimPowerSystems** blocks by selecting the source block, then holding down the **Ctrl** key, and left-clicking the destination block.



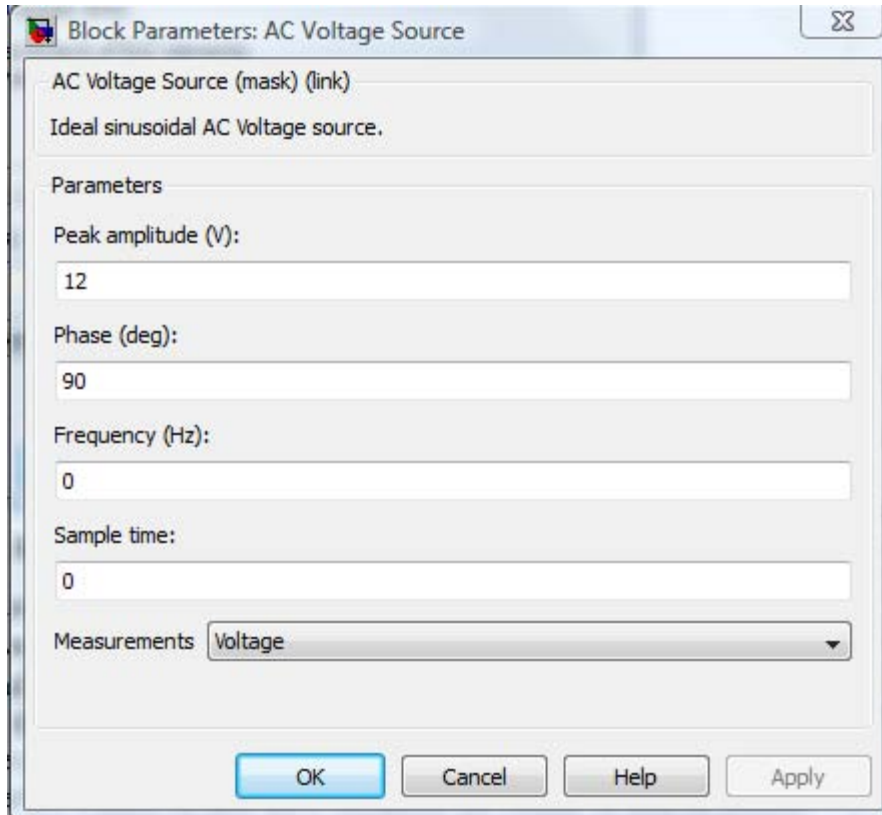


Figure C.9. Block parameter settings when using an AC Voltage Source block as a DC Voltage Source

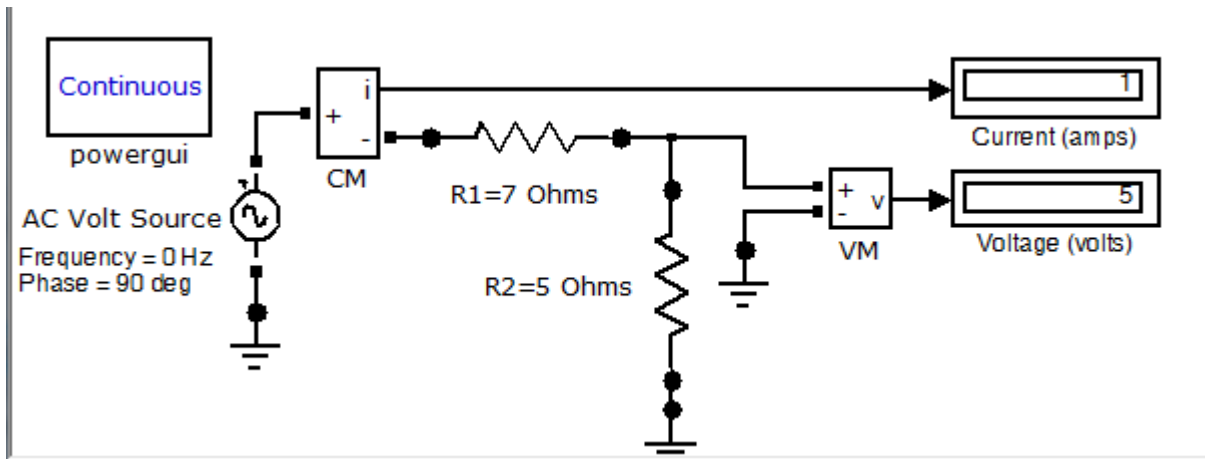


Figure C.10. Model with AC Voltage Source used as DC Voltage Source

A third option is to use a **Controlled Voltage Source** block with a **Constant** block set to the numerical value of the **DC voltage Source** as shown in the model of Figure C.11.

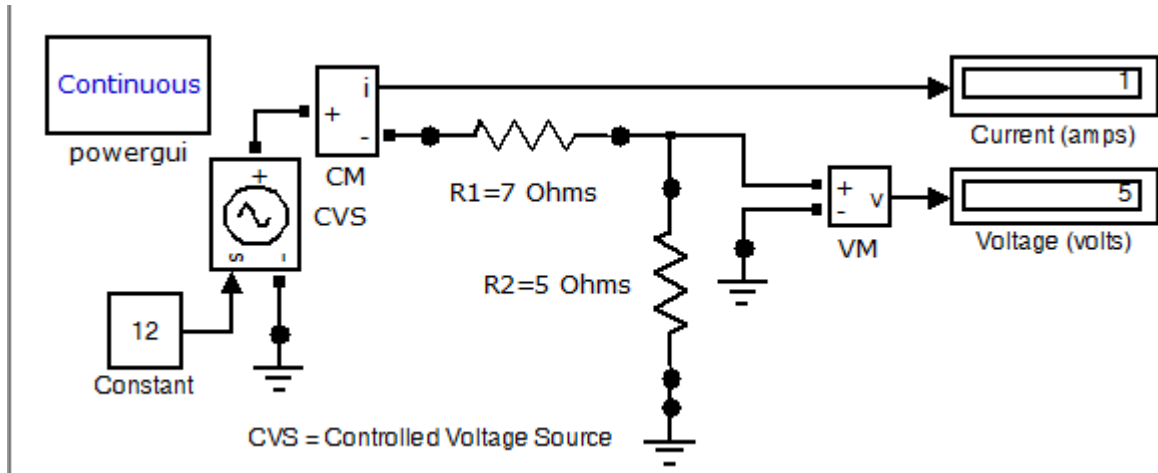


Figure C.11. Model with Controlled Voltage Source block

### Example C.2

Consider the AC electric circuit in Figure C.12

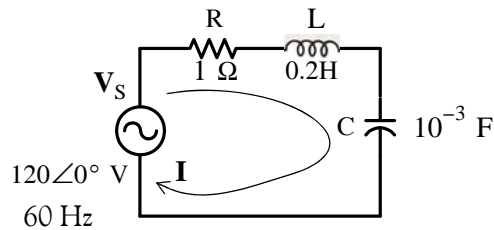


Figure C.12. Electric circuit for Example C.2

The current  $I$  and the voltage  $V_c$  across the capacitor are computed with MATLAB as follows:

```
Vs=120; f=60; R=1; L=0.2; C=10^(-3); XL=2*pi*f*L; XC=1/(2*pi*f*C);...
Z=sqrt(R^2+(XL-XC)^2); I=Vs/Z, Vc=XC*I
```

```
I =
    1.6494
Vc =
    4.3752
```

The **SimPowerSystems** model and the waveforms for the current  $I$  and the voltage  $V_c$  are shown in Figures C.13 and C.14 respectively.

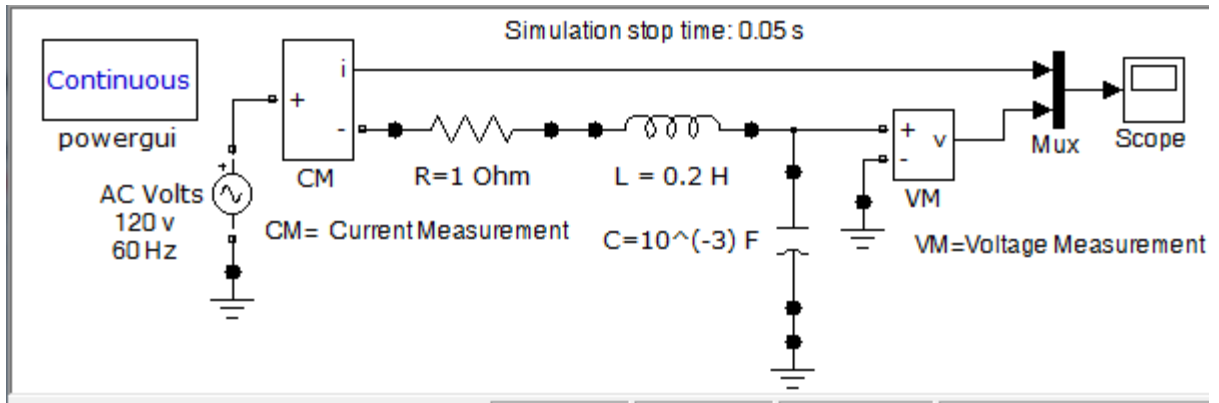


Figure C.13. SimPowerSystems model for the electric circuit in Figure C.12

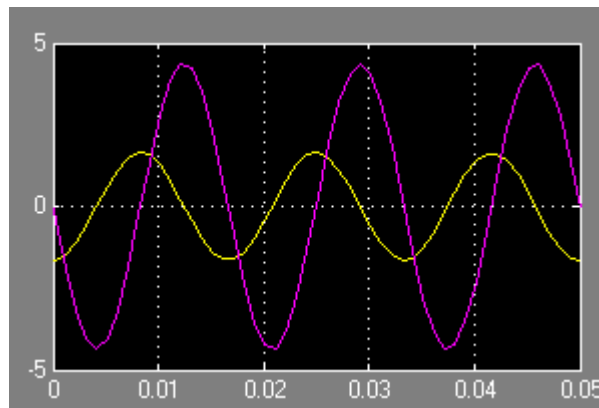


Figure C.14. Waveforms for the current  $I$  and voltage  $V_c$  across the capacitor in Figure C.12

The same results are obtained if we replace the applied AC voltage source block in the model of Figure C.13 with a **Controlled Voltage Source (CVS)** block as shown in Figure C.15.

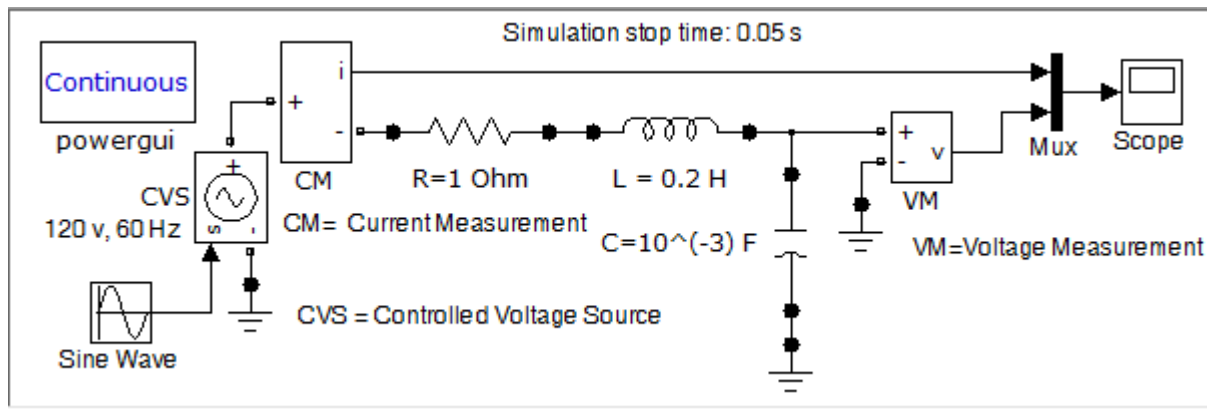


Figure C.15. The model in Figure C.13 with the AC Voltage Source block replaced with a CVS block

This appendix is a review of the algebra of complex numbers. The basic operations are defined and illustrated by several examples. Applications using Euler's identities are presented, and the exponential and polar forms are discussed and illustrated with examples.

### D.1 Definition of a Complex Number

In the language of mathematics, the square root of minus one is denoted as  $i$ , that is,  $i = \sqrt{-1}$ . In the electrical engineering field, we denote  $i$  as  $j$  to avoid confusion with current  $i$ . Essentially,  $j$  is an operator that produces a 90-degree counterclockwise rotation to any vector to which it is applied as a multiplying factor. Thus, if it is given that a vector  $A$  has the direction along the right side of the  $x$ -axis as shown in Figure D.1, multiplication of this vector by the operator  $j$  will result in a new vector  $jA$  whose magnitude remains the same, but it has been rotated counterclockwise by  $90^\circ$ .

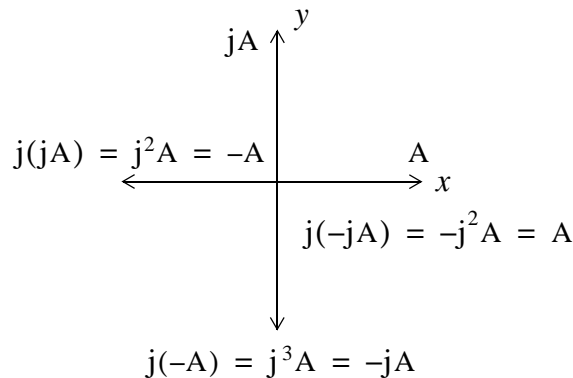


Figure D.1. The  $j$  operator

Also, another multiplication of the new vector  $jA$  by  $j$  will produce another  $90^\circ$  counterclockwise direction. In this case, the vector  $A$  has rotated  $180^\circ$  and its new value now is  $-A$ . When this vector is rotated by another  $90^\circ$  for a total of  $270^\circ$ , its value becomes  $j(-A) = -jA$ . A fourth  $90^\circ$  rotation returns the vector to its original position, and thus its value is again  $A$ . Therefore, we conclude that  $j^2 = -1$ ,  $j^3 = -j$ , and  $j^4 = 1$ .

---

## A Review of Complex Numbers

---

**Note:** In our subsequent discussion, we will denote the x-axis (abscissa) as the *real axis*, and the y-axis (ordinate) as the *imaginary axis* with the understanding that the “imaginary” axis is just as “real” as the real axis. In other words, the imaginary axis is just as important as the real axis.\*

An *imaginary number* is the product of a real number, say  $r$ , by the operator  $j$ . Thus,  $r$  is a real number and  $jr$  is an imaginary number.

A *complex number* is the sum (or difference) of a real number and an imaginary number. For example, the number  $A = a + jb$  where  $a$  and  $b$  are both real numbers, is a complex number. Then,  $a = \text{Re}\{A\}$  and  $b = \text{Im}\{A\}$  where  $\text{Re}\{A\}$  denotes real part of  $A$ , and  $b = \text{Im}\{A\}$  the imaginary part of  $A$ .

By definition, two complex numbers  $A$  and  $B$  where  $A = a + jb$  and  $B = c + jd$ , are equal if and only if their real parts are equal, and also their imaginary parts are equal. Thus,  $A = B$  if and only if  $a = c$  and  $b = d$ .

### D.2 Addition and Subtraction of Complex Numbers

The sum of two complex numbers has a real component equal to the sum of the real components, and an imaginary component equal to the sum of the imaginary components. For subtraction, we change the signs of the components of the subtrahend and we perform addition. Thus, if

$$A = a + jb \text{ and } B = c + jd$$

then

$$A + B = (a + c) + j(b + d)$$

and

$$A - B = (a - c) + j(b - d)$$

---

#### Example D.1

It is given that  $A = 3 + j4$ , and  $B = 4 - j2$ . Find  $A + B$  and  $A - B$

**Solution:**

$$A + B = (3 + j4) + (4 - j2) = (3 + 4) + j(4 - 2) = 7 + j2$$

and

$$A - B = (3 + j4) - (4 - j2) = (3 - 4) + j(4 + 2) = -1 + j6$$

---

\* We may think the real axis as the cosine axis and the imaginary axis as the sine axis.

---

### D.3 Multiplication of Complex Numbers

Complex numbers are multiplied using the rules of elementary algebra, and making use of the fact that  $j^2 = -1$ . Thus, if

$$A = a + jb \text{ and } B = c + jd$$

then

$$A \cdot B = (a + jb) \cdot (c + jd) = ac + jad + jbc + j^2bd$$

and since  $j^2 = -1$ , it follows that

$$\begin{aligned} A \cdot B &= ac + jad + jbc - bd \\ &= (ac - bd) + j(ad + bc) \end{aligned} \tag{D.1}$$

---

#### Example D.2

It is given that  $A = 3 + j4$  and  $B = 4 - j2$ . Find  $A \cdot B$

**Solution:**

$$A \cdot B = (3 + j4) \cdot (4 - j2) = 12 - j6 + j16 - j^28 = 20 + j10$$

---

The *conjugate* of a complex number, denoted as  $A^*$ , is another complex number with the same real component, and with an imaginary component of opposite sign. Thus, if  $A = a + jb$ , then  $A^* = a - jb$ .

---

#### Example D.3

It is given that  $A = 3 + j5$ . Find  $A^*$

**Solution:**

The conjugate of the complex number  $A$  has the same real component, but the imaginary component has opposite sign. Then,  $A^* = 3 - j5$

---

If a complex number  $A$  is multiplied by its conjugate, the result is a real number. Thus, if  $A = a + jb$ , then

$$A \cdot A^* = (a + jb)(a - jb) = a^2 - jab + jab - j^2b^2 = a^2 + b^2$$

---

## A Review of Complex Numbers

---

### Example D.4

It is given that  $A = 3 + j5$ . Find  $A \cdot A^*$

**Solution:**

$$A \cdot A^* = (3 + j5)(3 - j5) = 3^2 + 5^2 = 9 + 25 = 34$$

---

## D.4 Division of Complex Numbers

When performing division of complex numbers, it is desirable to obtain the quotient separated into a real part and an imaginary part. This procedure is called *rationalization of the quotient*, and it is done by multiplying the denominator by its conjugate. Thus, if  $A = a + jb$  and  $B = c + jd$ , then,

$$\begin{aligned} \frac{A}{B} &= \frac{a + jb}{c + jd} = \frac{(a + jb)(c - jd)}{(c + jd)(c - jd)} = \frac{A}{B} \cdot \frac{B^*}{B^*} = \frac{(ac + bd) + j(bc - ad)}{c^2 + d^2} \\ &= \frac{(ac + bd)}{c^2 + d^2} + j \frac{(bc - ad)}{c^2 + d^2} \end{aligned} \quad (D.2)$$

In (D.2), we multiplied both the numerator and denominator by the conjugate of the denominator to eliminate the  $j$  operator from the denominator of the quotient. Using this procedure, we see that the quotient is easily separated into a real and an imaginary part.

---

### Example D.5

It is given that  $A = 3 + j4$ , and  $B = 4 + j3$ . Find  $A/B$

**Solution:**

Using the procedure of (D.2), we obtain

$$\frac{A}{B} = \frac{3 + j4}{4 + j3} = \frac{(3 + j4)(4 - j3)}{(4 + j3)(4 - j3)} = \frac{12 - j9 + j16 + 12}{4^2 + 3^2} = \frac{24 + j7}{25} = \frac{24}{25} + j \frac{7}{25} = 0.96 + j0.28$$

---

## D.5 Exponential and Polar Forms of Complex Numbers

The relations

$$\boxed{e^{j\theta} = \cos\theta + j\sin\theta} \quad (D.3)$$

and

$$\boxed{e^{-j\theta} = \cos\theta - j\sin\theta} \quad (\text{D.4})$$

are known as the *Euler's identities*.

Multiplying (D.3) by the *real* positive constant  $C$  we obtain:

$$Ce^{j\theta} = C\cos\theta + jC\sin\theta \quad (\text{D.5})$$

This expression represents a complex number, say  $a + jb$ , and thus

$$Ce^{j\theta} = a + jb \quad (\text{D.6})$$

where the left side of (D.6) is the *exponential form*, and the right side is the *rectangular form*.

Equating real and imaginary parts in (D.5) and (D.6), we obtain

$$a = C\cos\theta \quad \text{and} \quad b = C\sin\theta \quad (\text{D.7})$$

Squaring and adding the expressions in (D.7), we obtain

$$a^2 + b^2 = (C\cos\theta)^2 + (C\sin\theta)^2 = C^2(\cos^2\theta + \sin^2\theta) = C^2$$

Then,

$$C^2 = a^2 + b^2$$

or

$$\boxed{C = \sqrt{a^2 + b^2}} \quad (\text{D.8})$$

Also, from (D.7)

$$\frac{b}{a} = \frac{C\sin\theta}{C\cos\theta} = \tan\theta$$

or

$$\boxed{\theta = \tan^{-1}\left(\frac{b}{a}\right)} \quad (\text{D.9})$$

To convert a complex number from rectangular to exponential form, we use the expression

$$\boxed{a + jb = \sqrt{a^2 + b^2} e^{j\left(\tan^{-1}\frac{b}{a}\right)}} \quad (\text{D.10})$$

To convert a complex number from exponential to rectangular form, we use the expressions

$$\boxed{\begin{aligned} Ce^{j\theta} &= C\cos\theta + jC\sin\theta \\ Ce^{-j\theta} &= C\cos\theta - jC\sin\theta \end{aligned}} \quad (\text{D.11})$$



---

## A Review of Complex Numbers

---

The *polar form* is essentially the same as the exponential form but the notation is different, that is,

$$\boxed{C e^{j\theta} = C \angle \theta} \quad (\text{D.12})$$

where the left side of (D.12) is the exponential form, and the right side is the polar form.

We must remember that *the phase angle  $\theta$  is always measured with respect to the positive real axis, and rotates in the counterclockwise direction.*

---

### Example D.6

Convert the following complex numbers to exponential and polar forms:

a.  $3 + j4$

b.  $-1 + j2$

c.  $-2 - j$

d.  $4 - j3$

**Solution:**

a. The real and imaginary components of this complex number are shown in Figure D.2.

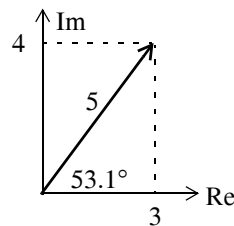


Figure D.2. The components of  $3 + j4$

Then,

$$3 + j4 = \sqrt{3^2 + 4^2} e^{j\left(\tan^{-1} \frac{4}{3}\right)} = 5e^{j53.1^\circ} = 5 \angle 53.1^\circ$$

Check with MATLAB:

```
x=3+j*4; magx=abs(x); thetax=angle(x)*180/pi; disp(magx); disp(thetax)
```

```
5
53.1301
```

Check with the Simulink **Complex to Magnitude–Angle** block\* shown in the Simulink model of Figure D.3.

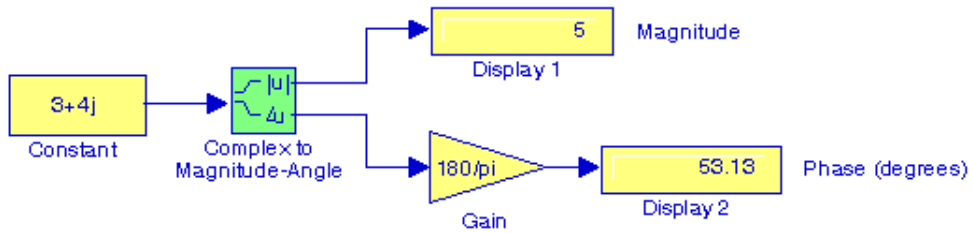


Figure D.3. Simulink model for Example D.6a

b. The real and imaginary components of this complex number are shown in Figure D.4.

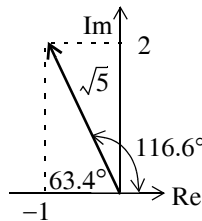


Figure D.4. The components of  $-1 + j2$

Then,

$$-1 + j2 = \sqrt{1^2 + 2^2} e^{j\left(\tan^{-1} \frac{2}{-1}\right)} = \sqrt{5} e^{j116.6^\circ} = \sqrt{5} \angle 116.6^\circ$$

Check with MATLAB:

```
y=-1+j*2; magy=abs(y); thetay=angle(y)*180/pi; disp(magy); disp(thetay)
```

```
2.2361
116.5651
```

c. The real and imaginary components of this complex number are shown in Figure D.5.

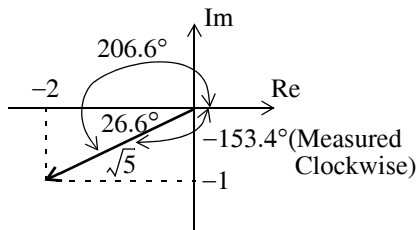


Figure D.5. The components of  $-2 - j$

\* For a detailed description and examples with this and other related transformation blocks, please refer to *Introduction to Simulink with Engineering Applications*, ISBN 978-1-934404-09-6.

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## A Review of Complex Numbers

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Then,

$$-2-j1 = \sqrt{2^2 + 1^2} e^{j\left(\tan^{-1} \frac{-1}{-2}\right)} = \sqrt{5} e^{j206.6^\circ} = \sqrt{5} \angle 206.6^\circ = \sqrt{5} e^{j(-153.4)^\circ} = \sqrt{5} \angle -153.4^\circ$$

Check with MATLAB:

```
v=-2-j*1; magv=abs(v); thetav=angle(v)*180/pi; disp(magv); disp(thetav)
2.2361
-153.4349
```

d. The real and imaginary components of this complex number are shown in Figure D.6.

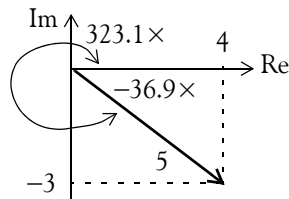


Figure D.6. The components of  $4 - j3$

Then,

$$4-j3 = \sqrt{4^2 + 3^2} e^{j\left(\tan^{-1} \frac{-3}{4}\right)} = 5 e^{j323.1^\circ} = 5 \angle 323.1^\circ = 5 e^{-j36.9^\circ} = 5 \angle -36.9^\circ$$

Check with MATLAB:

```
w=4-j*3; magw=abs(w); thetaw=angle(w)*180/pi; disp(magw); disp(thetaw)
5
-36.8699
```

---

### Example D.7

Express the complex number  $-2 \angle 30^\circ$  in exponential and in rectangular forms.

**Solution:**

We recall that  $-1 = j^2$ . Since each  $j$  rotates a vector by  $90^\circ$  counterclockwise, then  $-2 \angle 30^\circ$  is the same as  $2 \angle 30^\circ$  rotated counterclockwise by  $180^\circ$ . Therefore,

$$-2 \angle 30^\circ = 2 \angle (30^\circ + 180^\circ) = 2 \angle 210^\circ = 2 \angle -150^\circ$$

The components of this complex number are shown in Figure D.7.

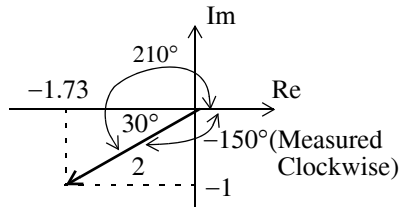


Figure D.7. The components of  $2\angle-150^\circ$

Then,

$$2\angle-150^\circ = 2e^{-j150^\circ} = 2(\cos 150^\circ - j\sin 150^\circ) = 2(-0.866 - j0.5) = -1.73 - j$$

**Note:** The rectangular form is most useful when we add or subtract complex numbers; however, the exponential and polar forms are most convenient when we multiply or divide complex numbers.

To multiply two complex numbers in exponential (or polar) form, we multiply the magnitudes and we add the phase angles, that is, if

$$A = M\angle\theta \quad \text{and} \quad B = N\angle\phi$$

then,

$$AB = MN\angle(\theta + \phi) = Me^{j\theta}Ne^{j\phi} = MNe^{j(\theta + \phi)} \quad (\text{D.13})$$

### Example D.8

Multiply  $A = 10\angle 53.1^\circ$  by  $B = 5\angle -36.9^\circ$

**Solution:**

Multiplication in polar form yields

$$AB = (10 \times 5)\angle[53.1^\circ + (-36.9^\circ)] = 50\angle 16.2^\circ$$

and multiplication in exponential form yields

$$AB = (10e^{j53.1^\circ})(5e^{-j36.9^\circ}) = 50e^{j(53.1^\circ - 36.9^\circ)} = 50e^{j16.2^\circ}$$

To divide one complex number by another when both are expressed in exponential or polar form, we divide the magnitude of the dividend by the magnitude of the divisor, and we subtract the phase angle of the divisor from the phase angle of the dividend, that is, if

$$A = M\angle\theta \quad \text{and} \quad B = N\angle\phi$$

---

## A Review of Complex Numbers

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then,

$$\boxed{\frac{A}{B} = \frac{M}{N} \angle(\theta - \phi) = \frac{Me^{j\theta}}{Ne^{j\phi}} = \frac{M}{N} e^{j(\theta - \phi)}} \quad (\text{D.14})$$

---

### Example D.9

Divide  $A = 10 \angle 53.1^\circ$  by  $B = 5 \angle -36.9^\circ$

**Solution:**

Division in polar form yields

$$\frac{A}{B} = \frac{10 \angle 53.1^\circ}{5 \angle -36.9^\circ} = 2 \angle [53.1^\circ - (-36.9^\circ)] = 2 \angle 90^\circ$$

Division in exponential form yields

$$\frac{A}{B} = \frac{10e^{j53.1^\circ}}{5e^{-j36.9^\circ}} = 2e^{j53.1^\circ} e^{j36.9^\circ} = 2e^{j90^\circ}$$

---

This appendix is an introduction to matrices and matrix operations. Determinants, Cramer's rule, and Gauss's elimination method are reviewed. Some definitions and examples are not applicable to the material presented in this text, but are included for subject continuity, and academic interest. They are discussed in detail in matrix theory textbooks. These are denoted with a dagger (†) and may be skipped.

### E.1 Matrix Definition

A *matrix* is a rectangular array of numbers such as those shown below.

$$\begin{bmatrix} 2 & 3 & 7 \\ 1 & -1 & 5 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 3 & 1 \\ -2 & 1 & -5 \\ 4 & -7 & 6 \end{bmatrix}$$

In general form, a matrix  $A$  is denoted as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \quad (\text{E.1})$$

The numbers  $a_{ij}$  are the *elements* of the matrix where the index  $i$  indicates the row, and  $j$  indicates the column in which each element is positioned. For instance,  $a_{43}$  indicates the element positioned in the fourth row and third column.

A matrix of  $m$  rows and  $n$  columns is said to be of  $m \times n$  *order matrix*.

If  $m = n$ , the matrix is said to be a *square matrix of order*  $m$  (or  $n$ ). Thus, if a matrix has five rows and five columns, it is said to be a square matrix of order 5.

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## Appendix E Matrices and Determinants

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In a square matrix, the elements  $a_{11}, a_{22}, a_{33}, \dots, a_{nn}$  are called the *main diagonal elements*. Alternately, we say that the matrix elements  $a_{11}, a_{22}, a_{33}, \dots, a_{nn}$ , are located on the *main diagonal*.

† The sum of the diagonal elements of a square matrix  $A$  is called the *trace*<sup>\*</sup> of  $A$ .

† A matrix in which every element is zero, is called a *zero matrix*.

### E.2 Matrix Operations

Two matrices  $A = [a_{ij}]$  and  $B = [b_{ij}]$  are equal, that is,  $A = B$ , if and only if

$$a_{ij} = b_{ij} \quad i = 1, 2, 3, \dots, m \quad j = 1, 2, 3, \dots, n \quad (\text{E.2})$$

Two matrices are said to be *conformable for addition (subtraction)*, if they are of the same order  $m \times n$ .

If  $A = [a_{ij}]$  and  $B = [b_{ij}]$  are conformable for addition (subtraction), their sum (difference) will be another matrix  $C$  with the same order as  $A$  and  $B$ , where each element of  $C$  is the sum (difference) of the corresponding elements of  $A$  and  $B$ , that is,

$$C = A \pm B = [a_{ij} \pm b_{ij}] \quad (\text{E.3})$$

---

#### Example E.1

Compute  $A + B$  and  $A - B$  given that

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 3 & 0 \\ -1 & 2 & 5 \end{bmatrix}$$

**Solution:**

$$A + B = \begin{bmatrix} 1+2 & 2+3 & 3+0 \\ 0-1 & 1+2 & 4+5 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 3 \\ -1 & 3 & 9 \end{bmatrix}$$

and

$$A - B = \begin{bmatrix} 1-2 & 2-3 & 3-0 \\ 0+1 & 1-2 & 4-5 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 3 \\ 1 & -1 & -1 \end{bmatrix}$$

---

\* Henceforth, all paragraphs and topics preceded by a dagger ( † ) may be skipped. These are discussed in matrix theory textbooks.

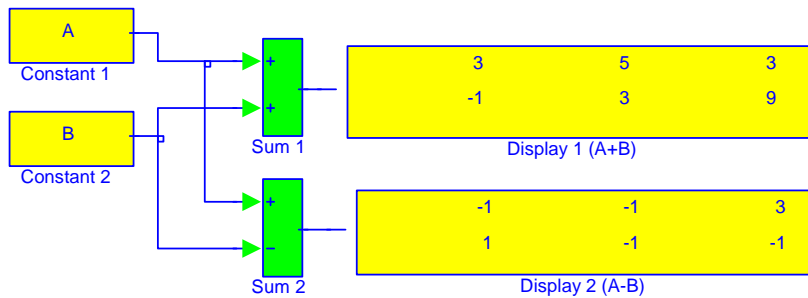
Check with MATLAB:

```
A=[1 2 3; 0 1 4]; B=[2 3 0; -1 2 5]; % Define matrices A and B
A+B, A-B % Add A and B, then Subtract B from A
```

```
ans =
     3     5     3
    -1     3     9

ans =
    -1    -1     3
     1    -1    -1
```

Check with Simulink:



Note: The elements of matrices A and B are specified in MATLAB's Command prompt

If  $k$  is any scalar (a positive or negative number), and not  $[k]$  which is a  $1 \times 1$  matrix, then multiplication of a matrix  $A$  by the scalar  $k$  is the multiplication of every element of  $A$  by  $k$ .

**Example E.2**

Multiply the matrix

$$A = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}$$

by

a.  $k_1 = 5$

b.  $k_2 = -3 + j2$



---

## Appendix E Matrices and Determinants

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**Solution:**

a.

$$k_1 \cdot A = 5 \times \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 5 \times 1 & 5 \times (-2) \\ 5 \times 2 & 5 \times 3 \end{bmatrix} = \begin{bmatrix} 5 & -10 \\ 10 & 15 \end{bmatrix}$$

b.

$$k_2 \cdot A = (-3 + j2) \times \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} (-3 + j2) \times 1 & (-3 + j2) \times (-2) \\ (-3 + j2) \times 2 & (-3 + j2) \times 3 \end{bmatrix} = \begin{bmatrix} -3 + j2 & 6 - j4 \\ -6 + j4 & -9 + j6 \end{bmatrix}$$

Check with MATLAB:

```
k1=5; k2=(-3 + 2*j);           % Define scalars k1 and k2
A=[1 -2; 2 3];                % Define matrix A
k1*A, k2*A                     % Multiply matrix A by scalars k1 and k2
```

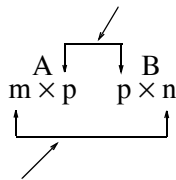
```
ans =
     5    -10
    10     15
```

```
ans =
-3.0000+ 2.0000i    6.0000- 4.0000i
-6.0000+ 4.0000i   -9.0000+ 6.0000i
```

---

Two matrices  $A$  and  $B$  are said to be *conformable for multiplication*  $A \cdot B$  in that order, only when the number of columns of matrix  $A$  is equal to the number of rows of matrix  $B$ . That is, the product  $A \cdot B$  (but not  $B \cdot A$ ) is conformable for multiplication only if  $A$  is an  $m \times p$  matrix and matrix  $B$  is an  $p \times n$  matrix. The product  $A \cdot B$  will then be an  $m \times n$  matrix. A convenient way to determine if two matrices are conformable for multiplication is to write the dimensions of the two matrices side-by-side as shown below.

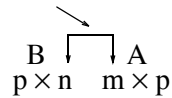
Shows that  $A$  and  $B$  are conformable for multiplication



Indicates the dimension of the product  $A \cdot B$

For the product  $B \cdot A$  we have:

Here, B and A are not conformable for multiplication



For matrix multiplication, the operation is row by column. Thus, to obtain the product  $A \cdot B$ , we multiply each element of a row of  $A$  by the corresponding element of a column of  $B$ ; then, we add these products.

**Example E.3**

Matrices  $C$  and  $D$  are defined as

$$C = [2 \ 3 \ 4] \text{ and } D = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

Compute the products  $C \cdot D$  and  $D \cdot C$

**Solution:**

The dimensions of matrices  $C$  and  $D$  are respectively  $1 \times 3$   $3 \times 1$ ; therefore the product  $C \cdot D$  is feasible, and will result in a  $1 \times 1$ , that is,

$$C \cdot D = [2 \ 3 \ 4] \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = [(2) \cdot (1) + (3) \cdot (-1) + (4) \cdot (2)] = [7]$$

The dimensions for  $D$  and  $C$  are respectively  $3 \times 1$   $1 \times 3$  and therefore, the product  $D \cdot C$  is also feasible. Multiplication of these will produce a  $3 \times 3$  matrix as follows:

$$D \cdot C = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} [2 \ 3 \ 4] = \begin{bmatrix} (1) \cdot (2) & (1) \cdot (3) & (1) \cdot (4) \\ (-1) \cdot (2) & (-1) \cdot (3) & (-1) \cdot (4) \\ (2) \cdot (2) & (2) \cdot (3) & (2) \cdot (4) \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 \\ -2 & -3 & -4 \\ 4 & 6 & 8 \end{bmatrix}$$

Check with MATLAB:

```
C=[2 3 4]; D=[1 -1 2]'; % Define matrices C and D. Observe that D is a column vector
C*D, D*C % Multiply C by D, then multiply D by C
```

```
ans =
     7
```

ans =

2	3	4
-2	-3	-4
4	6	8

Division of one matrix by another, is not defined. However, an analogous operation exists, and it will become apparent later in this chapter when we discuss the inverse of a matrix.

### E.3 Special Forms of Matrices

† A square matrix is said to be *upper triangular* when all the elements below the diagonal are zero. The matrix A of (E.4) is an upper triangular matrix. In an upper triangular matrix, not all elements above the diagonal need to be non-zero.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & \cdots & \cdots & \cdots \\ \cdots & \cdots & 0 & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_{mn} \end{bmatrix} \quad (\text{E.4})$$

† A square matrix is said to be *lower triangular*, when all the elements above the diagonal are zero. The matrix B of (E.5) is a lower triangular matrix. In a lower triangular matrix, not all elements below the diagonal need to be non-zero.

$$B = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & 0 \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \quad (\text{E.5})$$

† A square matrix is said to be *diagonal*, if all elements are zero, except those in the diagonal. The matrix C of (E.6) is a diagonal matrix.

$$C = \begin{bmatrix} a_{11} & 0 & 0 & \dots & 0 \\ 0 & a_{22} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & a_{nn} \end{bmatrix} \quad (E.6)$$

† A diagonal matrix is called a *scalar matrix*, if  $a_{11} = a_{22} = a_{33} = \dots = a_{nn} = k$  where  $k$  is a scalar. The matrix  $D$  of (E.7) is a scalar matrix with  $k = 4$ .

$$D = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \quad (E.7)$$

A scalar matrix with  $k = 1$ , is called an *identity matrix*  $I$ . Shown below are  $2 \times 2$ ,  $3 \times 3$ , and  $4 \times 4$  identity matrices.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (E.8)$$

The MATLAB **eye(n)** function displays an  $n \times n$  identity matrix. For example,

```
eye(4)    % Display a 4 by 4 identity matrix
```

```
ans =
     1     0     0     0
     0     1     0     0
     0     0     1     0
     0     0     0     1
```

Likewise, the **eye(size(A))** function, produces an identity matrix whose size is the same as matrix  $A$ . For example, let matrix  $A$  be defined as

```
A=[1 3 1; -2 1 -5; 4 -7 6]    % Define matrix A
```

```
A =
     1     3     1
    -2     1    -5
     4    -7     6
```

---

## Appendix E Matrices and Determinants

---

Then,  
`eye(size(A))`

displays

```
ans =  
    1    0    0  
    0    1    0  
    0    0    1
```

† The *transpose of a matrix*  $A$ , denoted as  $A^T$ , is the matrix that is obtained when the rows and columns of matrix  $A$  are interchangeE. For example, if

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \text{ then } A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \quad (\text{E.9})$$

In MATLAB, we use the apostrophe (') symbol to denote and obtain the transpose of a matrix. Thus, for the above example,

```
A=[1 2 3; 4 5 6] % Define matrix A
```

```
A =
```

```
    1    2    3  
    4    5    6
```

```
A' % Display the transpose of A
```

```
ans =
```

```
    1    4  
    2    5  
    3    6
```

† A *symmetric matrix*  $A$  is a matrix such that  $A^T = A$ , that is, the transpose of a matrix  $A$  is the same as  $A$ . An example of a symmetric matrix is shown below.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & -5 \\ 3 & -5 & 6 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & -5 \\ 3 & -5 & 6 \end{bmatrix} = A \quad (\text{E.10})$$

† If a matrix  $A$  has complex numbers as elements, the matrix obtained from  $A$  by replacing each element by its conjugate, is called the *conjugate of*  $A$ , and it is denoted as  $A^*$ , for example,

$$A = \begin{bmatrix} 1+j2 & j \\ 3 & 2-j3 \end{bmatrix} \quad A^* = \begin{bmatrix} 1-j2 & -j \\ 3 & 2+j3 \end{bmatrix}$$

MATLAB has two built-in functions which compute the complex conjugate of a number. The first, **conj(x)**, computes the complex conjugate of any complex number, and the second, **conj(A)**, computes the conjugate of a matrix A. Using MATLAB with the matrix A defined as above, we obtain

```
A = [1+2j j; 3 2-3j] % Define and display matrix A
A =
    1.0000 + 2.0000i      0 + 1.0000i
    3.0000              2.0000 - 3.0000i
conj_A=conj(A) % Compute and display the conjugate of A
conj_A =
    1.0000 - 2.0000i      0 - 1.0000i
    3.0000              2.0000 + 3.0000i
```

† A square matrix A such that  $A^T = -A$  is called *skew-symmetric*. For example,

$$A = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & -4 \\ 3 & 4 & 0 \end{bmatrix} \quad A^T = \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix} = -A$$

Therefore, matrix A above is skew symmetric.

† A square matrix A such that  $A^{T*} = A$  is called *Hermitian*. For example,

$$A = \begin{bmatrix} 1 & 1-j & 2 \\ 1+j & 3 & j \\ 2 & -j & 0 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & 1+j & 2 \\ 1-j & 3 & -j \\ 2 & j & 0 \end{bmatrix} \quad A^{T*} = \begin{bmatrix} 1 & 1+j & 2 \\ 1-j & 3 & -j \\ 2 & j & 0 \end{bmatrix} = A$$

Therefore, matrix A above is Hermitian.

† A square matrix A such that  $A^{T*} = -A$  is called *skew-Hermitian*. For example,

$$A = \begin{bmatrix} j & 1-j & 2 \\ -1-j & 3j & j \\ -2 & j & 0 \end{bmatrix} \quad A^T = \begin{bmatrix} j & -1-j & -2 \\ 1-j & 3j & j \\ 2 & j & 0 \end{bmatrix} \quad A^{T*} = \begin{bmatrix} -j & -1+j & -2 \\ 1+j & -3j & -j \\ 2 & -j & 0 \end{bmatrix} = -A$$

Therefore, matrix A above is skew-Hermitian.

### E.4 Determinants

Let matrix  $A$  be defined as the square matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \quad (\text{E.11})$$

then, the *determinant* of  $A$ , denoted as  $\det A$ , is defined as

$$\det A = a_{11}a_{22}a_{33}\cdots a_{nn} + a_{12}a_{23}a_{34}\cdots a_{n1} + a_{13}a_{24}a_{35}\cdots a_{n2} + \cdots - a_{n1}\cdots a_{22}a_{13}\cdots - a_{n2}\cdots a_{23}a_{14} - a_{n3}\cdots a_{24}a_{15} - \cdots \quad (\text{E.12})$$

The determinant of a square matrix of order  $n$  is referred to as *determinant of order  $n$* .

Let  $A$  be a determinant of order 2, that is,

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad (\text{E.13})$$

Then,

$$\det A = a_{11}a_{22} - a_{21}a_{12} \quad (\text{E.14})$$

---

#### Example E.4

Matrices  $A$  and  $B$  are defined as

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix}$$

Compute  $\det A$  and  $\det B$ .

**Solution:**

$$\det A = 1 \cdot 4 - 3 \cdot 2 = 4 - 6 = -2$$

$$\det B = 2 \cdot 0 - 2 \cdot (-1) = 0 - (-2) = 2$$

Check with MATLAB:

```
A=[1 2; 3 4]; B=[2 -1; 2 0];           % Define matrices A and B
det(A), det(B)                         % Compute the determinants of A and B
```

ans =  
-2

ans =  
2

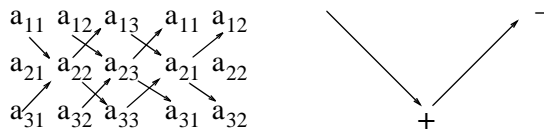
Let A be a matrix of order 3, that is,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad (\text{E.15})$$

then,  $\det A$  is found from

$$\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{11}a_{22}a_{33} - a_{11}a_{22}a_{33} - a_{11}a_{22}a_{33} - a_{11}a_{22}a_{33} \quad (\text{E.16})$$

A convenient method to evaluate the determinant of order 3, is to write the first two columns to the right of the  $3 \times 3$  matrix, and add the products formed by the diagonals from upper left to lower right; then subtract the products formed by the diagonals from lower left to upper right as shown on the diagram of the next page. When this is done properly, we obtain (E.16) above.



This method works only with second and third order determinants. To evaluate higher order determinants, we must first compute the *cofactors*; these will be defined shortly.

### Example E.5

Compute  $\det A$  and  $\det B$  if matrices A and B are defined as

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -3 & -4 \\ 1 & 0 & -2 \\ 0 & -5 & -6 \end{bmatrix}$$



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## Appendix E Matrices and Determinants

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Solution:

$$\det A = \begin{array}{cccccc} & 2 & 3 & 5 & 2 & 3 \\ & \swarrow & \nearrow & \swarrow & \nearrow & \swarrow \\ 1 & 0 & 1 & 1 & 0 & \\ & \swarrow & \nearrow & \swarrow & \nearrow & \swarrow \\ 2 & 1 & 0 & 2 & 1 & \end{array}$$

or

$$\det A = (2 \times 0 \times 0) + (3 \times 1 \times 1) + (5 \times 1 \times 1) - (2 \times 0 \times 5) - (1 \times 1 \times 2) - (0 \times 1 \times 3) = 11 - 2 = 9$$

Likewise,

$$\det B = \begin{array}{cccccc} & 2 & -3 & -4 & 2 & -3 \\ & \swarrow & \nearrow & \swarrow & \nearrow & \swarrow \\ 1 & 0 & -2 & 1 & -2 & \\ & \swarrow & \nearrow & \swarrow & \nearrow & \swarrow \\ 0 & -5 & -6 & 2 & -6 & \end{array}$$

or

$$\det B = [2 \times 0 \times (-6)] + [(-3) \times (-2) \times 0] + [(-4) \times 1 \times (-5)] - [0 \times 0 \times (-4)] - [(-5) \times (-2) \times 2] - [(-6) \times 1 \times (-3)] = 20 - 38 = -18$$

Check with MATLAB:

```
A=[2 3 5; 1 0 1; 2 1 0]; det(A) % Define matrix A and compute detA
```

```
ans =  
9
```

```
B=[2 -3 -4; 1 0 -2; 0 -5 -6]; det(B) % Define matrix B and compute detB
```

```
ans =  
-18
```

---

### E.5 Minors and Cofactors

Let matrix A be defined as the square matrix of order n as shown below.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \quad (\text{E.17})$$

If we remove the elements of its  $i$ th row, and  $j$ th column, the remaining  $n - 1$  square matrix is called the *minor of A*, and it is denoted as  $[M_{ij}]$ .

The signed minor  $(-1)^{i+j}[M_{ij}]$  is called the *cofactor* of  $a_{ij}$  and it is denoted as  $\alpha_{ij}$ .

---

**Example E.6**

Matrix A is defined as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad (\text{E.18})$$

Compute the minors  $[M_{11}]$ ,  $[M_{12}]$ ,  $[M_{13}]$  and the cofactors  $\alpha_{11}$ ,  $\alpha_{12}$  and  $\alpha_{13}$ .

**Solution:**

$$[M_{11}] = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} \quad [M_{12}] = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} \quad [M_{13}] = \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

and

$$\alpha_{11} = (-1)^{1+1}[M_{11}] = [M_{11}] \quad \alpha_{12} = (-1)^{1+2}[M_{12}] = -[M_{12}] \quad \alpha_{13} = [M_{13}] = (-1)^{1+3}[M_{13}]$$

The remaining minors

$$[M_{21}], [M_{22}], [M_{23}], [M_{31}], [M_{32}], [M_{33}]$$

and cofactors

$$\alpha_{21}, \alpha_{22}, \alpha_{23}, \alpha_{31}, \alpha_{32}, \text{ and } \alpha_{33}$$

are defined similarly.

---

**Example E.7**

Compute the cofactors of matrix A defined as

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & -4 & 2 \\ -1 & 2 & -6 \end{bmatrix} \quad (\text{E.19})$$

**Solution:**

$$\alpha_{11} = (-1)^{1+1} \begin{bmatrix} -4 & 2 \\ 2 & -6 \end{bmatrix} = 20 \quad \alpha_{12} = (-1)^{1+2} \begin{bmatrix} 2 & 2 \\ -1 & -6 \end{bmatrix} = 10 \quad (\text{E.20})$$

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$$\alpha_{13} = (-1)^{1+3} \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix} = 0 \quad \alpha_{21} = (-1)^{2+1} \begin{bmatrix} 2 & -3 \\ 2 & -6 \end{bmatrix} = 6 \quad (\text{E.21})$$

$$\alpha_{22} = (-1)^{2+2} \begin{bmatrix} 1 & -3 \\ -1 & -6 \end{bmatrix} = -9 \quad \alpha_{23} = (-1)^{2+3} \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} = -4 \quad (\text{E.22})$$

$$\alpha_{31} = (-1)^{3+1} \begin{bmatrix} 2 & -3 \\ -4 & 2 \end{bmatrix} = -8, \quad \alpha_{32} = (-1)^{3+2} \begin{bmatrix} 1 & -3 \\ 2 & 2 \end{bmatrix} = -8 \quad (\text{E.23})$$

$$\alpha_{33} = (-1)^{3+3} \begin{bmatrix} 1 & 2 \\ 2 & -4 \end{bmatrix} = -8 \quad (\text{E.24})$$

---

It is useful to remember that the signs of the cofactors follow the pattern below

$$\begin{array}{cccc} + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \end{array}$$

that is, the cofactors on the diagonals have the same sign as their minors.

Let  $A$  be a square matrix of any size; the value of the determinant of  $A$  is the sum of the products obtained by multiplying each element of *any* row or *any* column by its cofactor.

---

### Example E.8

Matrix  $A$  is defined as

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & -4 & 2 \\ -1 & 2 & -6 \end{bmatrix} \quad (\text{E.25})$$

Compute the determinant of  $A$  using the elements of the first row.

**Solution:**

$$\det A = 1 \begin{bmatrix} -4 & 2 \\ 2 & -6 \end{bmatrix} - 2 \begin{bmatrix} 2 & 2 \\ -1 & -6 \end{bmatrix} - 3 \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix} = 1 \times 20 - 2 \times (-10) - 3 \times 0 = 40$$

Check with MATLAB:

```
A=[1 2 -3; 2 -4 2; -1 2 -6]; det(A) % Define matrix A and compute detA
ans =
    40
```

We must use the above procedure to find the determinant of a matrix A of order 4 or higher. Thus, a fourth-order determinant can first be expressed as the sum of the products of the elements of its first row by its cofactor as shown below.

$$\begin{aligned}
 A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} &= a_{11} \begin{bmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{bmatrix} - a_{12} \begin{bmatrix} a_{12} & a_{13} & a_{14} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{bmatrix} \\
 &+ a_{31} \begin{bmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{42} & a_{43} & a_{44} \end{bmatrix} - a_{41} \begin{bmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \end{bmatrix}
 \end{aligned} \tag{E.26}$$

Determinants of order five or higher can be evaluated similarly.

**Example E.9**

Compute the value of the determinant of the matrix A defined as

$$A = \begin{bmatrix} 2 & -1 & 0 & -3 \\ -1 & 1 & 0 & -1 \\ 4 & 0 & 3 & -2 \\ -3 & 0 & 0 & 1 \end{bmatrix} \tag{E.27}$$

**Solution:**

Using the above procedure, we will multiply each element of the first column by its cofactor. Then,

$$\begin{aligned}
 A = & 2 \underbrace{\begin{bmatrix} 1 & 0 & -1 \\ 0 & 3 & -2 \\ 0 & 0 & 1 \end{bmatrix}}_{[a]} - (-1) \underbrace{\begin{bmatrix} -1 & 0 & -3 \\ 0 & 3 & -2 \\ 0 & 0 & 1 \end{bmatrix}}_{[b]} + 4 \underbrace{\begin{bmatrix} -1 & 0 & -3 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}}_{[c]} - (-3) \underbrace{\begin{bmatrix} -1 & 0 & -3 \\ 1 & 0 & -1 \\ 0 & 3 & -2 \end{bmatrix}}_{[d]}
 \end{aligned}$$

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## Appendix E Matrices and Determinants

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Next, using the procedure of Example E.5 or Example E.8, we find

$$[a] = 6, [b] = -3, [c] = 0, [d] = -36$$

and thus

$$\det A = [a] + [b] + [c] + [d] = 6 - 3 + 0 - 36 = -33$$

We can verify our answer with MATLAB as follows:

```
A=[2 -1 0 -3; -1 1 0 -1; 4 0 3 -2; -3 0 0 1]; delta = det(A)
```

```
delta =  
-33
```

---

Some useful properties of determinants are given below.

**Property 1:** *If all elements of one row or one column are zero, the determinant is zero. An example of this is the determinant of the cofactor [c] above.*

**Property 2:** *If all the elements of one row or column are m times the corresponding elements of another row or column, the determinant is zero. For example, if*

$$A = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 6 & 1 \\ 1 & 2 & 1 \end{bmatrix} \quad (\text{E.28})$$

then,

$$\det A = \begin{vmatrix} 2 & 4 & 1 \\ 3 & 6 & 1 \\ 1 & 2 & 1 \end{vmatrix} \begin{vmatrix} 2 & 4 \\ 3 & 6 \\ 1 & 2 \end{vmatrix} = 12 + 4 + 6 - 6 - 4 - 12 = 0 \quad (\text{E.29})$$

Here,  $\det A$  is zero because the second column in  $A$  is 2 times the first column.

Check with MATLAB:

```
A=[2 4 1; 3 6 1; 1 2 1]; det(A)
```

```
ans =  
0
```

**Property 3:** *If two rows or two columns of a matrix are identical, the determinant is zero. This follows from Property 2 with  $m = 1$ .*

### E.6 Cramer's Rule

Let us consider the systems of the three equations below:

$$\begin{aligned}
 a_{11}x + a_{12}y + a_{13}z &= A \\
 a_{21}x + a_{22}y + a_{23}z &= B \\
 a_{31}x + a_{32}y + a_{33}z &= C
 \end{aligned}
 \tag{E.30}$$

and let

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad
 D_1 = \begin{vmatrix} A & a_{12} & a_{13} \\ B & a_{22} & a_{23} \\ C & a_{32} & a_{33} \end{vmatrix} \quad
 D_2 = \begin{vmatrix} a_{11} & A & a_{13} \\ a_{21} & B & a_{23} \\ a_{31} & C & a_{33} \end{vmatrix} \quad
 D_3 = \begin{vmatrix} a_{11} & a_{12} & A \\ a_{21} & a_{22} & B \\ a_{31} & a_{32} & C \end{vmatrix}$$

Cramer's rule states that the unknowns  $x$ ,  $y$ , and  $z$  can be found from the relations

$$x = \frac{D_1}{\Delta} \quad y = \frac{D_2}{\Delta} \quad z = \frac{D_3}{\Delta}
 \tag{E.31}$$

provided that the determinant  $\Delta$  (delta) is not zero.

We observe that the numerators of (E.31) are determinants that are formed from  $\Delta$  by the substitution of the known values  $A$ ,  $B$ , and  $C$ , for the coefficients of the desired unknown.

Cramer's rule applies to systems of two or more equations.

If (E.30) is a homogeneous set of equations, that is, if  $A = B = C = 0$ , then,  $D_1$ ,  $D_2$ , and  $D_3$  are all zero as we found in Property 1 above. Then,  $x = y = z = 0$  also.

### Example E.10

Use Cramer's rule to find  $v_1$ ,  $v_2$ , and  $v_3$  if

$$\begin{aligned}
 2v_1 - 5 - v_2 + 3v_3 &= 0 \\
 -2v_3 - 3v_2 - 4v_1 &= 8 \\
 v_2 + 3v_1 - 4 - v_3 &= 0
 \end{aligned}
 \tag{E.32}$$

and verify your answers with MATLAB.

**Solution:**

Rearranging the unknowns  $v$ , and transferring known values to the right side, we obtain

$$\begin{aligned}
 2v_1 - v_2 + 3v_3 &= 5 \\
 -4v_1 - 3v_2 - 2v_3 &= 8 \\
 3v_1 + v_2 - v_3 &= 4
 \end{aligned}
 \tag{E.33}$$

By Cramer's rule,

$$\Delta = \begin{vmatrix} 2 & -1 & 3 \\ -4 & -3 & -2 \\ 3 & 1 & -1 \end{vmatrix} \begin{vmatrix} 2 & -1 \\ -4 & -3 \\ 3 & 1 \end{vmatrix} = 6 + 6 - 12 + 27 + 4 + 4 = 35$$

$$D_1 = \begin{vmatrix} 5 & -1 & 3 \\ 8 & -3 & -2 \\ 4 & 1 & -1 \end{vmatrix} \begin{vmatrix} 5 & -1 \\ 8 & -3 \\ 4 & 1 \end{vmatrix} = 15 + 8 + 24 + 36 + 10 - 8 = 85$$

$$D_2 = \begin{vmatrix} 2 & 5 & 3 \\ -4 & 8 & -2 \\ 3 & 4 & -1 \end{vmatrix} \begin{vmatrix} 2 & 5 \\ -4 & 8 \\ 3 & 4 \end{vmatrix} = -16 - 30 - 48 - 72 + 16 - 20 = -170$$

$$D_3 = \begin{vmatrix} 2 & -1 & 5 \\ -4 & -3 & 8 \\ 3 & 1 & 4 \end{vmatrix} \begin{vmatrix} 2 & -1 \\ -4 & -3 \\ 3 & 1 \end{vmatrix} = -24 - 24 - 20 + 45 - 16 - 16 = -55$$

Using relation (E.31) we obtain

$$x_1 = \frac{D_1}{\Delta} = \frac{85}{35} = \frac{17}{7} \quad x_2 = \frac{D_2}{\Delta} = \frac{-170}{35} = \frac{-34}{7} \quad x_3 = \frac{D_3}{\Delta} = \frac{-55}{35} = \frac{-11}{7} \quad (\text{E.34})$$

We will verify with MATLAB as follows:

% The following script will compute and display the values of  $v_1$ ,  $v_2$  and  $v_3$ .

```
format rat % Express answers in ratio form
B=[2 -1 3; -4 -3 -2; 3 1 -1]; % The elements of the determinant D of matrix B
delta=det(B); % Compute the determinant D of matrix B
d1=[5 -1 3; 8 -3 -2; 4 1 -1]; % The elements of D1
detd1=det(d1); % Compute the determinant of D1
d2=[2 5 3; -4 8 -2; 3 4 -1]; % The elements of D2
detd2=det(d2); % Compute the determinant of D2
d3=[2 -1 5; -4 -3 8; 3 1 4]; % The elements of D3
detd3=det(d3); % Compute the determinant of D3
v1=detd1/delta; % Compute the value of v1
v2=detd2/delta; % Compute the value of v2
v3=detd3/delta; % Compute the value of v3
%
disp('v1=');disp(v1); % Display the value of v1
disp('v2=');disp(v2); % Display the value of v2
disp('v3=');disp(v3); % Display the value of v3
```

$$\begin{aligned}v_1 &= 17/7 \\v_2 &= -34/7 \\v_3 &= -11/7\end{aligned}$$

These are the same values as in (E.34)

## E.7 Gaussian Elimination Method

We can find the unknowns in a system of two or more equations also by the *Gaussian elimination method*. With this method, the objective is to eliminate one unknown at a time. This can be done by multiplying the terms of any of the equations of the system by a number such that we can add (or subtract) this equation to another equation in the system so that one of the unknowns will be eliminated. Then, by substitution to another equation with two unknowns, we can find the second unknown. Subsequently, substitution of the two values found can be made into an equation with three unknowns from which we can find the value of the third unknown. This procedure is repeated until all unknowns are found. This method is best illustrated with the following example which consists of the same equations as the previous example.

### Example E.11

Use the Gaussian elimination method to find  $v_1$ ,  $v_2$ , and  $v_3$  of the system of equations

$$\begin{aligned}2v_1 - v_2 + 3v_3 &= 5 \\-4v_1 - 3v_2 - 2v_3 &= 8 \\3v_1 + v_2 - v_3 &= 4\end{aligned}\tag{E.35}$$

#### Solution:

As a first step, we add the first equation of (E.35) with the third to eliminate the unknown  $v_2$  and we obtain the equation

$$5v_1 + 2v_3 = 9\tag{E.36}$$

Next, we multiply the third equation of (E.35) by 3, and we add it with the second to eliminate  $v_2$ , and we obtain the equation

$$5v_1 - 5v_3 = 20\tag{E.37}$$

Subtraction of (E.37) from (E.36) yields



$$7v_3 = -11 \text{ or } v_3 = -\frac{11}{7} \quad (\text{E.38})$$

Now, we can find the unknown  $v_1$  from either (E.36) or (E.37). By substitution of (D.38) into (E.36) we obtain

$$5v_1 + 2 \cdot \left(-\frac{11}{7}\right) = 9 \text{ or } v_1 = \frac{17}{7} \quad (\text{E.39})$$

Finally, we can find the last unknown  $v_2$  from any of the three equations of (E.35). By substitution into the first equation we obtain

$$v_2 = 2v_1 + 3v_3 - 5 = \frac{34}{7} - \frac{33}{7} - \frac{35}{7} = -\frac{34}{7} \quad (\text{E.40})$$

These are the same values as those we found in Example E.10.

---

The Gaussian elimination method works well if the coefficients of the unknowns are small integers, as in Example E.11. However, it becomes impractical if the coefficients are large or fractional numbers.

### E.8 The Adjoint of a Matrix

Let us assume that  $A$  is an  $n$  square matrix and  $\alpha_{ij}$  is the cofactor of  $a_{ij}$ . Then *the adjoint of  $A$* , denoted as  $\text{adj}A$ , is defined as the  $n$  square matrix below.

$$\text{adj}A = \begin{bmatrix} \alpha_{11} & \alpha_{21} & \alpha_{31} & \cdots & \alpha_{n1} \\ \alpha_{12} & \alpha_{22} & \alpha_{32} & \cdots & \alpha_{n2} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} & \cdots & \alpha_{n3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{1n} & \alpha_{2n} & \alpha_{3n} & \cdots & \alpha_{nn} \end{bmatrix} \quad (\text{E.41})$$

We observe that the cofactors of the elements of the  $i$ th row (column) of  $A$  are the elements of the  $i$ th column (row) of  $\text{adj}A$ .

---

#### Example E.12

Compute  $\text{adj}A$  if Matrix  $A$  is defined as

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{bmatrix} \quad (\text{E.42})$$

**Solution:**

$$\text{adj}A = \begin{bmatrix} \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix} & -\begin{bmatrix} 2 & 3 \\ 4 & 3 \end{bmatrix} & \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} \\ -\begin{bmatrix} 1 & 4 \\ 1 & 3 \end{bmatrix} & \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix} & -\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} \\ \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} & -\begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix} & \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} -7 & 6 & -1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix}$$

## E.9 Singular and Non-Singular Matrices

An  $n$  square matrix  $A$  is called *singular* if  $\det A = 0$ ; if  $\det A \neq 0$ ,  $A$  is called *non-singular*.

### Example E.13

Matrix  $A$  is defined as

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix} \quad (\text{E.43})$$

Determine whether this matrix is singular or non-singular.

**Solution:**

$$\det A = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{vmatrix} = 1 \cdot \begin{vmatrix} 3 & 4 \\ 5 & 7 \end{vmatrix} - 2 \cdot \begin{vmatrix} 2 & 4 \\ 3 & 7 \end{vmatrix} + 3 \cdot \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} = 21 + 24 + 30 - 27 - 20 - 28 = 0$$

Therefore, matrix  $A$  is singular.

### E.10 The Inverse of a Matrix

If  $A$  and  $B$  are  $n$  square matrices such that  $AB = BA = I$ , where  $I$  is the identity matrix,  $B$  is called the *inverse* of  $A$ , denoted as  $B = A^{-1}$ , and likewise,  $A$  is called the inverse of  $B$ , that is,  $A = B^{-1}$

If a matrix  $A$  is non-singular, we can compute its inverse  $A^{-1}$  from the relation

$$\boxed{A^{-1} = \frac{1}{\det A} \text{adj}A} \quad (\text{E.44})$$

---

#### Example E.14

Matrix  $A$  is defined as

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{bmatrix} \quad (\text{E.45})$$

Compute its inverse, that is, find  $A^{-1}$

**Solution:**

Here,  $\det A = 9 + 8 + 12 - 9 - 16 - 6 = -2$ , and since this is a non-zero value, it is possible to compute the inverse of  $A$  using (E.44).

From Example E.12,

$$\text{adj}A = \begin{bmatrix} -7 & 6 & -1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix}$$

Then,

$$A^{-1} = \frac{1}{\det A} \text{adj}A = \frac{1}{-2} \begin{bmatrix} -7 & 6 & -1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 3.5 & -3 & 0.5 \\ -0.5 & 0 & 0.5 \\ -0.5 & 1 & -0.5 \end{bmatrix} \quad (\text{E.46})$$

Check with MATLAB:

```
A=[1 2 3; 1 3 4; 1 4 3], invA=inv(A) % Define matrix A and compute its inverse
```

```
A =  
    1    2    3  
    1    3    4  
    1    4    3
```

$$\begin{aligned} \text{invA} = \\ & \begin{bmatrix} 3.5000 & -3.0000 & 0.5000 \\ -0.5000 & 0 & 0.5000 \\ -0.5000 & 1.0000 & -0.5000 \end{bmatrix} \end{aligned}$$

---

Multiplication of a matrix  $A$  by its inverse  $A^{-1}$  produces the identity matrix  $I$ , that is,

$$AA^{-1} = I \quad \text{or} \quad A^{-1}A = I \tag{E.47}$$

---

### Example E.15

Prove the validity of (E.47) for the Matrix  $A$  defined as

$$A = \begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix}$$

**Proof:**

$$\det A = 8 - 6 = 2 \quad \text{and} \quad \text{adj}A = \begin{bmatrix} 2 & -3 \\ -2 & 4 \end{bmatrix}$$

Then,

$$A^{-1} = \frac{1}{\det A} \text{adj}A = \frac{1}{2} \begin{bmatrix} 2 & -3 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -3/2 \\ -1 & 2 \end{bmatrix}$$

and

$$AA^{-1} = \begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & -3/2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 4-3 & -6+6 \\ 2-2 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

---

## E.11 Solution of Simultaneous Equations with Matrices

Consider the relation

$$AX = B \tag{E.48}$$

where  $A$  and  $B$  are matrices whose elements are known, and  $X$  is a matrix (a column vector) whose elements are the unknowns. We assume that  $A$  and  $X$  are conformable for multiplication.

Multiplication of both sides of (E.48) by  $A^{-1}$  yields:

$$A^{-1}AX = A^{-1}B = IX = A^{-1}B \tag{E.49}$$

or

$$\boxed{X=A^{-1}B} \quad (E.50)$$

Therefore, we can use (E.50) to solve any set of simultaneous equations that have solutions. We will refer to this method as the *inverse matrix method of solution* of simultaneous equations.

---

### Example E.16

For the system of the equations

$$\begin{cases} 2x_1 + 3x_2 + x_3 = 9 \\ x_1 + 2x_2 + 3x_3 = 6 \\ 3x_1 + x_2 + 2x_3 = 8 \end{cases} \quad (E.51)$$

compute the unknowns  $x_1$ ,  $x_2$ , and  $x_3$  using the inverse matrix method.

#### Solution:

In matrix form, the given set of equations is  $AX = B$  where

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad B = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix} \quad (E.52)$$

Then,

$$X = A^{-1}B \quad (E.53)$$

or

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix} \quad (E.54)$$

Next, we find the determinant  $\det A$ , and the adjoint  $\text{adj}A$ .

$$\det A = 18 \quad \text{and} \quad \text{adj}A = \begin{bmatrix} 1 & -5 & 7 \\ 7 & 1 & -5 \\ -5 & 7 & 1 \end{bmatrix}$$

Therefore,

$$A^{-1} = \frac{1}{\det A} \text{adj}A = \frac{1}{18} \begin{bmatrix} 1 & -5 & 7 \\ 7 & 1 & -5 \\ -5 & 7 & 1 \end{bmatrix}$$

and with relation (E.53) we obtain the solution as follows:

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{18} \begin{bmatrix} 1 & -5 & 7 \\ 7 & 1 & -5 \\ -5 & 7 & 1 \end{bmatrix} \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix} = \frac{1}{18} \begin{bmatrix} 35 \\ 29 \\ 5 \end{bmatrix} = \begin{bmatrix} 35/18 \\ 29/18 \\ 5/18 \end{bmatrix} = \begin{bmatrix} 1.94 \\ 1.61 \\ 0.28 \end{bmatrix} \quad (\text{E.55})$$

To verify our results, we could use the MATLAB's **inv(A)** function, and then multiply  $A^{-1}$  by **B**. However, it is easier to use the *matrix left division* operation  $X = A \setminus B$ ; this is MATLAB's solution of  $A^{-1}B$  for the matrix equation  $A \cdot X = B$ , where matrix **X** is the same size as matrix **B**. For this example,

$A=[2 \ 3 \ 1; 1 \ 2 \ 3; 3 \ 1 \ 2]; B=[9 \ 6 \ 8]'; X=A \setminus B$

```
X =
    1.9444
    1.6111
    0.2778
```

### Example E.17

For the electric circuit of Figure E.1,

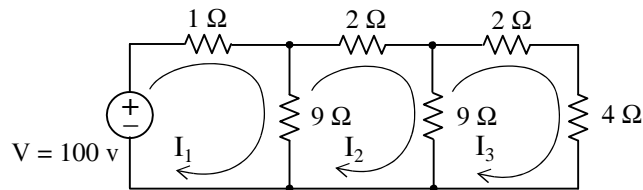


Figure E.1. Electric circuit for Example E.17

the loop equations are

$$\begin{aligned} 10I_1 - 9I_2 &= 100 \\ -9I_1 + 20I_2 - 9I_3 &= 0 \\ -9I_2 + 15I_3 &= 0 \end{aligned} \quad (\text{E.56})$$

---

## Appendix E Matrices and Determinants

---

Use the inverse matrix method to compute the values of the currents  $I_1$ ,  $I_2$ , and  $I_3$

### Solution:

For this example, the matrix equation is  $\mathbf{RI} = \mathbf{V}$  or  $\mathbf{I} = \mathbf{R}^{-1}\mathbf{V}$ , where

$$\mathbf{R} = \begin{bmatrix} 10 & -9 & 0 \\ -9 & 20 & -9 \\ 0 & -9 & 15 \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{I} = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}$$

The next step is to find  $\mathbf{R}^{-1}$ . It is found from the relation

$$\mathbf{R}^{-1} = \frac{1}{\det \mathbf{R}} \text{adj} \mathbf{R} \quad (\text{E.57})$$

Therefore, we must find the determinant and the adjoint of  $\mathbf{R}$ . For this example, we find that

$$\det \mathbf{R} = 975, \quad \text{adj} \mathbf{R} = \begin{bmatrix} 219 & 135 & 81 \\ 135 & 150 & 90 \\ 81 & 90 & 119 \end{bmatrix} \quad (\text{E.58})$$

Then,

$$\mathbf{R}^{-1} = \frac{1}{\det \mathbf{R}} \text{adj} \mathbf{R} = \frac{1}{975} \begin{bmatrix} 219 & 135 & 81 \\ 135 & 150 & 90 \\ 81 & 90 & 119 \end{bmatrix}$$

and

$$\mathbf{I} = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \frac{1}{975} \begin{bmatrix} 219 & 135 & 81 \\ 135 & 150 & 90 \\ 81 & 90 & 119 \end{bmatrix} \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix} = \frac{100}{975} \begin{bmatrix} 219 \\ 135 \\ 81 \end{bmatrix} = \begin{bmatrix} 22.46 \\ 13.85 \\ 8.31 \end{bmatrix}$$

Check with MATLAB:

```
R=[10 -9 0; -9 20 -9; 0 -9 15]; V=[100 0 0]'; I=R\V; fprintf(' \n');...  
fprintf('I1 = %4.2f \t', I(1)); fprintf('I2 = %4.2f \t', I(2)); fprintf('I3 = %4.2f \t', I(3)); fprintf(' \n')
```

```
I1 = 22.46    I2 = 13.85    I3 = 8.31
```

We can also use subscripts to address the individual elements of the matrix. Accordingly, the MATLAB script above could also have been written as:

```
R(1,1)=10; R(1,2)=-9;    % No need to make entry for A(1,3) since it is zero.  
R(2,1)=-9; R(2,2)=20; R(2,3)=-9; R(3,2)=-9; R(3,3)=15; V=[100 0 0]'; I=R\V; fprintf(' \n');...  
fprintf('I1 = %4.2f \t', I(1)); fprintf('I2 = %4.2f \t', I(2)); fprintf('I3 = %4.2f \t', I(3)); fprintf(' \n')
```

$$I_1 = 22.46 \quad I_2 = 13.85 \quad I_3 = 8.31$$

Spreadsheets also have the capability of solving simultaneous equations with real coefficients using the inverse matrix method. For instance, we can use Microsoft Excel's **MINVERSE** (Matrix Inversion) and **MMULT** (Matrix Multiplication) functions, to obtain the values of the three currents in Example E.17.

The procedure is as follows:

1. We begin with a blank spreadsheet and in a block of cells, say B3:D5, we enter the elements of matrix  $R$  as shown in Figure D.2. Then, we enter the elements of matrix  $V$  in G3:G5.
2. Next, we compute and display the inverse of  $R$ , that is,  $R^{-1}$ . We choose B7:D9 for the elements of this inverted matrix. We format this block for number display with three decimal places. With this range highlighted and making sure that the cell marker is in B7, we type the formula

`=MINVERSE(B3:D5)`

and we press the *Ctrl-Shift-Enter* keys simultaneously. We observe that  $R^{-1}$  appears in these cells.

3. Now, we choose the block of cells G7:G9 for the values of the current  $I$ . As before, we highlight them, and with the cell marker positioned in G7, we type the formula

`=MMULT(B7:D9,G3:G5)`

and we press the *Ctrl-Shift-Enter* keys simultaneously. The values of  $I$  then appear in G7:G9.

	A	B	C	D	E	F	G	H
1	Spreadsheet for Matrix Inversion and Matrix Multiplication							
2								
3		10	-9	0			100	
4	R=	-9	20	-9		V=	0	
5		0	-9	15			0	
6								
7		0.225	0.138	0.083			22.462	
8	$R^{-1}$ =	0.138	0.154	0.092		I=	13.846	
9		0.083	0.092	0.122			8.3077	
10								

Figure E.2. Solution of Example E.17 with a spreadsheet

### Example E.18

For the phasor circuit of Figure E.18



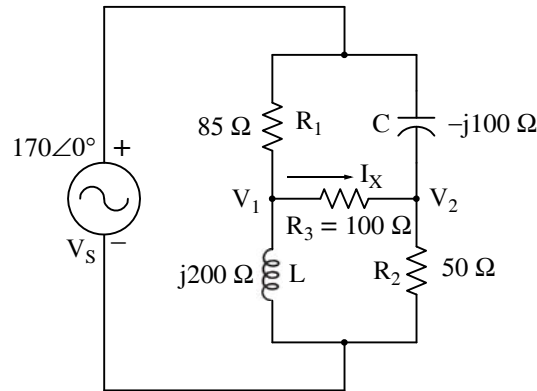


Figure E.3. Circuit for Example E.18

the current  $I_X$  can be found from the relation

$$I_X = \frac{V_1 - V_2}{R_3} \quad (\text{E.59})$$

and the voltages  $V_1$  and  $V_2$  can be computed from the nodal equations

$$\frac{V_1 - 170\angle 0^\circ}{85} + \frac{V_1 - V_2}{100} + \frac{V_1 - 0}{j200} = 0 \quad (\text{E.60})$$

and

$$\frac{V_2 - 170\angle 0^\circ}{-j100} + \frac{V_2 - V_1}{100} + \frac{V_2 - 0}{50} = 0 \quad (\text{E.61})$$

Compute, and express the current  $I_X$  in both rectangular and polar forms by first simplifying like terms, collecting, and then writing the above relations in matrix form as  $YV = I$ , where  $Y = \text{Admittance}$ ,  $V = \text{Voltage}$ , and  $I = \text{Current}$

### Solution:

The  $Y$  matrix elements are the coefficients of  $V_1$  and  $V_2$ . Simplifying and rearranging the nodal equations of (E.60) and (E.61), we obtain

$$\begin{aligned} (0.0218 - j0.005)V_1 - 0.01V_2 &= 2 \\ -0.01V_1 + (0.03 + j0.01)V_2 &= j1.7 \end{aligned} \quad (\text{E.62})$$

Next, we write (E.62) in matrix form as

$$\underbrace{\begin{bmatrix} 0.0218 - j0.005 & -0.01 \\ -0.01 & 0.03 + j0.01 \end{bmatrix}}_Y \underbrace{\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}}_V = \underbrace{\begin{bmatrix} 2 \\ j1.7 \end{bmatrix}}_I \quad (\text{E.63})$$

where the matrices  $Y$ ,  $V$ , and  $I$  are as indicated.

We will use MATLAB to compute the voltages  $V_1$  and  $V_2$ , and to do all other computations. The script is shown below.

```
Y=[0.0218-0.005j -0.01; -0.01 0.03+0.01j]; I=[2; 1.7j]; V=Y\I; % Define Y, I, and find V
fprintf('\n'); % Insert a line
disp('V1 = '); disp(V(1)); disp('V2 = '); disp(V(2)); % Display values of V1 and V2
```

```
V1 =
 1.0490e+002 + 4.9448e+001i
V2 =
 53.4162 + 55.3439i
```

Next, we find  $I_X$  from

```
R3=100; IX=(V(1)-V(2))/R3 % Compute the value of IX
```

```
IX =
 0.5149 - 0.0590i
```

This is the rectangular form of  $I_X$ . For the polar form we use the MATLAB script

```
magIX=abs(IX), thetaIX=angle(IX)*180/pi % Compute the magnitude and the angle in
degrees
```

```
magIX =
 0.5183
thetaIX =
 -6.5326
```

Therefore, in polar form,

$$I_X = 0.518 \angle -6.53^\circ$$

---

Spreadsheets have limited capabilities with complex numbers, and thus we cannot use them to compute matrices that include complex numbers in their elements as in Example E.18.

### E.12 Exercises

For Exercises 1, 2, and 3 below, the matrices A, B, C, and D are defined as:

$$A = \begin{bmatrix} 1 & -1 & -4 \\ 5 & 7 & -2 \\ 3 & -5 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 5 & 9 & -3 \\ -2 & 8 & 2 \\ 7 & -4 & 6 \end{bmatrix} \quad C = \begin{bmatrix} 4 & 6 \\ -3 & 8 \\ 5 & -2 \end{bmatrix} \quad D = \begin{bmatrix} 1 & -2 & 3 \\ -3 & 6 & -4 \end{bmatrix}$$

1. Perform the following computations, if possible. Verify your answers with MATLAB.

a.  $A + B$    b.  $A + C$    c.  $B + D$    d.  $C + D$

e.  $A - B$    f.  $A - C$    g.  $B - D$    h.  $C - D$

2. Perform the following computations, if possible. Verify your answers with MATLAB.

a.  $A \cdot B$    b.  $A \cdot C$    c.  $B \cdot D$    d.  $C \cdot D$

e.  $B \cdot A$    f.  $C \cdot A$    g.  $D \cdot A$    h.  $D \cdot C$

3. Perform the following computations, if possible. Verify your answers with MATLAB.

a.  $\det A$    b.  $\det B$    c.  $\det C$    d.  $\det D$    e.  $\det(A \cdot B)$    f.  $\det(A \cdot C)$

4. Solve the following systems of equations using Cramer's rule. Verify your answers with MATLAB.

$$\begin{array}{ll} x_1 - 2x_2 + x_3 = -4 & -x_1 + 2x_2 - 3x_3 + 5x_4 = 14 \\ a. \quad -2x_1 + 3x_2 + x_3 = 9 & b. \quad x_1 + 3x_2 + 2x_3 - x_4 = 9 \\ \quad 3x_1 + 4x_2 - 5x_3 = 0 & \quad 3x_1 - 3x_2 + 2x_3 + 4x_4 = 19 \\ & \quad 4x_1 + 2x_2 + 5x_3 + x_4 = 27 \end{array}$$

5. Repeat Exercise 4 using the Gaussian elimination method.

6. Solve the following systems of equations using the inverse matrix method. Verify your answers with MATLAB.

$$a. \begin{bmatrix} 1 & 3 & 4 \\ 3 & 1 & -2 \\ 2 & 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ 0 \end{bmatrix} \quad b. \begin{bmatrix} 2 & 4 & 3 & -2 \\ 2 & -4 & 1 & 3 \\ -1 & 3 & -4 & 2 \\ 2 & -2 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -14 \\ 7 \end{bmatrix}$$

---

## References and Suggestions for Further Study

A. The following publications by The MathWorks, are highly recommended for further study. They are available from The MathWorks, 3 Apple Hill Drive, Natick, MA, 01760, [www.mathworks.com](http://www.mathworks.com).

1. *Getting Started with MATLAB*<sup>®</sup>
2. *Using MATLAB*<sup>®</sup>
3. *Using MATLAB*<sup>®</sup> *Graphics*
4. *Using Simulink*<sup>®</sup>
5. *SimPowerSystems*<sup>®</sup> *for Use with Simulink*<sup>®</sup>
6. *Fixed-Point Toolbox*
7. *Simulink*<sup>®</sup> *Fixed-Point*
8. *Real-Time Workshop*
9. *Signal Processing Toolbox*
10. *Getting Started with Signal Processing Blockset*
10. *Signal Processing Blockset*
11. *Control System Toolbox*
12. *Stateflow*<sup>®</sup>

B. Other references indicated in text pages and footnotes throughout this text, are listed below.

1. *Mathematics for Business, Science, and Technology*, ISBN 978-1-934404-01-0
2. *Numerical Analysis Using MATLAB*<sup>®</sup> *and Excel*<sup>®</sup>, ISBN 978-1-934404-03-4
3. *Circuit Analysis II with MATLAB*<sup>®</sup> *Applications*, ISBN 0-9709511-5-9
4. *Signals and Systems with MATLAB*<sup>®</sup> *Computing and Simulink*<sup>®</sup> *Modeling*, ISBN 978-1-934404-11-9
5. *Electronic Devices and Amplifier Circuits with MATLAB*<sup>®</sup> *Applications*, ISBN 978-1-934404-13-3
6. *Digital Circuit Analysis and Design with Simulink Modeling and Introduction to CPLDs and FPGAs*, ISBN 978-1-934404-05-8

- 
7. *Introduction to Simulink<sup>®</sup> with Engineering Applications*, ISBN 978-1-934404-09-6
  8. *Introduction to Stateflow<sup>®</sup> with Applications*, ISBN 978-1-934404-07-2
  9. *Reference Data for Radio Engineers*, ISBN 0-672-21218-8, Howard W. Sams & Co.
  10. *Electronic Engineers' Handbook*, ISBN 0-07-020981-2, McGraw-Hill

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